Liquidity and Asset Prices: A Unified Framework

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Abstract

We examine how liquidity and asset prices are affected by the following market imperfections: asymmetric information, participation costs, transaction costs, leverage constraints, non-competitive behavior and search. Our model has three periods: agents are identical in the first, become heterogeneous and trade in the second, and consume asset payoffs in the third. We examine how imperfections in the second period affect different measures of illiquidity, as well as asset prices in the first period. Besides nesting multiple imperfections in a single model, we derive new results on the effects of each imperfection. Our results imply, in particular, that imperfections do not always raise expected returns, and can influence common measures of illiquidity in opposite directions.

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1 Introduction

Financial markets deviate, to varying degrees, from the perfect-market ideal in which there are no impediments to trade. A large and growing body of work has identified a variety of market imperfections, ranging from information asymmetries, to different forms of trading costs, to financial constraints. Most papers focus on a specific imperfection, relying on simplifications that are convenient in the context of that imperfection but vary substantially across imperfections. For example, models of trading costs typically assume life-cycle or risk-sharing motives to trade, while models of asymmetric information often rely on noise traders. Some asymmetric-information models further assume risk-neutral market makers who can take unlimited positions, while papers on other imperfections typically assume risk aversion or position limits. Missing from the literature is a systematic analysis of different imperfections within a single, unified framework. Beyond the obvious pedagogical advantages, such a framework could yield a better and more comprehensive understanding of how imperfections affect market behavior. Indeed, effects could be compared across imperfections, holding constant other assumptions such as trading motives and risk attitudes.

An additional limitation of the literature on market imperfections concerns the link with asset pricing. While the effects of imperfections on market liquidity have received much attention, the analysis of how imperfections affect expected asset returns has been more incomplete. This is partly because simplifications that are convenient for studying liquidity are not always suitable for pricing analysis. For example, in models with risk-neutral market-makers, expected returns are equal to the riskless rate regardless of the imperfection's severity. Likewise, models with exogenous noise traders cannot address how imperfections affect noise traders' willingness to invest. Links between imperfections and expected returns have been drawn in some cases. Yet, this has not been done systematically across imperfections, and not in a way that their effects can be compared.

In this paper, we develop a unified model to analyze how different imperfections affect market behavior. We consider the following imperfections: (1) asymmetric information, (2) participation costs, (3) transaction costs, (4) leverage constraints, (5) non-competitive behavior, and (6) search. We determine the effect of each imperfection on liquidity, price dynamics, and expected asset returns. We also compare effects across imperfections, and draw empirical implications for the measurement of liquidity, its link with expected returns, and its variation across assets and markets. Since the imperfections that we consider have been studied in the literature, some of our results are related to existing results. At the same time, because the effects of each imperfection on liquidity, price dynamics, and expected returns have not been fully addressed before—and not at all in some cases—most of our results are new.

Our model has three periods, $t = 0, 1, 2$. In Periods 0 and 1, risk-averse agents can trade a riskless and a risky asset that pay off in Period 2. In Period 0, agents are identical so no trade occurs. In Period 1, agents can be one of two types. Liquidity demanders receive an endowment correlated with the risky asset's payoff. They can hedge their endowment by trading with liquidity suppliers, who receive no endowment. Imperfections concern trade in Period 1. In the case of asymmetric information, liquidity demanders observe a private signal about the payoff of the risky asset. In the case of participation costs, agents must pay a cost to participate in the market. In the case of transaction costs, agents must pay a cost to trade (and the difference with participation costs is that the decision can be made conditional on trade size). In the case of leverage constraints, agents cannot fully commit to cover losses on their loans, and this limits leverage as a function of capital. In the case of non-competitive behavior, liquidity demanders take price impact into account, and can possibly also observe a private signal about asset payoff. In the case of search, agents are matched randomly with counterparties and bargain bilaterally over the price.[1]

We consider two measures of illiquidity, both commonly used in empirical studies. The first measure is lambda, defined as the regression coefficient of the price change between Periods 0 and 1 on liquidity demanders' signed volume in Period 1. This measure characterizes the price impact of volume, which has a transitory and a permanent component. The second measure is price reversal, defined as minus the autocovariance of price changes. This measure characterizes the importance of the transitory component in price, which in our model is entirely driven by volume. Both measures are positive even in the absence of imperfections. Indeed, because agents are risk-averse, liquidity demanders' trades move the price in Period 1 (implying that lambda is positive), and the movement is away from fundamental value (implying that price reversal is positive). We examine how each imperfection impacts the two measures of illiquidity and the expected return of the risky asset. To determine the effect on expected return, we examine how the price in Period 0 is influenced by the anticipation of imperfections in Period 1.

Table 1 summarizes the effects of each imperfection on market behavior. Results in dark (black) color are new, in the sense that either the question has not been asked in the literature, or the result is different than in previous papers. References to relevant papers are at the beginning of the section covering each imperfection.

A first observation from Table 1 is that imperfections do not always raise expected return. Consistent with previous papers, we find that expected return increases under participation costs

[1] Our list of imperfections does not include inventory costs, arising from the risk aversion of market makers (e.g., Stoll (1978)). Indeed, since our liquidity suppliers are risk averse, inventory costs are present even in the absence of the imperfections that we consider. Inventory costs increase, however, with imperfections such as participation costs, which reduce the measure of participating suppliers and raise their aggregate risk aversion.

Type of Imperfection	Impact of Imperfection: Lambda	Price Reversal	Expected Return
Asymmetric information	+	+/−	+
Participation costs	+	+	+
Transaction costs	+	+	+
Leverage constraints	+	+	+
Non-comp. behavior/Sym. info.	0	−	−
Non-comp. behavior/Asym. info.	+	−	+/−
Search	+/−	+/−	+/−

Table 1: **Impact of imperfections on illiquidity and expected returns.** "Lambda" is the regression coefficient of the price change between Periods 0 and 1 on the signed volume of liquidity demanders in Period 1; "Price Reversal" is minus the autocovariance of price changes; and "Expected Return" is the expected return of the risky asset between Periods 0 and 2. Results in dark (black) color are new; results in light (green) color are related to existing results.

and transaction costs. We further show that it increases under asymmetric information, comparing both to the case where the signal is public and the case where no agent observes the signal. Expected return also increases under leverage constraints. The intuition for these results is that agents are concerned that an endowment they receive in Period 1 increases the risk exposure they carry from Period 0. Because imperfections hamper agents' ability to modify their risk exposure, they reduce their willingness to hold the risky asset in Period 0, resulting in a low price and a high expected return. The effect can, however, reverse under non-competitive behavior. Indeed, since liquidity demanders can extract better terms of trade in Period 1, they are less concerned with the event where their risk exposure increases, and are therefore less averse to holding the asset in Period 0. The same is true under search if liquidity demanders hold most of the bargaining power in their bilateral meetings with suppliers.

A second observation from Table 1 is that imperfections can affect the two illiquidity measures in opposite directions. The effect on lambda is positive, except possibly under search. At the same time, the effect on price reversal is unambiguously positive only under participation costs, transaction costs and leverage constraints. The intuition for the discrepancy is that lambda measures the price impact per unit trade, while price reversal concerns the impact of the entire trade. Imperfections generally raise the price impact per unit trade, but because they also reduce trade size, the price impact of the entire trade can decrease. The second effect dominates under asymmetric information and non-competitive behavior.

The above results have a number of empirical implications. For example, many empirical studies seek to establish a link between illiquidity and expected asset returns. We show that the nature of this link depends crucially on the underlying cause of illiquidity: illiquidity caused by different imperfections can have opposite effects on expected returns. Furthermore, common measures of illiquidity do not always reflect the underlying imperfection: our results suggest that while lambda is generally a valid proxy, price reversal is valid only for certain imperfections.

Further implications follow by examining how changes in exogenous parameters, other than the imperfections themselves, affect the illiquidity measures and the expected return. We show that when the variance of liquidity demanders' hedging shock increases, price reversal and expected return increase, but lambda can increase or decrease depending on the imperfection. Our results suggest that the cross-sectional relationship between illiquidity and expected returns depends not only on the underlying imperfection but also on other sources of cross-sectional variation. Suppose, for example, that asymmetric information is the only imperfection. If it is also the main source of cross-sectional variation, then expected returns should be positively related to lambda. If, however, assets differ because of liquidity demanders' hedging needs and not because of asymmetric information, then expected returns should be negatively related to lambda because lambda decreases in the variance of the hedging shock. It is therefore important to control for sources of cross-sectional variation other than the imperfections themselves when linking illiquidity to expected returns.

Given the scope of this paper, the related literature is vast. Since our purpose here is not to survey the literature, but present a unified model and derive new results, we reference only the papers closest to our analysis. A more extensive and thorough review of the literature is left to a companion survey (Vayanos and Wang (2009)). Interested readers can also refer to existing surveys on liquidity, e.g., Amihud, Mendelson and Pedersen (2005), Biais, Glosten and Spatt (2005), and Cochrane (2005).

The rest of this paper is organized as follows. Section 2 presents the model and describes each imperfection. Section 3 treats the perfect-market benchmark, and Sections 4, 5, 6, 7, 8 and 9 treat asymmetric information, participation costs, transaction costs, leverage constraints, non-competitive behavior and search, respectively. Section 10 discusses empirical implications and Section 11 concludes.

2 Model

There are three periods, $t = 0, 1, 2$. The financial market consists of a riskless and a risky asset that pay off in terms of a consumption good in Period 2. The riskless asset is in supply of B shares and pays off one unit with certainty. The risky asset is in supply of $\bar{\theta}$ shares and pays off D units, where D has mean $\bar{D}$ and variance σ^2. Using the riskless asset as the numeraire, we denote by S_t the risky asset's price in Period t, where $S_2 = D$.

There is a measure one of agents, who derive utility from consumption in Period 2. Utility is exponential,

$$-\exp(-\alpha C_2), \tag{2.1}$$

where C_2 is consumption in Period 2, and $\alpha > 0$ is the coefficient of absolute risk aversion. Agents are identical in Period 0, and are endowed with the per capita supply of the riskless and the risky asset. They become heterogeneous in Period 1, and this generates trade. Because all agents have the same exponential utility, there is no preference heterogeneity. We instead introduce heterogeneity through agents' endowments and information.

A fraction π of agents receive an endowment $z(D - \bar{D})$ of the consumption good in Period 2, and the remaining fraction $1 - \pi$ receive no endowment.[2] The variable z has mean zero and variance σ_z^2, and is independent of D. While the endowment is received in Period 2, agents learn whether or not they will receive it before trade in Period 1, in an interim period $t = 1/2$. Only those agents who receive the endowment observe z, and they do so in Period 1. Since the endowment is correlated with D, it generates a hedging demand. When, for example, $z > 0$, the correlation is positive, and agents can hedge their endowment by reducing their holdings of the risky asset. We denote by W_t the wealth of an agent in Period t. Wealth in Period 2 is equal to consumption, i.e., $W_2 = C_2$.

For tractability, we assume that D and z are normal, and relax or modify this assumption only in Sections 6 (transaction costs) and 7 (leverage constraints). Under normality, the endowment $z(D - \bar{D})$ can take large negative values, and this can generate an infinitely negative expected utility. To guarantee that utility is finite, we assume that the variances of D and z satisfy the condition

$$\alpha^2\sigma^2\sigma_z^2 < 1. \tag{2.2}$$

In equilibrium, agents receiving an endowment initiate trades with others to share risk. Because the agents initiating trades can be thought of as consuming market liquidity, we refer to them as liquidity demanders and denote them by the subscript d. Moreover, we refer to z as the liquidity

[2] We assume that the endowment is perfectly correlated with D for simplicity; what matters for our analysis is that the correlation is non-zero.

shock. The agents who receive no endowment accommodate the trades of liquidity demanders, thus supplying liquidity. We refer to them as liquidity suppliers and denote them by the subscript s.

Because liquidity suppliers require compensation to absorb risk, the trades of liquidity demanders affect prices. Therefore, the price in Period 1 is influenced not only by the asset payoff, but also by the liquidity demanders' trades. Our measures of liquidity, defined in Section 3, are based on the price impact of these trades.

Liquidity is influenced by market imperfections. We define imperfections in reference to a perfect-market benchmark in which information is symmetric, participation and trade are costless, agents are competitive, and the market is centralized.[3] We consider six types of imperfections, all pertaining to trade in Period 1. We maintain the perfect-market assumption in Period 0 when determining the ex-ante effect of the imperfections, i.e., how the anticipation of imperfections in Period 1 impacts the Period 0 price.[4]

Asymmetric Information

In the perfect-market benchmark, all agents have the same information about the payoff of the risky asset. In practice, however, agents have access to different information sources, and can differ in their ability to process information. Such differences give rise to asymmetric information (Section 4). We assume that asymmetric information takes a simple form, where some agents observe a private signal s about the asset payoff D in Period 1. The signal is

$$s = D + \epsilon \tag{2.3}$$

where ϵ is normal with mean zero and variance σ_ϵ^2, and is independent of (D, z). We assume that only those agents who receive an endowment observe the signal, i.e., the set of informed agents coincides with that of liquidity demanders. Assuming that all liquidity demanders are informed is without loss of generality: even if they do not observe the signal, they can infer it perfectly from the price because they observe the liquidity shock. Asymmetric information can therefore exist only if some liquidity suppliers are uninformed. We assume that they are all uninformed for simplicity.

Participation Costs

In the perfect-market benchmark, all agents are present in the market in all periods. Thus,

[3]Our perfect-market benchmark has one market imperfection built in: agents cannot write contracts in Period 0 contingent on whether they are a liquidity demander or supplier in Period 1. Thus, the market in Period 0 is incomplete in the Arrow-Debreu sense. If agents could write complete contracts in Period 0, they would not need to trade in Period 1, in which case liquidity would not matter. In our model, complete contracts are infeasible because whether an agent is a liquidity demander or supplier is private information.

[4]Imperfections in Period 0 are not relevant in our model because agents are identical in that period and there is no trade.

a seller, for example, can have immediate access to the entire population of buyers. In practice, however, agents face costs of market participation. Such costs include buying trading infrastructure or membership of a financial exchange, having capital available on short notice, monitoring market movements, etc. To model costly participation (Section 5), we assume that agents must incur a cost c to trade in Period 1. Consistent with the notion that participation is an ex-ante decision, we assume that agents must decide whether or not to incur c in Period 1/2, after learning whether or not they will receive an endowment but before observing the price in Period 1. If the decision can be made contingent on the price in Period 1, then c is a fixed transaction cost rather than a participation cost. We consider transaction costs as a separate market imperfection.[5]

Transaction Costs

In addition to costs of market participation, agents typically pay costs when executing transactions. Transaction costs drive a wedge between the buying and selling price of an asset. They come in many types, e.g., brokerage commissions, exchange fees, transaction taxes, bid-ask spreads, price impact. Some types of transaction costs can be viewed as a consequence of other market imperfections: for example, Section 5 shows that costly participation can generate price-impact costs. Other types of costs, such as transaction taxes, can be viewed as more primitive. We assume (Section 6) that transaction costs concern trade in Period 1, and can be proportional or fixed. Proportional costs are proportional to transaction size, and for simplicity we assume that proportionality concerns the number of shares rather than the dollar value. Denoting by κ the cost per unit of shares traded and by θ_t the number of shares that an agent holds in Period $t = 0, 1$, proportional costs take the form $\kappa\,|\theta_1 - \theta_0|$. Fixed costs are independent of transaction size and take the form $\kappa 1_{\{\theta_1 \neq \theta_0\}}$, i.e., the agent pays $\kappa > 0$ when trading in Period 1.

Leverage Constraints

Agents' portfolios often involve leverage, i.e., borrow cash to establish a long position in a risky asset, or borrow a risky asset to sell it short. In the perfect-market benchmark, agents can borrow freely provided that they have enough resources to repay the loan. But as the Corporate Finance literature emphasizes, various frictions can limit agents' ability to borrow.

Since in our model consumption is allowed to be negative and unbounded from below, agents can repay a loan of any size by reducing consumption. Negative consumption can be interpreted as a costly activity that agents undertake in Period 2 to repay a loan. We derive a leverage constraint

[5] Our analysis can be extended to the case where participation is costly not only in Period 1 but also in Period 0. The cost to participate in Period 0 can be interpreted as an entry cost, e.g., learning about an asset. Entry costs reduce the measure of agents buying the asset in Period 0, and therefore lower the price. See, for example, Huang and Wang (2008a,b).

by assuming that agents cannot commit to reduce their consumption below a level $-A \leq 0$. This nests the case of full commitment assumed in the rest of this paper ($A = \infty$), and the case where agents can walk away from a loan rather than engaging in negative consumption ($A = 0$). Note that the same leverage constraint would arise if consumption below $-A$ is not feasible. Under the latter interpretation, however, the constraint would not constitute an imperfection: it would amount to redefining the utility function (2.1) as $-\infty$ when consumption is below $-A$. The two interpretations yield the same constraint and pricing implications, but differ in their welfare implications.[6]

Non-Competitive Behavior

In the perfect-market benchmark, all agents are competitive and have no effect on prices. In many markets, however, some agents are large relative to others and can influence prices. To model non-competitive behavior (Section 8), we assume that liquidity demanders behave as a single monopolist in Period 1. We consider both the case where liquidity demanders have no private information on asset payoffs, and the case where they observe the private signal (2.3).

Search

Both in the perfect-market benchmark and under the imperfections described so far, the market is organized as a centralized exchange. Many markets, however, have a more decentralized form of organization. For example, in over-the-counter markets, investors negotiate prices bilaterally with dealers. Locating suitable counter-parties in these markets can take time and involve search. To model decentralized markets (Section 9), we assume that agents do not meet in a centralized exchange in Period 1, but instead must search for counterparties. With some probability they meet a counterparty and bargain bilaterally over the price.

3 Perfect-Market Benchmark

In this section we solve the basic model described in Section 2, assuming no market imperfections. We first compute the equilibrium, going backwards from Period 1 to Period 0. We next construct measures of market liquidity in Period 1, and study how liquidity impacts the price dynamics and the price level in Period 0.

[6] While the leverage constraint in our model is linked to negative consumption, this is not the case in other settings. For example, in Gromb and Vayanos (2002) a leverage constraint arises because liquidity suppliers exploit price discrepancies between two correlated assets and cannot commit to use gains in one position to cover losses in the other.

3.1 Equilibrium

In Period 1, a liquidity demander chooses holdings θ_1^d of the risky asset to maximize the expected utility (2.1). Consumption in Period 2 is

$$C_2^d = W_1 + \theta_1^d(D - S_1) + z(D - \bar{D}),$$

i.e., wealth in Period 1, plus capital gains from the risky asset, plus the endowment. Therefore, expected utility is

$$-\mathrm{E}\exp\left\{-\alpha\left[W_1 + \theta_1^d(D - S_1) + z(D - \bar{D})\right]\right\}, \tag{3.4}$$

where the expectation is over D. Because D is normal, the expectation is equal to

$$-\exp\left\{-\alpha\left[W_1 + \theta_1^d(\bar{D} - S_1) - \tfrac{1}{2}\alpha\sigma^2(\theta_1^d + z)^2\right]\right\}. \tag{3.5}$$

A liquidity supplier chooses holdings θ_1^s of the risky asset to maximize the expected utility

$$-\exp\left\{-\alpha\left[W_1 + \theta_1^s(\bar{D} - S_1) - \tfrac{1}{2}\alpha\sigma^2(\theta_1^s)^2\right]\right\}, \tag{3.6}$$

which can be derived from (3.5) by setting $z = 0$. The solution to the optimization problems is straightforward and summarized in Proposition 3.1.

Proposition 3.1 *Agents' demand functions for the risky asset in Period 1 are*

$$\theta_1^s = \frac{\bar{D} - S_1}{\alpha\sigma^2}, \tag{3.7a}$$

$$\theta_1^d = \frac{\bar{D} - S_1}{\alpha\sigma^2} - z. \tag{3.7b}$$

Liquidity suppliers are willing to buy the risky asset as long as it trades below its expected payoff $\bar{D}$, and are willing to sell otherwise. Liquidity demanders have a similar price-elastic demand function, but are influenced by the liquidity shock z. When, for example, z is positive, liquidity demanders are willing to sell because their endowment is positively correlated with the asset.

Market clearing requires that the aggregate demand equals the asset supply $\bar{\theta}$:

$$(1 - \pi)\theta_1^s + \pi\theta_1^d = \bar{\theta}. \tag{3.8}$$

Substituting (3.7a) and (3.7b) into (3.8), we find

$$S_1 = \bar{D} - \alpha\sigma^2\left(\bar{\theta} + \pi z\right). \tag{3.9}$$

The price S_1 decreases in the liquidity shock z. When, for example, z is positive, liquidity demanders are willing to sell, and the price must drop so that the risk-averse liquidity suppliers are willing to buy.

In Period 0, all agents are identical. An agent choosing holdings θ_0 of the risky asset has wealth

$$W_1 = W_0 + \theta_0(S_1 - S_0) \tag{3.10}$$

in Period 1. The agent can be a liquidity supplier in Period 1 with probability $1-\pi$, or liquidity demander with probability π. Substituting θ_1^s from (3.7a), S_1 from (3.9), and W_1 from (3.10), we can write the expected utility (3.6) of a liquidity supplier in Period 1 as

$$-\exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0(\bar{\theta} + \pi z) + \tfrac{1}{2}\alpha\sigma^2(\bar{\theta} + \pi z)^2\right]\right\}. \tag{3.11}$$

The expected utility depends on the liquidity shock z since z affects the price S_1. We denote by U^s the expectation of (3.11) over z, and by U^d the analogous expectation for a liquidity demander. These expectations are agents' interim utilities in Period 1/2. An agent's expected utility in Period 0 is

$$U \equiv (1-\pi)U^s + \pi U^d. \tag{3.12}$$

Agents choose θ_0 to maximize U. The solution to this maximization problem coincides with the aggregate demand in Period 0, since all agents are identical in that period and are in measure one. In equilibrium, aggregate demand has to equal the asset supply $\bar{\theta}$, and this determines the price S_0 in Period 0.

Proposition 3.2 *The price in Period 0 is*

$$S_0 = \bar{D} - \alpha\sigma^2\bar{\theta} - \frac{\pi M}{1 - \pi + \pi M}\Delta_1\bar{\theta}, \tag{3.13}$$

where

$$M = \exp\left(\tfrac{1}{2}\alpha\Delta_2\bar{\theta}^2\right)\sqrt{\frac{1 + \Delta_0\pi^2}{1 + \Delta_0(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}}, \tag{3.14}$$

$$\Delta_0 = \alpha^2\sigma^2\sigma_z^2, \tag{3.15a}$$

$$\Delta_1 = \frac{\alpha\sigma^2\Delta_0\pi}{1 + \Delta_0(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}, \tag{3.15b}$$

$$\Delta_2 = \frac{\alpha\sigma^2\Delta_0}{1 + \Delta_0(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}. \tag{3.15c}$$

The first term in (3.13) is the asset's expected payoff in Period 2, the second term is a discount arising because the payoff is risky, and the third term is a discount due to illiquidity (i.e., low liquidity). In the next section we explain why illiquidity in Period 1 lowers the price in Period 0.

3.2 Illiquidity and its Effect on Price

We construct two measures of illiquidity, both based on the price impact of the liquidity demanders' trades in Period 1. The first measure is the coefficient of a regression of the price change between Periods 0 and 1 on the signed volume of liquidity demanders in Period 1:

$$\lambda \equiv \frac{\text{Cov}\left[S_1 - S_0, \pi(\theta_1^d - \bar{\theta})\right]}{\text{Var}\left[\pi(\theta_1^d - \bar{\theta})\right]}. \tag{3.16}$$

Intuitively, when λ is large, trades have large price impact and the market is illiquid. Eq. (3.9) implies that the price change between Periods 0 and 1 is

$$S_1 - S_0 = \bar{D} - \alpha\sigma^2\left(\bar{\theta} + \pi z\right) - S_0. \tag{3.17}$$

Eqs. (3.7b) and (3.9) imply that the signed volume of liquidity demanders is

$$\pi(\theta_1^d - \bar{\theta}) = -\pi(1-\pi)z. \tag{3.18}$$

Eqs. (3.16)-(3.18) imply that

$$\lambda = \frac{\alpha\sigma^2}{1-\pi}. \tag{3.19}$$

Illiquidity λ is higher when agents are more risk-averse (α large), the asset is riskier (σ^2 large), or liquidity suppliers are less numerous ($1-\pi$ small).

The second measure is based on the autocovariance of price changes. The liquidity demanders' trades in Period 1 cause the price to deviate from fundamental value, while the two coincide in Period 2. Therefore, price changes exhibit negative autocovariance, and more so when trades have large price impact. We use minus autocovariance

$$\gamma \equiv -\text{Cov}\left(S_2 - S_1, S_1 - S_0\right), \tag{3.20}$$

as a measure of illiquidity, and refer to it as price reversal, reserving the term illiquidity for λ. Eqs. (3.9), (3.17), (3.20) and $S_2 = D$ imply that

$$\gamma = -\text{Cov}\left[D - \bar{D} + \alpha\sigma^2\left(\bar{\theta} + \pi z\right), \bar{D} - \alpha\sigma^2\left(\bar{\theta} + \pi z\right) - S_0\right] = \alpha^2\sigma^4\sigma_z^2\pi^2. \tag{3.21}$$

Price reversal γ is higher when agents are more risk-averse, the asset is riskier, liquidity demanders are more numerous (π large), and liquidity shocks are larger (σ_z^2 large).[7]

The measures λ and γ have been derived within models focusing on specific imperfections, and have been widely used in empirical work ever since. Using our unified model, we examine the behavior of these measures across a variety of imperfections, and provide a broader perspective on their properties. We emphasize basic properties below, leaving a more detailed discussion and empirical implications to Section 10.

Kyle (1985) links λ to the degree of information asymmetry between an informed insider and uninformed market makers. In Kyle, market makers are risk neutral, and trades affect prices only because they contain information. Thus, the price impact, as measured by λ, reflects the amount of information that trades convey, and is permanent because the risk-neutral market makers set the price equal to their expectation of fundamental value. In general, as in our model, λ has both a transitory and a permanent component. The transitory component, present even in our perfect-market benchmark, arises because liquidity suppliers are risk averse and require a price movement away from fundamental value to absorb a liquidity shock. The permanent component arises only when information is asymmetric (Sections 4 and 8) for the same reasons as in Kyle.[8]

Roll (1984) links γ to the bid-ask spread, in a model where market orders cause the price to bounce between the bid and the ask. Grossman and Miller (1988) link γ to the price impact of liquidity shocks, in a model where risk-averse liquidity suppliers must incur a cost to participate in the market. In both models, price impact is transitory because information is symmetric. In our model, price impact has both a transitory and a permanent component, and γ isolates the effects of the transitory component. Note that besides being a measure of imperfections, γ provides a useful characterization of price dynamics: it measures the importance of the transitory component in price arising from temporary liquidity shocks, relative to the random-walk component arising from fundamentals.

Illiquidity in Period 1 lowers the price in Period 0 through the illiquidity discount, which is the third term in (3.13). To explain why the discount arises, consider the extreme case where trade in Period 1 is not allowed. In Period 0, agents know that with probability π they will receive an endowment in Period 2. The endowment amounts to a risky position in Period 1, the size of which is uncertain because it depends on z. Uncertainty about position size is costly (in utility terms) to

[7]The comparative statics of autocorrelation are similar to those of autocovariance. We use autocovariance rather than autocorrelation because normalizing by variance adds unnecessary complexity.

[8]An alternative definition of λ, which isolates the permanent component, involves the price change between Periods 0 and 2 rather than between Periods 0 and 1. This is because the transitory deviation between price and fundamental value in Period 1 disappears in Period 2.

risk-averse agents. Moreover, the effect is stronger when agents carry a large position from Period 0 because the cost of holding a position in Period 1 is convex in the overall size of the position. (The cost is the quadratic term in (3.5) and (3.6).) Therefore, uncertainty about z reduces agents' willingness to buy the asset in Period 0.

The intuition is similar when agents can trade in Period 1. Indeed, in the extreme case where trade is not allowed, the shadow price faced by liquidity demanders moves in response to z to the point where these agents are not willing to trade. When trade is allowed, the price movement is smaller, but non-zero. Therefore, uncertainty about z still reduces agents' willingness to buy the asset in Period 0. Moreover, the effect is weaker when trade is allowed in Period 1 than when it is not, and therefore corresponds to a discount driven by illiquidity.[9] Because the market imperfections studied in the following sections hinder trade in Period 1, they tend to raise the illiquidity discount in Period 0.

The illiquidity discount is the product of two terms. The first term, $\frac{\pi M}{1-\pi+\pi M}$, can be interpreted as the risk-neutral probability of being a liquidity demander: π is the true probability, and M is the ratio of marginal utilities of demanders and suppliers. The second term, $\Delta_1\bar{\theta}$, is the discount that an agent would require in Period 0 if he were certain to be a demander.

The illiquidity discount is higher when liquidity shocks are larger (σ_z^2 large) and occur with higher probability (π large). It is also higher when agents are more risk averse (α large), the asset is riskier (σ^2 large), and in larger supply ($\bar{\theta}$ large). Same comparative statics hold for the ratio of the illiquidity discount to the discount $\alpha\sigma^2\bar{\theta}$ driven by payoff risk. Thus, while risk aversion α, payoff risk σ^2, or asset supply $\bar{\theta}$ raise the risk discount, they have an even stronger impact on the illiquidity discount. For example, an increase in α raises not only the aversion of agents to the risk of receiving a liquidity shock, but also the shock's impact on price.

The parameter σ_z^2, which measures the magnitude of liquidity shocks, has different effects on the illiquidity measures and the illiquidity discount: it has no effect on λ, while it raises γ and the discount. The intuition is that λ measures the price impact per unit trade, while γ and S_0 concern the impact of the entire liquidity shock.

Proposition 3.3 *An increase in the variance σ_z^2 of liquidity shocks leaves illiquidity λ unchanged, raises price reversal γ, and lowers the price in Period 0.*

[9] The comparison of illiquidity discounts under trade and no trade follows from Proposition 4.6. See Footnote 12.

4 Asymmetric Information

In this section we assume that liquidity demanders observe the private signal (2.3) before trading in Period 1. Our analysis of equilibrium in Period 1 is closely related to Grossman and Stiglitz (1980) because we assume continua of informed and uninformed agents, and endow all informed agents with the same signal.[10] Our analysis of equilibrium in Period 0 is new, and so are the results on how asymmetric information affects the illiquidity discount and the price reversal γ.[11]

4.1 Equilibrium

The price in Period 1 incorporates the signal of liquidity demanders, and therefore reveals information to liquidity suppliers. To solve for equilibrium, we conjecture a price function (i.e., a relationship between the price and the signal), then determine how agents use their knowledge of the price function to learn about the signal and formulate demand functions, and finally confirm that the conjectured price function clears the market.

We conjecture a price function that is affine in the signal s and the liquidity shock z, i.e.,

$$S_1 = a + b(s - \bar{D} - cz) \tag{4.1}$$

for three constants (a, b, c). For expositional convenience, we set $\xi \equiv s - \bar{D} - cz$. We also refer to the price function as simply the price.

Agents use the price and their private information to form a posterior distribution about the asset payoff D. For a liquidity demander, the price conveys no additional information relative

[10] Grossman and Stiglitz model non-informational trading through exogenous shocks to the asset supply, while we model it through an endowment received by the informed. Modeling non-informational trading through random endowments dates back to Diamond and Verrecchia (1981), who solve a one-period model with a different information structure than Grossman and Stiglitz. (Agents receive conditionally independent signals with the same precision.) Wang (1994) solves an infinite-horizon model with continua of informed and uninformed agents, and models non-informational trading through a risky production opportunity available only to the informed.

[11] O'Hara (2003) and Easley and O'Hara (2004) study the effect of asymmetric information on expected returns in a multi-asset extension of Grossman and Stiglitz. They show that prices are lower and expected returns are higher when agents receive private signals than when signals are public. This comparison concerns prices in our Period 1. Moreover, it is driven not by asymmetric information *per se* but by the total amount of information agents have. Indeed, while prices in Period 1 are lower under asymmetric information than when signals are public (maximum total information), they are higher than under the alternative symmetric-information benchmark where no signals are observed (minimum total information). We instead compare prices in Period 0, to determine the ex-ante effect of the imperfection. This comparison is driven only by asymmetric information because prices are lower under asymmetric information than under either symmetric-information benchmark. Garleanu and Pedersen (2004) study the effect of asymmetric information on expected returns in a multi-period model with risk-neutral agents and unit demands. When probability distributions are symmetric (as they are in our model), they find no effect of asymmetric information on expected returns. Ellul and Pagano (2006) show that asymmetric information in the post-IPO stage can reduce the IPO price. The post-IPO stage, however, involves exogenous noise traders and an insider who is precluded from bidding for the IPO.

to observing the signal s. Given the joint normality of (D, ϵ), D remains normal conditional on $s = D + \epsilon$, with mean and variance

$$\mathrm{E}[D|s] = \bar{D} + \beta_s(s - \bar{D}), \tag{4.2a}$$

$$\sigma^2[D|s] = \beta_s \sigma_\epsilon^2, \tag{4.2b}$$

where $\beta_s \equiv \sigma^2/(\sigma^2 + \sigma_\epsilon^2)$. For a liquidity supplier, the only information is the price S_1, which is equivalent to observing ξ. Conditional on ξ (or S_1), D is normal with mean and variance

$$\mathrm{E}[D|S_1] = \bar{D} + \beta_\xi \xi = \bar{D} + \frac{\beta_\xi}{b}(S_1 - a), \tag{4.3a}$$

$$\sigma^2[D|S_1] = \beta_\xi(\sigma_\epsilon^2 + c^2\sigma_z^2), \tag{4.3b}$$

where $\beta_\xi \equiv \sigma^2/\sigma_\xi^2$ and $\sigma_\xi^2 \equiv \sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2$. Agents' optimization problems are as in Section 3, with the conditional distributions of D replacing the unconditional one. Proposition 4.1 summarizes the solution to these problems.

Proposition 4.1 *Agents' demand functions for the risky asset in Period 1 are*

$$\theta_1^s = \frac{\mathrm{E}[D|S_1] - S_1}{\alpha\sigma^2[D|S_1]}, \tag{4.4a}$$

$$\theta_1^d = \frac{\mathrm{E}[D|s] - S_1}{\alpha\sigma^2[D|s]} - z. \tag{4.4b}$$

Substituting (4.4a) and (4.4b) into the market-clearing equation (3.8), we find

$$(1-\pi)\frac{\mathrm{E}[D|S_1] - S_1}{\alpha\sigma^2[D|S_1]} + \pi\left(\frac{\mathrm{E}[D|s] - S_1}{\alpha\sigma^2[D|s]} - z\right) = \bar{\theta}. \tag{4.5}$$

The price (4.1) clears the market if (4.5) is satisfied for all values of (s, z). Substituting S_1, $\mathrm{E}[D|s]$, and $\mathrm{E}[D|S_1]$ from (4.1), (4.2a) and (4.3a), we can write (4.5) as an affine equation in (s, z). Therefore, (4.5) is satisfied for all values of (s, z) if the coefficients of (s, z) and of the constant term are equal to zero. This yields a system of three equations in (a, b, c), solved in Proposition 4.2.

Proposition 4.2 *The price in Period 1 is given by (4.1), where*

$$a = \bar{D} - \alpha(1-b)\sigma^2\bar{\theta}, \tag{4.6a}$$

$$b = \frac{\pi\beta_s\sigma^2[D|S_1] + (1-\pi)\beta_\xi\sigma^2[D|s]}{\pi\sigma^2[D|S_1] + (1-\pi)\sigma^2[D|s]}, \tag{4.6b}$$

$$c = \alpha\sigma_\epsilon^2. \tag{4.6c}$$

To determine the price in Period 0, we follow the same steps as in Section 3. The calculations are more complicated because expected utilities in Period 1 are influenced by two random variables (s, z) rather than only z. The price in Period 0, however, takes the same general form as in the perfect-market benchmark.

Proposition 4.3 *The price in Period 0 is given by (3.13), where M is given by (3.14),*

$$\Delta_0 = \frac{(b-\beta_\xi)^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)}{\sigma^2[D|S_1]\pi^2}, \tag{4.7a}$$

$$\Delta_1 = \frac{\alpha^3 b\sigma^2(\sigma^2+\sigma_\epsilon^2)\sigma_z^2}{1+\Delta_0(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}, \tag{4.7b}$$

$$\Delta_2 = \frac{\alpha^3\sigma^4\sigma_z^2\left[1+\frac{(\beta_s-b)^2(\sigma^2+\sigma_\epsilon^2)}{\sigma^2[D|s]}\right]}{1+\Delta_0(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}. \tag{4.7c}$$

4.2 Asymmetric Information and Illiquidity

We next examine how asymmetric information impacts the illiquidity measures and the illiquidity discount. We consider two symmetric-information benchmarks: the *no-information* case, where information is symmetric because no agent observes s, and the *full-information* case, where all agents observe s. The analysis in Section 3 concerns the no-information case, but can easily be extended to the full-information case (Appendix, Proposition A.1). Illiquidity λ and price reversal γ under full information are given by (3.19) and (3.21), respectively, where σ^2 is replaced by $\sigma^2[D|s]$.

Proposition 4.4 *Illiquidity λ under asymmetric information is*

$$\lambda = \frac{\alpha\sigma^2[D|S_1]}{(1-\pi)\left(1-\frac{\beta_\xi}{b}\right)}. \tag{4.8}$$

Illiquidity is highest under asymmetric information and lowest under full information. Moreover, illiquidity under asymmetric information increases when the private signal (2.3) becomes more precise, i.e., when σ_ϵ^2 decreases.

Under both symmetric and asymmetric information, illiquidity increases in the uncertainty faced by liquidity suppliers, measured by their conditional variance of the asset payoff. In addition to this *uncertainty* effect, a *learning* effect appears under asymmetric information: Because, for example, liquidity suppliers attribute selling pressure partly to a low signal, they require a larger price drop to buy. The learning effect corresponds to the term β_ξ/b in (4.8), which lowers the denominator and raises λ.

Because of the uncertainty effect, illiquidity under full information is lower than under no information, and illiquidity under asymmetric information tends to lie in-between. The learning effect raises illiquidity under asymmetric information, and works in the same direction as the uncertainty effect when comparing asymmetric to full information. The two effects work in opposite directions when comparing asymmetric to no information, but the learning effect dominates. Illiquidity is thus highest under asymmetric information.

Price reversal is not unambiguously highest under asymmetric information. Indeed, consider two extreme cases. If $\pi \approx 1$, i.e., almost all agents are liquidity demanders (informed), then the price processes under asymmetric and full information approximately coincide, and so do the price reversals. Since, in addition, liquidity suppliers face more uncertainty under no information than under full information, price reversal is highest under no information.

If instead $\pi \approx 0$, i.e., almost all agents are liquidity suppliers (uninformed), then illiquidity λ converges to infinity (order $1/\pi$) under asymmetric information. This is because the trading volume of liquidity demanders converges to zero, but the volume's informational content remains unchanged. Because of the high illiquidity, price reversal is highest under asymmetric information.

Proposition 4.5 *Price reversal γ under asymmetric information is*

$$\gamma = b(b - \beta_\xi)(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2). \tag{4.9}$$

Price reversal is lowest under full information. It is highest under asymmetric information if $\pi \approx 0$, and under no information if $\pi \approx 1$.

While illiquidity and price reversal are lower under full information than under no information, the comparison reverses for the illiquidity discount. This is because information reduces the scope for risk sharing, an effect originally shown in Hirshleifer (1971). Since risk sharing is better under no information, trade achieves larger gains, and the illiquidity discount is smaller.

Because of the Hirshleifer effect, the illiquidity discount under asymmetric information tends to lie between the full- and no-information discounts. At the same time, asymmetric information raises illiquidity in Period 1 because of the learning effect. The learning effect raises the discount and works in the same direction as the Hirshleifer effect when comparing asymmetric to no information. The two effects work in opposite directions when comparing asymmetric to full information, but the learning effect dominates. The illiquidity discount is thus highest under asymmetric information.[12]

Proposition 4.6 *The price in Period 0 is lowest under asymmetric information and highest under no information.*

The comparative statics with respect to the variance σ_z^2 of liquidity shocks are the same as in the perfect-market benchmark case, except for the illiquidity λ. Under asymmetric information, an increase in σ_z^2 lowers λ because liquidity shocks make prices less informative and attenuate learning.

Proposition 4.7 *An increase in the variance σ_z^2 of liquidity shocks lowers illiquidity λ, raises price reversal γ, and lowers the price in Period 0.*

5 Participation Costs

In this section we assume that agents must incur a cost c to participate in the market in Period 1. Our analysis of participation decisions and equilibrium in Period 1 is closely related to Grossman and Miller (1988), and of equilibrium in Period 0 to Huang and Wang (2008a,b).[13] Our result on how participation costs affect the illiquidity λ is new.

5.1 Equilibrium

The price in Period 1 is determined by the participating agents. We look for an equilibrium where all liquidity demanders participate, but only a fraction $\mu > 0$ of liquidity suppliers do. Market clearing requires that the aggregate demand of participating agents equals the asset supply held by these agents. Since in equilibrium agents enter Period 1 holding $\bar{\theta}$ shares of the risky asset, market clearing takes the form

$$(1-\pi)\mu\theta_1^s + \pi\theta_1^d = [(1-\pi)\mu + \pi]\,\bar{\theta}. \tag{5.1}$$

[12] Proposition 4.6 implies that the illiquidity discount under no trade is larger than in the perfect-market benchmark. Indeed, the perfect-market benchmark corresponds to the no-information case. On the other hand, no trade occurs in the full-information case if the signal (2.3) is perfect ($\sigma_\epsilon^2 = 0$) because there is no scope for risk sharing.

[13] Grossman and Miller assume participation costs for liquidity suppliers only, while we assume such costs for all agents. Huang and Wang's analysis is more general than ours in two respects. First, they assume no aggregate liquidity shocks and derive aggregate order imbalances as a consequence of participation costs. We assume instead an aggregate liquidity shock, in a spirit similar to Pagano (1989) and Allen and Gale (1994). Second, they consider general parameter values, while we limit attention to values under which liquidity demanders always participate.

Agents' demand functions are as in Section 3. Substituting (3.7a) and (3.7b) into (5.1), we find that the price in Period 1 is

$$S_1 = \bar{D} - \alpha\sigma^2 \left[\bar{\theta} + \frac{\pi}{(1-\pi)\mu + \pi} z\right]. \tag{5.2}$$

We next determine the measure μ of participating liquidity suppliers, assuming that all liquidity demanders participate. If a supplier participates, he submits the demand function (3.7a) in Period 1. Since participation entails a cost c, wealth in Period 1 is

$$W_1 = W_0 + \theta_0(S_1 - S_0) - c. \tag{5.3}$$

Using (3.7a), (5.2) and (5.3), we can compute the interim utility U^s of a participating supplier in Period 1/2. If the supplier does not participate, holdings in Period 1 are the same as in Period 0 ($\theta_1^s = \theta_0$), and wealth in Period 1 is given by (3.10). We denote by U^{sn} the interim utility of a non-participating supplier in Period 1/2.

The participation decision is derived by comparing U^s to U^{sn} for the equilibrium choice of θ_0, which is $\bar{\theta}$. If the participation cost c is below a threshold $\underline{c}$, then all suppliers participate ($\mu = 1$). If c is above $\underline{c}$ and below a larger threshold $\bar{c}$, then suppliers are indifferent between participating or not ($U^s = U^{sn}$), and only some participate ($0 < \mu < 1$). Increasing c within that region reduces the fraction μ of participating suppliers, while maintaining the indifference condition. This is because with fewer participating suppliers, competition becomes less intense, enabling the remaining suppliers to cover their increased participation cost. Finally, if c is above $\bar{c}$, then no suppliers participate ($\mu = 0$).

Proposition 5.1 *Suppose that all liquidity demanders participate. Then, the fraction of participating liquidity suppliers is*

$$\mu = 1, \qquad if \quad c \leq \underline{c} \equiv \frac{\log\left(1 + \alpha^2\sigma^2\sigma_z^2\pi^2\right)}{2\alpha}, \tag{5.4a}$$

$$\mu = \frac{\pi}{1-\pi}\left(\frac{\alpha\sigma\sigma_z}{\sqrt{e^{2\alpha c} - 1}} - 1\right), \qquad if \quad \underline{c} < c < \bar{c} \equiv \frac{\log\left(1 + \alpha^2\sigma^2\sigma_z^2\right)}{2\alpha}, \tag{5.4b}$$

$$\mu = 0, \qquad if \quad c \geq \bar{c}. \tag{5.4c}$$

We next determine the participation decisions of liquidity demanders, taking those of liquidity suppliers as given.

Proposition 5.2 *Suppose that a fraction $\mu > 0$ of liquidity suppliers participate. Then, a sufficient condition for all liquidity demanders to participate is*

$$(1-\pi)\mu \geq \pi. \tag{5.5}$$

Eq. (5.5) requires that the measure π of liquidity demanders does not exceed the measure $(1-\pi)\mu$ of participating suppliers. Intuitively, when demanders are the short side of the market, they stand to gain more from participation, and can therefore cover the participation cost (since suppliers do). Combining Propositions 5.1 and 5.2, we find:

Corollary 5.1 *An equilibrium where all liquidity demanders and a fraction $\mu > 0$ of liquidity suppliers participate exists under the sufficient conditions $\pi \leq 1/2$ and $c \leq \hat{c} \equiv \frac{\log\left(1+\frac{1}{4}\alpha^2\sigma^2\sigma_z^2\right)}{2\alpha}$.*

For $\pi \leq 1/2$ and $c \leq \hat{c}$, only two equilibria exist: the one in the corollary and the one where no agent participates. The same is true for π larger but close to 1/2, and for c larger but close to $\hat{c}$.[14] When, however, c exceeds a threshold in $(\hat{c}, \bar{c})$, the equilibrium in the corollary ceases to exist, and no-participation becomes the unique equilibrium.

To determine the price in Period 0, we follow the same steps as in Section 3. The price takes a form similar to that in the perfect-market benchmark.

Proposition 5.3 *The price in Period 0 is given by (3.13), where*

$$M = \exp\left(\tfrac{1}{2}\alpha\Delta_2\bar{\theta}^2\right)\sqrt{\frac{1+\Delta_0\frac{\pi^2}{[(1-\pi)\mu+\pi]^2}}{1+\Delta_0\frac{(1-\pi)^2\mu^2}{[(1-\pi)\mu+\pi]^2}-\alpha^2\sigma^2\sigma_z^2}}, \tag{5.6}$$

$$\Delta_1 = \frac{\alpha\sigma^2\Delta_0\frac{\pi}{(1-\pi)\mu+\pi}}{1+\Delta_0\frac{(1-\pi)^2\mu^2}{[(1-\pi)\mu+\pi]^2}-\alpha^2\sigma^2\sigma_z^2}, \tag{5.7a}$$

$$\Delta_2 = \frac{\alpha\sigma^2\Delta_0}{1+\Delta_0\frac{(1-\pi)^2\mu^2}{[(1-\pi)\mu+\pi]^2}-\alpha^2\sigma^2\sigma_z^2}, \tag{5.7b}$$

and Δ_0 is given by (3.15a).

[14] Other equilibria are ruled out by the following argument. Prices and trading profits in Period 1 depend only the relative measures of participating suppliers and demanders. Therefore, if participation occurs, the fraction of either suppliers or demanders must (generically) equal one. If the fraction of demanders is less than one, then the fraction of suppliers must equal one. This is a contradiction for $\pi \leq 1/2$ because of (5.5). It is also a contradiction for π larger but close to 1/2 because (5.5) is a sufficient condition: because liquidity demanders face the risk of liquidity shocks, they can benefit from participation more than suppliers even when they are the long side of the market. See Huang and Wang for a more detailed discussion of the nature of equilibrium under costly participation.

5.2 Participation Costs and Illiquidity

We next examine how participation costs impact the illiquidity measures and the illiquidity discount. Proceeding as in Section 3, we can compute the illiquidity λ and price reversal γ:

$$\lambda = \frac{\alpha\sigma^2}{(1-\pi)\mu}, \tag{5.8}$$

$$\gamma = \frac{\alpha^2\sigma^4\sigma_z^2\pi^2}{\left[(1-\pi)\mu+\pi\right]^2}. \tag{5.9}$$

Both measures are inversely related to the fraction μ of participating liquidity suppliers. Proposition 5.3 implies that the illiquidity discount is also inversely related to μ.

We derive comparative statics for the equilibrium in Corollary 5.1, and consider only the region $c > \underline{c}$, where the measure μ of participating suppliers is less than one. This is without loss of generality: in the region $c \leq \underline{c}$, where all suppliers participate, prices are not affected by the participation cost and are as in the perfect-market benchmark. When $c > \underline{c}$, an increase in the participation cost lowers μ, and therefore raises illiquidity, price reversal and the illiquidity discount.

Proposition 5.4 *An increase in the participation cost c raises illiquidity λ and price reversal γ, and lowers the price in Period 0.*

Consider next an increase in the variance σ_z^2 of liquidity shocks. Since liquidity supply becomes more profitable, there is more participation by suppliers and illiquidity λ decreases. Price reversal remains unchanged, however, because of two offsetting effects. Holding the measure of participating suppliers constant, an increase in σ_z^2 raises price reversal for the same reasons as in the perfect-market benchmark. At the same time, increased participation lowers price reversal. The effects exactly offset because the profits of participating suppliers depend on σ_z^2 only through the price reversal. Since profits in equilibrium must equal the participation cost, price reversal is independent of σ_z^2.

Proposition 5.5 *An increase in the variance σ_z^2 of liquidity shocks lowers illiquidity λ, leaves price reversal γ unchanged, and lowers the price in Period 0.*

6 Transaction Costs

In this section we assume that agents incur a transaction cost when trading in Period 1. The difference with the participation cost of the previous section is that the decision whether or not to incur the transaction cost is contingent on the price in Period 1. We mainly focus on the case where the transaction cost is proportional to transaction size, as measured by the number of shares, and consider the more complicated case of fixed costs at the end of this section. We assume that the liquidity shock z is drawn from a general distribution that is symmetric around zero with density $f(z)$; specializing to a normal distribution does not simplify the analysis. Our analysis is closest to Lo, Mamaysky and Wang (2004) because we examine how transaction costs affect prices in a setting where agents trade to share risk. Lo, Mamaysky and Wang assume fixed costs, while we focus on proportional costs.[15] Our results on how transaction costs affect the illiquidity λ and price reversal γ are new.

6.1 Equilibrium

Transaction costs generate a bid-ask spread in Period 1. An agent buying one share pays the price S_1 plus the transaction cost κ, and so faces an effective ask price $S_1 + \kappa$. Conversely, an agent selling one share receives S_1 but pays κ, and so faces an effective bid price $S_1 - \kappa$. The bid-ask spread is independent of transaction size because transaction costs are proportional. Because of the spread, trade occurs only if the liquidity shock z is sufficiently large. Suppose, for example, that $z > 0$, in which case liquidity demanders value the asset less than liquidity suppliers. If liquidity suppliers buy, their demand function is as in Section 3 (Eq. (3.7a)), but with $S_1 + \kappa$ taking the place of S_1, i.e.,

$$\theta_1^s = \frac{\bar{D} - S_1 - \kappa}{\alpha\sigma^2}. \tag{6.1}$$

Conversely, if liquidity demanders sell, their demand function is as in Section 3 (Eq. (3.7b)), but with $S_1 - \kappa$ taking the place of S_1, i.e.,

$$\theta_1^d = \frac{\bar{D} - S_1 + \kappa}{\alpha\sigma^2} - z. \tag{6.2}$$

[15] Equilibrium with proportional costs has mainly been studied in settings where agents trade because of life-cycle or consumption-smoothing motives, rather than risk sharing. See, for example, Amihud and Mendelson (1986), Vayanos (1998, 2004), Vayanos and Vila (1999), Huang (2002), and Acharya and Pedersen (2005) for life-cycle motives, and Aiyagari and Gertler (1991) and Heaton and Lucas (1996) for consumption-smoothing motives. See also Constantinides (1986) who derives general-equilibrium implications of transaction costs from a partial-equilibrium setting where an agent engages in dynamic portfolio rebalancing. The trading frequencies implied by the various motives differ: they are low for life cycle and consumption smoothing and higher for portfolio rebalancing and risk sharing.

Since in equilibrium agents enter Period 1 holding $\bar{\theta}$ shares of the risky asset, trade occurs if there exists a price S_1 such that $\theta_1^s > \bar{\theta}$ and $\theta_1^d < \bar{\theta}$. Using (6.1) and (6.2), we can write these conditions as

$$\kappa < \bar{D} - S_1 - \alpha\sigma^2\bar{\theta} < \alpha\sigma^2 z - \kappa.$$

Therefore, trade occurs if $z > \frac{2\kappa}{\alpha\sigma^2} \equiv \hat{\kappa}$, i.e., the liquidity shock z is large relative to the transaction cost κ. The price can be determined by substituting (6.1) and (6.2) into the market-clearing equation (3.8). Repeating the analysis for $z < 0$, we can derive the following proposition.

Proposition 6.1 *The equilibrium in Period 1 is as follows:*

- $|z| \leq \hat{\kappa}$: *Agents do not trade;*
- $|z| > \hat{\kappa}$: *All agents trade and the price is*

$$S_1 = \bar{D} - \alpha\sigma^2 \left[\bar{\theta} + \pi z + \hat{\kappa}\left(\tfrac{1}{2} - \pi\right)\mathrm{sign}(z)\right]. \tag{6.3}$$

The effect of transaction costs on the price depends on the relative measures of liquidity suppliers and demanders. Suppose, for example, that $z > 0$. In the absence of transaction costs, liquidity demanders sell and the price drops. Because transaction costs deter liquidity suppliers from buying, they tend to depress the price, amplifying the effect of z. At the same time, transaction costs deter liquidity demanders from selling, and this tends to raise the price, dampening the effect of z. The overall effect depends on agents' relative measures. If $\pi < 1/2$ (more suppliers than demanders), the impact on suppliers dominates, and transaction costs amplify the effect of z. The converse holds if $\pi > 1/2$. The price in Period 0 takes a form similar to that in the perfect-market benchmark.[16]

Proposition 6.2 *The price in Period 0 is given by (3.13), where*

$$M = \frac{\int_0^{\hat{\kappa}} \exp\left(\frac{1}{2}\alpha^2\sigma^2 z^2\right)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz + \int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}{\int_0^{\hat{\kappa}} f(z)dz + \int_{\hat{\kappa}}^{\infty}\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]f(z)dz}, \tag{6.4}$$

[16]Extending our analysis to fixed costs is more complicated because agents' optimization problems become non-convex. Non-convexity can give rise to multiple solutions, meaning that agents of the same type (suppliers or demanders) can fail to take the same action. Suppose, for example, that all agents start with the same position $\theta_0 = \bar{\theta}$ in Period 0. As with proportional costs, all agents trade in Period 1 if the liquidity shock z is large, while no agent trades if z is small. For intermediate values of z, however, some agents pay the fixed cost and trade, while others of the same type do not trade.

A further complication arising from non-convexity is that $\theta_0 = \bar{\theta}$ is not an equilibrium. Indeed, consider a deviation from $\theta_0 = \bar{\theta}$ in either direction. The trades that become profitable in the margin are those whose surplus equals the fixed cost. But while the net surplus of these trades is zero, the marginal surplus (i.e., the derivative with respect to θ_0) is non-zero. Thus, expected utility at $\theta_0 = \bar{\theta}$ has a local minimum and a kink, implying that identical agents in Period 0 choose different positions in equilibrium.

$$\Delta_1 = \frac{\alpha\sigma^2 \left[\int_0^{\hat{\kappa}} \exp\left(\frac{1}{2}\alpha^2\sigma^2 z^2\right) \text{sh}\left(\alpha^2\sigma^2\bar{\theta}z\right) z f(z)dz + \int_{\hat{\kappa}}^{\infty} \Gamma(z)\text{sh}(\alpha^2\sigma^2\bar{\theta}z)[\pi z + (1-\pi)\hat{\kappa}] f(z)dz\right]}{\bar{\theta}\left[\int_0^{\hat{\kappa}} \exp\left(\frac{1}{2}\alpha^2\sigma^2 z^2\right) \text{ch}(\alpha^2\sigma^2\bar{\theta}z) f(z)dz + \int_{\hat{\kappa}}^{\infty} \Gamma(z)\text{ch}(\alpha^2\sigma^2\bar{\theta}z) f(z)dz\right]}, \tag{6.5}$$

$$\Gamma(z) = \exp\left[\tfrac{1}{2}\alpha^2\sigma^2 z^2 - \tfrac{1}{2}\alpha^2\sigma^2(1-\pi)^2(z-\hat{\kappa})^2\right]. \tag{6.6}$$

6.2 Transaction Costs and Illiquidity

We next examine how transaction costs impact the illiquidity measures and the illiquidity discount. Because transaction costs deter liquidity suppliers from trading, they raise illiquidity λ. Note that λ rises even when transaction costs dampen the effect of the liquidity shock z on the price. Indeed, dampening occurs not because of enhanced liquidity supply, but because liquidity demanders scale back their trades.

Proposition 6.3 *Illiquidity λ is*

$$\lambda = \frac{\alpha\sigma^2}{1-\pi}\left[1 + \frac{\hat{\kappa}}{2\pi}\frac{\int_{\hat{\kappa}}^{\infty}(z-\hat{\kappa})f(z)dz}{\int_{\hat{\kappa}}^{\infty}(z-\hat{\kappa})^2 f(z)dz}\right], \tag{6.7}$$

and is higher than without transaction costs ($\kappa = 0$).

Defining price reversal γ involves a slight complication because for small values of z there is no trade in Period 1, and therefore the price S_1 is not uniquely defined. We define price reversal conditional on trade in Period 1. The empirical counterpart of our definition is that no-trade observations are dropped from the sample. Transaction costs affect price reversal both because they limit trade to large values of z, and because they impact the price conditional on trade occuring. The first effect raises price reversal. The second effect works in the same direction when transaction costs amplify the effect of z on the price, i.e., when $\pi < 1/2$.

Proposition 6.4 *Price reversal γ is*

$$\gamma = \alpha^2\sigma^4 \frac{\int_{\hat{\kappa}}^{\infty}\left[\pi z + \left(\frac{1}{2}-\pi\right)\hat{\kappa}\right]^2 f(z)dz}{\int_{\hat{\kappa}}^{\infty} f(z)dz}. \tag{6.8}$$

It is increasing in the transaction cost coefficient κ if $\pi \leq 1/2$.

Because transaction costs hinder trade in Period 1, a natural conjecture is that they raise the illiquidity discount. When, however, $\pi \approx 1$, transaction costs can lower the discount. The intuition is that for $\pi \approx 1$ liquidity suppliers are the short side of the market and stand to gain the most from trade. Therefore, transaction costs hurt them the most, and reduce the utility differential between suppliers and demanders. This lowers the risk-neutral probability of being a demander, and can lower the discount. Transaction costs always raise the discount when $\pi \leq 1/2$.

Proposition 6.5 *The price in Period 0 is decreasing in the transaction cost coefficient κ if $\pi \leq 1/2$.*

We can sharpen the results of Propositions 6.4 and 6.5 by assuming specific distributions for the liquidity shock z. When z is drawn from a two-point distribution, transaction costs raise price reversal γ for all values of π, but lower the illiquidity discount for $\pi \approx 1$. When z is normal, transaction costs raise γ for all values of π, and numerical calculations suggest that they also raise the discount for all values of π.

To derive comparative statics with respect to the variance σ_z^2 of z, we assume again specific distributions. When z is drawn from a two-point distribution, an increase in σ_z^2 lowers λ, while the effects on γ and the discount are as in the perfect-market benchmark. Same comparative statics on (λ, γ) hold when z is normal, and numerical calculations suggest same comparative statics on the discount. The intuition why λ decreases in σ_z^2 is that when liquidity shocks are large, the main determinant of λ is not the bid-ask spread, which is affected by transaction costs, but the suppliers' risk aversion. Since the relative importance of the bid-ask spread decreases when σ_z^2 increases, λ decreases. Proposition 6.6 summarizes the results in the case of a two-point distribution.

Proposition 6.6 *Suppose that z is drawn from a two-point distribution, and trade occurs in Period 1 ($\sigma_z > \hat{\kappa}$). Illiquidity λ and price reversal γ are increasing in the transaction cost coefficient κ. An increase in the variance σ_z^2 of liquidity shocks lowers illiquidity λ, raises price reversal γ, and lowers the price in Period 0.*

7 Leverage Constraints

In this section we assume that agents' leverage is limited as a function of their capital. We derive a leverage constraint from agents' inability to commit to cover losses on levered positions solely by reducing consumption. For simplicity, we assume that agents must be able to cover losses in full. To ensure that such commitment is possible despite the lower bound on consumption, we replace normal distributions by distributions with bounded support.[17] We denote the support of the asset payoff D by $[\bar{D}-b_D, \bar{D}+b_D]$ and that of the liquidity shock z by $[-b_z, b_z]$. We assume that D and z are distributed symmetrically around their respective means, D is positive (i.e., $\bar{D}-b_D \geq 0$), and agents receive a positive endowment B of the riskless asset in Period 0. Because our focus is on how the leverage constraint influences the supply of liquidity, we impose it on liquidity suppliers only. Our analysis is closest to Gromb and Vayanos (2002), who study the supply of liquidity by leverage-constrained agents.[18] In Gromb and Vayanos, liquidity is supplied by arbitrageurs who trade two correlated zero-supply assets across segmented markets. We assume instead one risky asset in positive supply, and add an ex-ante stage (Period 0) where all agents are identical. Our analysis of how leverage constraints affect the illiquidity discount (computed in the ex-ante stage before liquidity shocks occur) is new.

7.1 Equilibrium

In Period 1, a liquidity demander chooses holdings θ_1^d of the risky asset to maximize the expected utility (3.4). The expectation over D is

$$-\exp\left\{-\alpha\left[W_1+\theta_1^d(\bar{D}-S_1)-f(\theta_1^d+z)\right]\right\}, \tag{7.1}$$

where

$$f(\theta)\equiv\frac{\log \mathrm{E}\exp\left[-\alpha\theta(D-\bar{D})\right]}{\alpha}. \tag{7.2}$$

Eq. (7.1) generalizes (3.5), derived under normality, to any symmetric distribution. The function $f(\theta)$, equal to $\frac{1}{2}\alpha\theta^2$ under normality, is positive, symmetric around the y-axis, and convex.[19] Maximizing (7.1) over θ_1^d yields the demand function

$$\theta_1^d=\left(f'\right)^{-1}(\bar{D}-S_1)-z. \tag{7.3}$$

[17] The assumption that losses must be covered in full is also implicit in the perfect-market benchmark. Dropping this assumption and allowing for default would expand the set of payoffs beyond those achieved by the traded assets. Suppose, for example, that an agent borrows cash to buy the risky asset. If the agent can default, his payoff is that of a call option on the risky asset. See Geanakoplos (2003) for a general analysis of margin contracts and an example where allowing for default entails no loss of generality.

[18] See also Geanakoplos (2003) and Geanakoplos and Zame (2009) for a general formulation of equilibrium with collateral and margin contracts. Kyle and Xiong (2001) and Xiong (2001) consider settings where liquidity suppliers face no leverage constraints but have logarithmic preferences. Logarithmic preferences require that consumption is non-negative. At the same time, because the marginal utility at zero consumption is infinite, the leverage constraint implied by non-negative consumption never binds.

[19] The function $\alpha f(\theta)$ is the cumulant-generating function of $-\alpha(D-\bar{D})$. Cumulant-generating functions are convex. Symmetry follows because D is distributed symmetrically around $\bar{D}$. Positivity follows from $f(0)=0$, symmetry and convexity.

Since $f(\theta)$ is convex, the demand θ_1^d is a decreasing function of the price S_1.

A liquidity supplier chooses holdings θ_1^s of the risky asset to maximize the expected utility

$$-\exp\left\{-\alpha\left[W_1+\theta_1^s(\bar{D}-S_1)-f(\theta_1^s)\right]\right\}, \tag{7.4}$$

which can be derived from (7.1) by setting $z = 0$. The optimization is subject to a leverage constraint. Indeed, losses from investing in the risky asset can be covered by wealth W_1 or negative consumption. Since suppliers must be able to cover losses in full, and cannot commit to consume less than $-A$, losses cannot exceed $W_1 + A$, i.e.,

$$\theta_1^s(S_1 - D) \leq W_1 + A \quad \text{for all } D.$$

This yields the constraint

$$m|\theta_1^s| \leq W_1 + A, \tag{7.5}$$

where

$$m \equiv S_1 - \min_D D \quad \text{if} \quad \theta_1^s > 0, \tag{7.6a}$$

$$m \equiv \max_D D - S_1 \quad \text{if} \quad \theta_1^s < 0. \tag{7.6b}$$

The constraint (7.5) requires that a position of θ_1^s shares is backed by capital $m|\theta_1^s|$. This limits the size of the position as a function of the capital $W_1 + A$ available to suppliers in Period 1. Suppliers' capital is the sum of the capital W_1 that they physically own in Period 1, and the capital A that they can access through their commitment to consume $-A$ in Period 2. The parameter m is the required capital per share of levered position, and can be interpreted as a margin or haircut. The margin is equal to the maximum possible loss per share. For example, the margin (7.6a) for a long position does not exceed the asset price S_1, and is strictly smaller if the asset payoff D has a positive lower bound (i.e., $\min_D D = \bar{D} - b_D > 0$).[20]

Intuitively, the constraint (7.5) can bind when there is a large discrepancy between the price S_1 and the expected payoff $\bar{D}$, since this is when liquidity suppliers want to hold large positions. There is, however, a countervailing effect because of a decrease in the margin. When, for example, S_1 is low, suppliers want to hold large long positions, but the margin is small because the maximum possible loss is small. The required capital (position size times margin) increases in the discrepancy between S_1 and $\bar{D}$ under the sufficient condition

$$2\alpha\pi b_D b_z < 1, \tag{7.7}$$

[20] The margins (7.6a) for a long position and (7.6b) for a short position are finite because D has bounded support. Our analysis can accommodate short-sale constraints, i.e., infinite margins for short positions, by setting the upper bound of D to infinity.

which for simplicity we assume from now on.

Proposition 7.1 *The equilibrium in Period 1 has the following properties:*

- *The leverage constraint (7.5) never binds if*

$$B + A + \bar{\theta}(\bar{D} - b_D) - \pi b_z \left[b_D - f'(\bar{\theta} + \pi b_z)\right] \geq 0. \tag{7.8}$$

Otherwise, (7.5) binds for $z \in [-b_z, -\bar{z}) \cup (\underline{z}, b_z]$, *where* $0 < \underline{z} < \bar{z} \leq b_z$.

- *An increase in* z *lowers the price* S_1 *and raises the liquidity suppliers' position* θ_1^s. *When (7.5) does not bind,* $\theta_1^s = \bar{\theta} + \pi z$ *and*

$$S_1 = \bar{D} - f'(\bar{\theta} + \pi z). \tag{7.9}$$

The leverage constraint never binds if agents receive a large endowment B of the riskless asset in Period 0, or if they can commit to a large negative consumption $-A$ in Period 2. In both cases, the capital that they can access in Period 1 is large. If instead B and A are small, the constraint binds for large positive and possibly large negative values of the liquidity shock z. For example, when z is large and positive, the price S_1 is low and liquidity suppliers are constrained because they want to hold large long positions. Setting

$$K^* \equiv \pi b_z \left[b_D - f'(\bar{\theta} + \pi b_z)\right] - \bar{\theta}(\bar{D} - b_D),$$

we refer to the region $B + A > K^*$, where liquidity suppliers are well-capitalized and the constraint never binds, as the *abundant-capital* region, and to the region $B + A < K^*$, where the constraint binds for some values of z, as the *scarce-capital* region. Note that in both regions, the constraint does not bind for $z = 0$. Indeed, the unconstrained outcome for $z = 0$ is that liquidity suppliers maintain their endowments $\bar{\theta}$ of the risky asset and B of the riskless asset. Since this yields positive consumption, the constraint is met.

An increase in the liquidity shock z lowers the price S_1 and raises the liquidity suppliers' position θ_1^s. These results are the same as in the perfect-market benchmark of Section 3, but the intuition is more complicated when the leverage constraint binds. Suppose that capital is scarce (i.e., $B + A < K^*$), and z is large and positive, in which case suppliers hold long positions and are constrained. The intuition why they can buy more, despite the constraint, when z increases is as follows. Since the price S_1 decreases, suppliers realize a capital loss on the $\bar{\theta}$ shares of the risky asset that they carry from Period 0. This reduces their wealth in Period 1 and tightens the constraint. At the same time, a decrease in S_1 triggers an equal decrease in the margin (7.6a) for long positions, and loosens the constraint. This effect is equivalent to a capital gain on the θ_1^s shares that suppliers hold in Period 1. Because suppliers are net buyers for $z > 0$ (i.e., $\theta_1^s > \bar{\theta}$), the latter effect dominates, and suppliers can buy more in response to an increase in z.

To determine the price in Period 0, we make the simplifying assumption that the risk-aversion coefficient α is small. We denote by (σ^2, σ_z^2) the variances of (D, z), by $k \equiv \frac{\mathrm{E}[D-\bar{D}]^4}{\sigma^4} - 3$ the curtosis of D, by $F(z)$ the cumulative distribution function of z, and by $o(\alpha^n)$ terms smaller than α^n.

Proposition 7.2 *Suppose that α is small. The price in Period 0 is*

$$S_0 = \bar{D} - \alpha\sigma^2\bar{\theta} - \alpha^3\sigma^4\left[\left(1 + \tfrac{1}{2}k\right)\sigma_z^2\pi^2 + \tfrac{1}{6}k\bar{\theta}^2\right]\bar{\theta} + o(\alpha^3) \tag{7.10}$$

when capital is abundant, and

$$S_0 = \bar{D} - \alpha\sigma^2\bar{\theta} - \alpha\sigma^2(1-\pi)\left[\int_{\underline{z}}^{\overline{z}}(z - \underline{z})dF(z) + \int_{\overline{z}}^{b_z}(\overline{z} - \underline{z})dF(z)\right] + o(\alpha) \tag{7.11}$$

when capital is scarce.

7.2 Leverage Constraints and Illiquidity

We next examine how the leverage constraint impacts the illiquidity measures and the illiquidity discount. We compute these variables in the abundant-capital region (liquidity suppliers are well-capitalized and unconstrained by leverage for all values of the liquidity shock z), and compare with the scarce-capital region.

Proposition 7.3 *Suppose that α is small or z is drawn from a two-point distribution. Illiquidity λ is higher when capital is scarce than when it is abundant.*

Proposition 7.4 *Price reversal γ is higher when capital is scarce than when it is abundant.*

The intuition is as follows. When the liquidity shock z is close to zero, the constraint does not bind in both the abundant- and scarce-capital regions, and therefore price and volume are identical in the two regions. For larger values of z, the constraint binds when capital is scarce, impairing suppliers' ability to accommodate an increase in z. As a result, an increase in z has a larger effect on price and a smaller effect on volume. Since the effect on price is larger, so is the price reversal γ. Illiquidity λ is also larger because it measures the price impact per unit of volume. Note that λ measures an average price impact, i.e., the average slope of the relationship between price change and signed volume. This relationship exhibits an important non-linearity when capital is scarce: the slope increases for large values of z, which is when the constraint binds. This property distinguishes leverage constraints from other imperfections.

The illiquidity discount is higher when capital is scarce. This is because the leverage constraint binds asymmetrically: it is more likely to bind when liquidity demanders sell ($z > 0$) than when they buy ($z < 0$). Indeed, the constraint binds when the suppliers' position is large in absolute value—and a large position is more likely when suppliers buy in Period 1 because this adds to the long position $\bar{\theta}$ that they carry from Period 0. Since price movements in Period 1 are exacerbated when the constraint binds, and the constraint is more likely to bind when demanders sell, the average price in Period 1 is lower when capital is scarce. This yields a lower price in Period 0.

Proposition 7.5 *Suppose that α is small. The price in Period 0 is lower when capital is scarce than when it is abundant.*

We next consider an increase in the magnitude of liquidity shocks. We scale up all shocks uniformly, replacing z by ωz for a scalar $\omega > 1$.[21]

Proposition 7.6 *Suppose that α is small, and all liquidity shocks are multiplied by $\omega > 1$.*

- *If under the new distribution capital is abundant, then illiquidity λ remains the same (to the highest order in α), price reversal γ increases, and the price in Period 0 decreases.*
- *If under the new distribution capital is scarce, then illiquidity λ increases, price reversal γ increases, and the price in Period 0 decreases.*

The comparative statics when capital is abundant are the same as for the perfect-market benchmark of Section 3. When instead capital is scarce, an increase in the shocks' magnitude increases illiquidity. This result is different than for other imperfections, and is due to the non-linearity of the relationship between price change and signed volume: the relationship becomes stronger when the constraint binds, and the constraint is more likely to bind when shocks are larger.

Our analysis can be extended to the case where the leverage constraint (7.5) holds with a margin m that is constant, rather than a function of price as in (7.6a)-(7.6b). A constant margin

[21] Other sections consider an increase in the variance σ_z^2 of liquidity shocks, assuming a normal or a two-point distribution. This is equivalent to scaling up all shocks uniformly.

yields different implications for how liquidity suppliers respond to an increase in the liquidity shock z: while their position θ_1^s increases under the margin (7.6a)-(7.6b), it can decrease under a constant margin. Indeed, suppose that suppliers hold long positions and are constrained. An increase in z lowers S_1, triggering a capital loss and a tightening of the constraint. Under the margin (7.6a)-(7.6b), there is the countervailing and dominant effect that the margin decreases. This effect does not exist under a constant margin, and therefore suppliers are forced to sell. Liquidity suppliers thus consume liquidity: in response to selling pressure by demanders, they sell (to demanders). This yields *amplification*: following an increase in the liquidity shock z, the price drops, triggering sales by suppliers, amplifying the price drop, triggering more sales, etc. In particular, the price drop is larger than in the suppliers' absence.[22]

8 Non-Competitive Behavior

In this section we assume that liquidity demanders behave as a single monopolist in Period 1. We consider both the case where liquidity demanders have no private information on asset payoffs, and so information is symmetric, and the case where they observe the private signal (2.3), and so information is asymmetric. (The second case nests the first by setting the variance σ_ϵ^2 of the signal noise to infinity.) We show that strategic behavior by liquidity demanders influences the supply of liquidity, even though liquidity suppliers are competitive. The trading mechanism in Period 1 is that liquidity suppliers submit a demand function and liquidity demanders submit a market order, i.e., a price-inelastic demand function. Restricting liquidity demanders to trade by market order is without loss of generality: since they know all available information in Period 1, they know the demand function of liquidity suppliers. Our analysis of equilibrium in Period 1 is closely related to Bhattacharya and Spiegel (1991) because we assume that an informed monopolist with a hedging motive trades with competitive risk-averse agents.[23] Our analysis of equilibrium in Period 0 is new, and so are the results on how non-competitive behavior affects the illiquidity discount and the price reversal γ.

[22] Amplification can arise even in the presence of countervailing variation in margins, and for constraints derived endogenously in the spirit of (7.5). See, for example, Gromb and Vayanos (2002) and Geanakoplos (2003).

[23] Strategic behavior under asymmetric information has mainly been studied in a setting introduced by Kyle (1985), where strategic informed traders trade with competitive risk-neutral market makers and noise traders. Risk neutrality simplifies the derivations, but also eliminates any effect of illiquidity on expected returns. Indeed, expected returns are equal to the riskless rate because market makers are competitive and risk-neutral. See also Glosten and Milgrom (1985), Easley and O'Hara (1987) and Admati and Pfleiderer (1988) for other settings with competitive risk-neutral market makers and noise traders.

8.1 Equilibrium

We conjecture that the price in Period 1 has the same affine form (4.1) as in the competitive case, with possibly different constants (a, b, c). Given (4.1), the demand function of liquidity suppliers is (4.4a) as in the competitive case. Substituting (4.4a) into the market-clearing equation (3.8), and using (4.3a), yields the price in Period 1 as a function of the liquidity demanders' market order θ_1^d:

$$S_1(\theta_1^d) = \frac{\bar{D} - \frac{\beta_\xi}{b}a + \frac{\alpha\sigma^2[D|S_1]}{1-\pi}(\pi\theta_1^d - \bar{\theta})}{1 - \frac{\beta_\xi}{b}}. \tag{8.1}$$

Liquidity demanders choose θ_1^d to maximize the expected utility

$$-\mathrm{E}\exp\left\{-\alpha\left[W_1 + \theta_1^d\left(D - S_1(\theta_1^d)\right) + z(D - \bar{D})\right]\right\}. \tag{8.2}$$

The difference with the competitive case is that liquidity demanders behave as a single monopolist and take into account the impact of their order θ_1^d on the price S_1. Proposition 8.1 characterizes the solution to the liquidity demanders' optimization problem.

Proposition 8.1 *The liquidity demanders' market order in Period 1 satisfies*

$$\theta_1^d = \frac{\mathrm{E}[D|s] - S_1(\theta_1^d) - \alpha\sigma^2[D|s]z + \hat{\lambda}\bar{\theta}}{\alpha\sigma^2[D|s] + \hat{\lambda}}, \tag{8.3}$$

where $\hat{\lambda} \equiv \frac{dS_1(\theta_1^d)}{d\theta_1^d} = \frac{\alpha\pi\sigma^2[D|S_1]}{(1-\pi)\left(1-\frac{\beta_\xi}{b}\right)}$.

Eq. (8.3) determines θ_1^d implicitly because it includes θ_1^d in both the left- and the right-hand side. We write θ_1^d in the form (8.3) to facilitate the comparison with the competitive case. Indeed, the competitive counterpart of (8.3) is (4.4b), and can be derived by setting $\hat{\lambda}$ to zero. The parameter $\hat{\lambda}$ measures the price impact of liquidity demanders, and is closely related to the illiquidity λ. Because in equilibrium $\hat{\lambda} > 0$, the denominator of (8.3) is larger than that of (4.4b), and therefore θ_1^d is less sensitive to changes in $\mathrm{E}[D|s] - S_1$ and z than in the competitive case. Intuitively, because liquidity demanders take price impact into account, they trade less aggressively in response to their signal and their liquidity shock.

Substituting (4.4a) and (8.3) into the market-clearing equation (3.8), and proceeding as in Section 4, we find a system of three equations in (a, b, c). Proposition 8.2 solves this system.

Proposition 8.2 *The price in Period 1 is given by (4.1), where*

$$b = \frac{\pi\beta_s\sigma^2[D|S_1] + (1-\pi)\beta_\xi\sigma^2[D|s]}{2\pi\sigma^2[D|S_1] + (1-\pi)\sigma^2[D|s]}, \tag{8.4}$$

and (a, c) are given by (4.6a) and (4.6c), respectively. The linear equilibrium exists if $\sigma_\epsilon^2 > \hat{\sigma}_\epsilon^2$, where $\hat{\sigma}_\epsilon^2$ is the positive solution of

$$\alpha^2\hat{\sigma}_\epsilon^4\sigma_z^2 = \sigma^2 + \hat{\sigma}_\epsilon^2. \tag{8.5}$$

The price in the competitive market in Period 0 can be determined through similar steps as in previous sections.

Proposition 8.3 *The price in Period 0 is given by (3.13), where*

$$M = \exp\left(\tfrac{1}{2}\alpha\Delta_2\bar{\theta}^2\right)\sqrt{\frac{1+\Delta_0\pi^2}{1+\Delta_0\left(1+\frac{2\hat{\lambda}}{\alpha\sigma^2[D|s]}\right)(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}}, \tag{8.6}$$

$$\Delta_1 = \frac{\alpha^3 b\sigma^2(\sigma^2+\sigma_\epsilon^2)\sigma_z^2}{1+\Delta_0\left(1+\frac{2\hat{\lambda}}{\alpha\sigma^2[D|s]}\right)(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}, \tag{8.7a}$$

$$\Delta_2 = \frac{\alpha^3\sigma^4\sigma_z^2\left[1+\frac{\alpha(\beta_s-b)^2(\sigma^2+\sigma_\epsilon^2)\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}\right]}{1+\Delta_0\left(1+\frac{2\hat{\lambda}}{\alpha\sigma^2[D|s]}\right)(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}, \tag{8.7b}$$

and Δ_0 is given by (4.7a).

8.2 Non-Competitive Behavior and Illiquidity

We next examine how non-competitive behavior impacts the illiquidity measures and the illiquidity discount.

Proposition 8.4 *Illiquidity λ is given by (4.8). It is the same as under competitive behavior when information is symmetric, and higher when information is asymmetric.*

Although illiquidity is given by the same equation as under competitive behavior, it is higher when behavior is non-competitive because the coefficient b is smaller. Intuitively, when liquidity demanders take price impact into account, they trade less aggressively in response to their signal and their liquidity shock. This reduces the size of both information- and liquidity-generated trades. The relative size of the two types of trades remains the same, and so does price informativeness, measured by the signal-to-noise ratio. Monopoly trades thus have the same informational content as competitive trades, but are smaller in size. As a result, the signal per trade size is higher, and so is the price impact of trades and the illiquidity λ. Non-competitive behavior has no effect on illiquidity when information is symmetric because trades have no informational content.

An increase in information asymmetry, through a reduction in the variance σ_ϵ^2 of the signal noise, generates an illiquidity spiral. Because illiquidity increases, liquidity demanders scale back their trades. This raises the signal per trade size, further increasing illiquidity. When information asymmetry becomes severe, illiquidity becomes infinite and trade ceases, leading to a market breakdown. This occurs when $\sigma_\epsilon^2 \leq \hat{\sigma}_\epsilon^2$, i.e., for values of σ_ϵ^2 such that the equilibrium of Proposition 8.2 does not exist. Non-competitive behavior is essential for the non-existence of an equilibrium with trade because such an equilibrium always exists under competitive behavior.[24]

Proposition 8.5 *Price reversal γ is given by (4.9), and is lower than under competitive behavior.*

Although price reversal is given by the same equation as under competitive behavior, it is lower when behavior is non-competitive because the coefficient b is smaller. Intuitively, price reversal arises because the liquidity demanders' trades in Period 1 cause the price to deviate from fundamental value. Under non-competitive behavior, these trades are smaller and so is price reversal. Note that non-competitive behavior has opposite effects on the two illiquidity measures: illiquidity λ increases but price reversal γ decreases.

While illiquidity λ is higher under non-competitive behavior, the illiquidity discount can be lower. This is because liquidity demanders scale back their trades, rendering the price less responsive to their liquidity shock and obtaining better insurance against the shock. This effect drives the illiquidity discount below the competitive value when information is symmetric. When information is asymmetric, the comparison can reverse. This is because the scaling back of trades generates the spiral of increasing illiquidity, and this reduces the insurance received by liquidity demanders.

Proposition 8.6 *The price in Period 0 is higher than under competitive behavior when information is symmetric, but can be lower when information is asymmetric.*

[24] There exist settings, however, where asymmetric information leads to market breakdowns even with competitive agents. See Akerlof (1970) for a setting where agents trade heterogeneous goods of different qualities, and Glosten and Milgrom (1985) for an asset-market setting.

The comparative statics with respect to the variance σ_z^2 of liquidity shocks are the same as under competitive behavior.

Proposition 8.7 *An increase in the variance σ_z^2 of liquidity shocks leaves illiquidity λ unchanged under symmetric information but lowers it under asymmetric information. It raises price reversal and lowers the price in Period 0.*

9 Search

In this section we assume that agents do not meet in a centralized exchange in Period 1, but instead must search for counterparties. When a liquidity demander meets a supplier, they bargain bilaterally over the terms of trade, i.e., the number of shares traded and the share price. We assume that bargaining leads to an efficient outcome, and denote by $\phi \in [0, 1]$ the fraction of transaction surplus appropriated by suppliers. We denote by N the measure of bilateral meetings between demanders and suppliers. This parameter characterizes the efficiency of the search process, and is bounded by $\min\{\pi, 1-\pi\}$ since there cannot be more meetings than demanders or suppliers. Assuming that all meetings are equally likely, the probability of a demander meeting a supplier is $\pi^d \equiv N/\pi$, and of a supplier meeting a demander is $\pi^s \equiv N/(1-\pi)$. Our analysis is closest to Lagos, Rocheteau and Weill (2009), who study asset-market search in a continuous-time model where agents can hold arbitrary positions and there are aggregate liquidity shocks.[25] Our results on how the search friction affects the illiquidity λ and price reversal γ are new.[26]

9.1 Equilibrium

Prices in Period 1 are determined through pairwise bargaining between liquidity demanders and suppliers. Agents' outside option is not to trade and retain their positions from Period 0, which in equilibrium are equal to $\bar{\theta}$. The consumption in Period 2 of a liquidity supplier who does not trade in Period 1 is $C_2^{sn} = W_0 + \bar{\theta}(D - S_0)$. This generates a certainty equivalent

$$CEQ^{sn} = W_0 + \bar{\theta}(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2, \tag{9.1}$$

[25] See also Duffie, Garleanu and Pedersen (2008) and Weill (2007), who study aggregate shocks under the assumption that the positions of some agents can take one of two values. Most search models of asset markets assume deterministic steady states and no aggregate shocks, following Duffie, Garleanu and Pedersen (2005).

[26] Duffie, Garleanu and Pedersen (2008) and Weill (2007) show in the context of numerical examples that prices recover more slowly from shocks in a search market than in a centralized market. Lagos, Rocheteau and Weill (2009) show that the speed of recovery is non-monotonic in the search friction. None of these papers, however, relates the speed of recovery to λ or γ.

where the first two terms are the expected consumption, and the third a risk adjustment quadratic in position size. If the supplier buys x shares at price S_1, the certainty equivalent becomes

$$CEQ^s = W_0 + \bar{\theta}(\bar{D} - S_0) + x(\bar{D} - S_1) - \tfrac{1}{2}\alpha\sigma^2(\bar{\theta} + x)^2 \tag{9.2}$$

because the position becomes $\bar{\theta} + x$. Likewise, the certainty equivalent of a liquidity demander who does not trade in Period 1 is

$$CEQ^{dn} = W_0 + \bar{\theta}(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2(\bar{\theta} + z)^2, \tag{9.3}$$

and if the demander sells x shares at price S_1, the certainty equivalent becomes

$$CEQ^d = W_0 + \bar{\theta}(\bar{D} - S_0) - x(\bar{D} - S_1) - \tfrac{1}{2}\alpha\sigma^2(\bar{\theta} + z - x)^2. \tag{9.4}$$

Under efficient bargaining, x maximizes the sum of certainty equivalents $CEQ^s + CEQ^d$. The maximization yields $x = z/2$, i.e., the liquidity shock is shared equally between the two agents. The price S_1 is such that the supplier receives a fraction ϕ of the transaction surplus, i.e.,

$$CEQ^s - CEQ^{sn} = \phi\left(CEQ^s + CEQ^d - CEQ^{sn} - CEQ^{dn}\right). \tag{9.5}$$

Proposition 9.1 *When a supplier and a demander meet in Period 1, the supplier buys $z/2$ shares at the price*

$$S_1 = \bar{D} - \alpha\sigma^2\left[\bar{\theta} + \tfrac{1}{4}z(1 + 2\phi)\right]. \tag{9.6}$$

Eq. (9.6) implies that the impact of the liquidity shock z on the price in Period 1 increases in the liquidity suppliers' bargaining power ϕ. When, for example, $z > 0$, liquidity demanders need to sell, and greater bargaining power by suppliers results in a lower price. Comparing (9.6) to its centralized-market counterpart (3.9) reveals an important difference: price impact in the search market depends on the distribution of bargaining power within a meeting, characterized by the parameter ϕ, while price impact in the centralized market depends on aggregate demand-supply conditions, characterized by the measures $(\pi, 1 - \pi)$ of demanders and suppliers.[27] The price in the centralized market in Period 0 can be determined through similar steps as in previous sections.

[27] That ϕ is the sole determinant of price impact in the search market is a special feature of our model, where search occurs only within one period. When instead agents' outside option is to search again (as in, e.g., Duffie, Garleanu and Pedersen (2005, 2008)), price impact is influenced not only by ϕ, but also by the measures of liquidity demanders and suppliers and the efficiency of the search process.

Proposition 9.2 *The price in Period 0 is*

$$S_0 = \bar{D} - \alpha\sigma^2\bar{\theta} - \frac{\frac{N(1+\phi)}{2G_2^{\frac{3}{2}}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_2}\right) + \frac{\pi-N}{G_3^{\frac{3}{2}}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_3}\right)}{\frac{N}{\sqrt{G_1}} + 1 - \pi - N + \frac{N}{\sqrt{G_2}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_2}\right) + \frac{\pi-N}{\sqrt{G_3}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_3}\right)}\alpha^3\sigma^4\sigma_z^2\bar{\theta}, \quad (9.7)$$

where

$$G_1 = 1 + \tfrac{1}{2}\phi\alpha^2\sigma^2\sigma_z^2,$$

$$G_2 = 1 - \tfrac{1}{2}(1+\phi)\alpha^2\sigma^2\sigma_z^2,$$

$$G_3 = 1 - \alpha^2\sigma^2\sigma_z^2.$$

9.2 Search and Illiquidity

We next examine how the search friction impacts the illiquidity measures and the illiquidity discount. We perform two related but distinct exercises: compare the search market with the centralized market of Section 3, and vary the measure N of meetings between liquidity demanders and suppliers.

When N decreases, the search process becomes less efficient and trading volume decreases. At the same time, the price in each meeting remains the same because it depends only on the distribution of bargaining power within the meeting. Since illiquidity λ measures the price impact of volume, it increases. One would expect that λ in the search market is higher than in the centralized market because only a fraction of suppliers are involved in bilateral meetings and provide liquidity ($N \leq 1 - \pi$). Proposition 9.3 confirms this result when bargaining power is symmetric ($\phi = 1/2$). The result is also true when suppliers have more bargaining power than demanders ($\phi > 1/2$) because the liquidity shock has then larger price impact. Moreover, the result extends to all values of ϕ when less than half of suppliers are involved in meetings ($N \leq (1-\pi)/2$).

Proposition 9.3 *Illiquidity* λ *is*

$$\lambda = \frac{\alpha\sigma^2(1+2\phi)}{2N}, \quad (9.8)$$

and increases when the measure N *of meetings decreases. It is higher than in the centralized market if* $\phi + 1/2 \geq N/(1-\pi)$.

Because the price in the search market is independent of N, so is the price reversal γ. Moreover, γ in the search market is higher than in the centralized market if ϕ is large relative to π.

Proposition 9.4 *Price reversal γ is*

$$\gamma = \frac{\alpha^2\sigma^4\sigma_z^2(1+2\phi)^2}{16}, \tag{9.9}$$

and is independent of the measure N of meetings. It is higher than in the centralized market if $\phi + 1/2 \geq 2\pi$.

When the measure N of meetings decreases, agents are less likely to trade in Period 1, and a natural conjecture is that the illiquidity discount increases. Proposition 9.5 confirms this conjecture under the sufficient condition $\phi \leq 1/2$. Intuitively, if $\phi \approx 1$, a decrease in the measure of meetings does not affect liquidity demanders because they extract no surplus from a meeting. Since, however, liquidity suppliers become worse off, the risk-neutral probability of being a demander decreases, and the price can increase.[28]

Proposition 9.5 *A decrease in the measure N of meetings lowers the price in Period 0 if $\phi \leq 1/2$.*

The comparative statics with respect to the variance σ_z^2 of liquidity shocks are as in the case of a centralized market.

Proposition 9.6 *An increase in the variance σ_z^2 of liquidity shocks leaves illiquidity λ unchanged, raises price reversal γ, and lowers the price in Period 0.*

10 Empirical Implications

In this section we explore implications of our model for empirical studies of liquidity.

10.1 Liquidity and Expected Returns

The concept of liquidity is central to certain areas of finance such as market microstructure or optimal trade execution. Yet, its importance for asset valuation remains unclear. Many empirical studies seek to establish a link between liquidity and expected asset returns.[29] The basic premise in these studies is that illiquidity is positively related to expected returns. Our analysis shows that

[28] The illiquidity discount in the search market is higher than in the centralized market if ϕ is large relative to π. This property is the same as for λ and γ, but the calculations are more complicated.

[29] The survey by Amihud, Mendelson and Pedersen (2006) includes detailed references.

the nature of this relationship depends crucially on the underlying cause of illiquidity. Indeed, while imperfections such as asymmetric information, participation costs, transaction costs, and leverage constraints raise expected returns, other imperfections such as non-competitive behavior and search can have the opposite effect. Since many imperfections can exist simultaneously in the market, the relationship between illiquidity and expected returns can become ambiguous. Identifying the main imperfection in specific contexts could help better estimate this relationship.

Type of Imperfection	Impact of Variance of Liquidity Shocks		
	Lambda	Price Reversal	Expected Return
Perfect-market benchmark	0	+	+
Asymmetric information	−	+	+
Participation costs	−	0	+
Transaction costs	−	+	+
Leverage constraints	+	+	+
Non-comp. behavior/Sym. info.	0	+	+
Non-comp. behavior/Asym. info.	−	+	+
Search	0	+	+

Table 2: **Impact of the variance of liquidity shocks on illiquidity and expected returns.** "Lambda" is the regression coefficient of the price change between Periods 0 and 1 on the signed volume of liquidity demanders in Period 1; "Price Reversal" is minus the autocovariance of price changes; and "Expected Return" is the expected return of the risky asset between Periods 0 and 2.

Even when the theoretical relationship between illiquidity and expected returns is unambiguous, confirming this relationship empirically in a cross-section of assets can be challenging. This is because cross-sectional variation can be driven by factors other than the imperfections themselves. For example, Table 2 summarizes how the variance σ_z^2 of liquidity shocks influences illiquidity and expected returns. Under all six imperfections, larger σ_z^2 leads to higher expected returns. The impact on lambda, however, is negative under asymmetric information, participation costs, transaction costs and non-competitive behavior. To explain why this might complicate cross-sectional tests, suppose, for example, that asymmetric information is the only imperfection. If it is also the main source of cross-sectional variation, then Table 1 implies a positive relationship between lambda and expected returns. If, however, asymmetric information is the same across assets and differences arise because of σ_z^2, then Table 2 implies a negative relationship. The same is true under participation costs, transaction costs, and non-competitive behavior. Therefore, our results on how factors other than the imperfections affect illiquidity and expected returns are relevant for cross-sectional tests. Knowing the effects of these factors, and finding suitable empirical controls, could help identify more precisely the effects of illiquidity.

10.2 Measures of Liquidity

A key question when studying liquidity is how to measure it empirically. We consider two widely used measures. The first is λ, defined as the regression coefficient of price changes on signed volume, and based on the idea that trades in illiquid markets should have large price impact. The second is γ, defined as minus the autocovariance of price changes, and based on the idea that trades in illiquid markets should generate large transitory deviations between price and fundamental value.[30] The measures λ and γ have been linked to illiquidity within models focusing on specific imperfections: λ in Kyle (1985), and γ in Roll (1984) and Grossman and Miller (1988). Using our unified model, we examine the behavior of these measures across a variety of imperfections.

In our analysis, λ captures not only the permanent component of price impact, driven by the information that trades convey (as in Kyle), but also the transitory component, driven by the risk aversion of liquidity suppliers. In this sense, λ overlaps with γ, which isolates the transitory component.[31] We further show that λ reflects the underlying imperfections more accurately than γ. Indeed, λ increases in the imperfections' presence, except possibly under search, while γ can decrease under asymmetric information, non-competitive behavior and search.

The benefits of λ relative to γ must be set against some drawbacks. First, λ might not reflect a causal effect of volume on prices. For example, if public news cause both volume and prices, λ can be positive even in the absence of a causal effect of volume on prices.[32] Second, estimating λ requires information on signed trades that might not be available, while estimating γ requires information only on transaction prices. Putting these issues aside, a broad implication of our analysis is that the validity of a measure of illiquidity can depend of the underlying imperfection.

Both λ and γ are unconditional measures: λ measures the average slope of the relationship between price change and signed volume, and γ measures the unconditional autocovariance. Our

[30]Measures closely related to λ are, for example, the regression-based measure of Glosten and Harris (1988) and Sadka (2006), and the ratio of average absolute returns to trading volume of Amihud (2002). Measures closely related to γ, are, for example, the bid-ask spread measure of Roll (1984), the Gibbs estimate of Hasbrouck (2006), the price reversal measure of Bao, Pan and Wang (2008), and the price reversal conditional on signed volume of Campbell, Grossman and Wang (1993).

[31]The overlap is larger between λ and the conditional price reversal of Campbell, Grossman and Wang (1993) because both measures condition on signed volume.

[32]The causality problem does not arise in our model. Indeed, volume is generated by shocks observable only to liquidity demanders, such as the liquidity shock z and the signal s. Since these shocks can affect prices only through the liquidity demanders' trades, λ measures correctly the price impact of these trades.

analysis has further implications for conditional measures. Consider, for example, the conditional λ, defined as the sensitivity of price to signed volume conditional on signed volume. Under asymmetric information, participation costs, non-competitive behavior and search, the relationship between price and signed volume is linear, and therefore conditional and unconditional λ's coincide. Under leverage constraints, however, the price is more sensitive to signed volume for large values of volume because this is when constraints bind. The opposite is true under transaction costs. Indeed, taking the price for zero volume to be the mid-point of the bid-ask spread, the price jumps discontinuously to the ask following arbitrarily small buy volume, and then increases continuously (thus becoming less sensitive to volume). Therefore, λ conditional on large volume is larger than unconditional λ under leverage constraints and smaller under transaction costs. These properties could help test for the presence of specific imperfections, or could themselves be tested in contexts where the imperfections can be identified.

10.3 Liquidity Across Assets and Markets

Liquidity varies considerably across assets and markets. For example, large stocks are more liquid than small stocks, on-the-run (i.e., newly issued) government bonds are more liquid than off-the-run bonds, highly rated corporate bonds are more liquid than lower rated bonds, and government bonds are more liquid than stocks, which are in turn more liquid than corporate bonds. Spiegel (2008) summarizes this evidence and argues that it poses a challenge to existing theories. For example, why are on-the-run bonds more liquid than off-the-run bonds with similar payoffs? And if stocks are less liquid than government bonds because they are riskier or more prone to asymmetric information, why are they more liquid than corporate bonds, where the same comparisons apply?

The variation in liquidity across assets and markets is hard to explain based on a single imperfection; multiple imperfections are likely to be at work, with their relative significance differing across markets. Because our model incorporates multiple imperfections, it has the potential to explain the variation in liquidity within a single framework. Consistent with previous work, we find that riskier assets are less liquid, and so are assets more prone to asymmetric information.[33] This helps explain why large stocks are more liquid than small stocks, highly rated corporate bonds are more liquid than lower rated bonds, and government bonds are more liquid than stocks: in each case the less liquid asset is riskier and more prone to asymmetric information.

The higher liquidity of on- relative to off-the-run bonds could be traced to a larger pool of

[33] Illiquidity λ increases in the variance σ^2 of asset payoffs even in the perfect-market benchmark (Eq. (3.19)). It also increases in the degree of information asymmetry, as measured by the precision of the private signal (Proposition 4.4).

liquidity suppliers. Indeed, as time from issuance increases, a bond migrates from the portfolios of dealers to those of end-investors. Since costs of market participation are larger for end-investors than for dealers, off-the-run bonds have a smaller pool of liquidity suppliers and are therefore less liquid (Proposition 6.3).[34]

The lower liquidity of corporate bonds relative to stocks might be because they are traded in a less competitive market. Indeed, while participation by individual investors in the stock market is significant, the corporate-bond market is dominated by large institutions. As a consequence, market power is more important for corporate bonds, implying lower liquidity (Proposition 8.4).[35]

11 Conclusion

We develop a unified model to examine how market imperfections affect liquidity and expected asset returns. Our model encompasses the following imperfections: asymmetric information, participation costs, transaction costs, leverage constraints, non-competitive behavior and search. Besides nesting these imperfections in a single model, we derive new results on the effects of each imperfection. Our results imply, in particular, that imperfections do not always raise expected returns, and can influence common measures of illiquidity in opposite directions.

One extension of our analysis is to consider interactions between imperfections. A natural interaction, studied in this paper, is between non-competitive behavior and asymmetric information. Other interactions, e.g., between asymmetric information and participation costs, could be studied as well. A related and more fundamental extension is to explore the economic links between imperfections. For example, if participation costs are costs to monitor market information, can costly participation be linked to asymmetric information? Such an extension could provide more guidance on the nature of different imperfections and their relative significance.

Another extension is to allow for multiple risky assets and additional trading periods. By introducing multiple assets, we can study more explicitly the cross-sectional relationship between liquidity and expected returns. By introducing additional trading periods, we can further study how liquidity changes over time and liquidity risk is priced.[36]

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[34] Vayanos and Weill (2008) flesh out a related explanation that emphasizes search rather than participation costs.

[35] The low participation by individual investors in the corporate-bond market could be explained partly by that market's low liquidity. At the same time, there are corporate-finance benefits to holding concentrated stakes in corporate bonds, e.g., bargaining power relative to holders of other bond issues of the same company during bankruptcy.

Note that our analysis focuses on non-competitive behavior by liquidity demanders. Non-competitive behavior by liquidity suppliers (e.g., dealers) could also be an important driver of the low liquidity of corporate bonds.

[36] See Acharya and Pedersen (2005) for a model of liquidity risk, in which the market imperfection is transaction costs that vary exogenously over time.

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Appendix

A Perfect-Market Benchmark

We start with a useful lemma.

Lemma A.1 *Let x be an $n \times 1$ normal vector with mean zero and covariance matrix Σ, A a scalar, B an $n \times 1$ vector, C an $n \times n$ symmetric matrix, I the $n \times n$ identity matrix, and $|M|$ the determinant of a matrix M. Then,*

$$\mathrm{E}_x \exp\left\{-\alpha\left[A + B'x + \tfrac{1}{2}x'Cx\right]\right\} = \exp\left\{-\alpha\left[A - \tfrac{1}{2}\alpha B'\Sigma(I+\alpha C\Sigma)^{-1}B\right]\right\}\frac{1}{\sqrt{|I+\alpha C\Sigma|}}. \quad \text{(A.1)}$$

Proof: When $C = 0$, (A.1) gives the moment-generating function of the normal distribution. We can always assume $C = 0$ by also assuming that x is a normal vector with mean 0 and covariance matrix $\Sigma(I + \alpha C\Sigma)^{-1}$. ∎

Proof of Proposition 3.1: Eqs. (3.7a) and (3.7b) follow by maximizing the term inside the exponential in (3.6) and (3.5), respectively. ∎

Proof of Proposition 3.2: We first compute the interim utilities U^s and U^d of liquidity suppliers and demanders in Period 1/2. The utility U^s is the expectation of (3.11) over z. To compute this expectation, we use Lemma A.1 and set

$$x \equiv z,$$

$$\Sigma \equiv \sigma_z^2,$$

$$A \equiv W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2,$$

$$B \equiv \alpha\sigma^2\pi(\bar{\theta} - \theta_0),$$

$$C \equiv \alpha\sigma^2\pi^2.$$

Eq. (A.1) implies that

$$\begin{aligned} U^s &= -\exp\left(-\alpha F^s\right)\frac{1}{\sqrt{1+\alpha^2\sigma^2\sigma_z^2\pi^2}} \\ &= -\exp\left(-\alpha F^s\right)\frac{1}{\sqrt{1+\Delta_0\pi^2}}, \end{aligned} \quad \text{(A.2)}$$

where Δ_0 is given by (3.15a) and

$$F^s = W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2 - \frac{\alpha^3\sigma^4\sigma_z^2\pi^2(\theta_0 - \bar{\theta})^2}{2\,(1 + \alpha^2\sigma^2\sigma_z^2\pi^2)}$$

$$= W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2 + \frac{\alpha\sigma^2(\theta_0 - \bar{\theta})^2}{2\,(1 + \alpha^2\sigma^2\sigma_z^2\pi^2)}. \tag{A.3}$$

To compute U^d, we derive the counterpart of (3.11) for a liquidity demander. Substituting θ_1^d from (3.7b), S_1 from (3.9), and W_1 from (3.10), we can write the expected utility (3.5) of a liquidity demander in Period 1 as

$$-\exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2(\theta_0 + z)(\bar{\theta} + \pi z) + \tfrac{1}{2}\alpha\sigma^2(\bar{\theta} + \pi z)^2\right]\right\}. \tag{A.4}$$

The utility U^d is the expectation of (A.4) over z. To compute this expectation, we use Lemma A.1 and set

$$x \equiv z,$$

$$\Sigma \equiv \sigma_z^2,$$

$$A \equiv W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2,$$

$$B \equiv -\alpha\sigma^2\left[\pi\theta_0 + (1-\pi)\bar{\theta}\right],$$

$$C \equiv -\alpha\sigma^2(2\pi - \pi^2).$$

Eq. (A.1) implies that

$$U^d = -\exp\left(-\alpha F^d\right)\frac{1}{\sqrt{1 - \alpha^2\sigma^2\sigma_z^2(2\pi - \pi^2)}}$$

$$= -\exp\left(-\alpha F^d\right)\frac{1}{\sqrt{1 + \Delta_0(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}}, \tag{A.5}$$

where

$$F^d = W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2 - \frac{\alpha^3\sigma^4\sigma_z^2\left[\pi\theta_0 + (1-\pi)\bar{\theta}\right]^2}{2\left[1 - \alpha^2\sigma^2\sigma_z^2(2\pi - \pi^2)\right]}. \tag{A.6}$$

An agent in Period 0 chooses θ_0 to maximize

$$U = (1-\pi)U^s + \pi U^d.$$

The first-order condition is

$$(1-\pi)\exp\left(-\alpha F^s\right)\frac{dF^s}{d\theta_0}\frac{1}{\sqrt{1 + \Delta_0\pi^2}} + \pi\exp\left(-\alpha F^d\right)\frac{dF^d}{d\theta_0}\frac{1}{\sqrt{1 + \Delta_0(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}} = 0, \tag{A.7}$$

and characterizes a maximum since U is concave. In equilibrium, (A.7) is satisfied for $\theta_0 = \bar{\theta}$. Moreover, (A.23) and (A.27) imply that when $\theta_0 = \bar{\theta}$,

$$\frac{dF^s}{d\theta_0} = \bar{D} - S_0 - \alpha\sigma^2\bar{\theta}, \tag{A.8}$$

$$F^s = W_0 + \bar{\theta}(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2, \tag{A.9}$$

$$\frac{dF^d}{d\theta_0} = \frac{dF^s}{d\theta_0} - \Delta_1\bar{\theta}, \tag{A.10}$$

$$F^d = F^s - \tfrac{1}{2}\Delta_2\bar{\theta}^2, \tag{A.11}$$

where Δ_1 is given by (3.15b) and Δ_2 by (3.15c). Substituting (A.8)-(A.11) into (A.7), and solving for S_0, we find (3.13). ∎

Proof of Proposition 3.3: Eq. (3.19) implies that λ is independent of σ_z^2. Eq. (3.21) implies that γ is increasing in σ_z^2. Eqs. (3.14), (3.15a), (3.15b) and (3.15c) imply that (M, Δ_1, Δ_2) are increasing in σ_z^2. Therefore, (3.13) implies that S_0 is decreasing in σ_z^2. ∎

Proposition A.1 determines the equilibrium in the full-information case.

Proposition A.1 *In the full-information case, agents' demand functions in Period 1 are*

$$\theta_1^s = \frac{E[D|s] - S_1}{\alpha\sigma^2[D|s]}, \tag{A.12}$$

$$\theta_1^d = \frac{E[D|s] - S_1}{\alpha\sigma^2[D|s]} - z, \tag{A.13}$$

the price in Period 1 is

$$S_1 = E[D|s] - \alpha\sigma^2[D|s]\left(\bar{\theta} + \pi z\right), \tag{A.14}$$

and the price in Period 0 is given by (3.13), where M is given by (3.14) and

$$\Delta_0 = \alpha^2\sigma^2[D|s]\sigma_z^2, \tag{A.15}$$

$$\Delta_1 = \frac{\alpha^3\sigma^4\sigma_z^2\left[1 - \frac{\sigma_\epsilon^2}{\sigma^2+\sigma_\epsilon^2}(1-\pi)\right]}{1 + \Delta_0(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}, \tag{A.16}$$

$$\Delta_2 = \frac{\alpha^3\sigma^4\sigma_z^2}{1 + \Delta_0(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2}. \tag{A.17}$$

Proof: In Period 1, a liquidity demander chooses holdings θ_1^d of the risky asset to maximize the expected utility

$$-\mathrm{E}\exp\left\{-\alpha\left[W_1 + \theta_1^d(D - S_1) + z(D - \bar{D})\right]\right\},$$

where the expectation is over D and conditional on s. Because of normality, the expectation is equal to

$$-\exp\left\{-\alpha\left[W_1+\theta_1^d\left(\mathrm{E}[D|s]-S_1\right)+z\left(\mathrm{E}[D|s]-\bar{D}\right)-\tfrac{1}{2}\alpha\sigma^2[D|s](\theta_1^d+z)^2\right]\right\}. \tag{A.18}$$

A liquidity supplier chooses holdings θ_1^s of the risky asset to maximize the expected utility

$$-\exp\left\{-\alpha\left[W_1+\theta_1^s\left(\mathrm{E}[D|s]-S_1\right)-\tfrac{1}{2}\alpha\sigma^2[D|s](\theta_1^s)^2\right]\right\}. \tag{A.19}$$

which can be derived from (A.18) by setting $z=0$. The solution to the optimization problems is straightforward and yields the demand functions (A.12) and (A.13). Substituting (A.12) and (A.13) into the market-clearing equation (3.8), we find that the price in Period 1 is given by (A.14).

Substituting W_1 from (3.10), θ_1^s from (A.12), S_1 from (A.14), and $\mathrm{E}[D|s]$ from (4.3a), we can write the expected utility (A.19) of a liquidity supplier in Period 1 as

$$-\exp\left\{-\alpha\left[W_0+\theta_0(\bar{D}-S_0)+\theta_0\left[\beta_s(s-\bar{D})-\alpha\sigma^2[D|s](\bar{\theta}+\pi z)\right]+\tfrac{1}{2}\alpha\sigma^2[D|s](\bar{\theta}+\pi z)^2\right]\right\}. \tag{A.20}$$

Substituting W_1 from (3.10), θ_1^d from (A.13), S_1 from (A.14), and $\mathrm{E}[D|s]$ from (4.3a), we can write the expected utility (A.18) of a liquidity demander in Period 1 as

$$-\exp\left\{-\alpha\left[W_0+\theta_0(\bar{D}-S_0)+(\theta_0+z)\left[\beta_s(s-\bar{D})-\alpha\sigma^2[D|s](\bar{\theta}+\pi z)\right]+\tfrac{1}{2}\alpha\sigma^2[D|s](\bar{\theta}+\pi z)^2\right]\right\}. \tag{A.21}$$

We next compute the expectations of (A.20) and (A.21) over (s,z), i.e., the interim utilities U^s and U^d of liquidity suppliers and demanders in Period 1/2. To compute U^s, we use Lemma A.1 and set

$$x \equiv \begin{bmatrix} s-\bar{D} \\ z \end{bmatrix}$$

$$\Sigma \equiv \begin{bmatrix} \sigma^2+\sigma_\epsilon^2 & 0 \\ 0 & \sigma_z^2 \end{bmatrix}$$

$$A \equiv W_0+\theta_0(\bar{D}-S_0)-\alpha\sigma^2[D|s]\theta_0\bar{\theta}+\tfrac{1}{2}\alpha\sigma^2[D|s]\bar{\theta}^2$$

$$B \equiv \begin{bmatrix} \beta_s\theta_0 \\ \alpha\sigma^2[D|s]\pi(\bar{\theta}-\theta_0) \end{bmatrix}$$

$$C \equiv \begin{bmatrix} 0 & 0 \\ 0 & \alpha\sigma^2[D|s]\pi^2 \end{bmatrix}.$$

Since

$$I + \alpha C\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \alpha^2\sigma^2[D|s]\sigma_z^2\pi^2 \end{bmatrix},$$

(A.1) implies that

$$U^s = -\exp(-\alpha F^s)\frac{1}{\sqrt{1+\alpha^2\sigma^2[D|s]\sigma_z^2\pi^2}}, \tag{A.22}$$

where

$$F^s = W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2[D|s]\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2[D|s]\bar{\theta}^2 - \tfrac{1}{2}\alpha\beta_s^2(\sigma^2+\sigma_\epsilon^2)\theta_0^2 - \frac{\alpha^3\sigma^4[D|s]\sigma_z^2\pi^2(\theta_0-\bar{\theta})^2}{2\left[1+\alpha^2\sigma^2[D|s]\sigma_z^2\pi^2\right]}.$$

Noting that

$$-\alpha\sigma^2[D|s]\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2[D|s]\bar{\theta}^2 - \tfrac{1}{2}\alpha\beta_s^2(\sigma^2+\sigma_\epsilon^2)\theta_0^2 = -\tfrac{1}{2}\alpha\sigma^2\theta_0^2 + \tfrac{1}{2}\alpha\sigma^2[D|s](\theta_0-\bar{\theta})^2,$$

we can write F^s as

$$F^s = W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2 + \frac{\alpha\sigma^2[D|s](\theta_0-\bar{\theta})^2}{2\left[1+\alpha^2\sigma^2[D|s]\sigma_z^2\pi^2\right]}. \tag{A.23}$$

To compute U^d, we use Lemma A.1 and set

$$x \equiv \begin{bmatrix} s - \bar{D} \\ z \end{bmatrix}$$

$$\Sigma \equiv \begin{bmatrix} \sigma^2 + \sigma_\epsilon^2 & 0 \\ 0 & \sigma_z^2 \end{bmatrix}$$

$$A \equiv W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2[D|s]\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2[D|s]\bar{\theta}^2$$

$$B \equiv \begin{bmatrix} \beta_s\theta_0 \\ -\alpha\sigma^2[D|s]\left[\pi\theta_0 + (1-\pi)\bar{\theta}\right] \end{bmatrix}$$

$$C \equiv \begin{bmatrix} 0 & \beta_s \\ \beta_s & -\alpha\sigma^2[D|s](2\pi - \pi^2) \end{bmatrix}.$$

Using (4.2b) and the definition of β_s, we find

$$I + \alpha C\Sigma = \begin{bmatrix} 1 & \alpha\beta_s\sigma_z^2 \\ \alpha\sigma^2 & 1 - \alpha^2\sigma^2[D|s]\sigma_z^2(2\pi-\pi^2) \end{bmatrix},$$

$$|I + \alpha C\Sigma| = 1 + \alpha^2\sigma^2[D|s]\sigma_z^2(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2, \tag{A.24}$$

$$\Sigma(I+\alpha C\Sigma)^{-1} = \frac{1}{|I+\alpha C\Sigma|}\begin{bmatrix} \left[1-\alpha^2\sigma^2[D|s]\sigma_z^2(2\pi-\pi^2)\right](\sigma^2+\sigma_\epsilon^2) & -\alpha\sigma^2\sigma_z^2 \\ -\alpha\sigma^2\sigma_z^2 & \sigma_z^2 \end{bmatrix}. \tag{A.25}$$

Eqs. (A.1), (A.24) and (A.25) imply that

$$U^d = -\exp\left(-\alpha F^d\right)\frac{1}{\sqrt{1+\alpha^2\sigma^2[D|s]\sigma_z^2(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}}, \tag{A.26}$$

where

$$\begin{aligned}F^d =& W_0+\theta_0(\bar{D}-S_0)-\alpha\sigma^2[D|s]\theta_0\bar{\theta}+\tfrac{1}{2}\alpha\sigma^2[D|s]\bar{\theta}^2\\ &-\frac{\alpha}{2\left[1+\alpha^2\sigma^2[D|s]\sigma_z^2(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2\right]}\left\{\beta_s^2\left[1-\alpha^2\sigma^2[D|s]\sigma_z^2(2\pi-\pi^2)\right](\sigma^2+\sigma_\epsilon^2)\theta_0^2\right.\\ &\left.+2\alpha^2\beta_s\sigma^2[D|s]\sigma^2\sigma_z^2\left[\pi\theta_0+(1-\pi)\bar{\theta}\right]\theta_0+\alpha^2\sigma^4[D|s]\sigma_z^2\left[\pi\theta_0+(1-\pi)\bar{\theta}\right]^2\right\}.\end{aligned}$$

Noting that

$$-\alpha\sigma^2[D|s]\theta_0\bar{\theta}+\tfrac{1}{2}\alpha\sigma^2[D|s]\bar{\theta}^2=-\alpha\sigma^2\theta_0\bar{\theta}+\tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2+\alpha\beta_s\sigma^2\theta_0\bar{\theta}-\tfrac{1}{2}\alpha\beta_s\sigma^2\bar{\theta}^2,$$

we can write F^d as

$$\begin{aligned}F^d =& W_0+\theta_0(\bar{D}-S_0)-\alpha\sigma^2\theta_0\bar{\theta}+\tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2\\ &-\frac{\alpha\left[\beta_s\sigma^2(1-\alpha^2\sigma^2\sigma_z^2)(\theta_0-\bar{\theta})^2+\alpha^2\beta_s\sigma^4\sigma_z^2\theta_0^2+\alpha^2\sigma^2\sigma^2[D|s]\sigma_z^2\left[\pi\theta_0+(1-\pi)\bar{\theta}\right]^2\right]}{2\left[1+\alpha^2\sigma^2[D|s]\sigma_z^2(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2\right]}.\end{aligned} \tag{A.27}$$

Eqs. (A.22) and (A.26) take the form (A.2) and (A.5), with Δ_0 given by (A.15). Moreover, (A.23) and (A.27) imply that when $\theta_0=\bar{\theta}$, $(dF^s/d\theta_0, F^s, dF^d/d\theta_0, F^d)$ are given by (A.8)-(A.11), with (Δ_1,Δ_2) given by (A.16) and (A.17). Since the equations for $(U^s, U^d, dF^s/d\theta_0, F^s, dF^d/d\theta_0, F^d)$ take the same form as in Proposition 3.2, the same applies to S_0. ∎

B Asymmetric Information

Proof of Proposition 4.1: Same arguments as in the proof of Proposition A.1 imply that a liquidity demander chooses holdings θ_1^d to maximize (A.18), and a liquidity supplier chooses holdings θ_1^s to maximize

$$-\exp\left\{-\alpha\left[W_1+\theta_1^s\left(\mathrm{E}[D|S_1]-S_1\right)-\tfrac{1}{2}\alpha\sigma^2[D|S_1](\theta_1^s)^2\right]\right\}. \tag{B.1}$$

The solution to the optimization problems is straightforward and yields the demand functions (4.4a) and (4.4b). ∎

Proof of Proposition 4.2: Substituting $\mathrm{E}[D|s]$ from (4.2a) and $\mathrm{E}[D|S_1]$ from (4.3a), we can write (4.5) as

$$(1-\pi)\frac{\bar{D}+\frac{\beta_\xi}{b}(S_1-a)-S_1}{\alpha\sigma^2[D|S_1]}+\pi\left[\frac{\bar{D}+\beta_s(s-\bar{D})-S_1}{\alpha\sigma^2[D|s]}-z\right]=\bar{\theta}$$

$$\Leftrightarrow(1-\pi)\frac{\bar{D}+\frac{\beta_\xi}{b}(S_1-a)-S_1}{\alpha\sigma^2[D|S_1]}+\pi\left[\frac{\bar{D}+\frac{\beta_s}{b}(S_1-a)+\beta_s cz-S_1}{\alpha\sigma^2[D|s]}-z\right]=\bar{\theta}, \tag{B.2}$$

where the second step follows from (4.1). Eq. (B.2) can be viewed as an affine equation in the variables (S_1-a,z). Setting terms in S_1-a to zero, we find

$$(1-\pi)\frac{\frac{\beta_\xi}{b}-1}{\alpha\sigma^2[D|S_1]}+\pi\frac{\frac{\beta_s}{b}-1}{\alpha\sigma^2[D|s]}=0, \tag{B.3}$$

which yields (4.6b). Setting terms in z to zero, and using (4.2b), we find (4.6c). Setting constant terms to zero, we find

$$(1-\pi)\frac{\bar{D}-a}{\alpha\sigma^2[D|S_1]}+\pi\frac{\bar{D}-a}{\alpha\sigma^2[D|s]}=\bar{\theta}$$

$$\Leftrightarrow(1-\pi)\frac{\bar{D}-a}{\alpha\sigma^2[D|S_1]}+\pi\left[\bar{\theta}+\frac{\bar{D}-a-\alpha\sigma^2[D|s]\bar{\theta}}{\alpha\sigma^2[D|s]}\right]=\bar{\theta}. \tag{B.4}$$

Using (B.3), we can write (B.4) as

$$(1-\pi)\frac{\bar{D}-a}{\alpha\sigma^2[D|S_1]}+\pi\bar{\theta}-(1-\pi)\frac{\frac{\beta_\xi}{b}-1}{\frac{\beta_s}{b}-1}\frac{\bar{D}-a-\alpha\sigma^2[D|s]\bar{\theta}}{\alpha\sigma^2[D|S_1]}=\bar{\theta}$$

$$\Leftrightarrow\bar{D}-a=\alpha\frac{\sigma^2[D|S_1](\beta_s-b)+\sigma^2[D|s](b-\beta_\xi)}{\beta_s-\beta_\xi}\bar{\theta}. \tag{B.5}$$

Using (4.2b), (4.3b) and the definitions of (β_s,β_ξ), we can write (B.5) as (4.6a). ∎

Proof of Proposition 4.3: We first compute the expected utilities of liquidity suppliers and demanders in Period 1. Substituting W_1 from (3.10), θ_1^s from (4.4a), S_1 from (4.1), and $E(D|S_1)$ from (4.3a), we can write the expected utility (B.1) of a liquidity supplier as

$$-\exp\left\{-\alpha\left[W_0+\theta_0(a+b\xi-S_0)+\frac{\left[\bar{D}+\beta_\xi\xi-(a+b\xi)\right]^2}{2\alpha\sigma^2[D|S_1]}\right]\right\}. \tag{B.6}$$

Substituting $\mathrm{E}[D|s]-S_1$ from (4.4b), we can write the expected utility (A.18) of a liquidity demander as

$$-\exp\left\{-\alpha\left[W_1+z\left(\mathrm{E}[D|s]-\bar{D}\right)+\tfrac{1}{2}\alpha\sigma^2[D|s]\left[(\theta_1^d)^2-z^2\right]\right]\right\}$$

$$=-\exp\left\{-\alpha\left[W_1+\beta_s\xi z+\tfrac{1}{2}\alpha\sigma^2[D|s]\left[(\theta_1^d)^2+z^2\right]\right]\right\}, \tag{B.7}$$

where the second step follows from (4.2a), (4.2b), (4.6c) and the definition of ξ. Using (4.2a), (4.2b), (4.6c) and the definition of ξ, we can write (4.4b) as

$$\theta_1^d = \frac{\bar{D} + \beta_s \xi - S_1}{\alpha\sigma^2[D|s]}. \tag{B.8}$$

Substituting W_1 from (3.10), θ_1^d from (B.8), and S_1 from (4.1), we can write (B.7) as

$$-\exp\left\{-\alpha\left[W_0 + \theta_0(a + b\xi - S_0) + \beta_s\xi z + \frac{\left[\bar{D} + \beta_s\xi - (a + b\xi)\right]^2}{2\alpha\sigma^2[D|s]} + \tfrac{1}{2}\alpha\sigma^2[D|s]z^2\right]\right\}. \tag{B.9}$$

We next compute the expectations of (B.6) and (B.9) over (s, z), i.e., the interim utilities U^s and U^d of liquidity suppliers and demanders in Period 1/2. To compute U^s, we use Lemma A.1 and set

$$x \equiv \xi$$

$$\Sigma \equiv \sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2$$

$$A \equiv W_0 + \theta_0(a - S_0) + \frac{(\bar{D} - a)^2}{2\alpha\sigma^2[D|S_1]}$$

$$B \equiv b\theta_0 - \frac{(\bar{D} - a)(b - \beta_\xi)}{\alpha\sigma^2[D|S_1]}$$

$$C \equiv \frac{(b - \beta_\xi)^2}{\alpha\sigma^2[D|S_1]}.$$

Eq. (A.1) implies that

$$U^s = -\exp\left(-\alpha F^s\right)\frac{1}{\sqrt{1 + \frac{(b-\beta_\xi)^2}{\sigma^2[D|S_1]}(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)}}, \tag{B.10}$$

where

$$\begin{aligned} F^s &= W_0 + \theta_0(a - S_0) + \frac{(\bar{D} - a)^2}{2\alpha\sigma^2[D|S_1]} - \frac{\alpha\left[b\theta_0 - \frac{(\bar{D}-a)(b-\beta_\xi)}{\alpha\sigma^2[D|S_1]}\right]^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)}{2\left[1 + \frac{(b-\beta_\xi)^2}{\sigma^2[D|S_1]}(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)\right]} \\ &= \theta_0(\bar{D} - S_0) - \frac{\alpha b^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)\theta_0^2 + 2(\bar{D} - a)\left[1 - \frac{\beta_\xi(b-\beta_\xi)}{\sigma^2[D|S_1]}(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)\right]\theta_0 - \frac{(\bar{D}-a)^2}{\alpha\sigma^2[D|S_1]}}{2\left[1 + \frac{(b-\beta_\xi)^2}{\sigma^2[D|S_1]}(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)\right]}. \end{aligned} \tag{B.11}$$

Substituting $\bar{D}-a$ from (4.6a) into (B.11), and using (4.3b) and the definition of β_ξ, we find

$$F^s = W_0 + \theta_0(\bar{D}-S_0) - \frac{\alpha\left[b^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)\theta_0^2 + \frac{(1-b)^2\sigma^4}{\sigma^2[D|S_1]}(2\theta_0-\bar{\theta})\bar{\theta}\right]}{2\left[1+\frac{(b-\beta_\xi)^2}{\sigma^2[D|S_1]}(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)\right]}. \tag{B.12}$$

Eq. (4.3b) and the definition of β_ξ imply that for all b,

$$\frac{(1-b)^2\sigma^4}{\sigma^2[D|S_1]} + b^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2) = \sigma^2 + \frac{(b-\beta_\xi)^2\sigma^2}{\sigma^2[D|S_1]}(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2). \tag{B.13}$$

Using (B.13), we can write (B.12) as

$$F^s = W_0 + \theta_0(\bar{D}-S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2 + \frac{\alpha\frac{(1-b)^2\sigma^4}{\sigma^2[D|S_1]}(\theta_0-\bar{\theta})^2}{2\left[1+\frac{(b-\beta_\xi)^2}{\sigma^2[D|S_1]}(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)\right]}. \tag{B.14}$$

To compute U^d, we use Lemma A.1 and set

$$x \equiv \begin{bmatrix} \xi \\ z \end{bmatrix}$$

$$\Sigma \equiv \begin{bmatrix} \sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2 & -c\sigma_z^2 \\ -c\sigma_z^2 & \sigma_z^2 \end{bmatrix}$$

$$A \equiv W_0 + \theta_0(a-S_0) + \frac{(\bar{D}-a)^2}{2\alpha\sigma^2[D|s]}$$

$$B \equiv \begin{bmatrix} b\theta_0 + \frac{(\bar{D}-a)(\beta_s-b)}{\alpha\sigma^2[D|s]} \\ 0 \end{bmatrix}$$

$$C \equiv \begin{bmatrix} \frac{(\beta_s-b)^2}{\alpha\sigma^2[D|s]} & \beta_s \\ \beta_s & \alpha\sigma^2[D|s] \end{bmatrix}.$$

Using (4.2b), (4.6c) and the definition of β_s, we find

$$I+\alpha C\Sigma = \begin{bmatrix} 1+\frac{(\beta_s-b)^2}{\sigma^2[D|s]}(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)-\alpha\beta_s c\sigma_z^2 & -\frac{(\beta_s-b)^2}{\sigma^2[D|s]}c\sigma_z^2+\alpha\beta_s\sigma_z^2 \\ \alpha\sigma^2 & 1 \end{bmatrix},$$

$$|I+\alpha C\Sigma| = 1 + \frac{(\beta_s-b)^2}{\sigma^2[D|s]}(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2, \tag{B.15}$$

$$\left[\Sigma(I+\alpha C\Sigma)^{-1}\right]_{(1,1)} = \frac{(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2)}{|I+\alpha C\Sigma|}, \tag{B.16}$$

where the subscript (1,1) refers to the term in the first row and column of a matrix. Eqs. (A.1), (B.15), and (B.16) imply that

$$U^d = -\exp\left(-\alpha F^d\right) \frac{1}{\sqrt{1 + \frac{(\beta_s - b)^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2}}, \tag{B.17}$$

where

$$F^d = W_0 + \theta_0(a - S_0) + \frac{(\bar{D} - a)^2}{2\alpha\sigma^2[D|s]} - \frac{\alpha\left[b\theta_0 + \frac{(\bar{D}-a)(\beta_s - b)}{\alpha\sigma^2[D|s]}\right]^2(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2)}{2\left[1 + \frac{(\beta_s - b)^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2\right]}. \tag{B.18}$$

Substituting $\bar{D} - a$ from (4.6a) into (B.18), we find

$$\begin{aligned} F^d =& W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2 \\ &+ \alpha\left\{b\sigma^2\theta_0\bar{\theta} - \tfrac{1}{2}\sigma^2\bar{\theta}^2 + \frac{(1-b)^2\sigma^4}{2\sigma^2[D|s]}\bar{\theta}^2 - \frac{\left[b\theta_0 + \frac{(1-b)(\beta_s - b)\sigma^2}{\sigma^2[D|s]}\right]^2(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2)}{2\left[1 + \frac{(\beta_s - b)^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2\right]}\right\}. \end{aligned} \tag{B.19}$$

Using (4.2b) and the definition of β_s, we can write (B.19) as

$$\begin{aligned} F^d =& W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2 \\ &- \frac{\alpha\left\{b^2(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2)\theta_0^2 + 2b(\sigma^2 + \sigma_\epsilon^2)\left[\alpha^2\sigma^2\sigma_z^2 - b(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2)\right]\theta_0\bar{\theta} + X\bar{\theta}^2\right\}}{2\left[1 + \frac{(\beta_s - b)^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2\right]}, \end{aligned} \tag{B.20}$$

where

$$X \equiv \left[\sigma^2 - \frac{(1-b)^2\sigma^4}{\sigma^2[D|s]}\right](1 - \alpha^2\sigma^2\sigma_z^2) + \frac{(\beta_s - b)^2\sigma^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2).$$

Eq. (4.2b) and the definition of β_s imply that for all b,

$$\frac{(1-b)^2\sigma^4}{\sigma^2[D|s]} + b^2(\sigma^2 + \sigma_\epsilon^2) = \sigma^2 + \frac{(\beta_s - b)^2\sigma^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2). \tag{B.21}$$

Using (B.21) to eliminate the term in $\sigma^2 - \frac{(1-b)^2\sigma^4}{\sigma^2[D|s]}$ in the definition of X, and substituting X into (B.20), we find

$$\begin{aligned} F^d =& W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2 \\ &- \frac{\alpha\left\{b^2(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2)(\theta_0 - \bar{\theta})^2 + \alpha^2(\sigma^2 + \sigma_\epsilon^2)^2\sigma_z^2\left[\frac{2b\sigma^2\theta_0\bar{\theta}}{\sigma^2 + \sigma_\epsilon^2} + \left[\frac{(\beta_s - b)^2\sigma^2}{\sigma^2[D|s]} - b^2\right]\bar{\theta}^2\right]\right\}}{2\left[1 + \frac{(\beta_s - b)^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2\right]}. \end{aligned} \tag{B.22}$$

Eqs. (B.10) and (B.17) take the form (A.2) and (A.5), with Δ_0 given by (4.7a). In the case of (B.10), this follows directly from (4.7a). In the case of (B.17), this is because

$$
\begin{aligned}
&\frac{(\beta_s - b)^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) \\
=&\frac{(b - \beta_\xi)^2\sigma^2[D|s](1-\pi)^2}{\sigma^4[D|S_1]\pi^2}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) \\
=&\frac{(b - \beta_\xi)^2\sigma^2[D|s](\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2)(1-\pi)^2}{\sigma^4[D|S_1]\sigma_\epsilon^2\pi^2} \\
=&\frac{(b - \beta_\xi)^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)(1-\pi)^2}{\sigma^2[D|S_1]\pi^2} = \Delta_0(1-\pi)^2,
\end{aligned}
\tag{B.23}
$$

where the first step follows from (B.3), the second from (4.6c), the third from (4.2b), (4.3b) and the definitions of (β_s, β_ξ), and the fourth from (4.7a). Eqs. (B.14), (B.22) and (B.23) imply that when $\theta_0 = \bar{\theta}$, $(dF^s/d\theta_0, F^s, dF^d/d\theta_0, F^d)$ are given by (A.8)-(A.11), with (Δ_1, Δ_2) given by (4.7b) and (4.7c). Since the equations for $(U^s, U^d, dF^s/d\theta_0, F^s, dF^d/d\theta_0, F^d)$ take the same form as in Proposition 3.2, the same applies to S_0. ∎

Proof of Proposition 4.4: The price change between Periods 0 and 1 is $S_1 - S_0$. The signed volume of liquidity demanders is

$$
\begin{aligned}
\pi(\theta_1^d - \bar{\theta}) &= -(1-\pi)(\theta_1^s - \bar{\theta}) \\
&= -(1-\pi)\left(\frac{\mathrm{E}[D|S_1] - S_1}{\alpha\sigma^2[D|S_1]} - \bar{\theta}\right) \\
&= -(1-\pi)\left[\frac{\bar{D} + \frac{\beta_\xi}{b}(S_1 - a) - S_1}{\alpha\sigma^2[D|S_1]} - \bar{\theta}\right],
\end{aligned}
$$

where the first step follows from (3.8), the second from (4.4a), and the third from (4.3a). Substituting into (3.16), and noting that variation in the numerator and denominator arises because of S_1, we find (4.8).

Illiquidity under no information is given by (3.19), under full information by

$$
\lambda = \frac{\alpha\sigma^2[D|s]}{1-\pi}, \tag{B.24}
$$

and under asymmetric information by (4.8). We can write (4.8) as

$$\lambda = \frac{\alpha\left(\pi\beta_s\sigma^2[D|S_1] + (1-\pi)\beta_\xi\sigma^2[D|s]\right)}{(\beta_s-\beta_\xi)\pi(1-\pi)}$$
$$= \frac{\alpha\sigma^2(\sigma_\epsilon^2 + c^2\sigma_z^2\pi)}{c^2\sigma_z^2\pi(1-\pi)}, \tag{B.25}$$

where the first step follows from (4.6b), and the second from (4.2b), (4.3b), and the definitions of (β_s, β_ξ). Eqs. (3.19), (B.24) and (B.25) imply that illiquidity is highest under asymmetric information and lowest under full information. Moreover, (4.6c) and (B.25) imply that illiquidity under asymmetric information increases when σ_ϵ^2 decreases. ∎

Proof of Proposition 4.5: Eqs. (3.20) and (4.1) imply that

$$\gamma = -\text{Cov}\left[D - a - b(s - \bar{D} - cz), a + b(s - \bar{D} - cz) - S_0\right]$$
$$= -\text{Cov}\left[(1-b)(D-\bar{D}) - b\epsilon + bcz, b(D-\bar{D}) + b\epsilon - bcz\right]$$
$$= -b\left[\sigma^2 - b(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)\right]. \tag{B.26}$$

Using the definition of β_ξ, we can write (B.26) as (4.9).

Price reversal under no information is given by (3.21), under full information by

$$\gamma = \alpha^2\sigma^4[D|s]\sigma_z^2\pi^2, \tag{B.27}$$

and under asymmetric information by (4.9). Substituting b from (4.6b), $\sigma^2[D|s]$ from (4.2b), $\sigma^2[D|S_1]$ from (4.3b), and using the definitions of (β_s, β_ξ), we can write (4.9) as

$$\gamma = \frac{\sigma^4(\sigma_\epsilon^2 + c^2\sigma_z^2)(\sigma_\epsilon^2 + c^2\sigma_z^2\pi)c^2\sigma_z^2\pi}{\left[\sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2) + \sigma^2c^2\sigma_z^2\pi\right]^2}. \tag{B.28}$$

Price reversal under full information is lower than under no information because $\sigma^2 > \sigma^2[D|s]$, and lower than under asymmetric information if

$$\frac{\sigma^4(\sigma_\epsilon^2 + c^2\sigma_z^2)(\sigma_\epsilon^2 + c^2\sigma_z^2\pi)c^2\sigma_z^2\pi}{\left[\sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2) + \sigma^2c^2\sigma_z^2\pi\right]^2} > \frac{\alpha^2\sigma^4\sigma_\epsilon^4\sigma_z^2\pi^2}{(\sigma^2+\sigma_\epsilon^2)^2}$$
$$\Leftrightarrow 1 > \frac{\left[\sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2) + \sigma^2c^2\sigma_z^2\pi\right]^2\pi}{(\sigma^2+\sigma_\epsilon^2)^2(\sigma_\epsilon^2 + c^2\sigma_z^2)(\sigma_\epsilon^2 + c^2\sigma_z^2\pi)}, \tag{B.29}$$

where the second step follows from (4.6c). Eq. (B.29) holds because the right-hand side is increasing in π and equal to one for $\pi = 1$. Since for $\pi = 1$, price reversals under asymmetric and full information coincide, they are lower than under no information. For $\pi \approx 0$, price reversal is of order π^2 under no information and of order π under asymmetric information. ∎

Lemma B.1 compares the parameters (Δ_0, Δ_2) under symmetric and asymmetric information. For expositional convenience, we use the following superscripts for $\{\Delta_j\}_{j=0,1,2}$ and M: ni under no information, fi under full information, and ai under asymmetric information.

Lemma B.1 $\Delta_0^{ni} > \Delta_0^{fi} > \Delta_0^{ai}$ *and* $\Delta_2^{ni} < \Delta_2^{fi} < \Delta_2^{ai}$.

Proof: Substituting b from (4.6b), $\sigma^2[D|s]$ from (4.2b), $\sigma^2[D|S_1]$ from (4.3b), and using the definitions of (β_s, β_ξ), we can write (4.7a) as

$$\Delta_0^{ai} = \frac{\sigma^2 c^4 \sigma_z^4(\sigma_\epsilon^2 + c^2\sigma_z^2)}{[\sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2) + \sigma^2 c^2 \sigma_z^2 \pi]^2}. \tag{B.30}$$

Eqs. (3.15a) and (A.15) imply that $\Delta_0^{ni} > \Delta_0^{fi}$. Eqs. (A.15) and (B.30) imply that $\Delta_0^{fi} > \Delta_0^{ai}$ if

$$\begin{aligned} &\alpha^2\sigma^2[D|s]\sigma_z^2 > \frac{\sigma^2 c^4 \sigma_z^4(\sigma_\epsilon^2 + c^2\sigma_z^2)}{[\sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2) + \sigma^2 c^2 \sigma_z^2 \pi]^2} \\ \Leftrightarrow &1 > \frac{\sigma_\epsilon^2 c^2 \sigma_z^2(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2)}{[\sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2) + \sigma^2 c^2 \sigma_z^2 \pi]^2}, \end{aligned} \tag{B.31}$$

where the second step follows from (4.2b) and (4.6c). Eq. (B.31) holds for all $\pi \in [0,1]$ if it holds for $\pi = 0$, i.e.,

$$\begin{aligned} &\sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)^2 > c^2\sigma_z^2(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2) \\ \Leftrightarrow &\sigma_\epsilon^2\sigma^4 + 2\sigma_\epsilon^2\sigma^2(\sigma_\epsilon^2 + c^2\sigma_z^2) + \sigma_\epsilon^2(\sigma_\epsilon^2 + c^2\sigma_z^2)^2 - c^2\sigma_z^2(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2) > 0 \\ \Leftrightarrow &\sigma_\epsilon^2\sigma^4 + (\sigma_\epsilon^2 + c^2\sigma_z^2)\left[2\sigma_\epsilon^2\sigma^2 + \sigma_\epsilon^4(1 - \alpha^2\sigma^2\sigma_z^2)\right] > 0, \end{aligned} \tag{B.32}$$

where the last step follows from (4.6c). Eq. (B.32) holds because of (2.2).

Eq. (3.15c) implies that

$$\left[1 + \Delta_0^{ni}(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2\right]\Delta_2^{ni} = \alpha^3\sigma^4\sigma_z^2. \tag{B.33}$$

Eq. (A.17) implies that

$$\left[1 + \Delta_0^{fi}(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2\right]\Delta_2^{fi} = \alpha^3\sigma^4\sigma_z^2. \tag{B.34}$$

Eq. (4.7c) implies that

$$\left[1 + \Delta_0^{ai}(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2\right]\Delta_2^{ai} = \alpha^3\sigma^4\sigma_z^2\left[1 + \frac{(\beta_s - b)^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2)\right]. \tag{B.35}$$

Since $\Delta_0^{ni} > \Delta_0^{fi}$, (B.33) and (B.34) imply that $\Delta_2^{ni} < \Delta_2^{fi}$. Since $\Delta_0^{fi} > \Delta_0^{ai}$, (B.34) and (B.35) imply that $\Delta_2^{fi} < \Delta_2^{ai}$. ∎

Proof of Proposition 4.6: To show the ranking for S_0, we must show the reverse ranking for the illiquidity discount in (3.13), i.e.,

$$\frac{\pi M^{ni}}{1-\pi+\pi M^{ni}}\Delta_1^{ni} < \frac{\pi M^{fi}}{1-\pi+\pi M^{fi}}\Delta_1^{fi} < \frac{\pi M^{ai}}{1-\pi+\pi M^{ai}}\Delta_1^{ai}. \tag{B.36}$$

Since $\Delta_2^{ni} < \Delta_2^{fi} < \Delta_2^{ai}$, (B.36) holds if it does so when $\{\Delta_2^j\}_{j=ni,fi,ai}$ are replaced by zero. Using (3.14), we can write the latter condition as

$$\left(\frac{1-\pi}{\pi\hat{M}^{ni}}+1\right)\frac{1}{\Delta_1^{ni}} > \left(\frac{1-\pi}{\pi\hat{M}^{fi}}+1\right)\frac{1}{\Delta_1^{fi}} > \left(\frac{1-\pi}{\pi\hat{M}^{ai}}+1\right)\frac{1}{\Delta_1^{ai}}. \tag{B.37}$$

where

$$\hat{M}^j \equiv \sqrt{\frac{1+\Delta_0^j\pi^2}{1+\Delta_0^j(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}},$$

for $j = ni, fi, ai$. Eq. (3.15b) implies that

$$\left[1+\Delta_0^{ni}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2\right]\Delta_1^{ni} = \alpha^3\sigma^4\sigma_z^2\pi. \tag{B.38}$$

Eq. (A.16) implies that

$$\left[1+\Delta_0^{fi}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2\right]\Delta_1^{fi} = \alpha^3\sigma^4\sigma_z^2\left[1-\frac{\sigma_\epsilon^2}{\sigma^2+\sigma_\epsilon^2}(1-\pi)\right]. \tag{B.39}$$

Eq. (4.7b) implies that

$$\begin{aligned}\left[1+\Delta_0^{ai}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2\right]\Delta_1^{ai} &= \alpha^3 b\sigma^2(\sigma^2+\sigma_\epsilon^2)\sigma_z^2\\ &= \frac{\alpha^3\sigma^4\sigma_z^2(\sigma^2+\sigma_\epsilon^2)(\sigma_\epsilon^2+c^2\sigma_z^2\pi)}{\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi},\end{aligned} \tag{B.40}$$

where the second step follows from (4.2b), (4.3b), (4.6b) and the definitions of (β_s, β_ξ). Eqs. (B.38)-(B.40) and $\Delta_0^{ni} > \Delta_0^{fi} > \Delta_0^{ai}$ imply that a sufficient condition for (B.37) is

$$\left(\frac{1-\pi}{\pi\hat{M}^{ni}}+1\right)\frac{1}{\pi} > \left(\frac{1-\pi}{\pi\hat{M}^{fi}}+1\right)\frac{1}{1-\frac{\sigma_\epsilon^2}{\sigma^2+\sigma_\epsilon^2}(1-\pi)} > \left(\frac{1-\pi}{\pi\hat{M}^{ai}}+1\right)\frac{\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi}{(\sigma^2+\sigma_\epsilon^2)(\sigma_\epsilon^2+c^2\sigma_z^2\pi)}.$$

$$\tag{B.41}$$

We can write the first inequality in (B.41) as

$$
\begin{aligned}
&\frac{1-\frac{\sigma_\epsilon^2}{\sigma^2+\sigma_\epsilon^2}(1-\pi)}{\pi}\left(\frac{1-\pi}{\pi\hat{M}^{ni}}+1\right) > \frac{1-\pi}{\pi\hat{M}^{fi}}+1 \\
\Leftrightarrow &\left(1+\frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}\frac{1-\pi}{\pi}\right)\left(\frac{1-\pi}{\pi\hat{M}^{ni}}+1\right) > \frac{1-\pi}{\pi\hat{M}^{fi}}+1.
\end{aligned}\tag{B.42}
$$

A sufficient condition for (B.42) is

$$
\begin{aligned}
&\frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}+\frac{1}{\hat{M}^{ni}} > \frac{1}{\hat{M}^{fi}} \\
\Leftrightarrow &\frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2} > \sqrt{\frac{1+\Delta_0^{fi}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{fi}\pi^2}}-\sqrt{\frac{1+\Delta_0^{ni}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{ni}\pi^2}} \\
\Leftrightarrow &\frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2} > \frac{\frac{1+\Delta_0^{fi}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{fi}\pi^2}-\frac{1+\Delta_0^{ni}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{ni}\pi^2}}{\sqrt{\frac{1+\Delta_0^{fi}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{fi}\pi^2}}+\sqrt{\frac{1+\Delta_0^{ni}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{ni}\pi^2}}} \\
\Leftrightarrow &\frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2} > \frac{\left(\Delta_0^{ni}-\Delta_0^{fi}\right)\left[(1-\alpha^2\sigma^2\sigma_z^2)\pi^2-(1-\pi)^2\right]}{\left[\sqrt{\frac{1+\Delta_0^{fi}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{fi}\pi^2}}+\sqrt{\frac{1+\Delta_0^{ni}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{ni}\pi^2}}\right]\left(1+\Delta_0^{fi}\pi^2\right)\left(1+\Delta_0^{ni}\pi^2\right)}.
\end{aligned}\tag{B.43}
$$

Eqs. (3.15a), (A.15) and the non-negativity of $(\Delta_0^{ni}, \Delta_0^{fi})$ imply that a sufficient condition for (B.43) is

$$
\frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2} > \frac{\alpha^2\sigma^2\sigma_z^2\frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}(1-\alpha^2\sigma^2\sigma_z^2)\pi^2}{2\sqrt{1-\alpha^2\sigma^2\sigma_z^2}}.\tag{B.44}
$$

Eq. (B.44) holds because of (2.2).

We can write the second inequality in (B.41) as

$$
\begin{aligned}
&\frac{(\sigma^2+\sigma_\epsilon^2)(\sigma_\epsilon^2+c^2\sigma_z^2\pi)}{\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right]\left[1-\frac{\sigma_\epsilon^2}{\sigma^2+\sigma_\epsilon^2}(1-\pi)\right]}\left(\frac{1-\pi}{\pi\hat{M}^{fi}}+1\right) > \frac{1-\pi}{\pi\hat{M}^{ai}}+1 \\
\Leftrightarrow &\left\{1+\frac{\sigma_\epsilon^2\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2)-\sigma^2c^2\sigma_z^2(1-\pi)\right](1-\pi)}{\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\pi\sigma^2c^2\sigma_z^2\right](\sigma^2+\pi\sigma_\epsilon^2)}\right\}\left(\frac{1-\pi}{\pi\hat{M}^{fi}}+1\right) > \frac{1-\pi}{\pi\hat{M}^{ai}}+1.
\end{aligned}\tag{B.45}
$$

A sufficient condition for (B.45) is

$$\frac{\sigma_\epsilon^2\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2)-\sigma^2c^2\sigma_z^2(1-\pi)\right]\pi}{\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](\sigma^2+\sigma_\epsilon^2\pi)}+\frac{1}{\hat{M}^{fi}}>\frac{1}{\hat{M}^{ai}}$$

$$\Leftrightarrow\frac{\sigma_\epsilon^2\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2)-\sigma^2c^2\sigma_z^2(1-\pi)\right]\pi}{\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](\sigma^2+\sigma_\epsilon^2\pi)}$$

$$>\frac{(\Delta_0^{fi}-\Delta_0^{ai})\left[(1-\alpha^2\sigma^2\sigma_z^2)\pi^2-(1-\pi)^2\right]}{\left[\sqrt{\frac{1+\Delta_0^{ai}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{ai}\pi^2}}+\sqrt{\frac{1+\Delta_0^{fi}(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}{1+\Delta_0^{fi}\pi^2}}\right]\left(1+\Delta_0^{ai}\pi^2\right)\left(1+\Delta_0^{fi}\pi^2\right)}, \qquad \text{(B.46)}$$

where the intermediate steps are as for (B.43). Eqs. (4.6c), (4.7a), (A.15) and the non-negativity of $(\Delta_0^{fi},\Delta_0^{ai})$ imply that a sufficient condition for (B.46) is

$$\frac{\sigma_\epsilon^2\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2)-\sigma^2c^2\sigma_z^2(1-\pi)\right]\pi}{\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](\sigma^2+\sigma_\epsilon^2\pi)}$$

$$>\frac{\alpha^2\sigma^2\sigma_z^2(1-\alpha^2\sigma^2\sigma_z^2)\pi^2}{2\sqrt{1-\alpha^2\sigma^2\sigma_z^2}}\left[\frac{\sigma_\epsilon^2}{\sigma^2+\sigma_\epsilon^2}-\frac{\sigma_\epsilon^4c^2\sigma_z^2(\sigma_\epsilon^2+c^2\sigma_z^2)}{\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right]^2}\right]. \qquad \text{(B.47)}$$

A sufficient condition for (B.47) is

$$\frac{2\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2)-\sigma^2c^2\sigma_z^2(1-\pi)\right]}{\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](\sigma^2+\sigma_\epsilon^2)}>\left[\frac{1}{\sigma^2+\sigma_\epsilon^2}-\frac{\sigma_\epsilon^2c^2\sigma_z^2(\sigma_\epsilon^2+c^2\sigma_z^2)}{\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right]^2}\right]\pi, \qquad \text{(B.48)}$$

which is derived from (B.47) by using (2.2) and replacing the term $\sigma^2+\sigma_\epsilon^2\pi$ in the denominator of the left-hand side by $\sigma^2+\sigma_\epsilon^2$. Multiplying by the smallest common denominator, we can write (B.48) as

$$\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2)\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](2-\pi)$$
$$>\sigma^2c^2\sigma_z^2\left\{2\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](1-\pi)\right.$$
$$\left.+(\sigma_\epsilon^2+\sigma^2\pi)(\sigma_\epsilon^2+c^2\sigma_z^2\pi)\pi-\sigma_\epsilon^2(\sigma_\epsilon^2+c^2\sigma_z^2)\pi(1-\pi)\right\}. \qquad \text{(B.49)}$$

A sufficient condition for (B.49) is

$$\sigma_\epsilon^4\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](2-\pi)$$
$$>\sigma^2c^2\sigma_z^2\left\{2\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](1-\pi)+(\sigma_\epsilon^2+\sigma^2\pi)(\sigma_\epsilon^2+c^2\sigma_z^2\pi)\pi\right\}. \qquad \text{(B.50)}$$

Eqs. (2.2) and (4.6c) imply that a sufficient condition for (B.50) is

$$\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](2-\pi)$$
$$>\left\{2\left[\sigma_\epsilon^2(\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2)+\sigma^2c^2\sigma_z^2\pi\right](1-\pi)+(\sigma_\epsilon^2+\sigma^2\pi)(\sigma_\epsilon^2+c^2\sigma_z^2\pi)\pi\right\},$$

which obviously holds. ∎

Proof of Proposition 4.7: Eq. (B.25) implies that λ is decreasing in σ_z^2. Eq. (B.28) implies that γ is increasing in σ_z^2. Eq. (B.30) implies that Δ_0 is increasing in σ_z^2, and

$$1 + \Delta_0(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2$$

is decreasing in σ_z^2. Since the left-hand side of (B.40) is increasing in σ_z^2, so is Δ_1. Eqs. (B.23) and (B.30) imply that

$$\frac{(\beta_s - b)^2}{\sigma^2[D|s]}(\sigma^2 + \sigma_\epsilon^2) = \frac{\sigma^2\sigma_\epsilon^2 c^4\sigma_z^4(1-\pi)^2}{[\sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2) + \sigma^2c^2\sigma_z^2\pi]^2}. \tag{B.51}$$

Since the left-hand side of (B.51) is increasing in σ_z^2, so are the left-hand side of (B.35), Δ_2 and M. Therefore, (3.13) implies that S_0 is decreasing in σ_z^2. ∎

C Participation Costs

Proof of Proposition 5.1: Since the price S_1 is as in Section 3, with $\frac{\pi}{(1-\pi)\mu+\pi}$ taking the place of π, the interim utility U^s of a participating supplier can be derived from (A.2) by making the same substitution and subtracting c. That is,

$$U^s = -\exp\left(-\alpha F^s\right)\frac{1}{\sqrt{1 + \frac{\alpha^2\sigma^2\sigma_z^2\pi^2}{[(1-\pi)\mu+\pi]^2}}}, \tag{C.1}$$

where

$$F^s = W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2 + \frac{\alpha\sigma^2(\theta_0 - \bar{\theta})^2}{2\left[1 + \frac{\alpha^2\sigma^2\sigma_z^2\pi^2}{[(1-\pi)\mu+\pi]^2}\right]} - c.$$

The interim utility U^{sn} of a non-participating supplier can be derived from (C.1) by noting that non-participation is equivalent to participation in a fictitious market where all agents choose θ_0 in Period 0, receive no endowment, and pay no cost to participate. Setting $(\pi, \bar{\theta}, c) = (0, \theta_0, 0)$ in (C.1), we find

$$U^{sn} = -\exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2\right]\right\}. \tag{C.2}$$

In equilibrium, suppliers enter Period 1/2 holding $\theta_0 = \bar{\theta}$ shares, and are willing to participate if $U^s \geq U^{sn}$. Setting $\theta_0 = \bar{\theta}$ in (C.1) and (C.2), we can write condition $U^s \geq U^{sn}$ as

$$\exp(2\alpha c) \leq 1 + \frac{\alpha^2\sigma^2\sigma_z^2\pi^2}{[(1-\pi)\mu+\pi]^2}. \tag{C.3}$$

If $c \leq \underline{c}$, (C.3) holds for $\mu = 1$, and all suppliers participate. If $\underline{c} < c < \bar{c}$, (C.3) holds as an equality for μ given in (5.4b). This value of μ is in $(0,1)$ and coincides with the measure of participating suppliers. If $c \geq \bar{c}$, (C.3) does not hold for any $\mu \in [0,1]$, and no suppliers participate. ∎

Proof of Proposition 5.2: The interim utility U^d of a participating demander can be derived from (A.5) by replacing π by $\frac{\pi}{(1-\pi)\mu+\pi}$ and subtracting c. That is,

$$U^d = -\exp\left(-\alpha F^d\right) \frac{1}{\sqrt{1-\alpha^2\sigma^2\sigma_z^2\frac{2\pi(1-\pi)\mu+\pi^2}{[(1-\pi)\mu+\pi]^2}}}, \tag{C.4}$$

where

$$F^d = W_0+\theta_0(\bar{D}-S_0)-\alpha\sigma^2\frac{\alpha^2\sigma^2\sigma_z^2\frac{\pi^2}{[(1-\pi)\mu+\pi]^2}\theta_0^2+2\left[1-\alpha^2\sigma^2\sigma_z^2\frac{\pi}{(1-\pi)\mu+\pi}\right]\theta_0\bar{\theta}-(1-\alpha^2\sigma^2\sigma_z^2)\bar{\theta}^2}{2\left[1-\alpha^2\sigma^2\sigma_z^2\frac{2\pi(1-\pi)\mu+\pi^2}{[(1-\pi)\mu+\pi]^2}\right]}-c.$$

The interim utility U^{dn} of a non-participating demander can be derived from (C.4) by noting that non-participation is equivalent to participation in a fictitious market where all agents choose θ_0 in Period 0, receive an endowment, and pay no cost to participate. Setting $(\pi, \bar{\theta}, c) = (1, \theta_0, 0)$ in (C.4), we find

$$U^{dn} = -\exp\left\{-\alpha\left[W_0+\theta_0(\bar{D}-S_0)-\frac{\alpha\sigma^2\theta_0^2}{2(1-\alpha^2\sigma^2\sigma_z^2)}\right]\right\}\frac{1}{\sqrt{1-\alpha^2\sigma^2\sigma_z^2}}. \tag{C.5}$$

In equilibrium, demanders enter Period 1/2 holding $\theta_0 = \bar{\theta}$ shares, and are willing to participate if $U^d \geq U^{dn}$. We next show that (5.5) ensures $U^d > U^{dn}$ for all θ_0 (and not only for $\theta_0 = \bar{\theta}$). Using (C.4) and (C.5), we can write $U^d > U^{dn}$ as

$$\exp\left\{2\alpha\left[\theta_0(\bar{D}-S_0)-\frac{\alpha\sigma^2\theta_0^2}{2(1-\alpha^2\sigma^2\sigma_z^2)}-F^d\right]\right\} < \frac{1-\alpha^2\sigma^2\sigma_z^2\frac{2\pi(1-\pi)\mu+\pi^2}{[(1-\pi)\mu+\pi]^2}}{1-\alpha^2\sigma^2\sigma_z^2}. \tag{C.6}$$

Since a fraction $\mu > 0$ of suppliers participate, (C.3) holds, and a sufficient condition for (C.6) is

$$\exp\left\{2\alpha\left[\theta_0(\bar{D}-S_0)-\frac{\alpha\sigma^2\theta_0^2}{2(1-\alpha^2\sigma^2\sigma_z^2)}-F^d-c\right]\right\} < \frac{1-\alpha^2\sigma^2\sigma_z^2\frac{2\pi(1-\pi)\mu+\pi^2}{[(1-\pi)\mu+\pi]^2}}{(1-\alpha^2\sigma^2\sigma_z^2)\left[1+\frac{\alpha^2\sigma^2\sigma_z^2\pi^2}{[(1-\pi)\mu+\pi]^2}\right]}. \tag{C.7}$$

Since

$$\theta_0(\bar{D}-S_0) - \frac{\alpha\sigma^2\theta_0^2}{2(1-\alpha^2\sigma^2\sigma_z^2)} - F^d - c = -\frac{\alpha^2\sigma^2\left\{\left[1-\alpha^2\sigma^2\sigma_z^2\right]\bar{\theta} - \left[1-\frac{\alpha^2\sigma^2\sigma_z^2\pi}{(1-\pi)\mu+\pi}\right]\theta_0\right\}^2}{2\left(1-\alpha^2\sigma^2\sigma_z^2\right)\left[1-\alpha^2\sigma^2\sigma_z^2\frac{2\pi(1-\pi)\mu+\pi^2}{[(1-\pi)\mu+\pi]^2}\right]} \leq 0,$$

a sufficient condition for (C.7) is

$$\begin{aligned}
&\frac{1-\alpha^2\sigma^2\sigma_z^2\frac{2\pi(1-\pi)\mu+\pi^2}{[(1-\pi)\mu+\pi]^2}}{\left(1-\alpha^2\sigma^2\sigma_z^2\right)\left[1+\frac{\alpha^2\sigma^2\sigma_z^2\pi^2}{[(1-\pi)\mu+\pi]^2}\right]} > 1 \\
\Leftrightarrow &1-\alpha^2\sigma^2\sigma_z^2\left[1-\frac{(1-\pi)^2\mu^2}{[(1-\pi)\mu+\pi]^2}\right] > \left(1-\alpha^2\sigma^2\sigma_z^2\right)\left[1+\frac{\alpha^2\sigma^2\sigma_z^2\pi^2}{[(1-\pi)\mu+\pi]^2}\right] \\
\Leftrightarrow &\frac{(1-\pi)^2\mu^2}{[(1-\pi)\mu+\pi]^2} > \frac{\left(1-\alpha^2\sigma^2\sigma_z^2\right)\pi^2}{[(1-\pi)\mu+\pi]^2}.
\end{aligned} \tag{C.8}$$

Eq. (C.8) holds because of (5.5). ∎

Proof of Corollary 5.1: Since $\pi \leq 1/2$, $\underline{c} \leq \hat{c} < \bar{c}$. If $\underline{c} < c \leq \hat{c}$, and all liquidity demanders participate, then the fraction $\mu \in (0,1)$ of participating suppliers is given by (5.4b). Conversely, (5.5) holds for that μ because $c \leq \hat{c}$, and therefore all liquidity demanders participate. If instead $c \leq \underline{c}$, and all liquidity demanders participate, then all liquidity suppliers do. Conversely, (5.5) holds for $\mu = 1$ because $\pi \leq 1/2$, and therefore all liquidity demanders participate. ∎

Proof of Proposition 5.3: An agent in Period 0 chooses θ_0 to maximize

$$U = (1-\pi)\max\{U^s, U^{sn}\} + \pi\max\{U^d, U^{dn}\}.$$

Proposition 5.2 implies that $U^d > U^{dn}$ for all θ_0. Moreover, (C.1) and (C.2) imply that if $U^s \geq U^{sn}$ for $\theta_0 = \bar{\theta}$, then $U^s > U^{sn}$ for all $\theta_0 \neq \bar{\theta}$. Therefore, the agent maximizes

$$U = (1-\pi)U^s + \pi U^d.$$

The function U is concave in θ_0 and the first-order condition characterizes a maximum. The price S_0 can be derived as in Proposition 3.2. It given by (3.13), but with $\pi/[(1-\pi)\mu+\pi]$ taking the place of π when evaluating (Δ_1, Δ_2, M). ∎

Proof of Proposition 5.4: Eqs. (5.8) and (5.9) imply that a decrease in μ raises λ and γ. Eqs. (5.6), (5.7a) and (5.7b) imply that a decrease in μ raises (Δ_1, M), and therefore lowers S_0 from (3.13). When $c > \underline{c}$, Proposition 5.1 implies that an increase in c lowers μ. ∎

Proof of Proposition 5.5: When $c > \underline{c}$, Proposition 5.1 implies that an increase in σ_z^2 raises μ. Eq. (5.8) implies that this lowers λ. Eq. (5.4b) implies that when $c > \underline{c}$,

$$(1-\pi)\mu + \pi = \frac{\alpha\sigma\sigma_z}{\sqrt{e^{2\alpha c}-1}}. \tag{C.9}$$

Eqs. (5.9) and (C.9) imply that γ is independent of σ_z^2. Eqs. (5.6), (5.7a), (5.7b) and (C.9) imply that (Δ_1, Δ_2, M) are increasing in σ_z^2. Therefore, (3.13) implies that S_0 is decreasing in σ_z^2. ∎

D Transaction Costs

Proof of Proposition 6.1: The results for $z > 0$ follow from the arguments prior to the proposition's statement. Suppose next that $z < 0$. If liquidity suppliers sell, their demand function is

$$\theta_1^s = \frac{\bar{D} - S_1 + \kappa}{\alpha\sigma^2}, \tag{D.1}$$

and if liquidity demanders buy, their demand function is

$$\theta_1^d = \frac{\bar{D} - S_1 - \kappa}{\alpha\sigma^2} - z. \tag{D.2}$$

Since in equilibrium agents enter Period 1 holding $\bar{\theta}$ shares of the risky asset, trade occurs if there exists a price S_1 such that $\theta_1^s < \bar{\theta}$ and $\theta_1^d > \bar{\theta}$. Using (D.1) and (D.2), we can write these conditions as

$$\alpha\sigma^2 z + \kappa < \bar{D} - S_1 - \alpha\sigma^2\bar{\theta} < -\kappa.$$

Therefore, trade occurs if $z < -\frac{2\kappa}{\alpha\sigma^2}$. Substituting (D.1) and (D.2) into (3.8), we find (6.3). ∎

Proof of Proposition 6.2: We first compute the expected utility of a liquidity supplier in Period 1, assuming that the agent enters that period holding θ_0 shares. If $|z| \le \hat{\kappa}$, there is no trade in Period 1, and expected utility is

$$-\exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2\right]\right\}.$$

Suppose next that $z > \hat{\kappa}$. Eqs. (6.1) and (6.3) imply that if the supplier buys, he establishes a position

$$\theta_1^{sb+} = \frac{\bar{D} - S_1 - \kappa}{\alpha\sigma^2} = \bar{\theta} + \pi(z - \hat{\kappa}) \tag{D.3}$$

and receives expected utility

$$
\begin{aligned}
&- \exp\left\{-\alpha\left[W_0 - \theta_0 S_0 - (\theta_1^{sb+} - \theta_0)(S_1 + \kappa) + \theta_1^{sb+}\bar{D} - \tfrac{1}{2}\alpha\sigma^2(\theta_1^{sb+})^2\right]\right\} \\
= &- \exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2 + \tfrac{1}{2}\alpha\sigma^2(\theta_0 - \theta_1^{sb+})^2\right]\right\},
\end{aligned}
$$

where the second step follows from (D.3). Conversely, (6.3) and (D.1) imply that if the supplier sells, he establishes a position

$$
\theta_1^{ss+} = \frac{\bar{D} - S_1 + \kappa}{\alpha\sigma^2} = \bar{\theta} + \pi z + (1 - \pi)\hat{\kappa} \tag{D.4}
$$

and receives expected utility

$$
\begin{aligned}
&- \exp\left\{-\alpha\left[W_0 - \theta_0 S_0 - (\theta_1^{ss+} - \theta_0)(S_1 - \kappa) + \theta_1^{ss+}\bar{D} - \tfrac{1}{2}\alpha\sigma^2(\theta_1^{ss+})^2\right]\right\} \\
= &- \exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2 + \tfrac{1}{2}\alpha\sigma^2(\theta_0 - \theta_1^{ss+})^2\right]\right\},
\end{aligned}
$$

where the second step follows from (D.4). To nest the two cases, as well as the case where the supplier decides not to trade, we can write expected utility as

$$
- \exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2\right]\right\} G^+
$$

where

$$
\begin{aligned}
G^+ &= \exp\left[-\tfrac{1}{2}\alpha^2\sigma^2(\theta_0 - \theta_1^{sb+})^2\right] \quad \text{if} \quad \theta_0 < \theta_1^{sb+}, \\
G^+ &= 1 \quad \text{if} \quad \theta_1^{sb+} \leq \theta_0 \leq \theta_1^{ss+}, \\
G^+ &= \exp\left[-\tfrac{1}{2}\alpha^2\sigma^2(\theta_0 - \theta_1^{ss+})^2\right] \quad \text{if} \quad \theta_0 > \theta_1^{ss+}.
\end{aligned}
$$

Similar calculations imply that expected utility when $z < -\hat{\kappa}$ is

$$
- \exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2\right]\right\} G^-,
$$

where

$$
\begin{aligned}
G^- &= \exp\left[-\tfrac{1}{2}\alpha^2\sigma^2(\theta_0 - \theta_1^{sb-})^2\right] \quad \text{if} \quad \theta_0 < \theta_1^{sb-}, \\
G^- &= 1 \quad \text{if} \quad \theta_1^{sb-} \leq \theta_0 \leq \theta_1^{ss-}, \\
G^- &= \exp\left[-\tfrac{1}{2}\alpha^2\sigma^2(\theta_0 - \theta_1^{ss-})^2\right] \quad \text{if} \quad \theta_0 > \theta_1^{ss-},
\end{aligned}
$$

and

$$
\begin{aligned}
\theta_1^{sb-} &= \bar{\theta} + \pi z - (1 - \pi)\hat{\kappa}, \\
\theta_1^{ss-} &= \bar{\theta} + \pi(z + \hat{\kappa}).
\end{aligned}
$$

The supplier's interim utility in Period 1/2 is

$$U^s = -\exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2\right]\right\}\left[\int_{-\hat{\kappa}}^{\hat{\kappa}} f(z)dz + \int_{\hat{\kappa}}^{\infty} G^+ f(z)dz + \int_{-\infty}^{-\hat{\kappa}} G^- f(z)dz\right].$$

To compute the equilibrium price S_0, we need to evaluate the derivative $dU^s/d\theta_0$ at $\theta_0 = \bar{\theta}$. Using the symmetry of z around zero, and noting that $\theta_1^{ss-} < \bar{\theta} < \theta_1^{sb+}$ (i.e., a supplier holding $\theta_0 = \bar{\theta}$ shares buys if $z > \hat{\kappa}$ and sells if $z < -\hat{\kappa}$), we find

$$\left.\frac{dU^s}{d\theta_0}\right|_{\theta_0=\bar{\theta}} = 2\alpha \exp\left\{-\alpha\left[W_0 + \bar{\theta}(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2\right]\right\}\left(\bar{D} - S_0 - \alpha\sigma^2\bar{\theta}\right)N_1, \tag{D.5}$$

where

$$N_1 = \int_0^{\hat{\kappa}} f(z)dz + \int_{\hat{\kappa}}^{\infty} \exp\left[-\tfrac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right] f(z)dz.$$

Similar calculations for a liquidity demander yield

$$\left.\frac{dU^d}{d\theta_0}\right|_{\theta_0=\bar{\theta}} = 2\alpha \exp\left\{-\alpha\left[W_0 + \bar{\theta}(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2\right]\right\}\left[\left(\bar{D} - S_0 - \alpha\sigma^2\bar{\theta}\right)N_2 - N_3\right], \tag{D.6}$$

where

$$N_2 = \int_0^{\hat{\kappa}} \exp\left(\tfrac{1}{2}\alpha^2\sigma^2 z^2\right)\text{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz + \int_{\hat{\kappa}}^{\infty}\Gamma(z)\text{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz,$$

$$N_3 = \alpha\sigma^2\left[\int_0^{\hat{\kappa}} \exp\left(\tfrac{1}{2}\alpha^2\sigma^2 z^2\right)\text{sh}\left(\alpha^2\sigma^2\bar{\theta}z\right)zf(z)dz + \int_{\hat{\kappa}}^{\infty}\Gamma(z)\text{sh}(\alpha^2\sigma^2\bar{\theta}z)[\pi z + (1-\pi)\hat{\kappa}]f(z)dz\right].$$

The first-order condition in Period 0 is

$$\left.\frac{dU}{d\theta_0}\right|_{\theta_0=\bar{\theta}} = (1-\pi)\left.\frac{dU^s}{d\theta_0}\right|_{\theta_0=\bar{\theta}} + \pi\left.\frac{dU^d}{d\theta_0}\right|_{\theta_0=\bar{\theta}} = 0, \tag{D.7}$$

and characterizes a maximum because (U^s, U^d) are concave in θ_0. Substituting (D.5) and (D.6) into (D.7), we find (3.13) with $M = N_2/N_1$ and $\Delta_1\bar{\theta} = N_3/N_2$. ∎

Proof of Proposition 6.3: Eqs. (6.2) and (6.3) imply that the signed volume of liquidity demanders is

$$\pi(\theta_1^d - \bar{\theta}) = -\pi(1-\pi)(z-\hat{\kappa})$$

when $z > \hat{\kappa} > 0$, and

$$\pi(\theta_1^d - \bar{\theta}) = -\pi(1-\pi)(z+\hat{\kappa})$$

when $z < -\hat{\kappa} < 0$. Since signed volume is distributed symmetrically around zero, its variance is

$$\text{Var}\left[\pi(\theta_1^d - \bar{\theta})\right] = E\left[\pi^2(\theta_1^d - \bar{\theta})^2\right] = 2\pi^2(1-\pi)^2 \int_{\hat{\kappa}}^{\infty} (z-\hat{\kappa})^2 f(z)dz \tag{D.8}$$

and its covariance with the price change is

$$\text{Cov}\left[S_1 - S_0, \pi(\theta_1^d - \bar{\theta})\right] = E\left[S_1 \pi(\theta_1^d - \bar{\theta})\right]. \tag{D.9}$$

Since S_1 is distributed symmetrically around its mean and takes the form (6.3), (D.9) becomes

$$2\alpha\sigma^2\pi(1-\pi)\int_{\hat{\kappa}}^{\infty} \left[\pi z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa}\right](z-\hat{\kappa})f(z)dz. \tag{D.10}$$

Dividing (D.9) by (D.8), we find (6.7). Since the integrals in (6.7) are positive, illiquidity is higher than when $\hat{\kappa} = 0$. ∎

Proof of Proposition 6.4: Eqs. (3.20) and (6.3) imply that

$$\begin{aligned}
\gamma &= -\text{Cov}\left\{D - \bar{D} + \alpha\sigma^2\left[\bar{\theta} + \pi z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa}\,\text{sign}(z)\right], \bar{D} - \alpha\sigma^2\left[\bar{\theta} + \pi z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa}\,\text{sign}(z)\right] - S_0\right\} \\
&= \text{Var}\left\{\alpha\sigma^2\left[\pi z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa}\,\text{sign}(z)\right]\right\} \\
&= \alpha^2\sigma^4 E\left[\pi z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa}\,\text{sign}(z)\right]^2,
\end{aligned} \tag{D.11}$$

where the last step uses the symmetry of the distribution of z. The expectation (D.11) is conditional on trade in Period 1, i.e., $|z| > \hat{\kappa}$. Using this fact and the symmetry of the distribution of z, we find (6.8). The derivative of (6.8) with respect to $\hat{\kappa}$ has the same sign as

$$f(\hat{\kappa})\left[\frac{\int_{\hat{\kappa}}^{\infty}\left[\pi z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa}\right]^2 f(z)dz}{\int_{\hat{\kappa}}^{\infty} f(z)dz} - \tfrac{1}{4}\hat{\kappa}^2\right] + (1-2\pi)\int_{\hat{\kappa}}^{\infty}\left[\pi z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa}\right] f(z)dz. \tag{D.12}$$

Since

$$\pi z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa} > \tfrac{1}{2}\hat{\kappa}$$

for $z > \hat{\kappa}$, the first term in (D.12) is positive. Therefore, (D.12) is positive if $\pi \leq 1/2$. ∎

Proof of Proposition 6.5: The price S_0 is decreasing in $\hat{\kappa}$ if (M, Δ_1) are increasing in $\hat{\kappa}$. The parameter M is increasing in $\hat{\kappa}$ if $d\log(N_2)/d\hat{\kappa} > d\log(N_1)/d\hat{\kappa}$. Using the definitions of (N_1, N_2), we can write this equation as

$$\frac{\alpha^2\sigma^2(1-\pi)^2\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)(z-\hat{\kappa})f(z)dz}{\int_0^{\hat{\kappa}}\exp\left(\frac{1}{2}\alpha^2\sigma^2z^2\right)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz+\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}$$
$$>\frac{\alpha^2\sigma^2\pi^2\int_{\hat{\kappa}}^{\infty}\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right](z-\hat{\kappa})f(z)dz}{\int_0^{\hat{\kappa}}f(z)dz+\int_{\hat{\kappa}}^{\infty}\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]f(z)dz}. \tag{D.13}$$

When $\pi \leq 1/2$, a sufficient condition for (D.13) is

$$\frac{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)(z-\hat{\kappa})f(z)dz}{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}>\frac{\int_{\hat{\kappa}}^{\infty}\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right](z-\hat{\kappa})f(z)dz}{\int_{\hat{\kappa}}^{\infty}\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]f(z)dz} \tag{D.14}$$

and

$$\frac{\int_0^{\hat{\kappa}}\exp\left(\frac{1}{2}\alpha^2\sigma^2z^2\right)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}<\frac{\int_0^{\hat{\kappa}}f(z)dz}{\int_{\hat{\kappa}}^{\infty}\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]f(z)dz}. \tag{D.15}$$

Eq. (D.14) can be written as

$$E_{g_1}(z-\hat{\kappa}) > E_{g_2}(z-\hat{\kappa}), \tag{D.16}$$

where (E_{g_1}, E_{g_2}) refer to expectation under the probability densities

$$g_1(z) = \frac{\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)}{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}$$
$$g_2(z) = \frac{\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]f(z)}{\int_{\hat{\kappa}}^{\infty}\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]f(z)dz},$$

defined in $[\hat{\kappa}, \infty)$. Since the likelihood ratio

$$\frac{g_1(z)}{g_2(z)} = \exp\left\{\frac{1}{2}\alpha^2\sigma^2\left[z^2-(z-\hat{\kappa})^2+2\pi(z-\hat{\kappa})^2\right]\right\}\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)\frac{\int_{\hat{\kappa}}^{\infty}\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]f(z)dz}{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}$$

is increasing in z, the distribution associated to g_1 first-order stochastically dominates that associated to g_2, and (D.16) holds. Eq. (D.15) can be written as

$$\frac{\int_0^{\hat{\kappa}}\exp\left(\frac{1}{2}\alpha^2\sigma^2z^2\right)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}{\int_0^{\hat{\kappa}}f(z)dz}<\frac{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}{\int_{\hat{\kappa}}^{\infty}\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]f(z)dz}$$
$$\Leftrightarrow E_{h_1}K_1(z) < E_{h_2}K_2(z), \tag{D.17}$$

where (E_{h_1}, E_{h_2}) refer to expectation under the probability densities

$$h_1(z) = \frac{f(z)}{\int_0^{\hat{\kappa}} f(z)dz},$$

$$h_2(z) = \frac{\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right] f(z)}{\int_{\hat{\kappa}}^{\infty} \exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right] f(z)dz},$$

defined in $[0, \hat{\kappa}]$ and $[\hat{\kappa}, \infty)$, respectively, and

$$K_1(z) = \exp\left(\tfrac{1}{2}\alpha^2\sigma^2 z^2\right) \mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z),$$

$$K_2(z) = \frac{\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)}{\exp\left[-\frac{1}{2}\alpha^2\sigma^2\pi^2(z-\hat{\kappa})^2\right]} = \exp\left\{\tfrac{1}{2}\alpha^2\sigma^2\left[z^2-(z-\hat{\kappa})^2+2\pi(z-\hat{\kappa})^2\right]\right\}\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z).$$

Since the functions $K_1(z)$ and $K_2(z)$ are increasing in z, a sufficient condition for (D.17) is $K_1(\hat{\kappa}) \leq K_2(\hat{\kappa})$, which holds.

The parameter Δ_1 is increasing in $\hat{\kappa}$ if $d\log(N_3)/d\hat{\kappa} > d\log(N_2)/d\hat{\kappa}$. Using the definitions of (N_2, N_3), we can write this equation as

$$\frac{(1-\pi)\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{sh}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz + \alpha^2\sigma^2(1-\pi)^2\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{sh}(\alpha^2\sigma^2\bar{\theta}z)[\pi z+(1-\pi)\hat{\kappa}](z-\hat{\kappa})f(z)dz}{\int_0^{\hat{\kappa}}\exp\left(\frac{1}{2}\alpha^2\sigma^2z^2\right)\mathrm{sh}\left(\alpha^2\sigma^2\bar{\theta}z\right)zf(z)dz + \int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{sh}(\alpha^2\sigma^2\bar{\theta}z)[\pi z+(1-\pi)\hat{\kappa}]f(z)dz}$$

$$> \frac{\alpha^2\sigma^2(1-\pi)^2\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)(z-\hat{\kappa})f(z)dz}{\int_0^{\hat{\kappa}}\exp\left(\frac{1}{2}\alpha^2\sigma^2z^2\right)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz + \int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}. \tag{D.18}$$

A sufficient condition for (D.18) is

$$\frac{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{sh}(\alpha^2\sigma^2\bar{\theta}z)[\pi z+(1-\pi)\hat{\kappa}](z-\hat{\kappa})f(z)dz}{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{sh}(\alpha^2\sigma^2\bar{\theta}z)[\pi z+(1-\pi)\hat{\kappa}]f(z)dz} > \frac{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)(z-\hat{\kappa})f(z)dz}{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz} \tag{D.19}$$

and

$$\frac{\int_0^{\hat{\kappa}}\exp\left(\frac{1}{2}\alpha^2\sigma^2z^2\right)\mathrm{sh}\left(\alpha^2\sigma^2\bar{\theta}z\right)zf(z)dz}{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{sh}(\alpha^2\sigma^2\bar{\theta}z)[\pi z+(1-\pi)\hat{\kappa}]f(z)dz} < \frac{\int_0^{\hat{\kappa}}\exp\left(\frac{1}{2}\alpha^2\sigma^2z^2\right)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}{\int_{\hat{\kappa}}^{\infty}\Gamma(z)\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}z)f(z)dz}. \tag{D.20}$$

The proof for (D.19) and (D.20) is as for (D.14) and (D.15). ∎

Proof of Proposition 6.6: When z is drawn from a two-point distribution, it takes the two values $\pm\sigma_z$. Eq. (6.7) implies that illiquidity is

$$\frac{\alpha\sigma^2}{1-\pi}\left[1+\frac{\hat{\kappa}}{2\pi(\sigma_z-\hat{\kappa})}\right],$$

and is increasing in $\hat{\kappa}$ and decreasing in σ_z^2. Eq. (6.8) implies that price reversal is

$$\gamma = \alpha^2\sigma^4\left[\pi\sigma_z + \left(\tfrac{1}{2} - \pi\right)\hat{\kappa}\right]^2,$$

and is increasing in $\hat{\kappa}$ and σ_z^2. Eqs. (6.4)-(6.6) imply that

$$M = \exp\left[\tfrac{1}{2}\alpha^2\sigma^2\sigma_z^2 - \tfrac{1}{2}\alpha^2\sigma^2(1-2\pi)(\sigma_z - \hat{\kappa})^2\right]\mathrm{ch}(\alpha^2\sigma^2\bar{\theta}\sigma_z),$$

$$\Delta_1 = \frac{\alpha\sigma^2}{\bar{\theta}}\mathrm{th}(\alpha^2\sigma^2\bar{\theta}\sigma_z)[\pi\sigma_z + (1-\pi)\hat{\kappa}].$$

Since (M, Δ_1) are increasing in σ_z^2, (3.13) implies that S_0 is decreasing in σ_z^2. ∎

E Leverage Constraints

Proof of Proposition 7.1: When (7.5) does not bind, the demand function of a liquidity supplier is

$$\theta_1^s = (f')^{-1}(\bar{D} - S_1). \tag{E.1}$$

Eqs. (7.3) and (E.1) imply that $\theta_1^s = \theta_1^d + z$. Combining with the market-clearing equation (3.8), we find $\theta_1^s = \bar{\theta} + \pi z$ and (7.9). To determine whether or not (7.5) binds, we examine whether the unconstrained solution meets (7.5). In the case $\bar{\theta} + \pi z \geq 0$, (7.5) is met if

$$\begin{aligned} & W_0 + \bar{\theta}(S_1 - S_0) + A > (\bar{\theta} + \pi z)\left(S_1 - \bar{D} + b_D\right) \\ \Leftrightarrow & B + A + \bar{\theta}S_1 > (\bar{\theta} + \pi z)\left(S_1 - \bar{D} + b_D\right) \\ \Leftrightarrow & G^+(z) \equiv B + A + \bar{\theta}(\bar{D} - b_D) - \pi z\left[b_D - f'(\bar{\theta} + \pi z)\right] > 0, \end{aligned}$$

where the first step follows from (3.10) and because in equilibrium $\theta_0 = \bar{\theta}$, and the third step follows from (7.9). Similar calculations imply that in the case $\bar{\theta} + \pi z < 0$, (7.5) is met if

$$G^-(z) \equiv B + A + \bar{\theta}(\bar{D} + b_D) + \pi z\left[b_D + f'(\bar{\theta} + \pi z)\right] > 0.$$

To show the first bullet point in the proposition, it suffices to show that (i) $G^+(z)$ is decreasing in z, (ii) $G^-(z)$ is increasing in z, and (iii) $G^-(-z) > G^+(z)$ for $z > \bar{\theta}/\pi$. We next explain why (i)-(iii) are sufficient conditions, distinguishing between the case $\bar{\theta} - \pi b_z \geq 0$, where suppliers hold long positions for all values of z, and the case $\bar{\theta} - \pi b_z < 0$, where they hold short positions for large negative values of z.

Case $\bar{\theta} - \pi b_z \geq 0$: If (7.8) is met, i.e., $G^+(b_z) \geq 0$, then $G^+(z) > 0$ for all $z \in [-b_z, b_z)$ because of (i). Thus, the unconstrained solution meets (7.5). If instead (7.8) is not met, i.e., $G^+(b_z) < 0$, then $G^+(0) > 0$ implies that there exists $\underline{z} \in (0, b_z)$ such that $G^+(\underline{z}) = 0$. Moreover, (i) implies that $G^+(z) > 0$ for $z \in [-b_z, \underline{z})$ and $G^+(z) < 0$ for $z \in (\underline{z}, b_z]$. Thus, the unconstrained solution meets (7.5) for $z \in [-b_z, \underline{z}]$.

Case $\bar{\theta} - \pi b_z < 0$: If (7.8) is met, then $G^+(z) > 0$ for all $z \in [-\bar{\theta}/\pi, b_z)$ because of (i), and $G^-(z) > 0$ for all $z \in [-b_z, -\bar{\theta}/\pi)$ because of (iii). Thus, the unconstrained solution meets (7.5). If instead (7.8) is not met, then there exists $\underline{z} \in (0, b_z)$ such that $G^+(\underline{z}) = 0$. Moreover, (i) implies that $G^+(z) > 0$ for $z \in [-\bar{\theta}/\pi, \underline{z})$ and $G^+(z) < 0$ for $z \in (\underline{z}, b_z]$. If, in addition, $G^-(-b_z) < 0$, then there exists $\overline{z} \in (\bar{\theta}/\pi, b_z)$ such that $G^-(-\overline{z}) = 0$. If instead $G^-(-b_z) \geq 0$, we set $\overline{z} \equiv b_z$. In both cases, (ii) implies that $G^-(z) < 0$ for $z \in [-b_z, -\overline{z})$ and $G^-(z) > 0$ for $z \in (-\overline{z}, -\bar{\theta}/\pi)$. Thus, the unconstrained solution meets (7.5) for $z \in [-\overline{z}, \underline{z}]$. If $\overline{z} = b_z$, then $\overline{z} > \underline{z}$, and if $\overline{z} < b_z$, then $\overline{z} > \underline{z}$ because (iii) implies that $0 = G^-(-\overline{z}) > G^+(\overline{z})$.

To show (i)-(iii), we establish some properties of the function $f(\theta)$. Symmetry around the y-axis implies that

$$f(\theta) = \frac{\log \mathrm{E} \exp\left[\alpha\theta(D - \bar{D})\right]}{\alpha}. \tag{E.2}$$

Using (E.2), we find

$$f'(\theta) = \frac{\mathrm{E}\left\{(D - \bar{D}) \exp\left[\alpha\theta(D - \bar{D})\right]\right\}}{\mathrm{E} \exp\left[\alpha\theta(D - \bar{D})\right]} = \mathrm{E}_\theta(D - \bar{D}), \tag{E.3}$$

$$\begin{aligned} f''(\theta) &= \alpha \frac{\mathrm{E}\left\{(D - \bar{D})^2 \exp\left[\alpha\theta(D - \bar{D})\right]\right\}}{\mathrm{E} \exp\left[\alpha\theta(D - \bar{D})\right]} - \alpha \left(\frac{\mathrm{E}\left\{(D - \bar{D}) \exp\left[\alpha\theta(D - \bar{D})\right]\right\}}{\mathrm{E} \exp\left[\alpha\theta(D - \bar{D})\right]}\right)^2 \\ &= \alpha \mathrm{E}_\theta(D - \bar{D})^2 - \alpha \left[\mathrm{E}_\theta(D - \bar{D})\right]^2, \end{aligned} \tag{E.4}$$

where E_θ denotes expectation with respect to a measure with Radon-Nikodym derivative

$$\frac{\exp\left[\alpha\theta(D - \bar{D})\right]}{\mathrm{E}\left\{\exp\left[\alpha\theta(D - \bar{D})\right]\right\}}$$

relative to the true measure. Eq. (E.3) implies that

$$|f'(\theta)| < b_D. \tag{E.5}$$

Symmetry around the y-axis and convexity imply that $f'(\theta)$ has the same sign as θ.

We next show (i)-(iii). The function $G^+(z)$ is decreasing in z if

$$b_D - f'(\bar{\theta} + \pi z) - \pi z f''(\bar{\theta} + \pi z) > 0. \tag{E.6}$$

To show (E.6), we will show more generally that

$$b_D - f'(\theta) - \pi z f''(\theta) > 0. \tag{E.7}$$

Using (E.3) and (E.4), we can write (E.7) as

$$\begin{aligned} & b_D - E_\theta(D - \bar{D}) - \alpha\pi z \left\{ \mathrm{E}_\theta(D - \bar{D})^2 - \left[\mathrm{E}_\theta(D - \bar{D})\right]^2 \right\} > 0 \\ \Leftrightarrow & b_D - \sqrt{\mathrm{E}_\theta(D - \bar{D})^2} \\ & + \left[\sqrt{\mathrm{E}_\theta(D - \bar{D})^2} - \mathrm{E}_\theta(D - \bar{D})\right] \left\{1 - \alpha\pi z \left[\sqrt{\mathrm{E}_\theta(D - \bar{D})^2} + \mathrm{E}_\theta(D - \bar{D})\right]\right\} > 0. \end{aligned} \tag{E.8}$$

Since

$$\left|\mathrm{E}_\theta(D - \bar{D})\right| < \sqrt{\mathrm{E}_\theta(D - \bar{D})^2} \leq b_D,$$

(E.8) holds under the sufficient condition $2\alpha\pi b_D z < 1$, which in turn holds for $z \in [-b_z, b_z]$ because of (7.7). The function $G^-(z)$ is increasing in z if

$$b_D + f'(\bar{\theta} + \pi z) + \pi z f''(\bar{\theta} + \pi z) > 0. \tag{E.9}$$

Eq. (E.9) follows from (E.7) by replacing $\bar{\theta} + \pi z$ by its opposite and using the symmetry of $f(\theta)$ around the y-axis. Finally,

$$\begin{aligned} G^-(-z) - G^+(z) &= 2\bar{\theta} b_D - \pi z \left[f'(\bar{\theta} + \pi z) + f'(\bar{\theta} - \pi z)\right] \\ &= 2\bar{\theta} b_D - \pi z \left[f'(\bar{\theta} + \pi z) - f'(-\bar{\theta} + \pi z)\right] \\ &= 2\bar{\theta}\left[b_D - \pi z f''(\hat{\theta})\right] > 0, \end{aligned}$$

where the second step follows from the symmetry of $f(\theta)$ around the y-axis, the third from the intermediate-value theorem, and the fourth from (E.7) and $\hat{\theta} > 0$.

To show the second bullet point in the proposition, it suffices to show the comparative statics when the constraint binds, i.e., for $z \in [-b_z, -\overline{z}) \cup (\underline{z}, b_z]$. Consider first the case $z \in (\underline{z}, b_z]$. Since (7.5) holds with equality and $\theta_1^s \geq \bar{\theta} + \pi\underline{z} > \bar{\theta} > 0$,

$$\begin{aligned} & W_0 + \bar{\theta}(S_1 - S_0) + A = \theta_1^s \left(S_1 - \bar{D} + b_D\right) \\ \Rightarrow & B + A + \bar{\theta}(\bar{D} - b_D) - (\theta_1^s - \bar{\theta})\left[b_D - f'(\theta_1^d + z)\right] = 0, \end{aligned} \tag{E.10}$$

where the second step follows from (7.3). Differentiating implicitly with respect to z and using (3.8), we find

$$\frac{d\theta_1^s}{dz} = \frac{(\theta_1^s - \bar{\theta})f''(\theta_1^d + z)}{b_D - f'(\theta_1^d + z) + \frac{(1-\pi)(\theta_1^s - \bar{\theta})}{\pi} f''(\theta_1^d + z)}. \tag{E.11}$$

Eqs. (E.5), (E.11), $f''(\theta) > 0$ and $\theta_1^s > \bar{\theta}$ imply that θ_1^s is increasing in z. Moreover, (E.11) implies that

$$\frac{d\theta_1^s}{dz} < \pi \Leftrightarrow b_D - f'(\theta_1^d + z) - (\theta_1^s - \bar{\theta})f''(\theta_1^d + z) > 0. \tag{E.12}$$

Since $\theta_1^s - \bar{\theta} < \pi z$, the second inequality in (E.12) follows from (E.7). Eq. (3.8) and the first inequality in (E.12) imply that

$$\frac{d(\theta_1^d + z)}{dz} > \pi, \tag{E.13}$$

and (7.3) implies that S_1 is decreasing in z. Consider next the case $z \in [-b_z, -\overline{z})$. The counterpart of (E.10) is

$$B + A + \bar{\theta}(\bar{D} + b_D) + (\theta_1^s - \bar{\theta})\left[b_D + f'(\theta_1^d + z)\right] = 0, \tag{E.14}$$

and of (E.11) is

$$\frac{d\theta_1^s}{dz} = \frac{(\bar{\theta} - \theta_1^s)f''(\theta_1^d + z)}{b_D + f'(\theta_1^d + z) + \frac{(1-\pi)(\bar{\theta} - \theta_1^s)}{\pi} f''(\theta_1^d + z)}. \tag{E.15}$$

Using (E.15) and proceeding as in the case $z \in (\underline{z}, b_z]$, we find that θ_1^s is increasing in z and S_1 is decreasing in z. ∎

Proof of Proposition 7.2: We first determine S_0 in the abundant-capital region. Substituting S_1 from (7.3) and W_1 from (3.10), we can write the expected utility (7.4) of a liquidity supplier and (7.1) of a liquidity demander in Period 1 as

$$-\exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) + (\theta_1^s - \theta_0)f'(\theta_1^d + z) - f(\theta_1^s)\right]\right\}, \tag{E.16}$$

$$-\exp\left\{-\alpha\left[W_0 + \theta_0(\bar{D} - S_0) + (\theta_1^d - \theta_0)f'(\theta_1^d + z) - f(\theta_1^d + z)\right]\right\}, \tag{E.17}$$

respectively. Taking expectations of (E.16) and (E.17) over z yields the interim utilities (U^s, U^d) in Period 1/2. Agents choose θ_0 to maximize $U = (1-\pi)U^s + \pi U^d$. Setting $\theta_0 = \bar{\theta}$ in the first-order condition and solving for S_0, we find

$$S_0 = \bar{D} - \frac{\mathrm{E}\left\{\left[(1-\pi)G^s + \pi G^d\right] f'(\theta_1^d + z)\right\}}{\mathrm{E}\left[(1-\pi)G^s + \pi G^d\right]}, \tag{E.18}$$

where

$$G^s \equiv \exp\left\{-\alpha\left[(\theta_1^s - \bar{\theta})f'(\theta_1^d + z) - f(\theta_1^s)\right]\right\},$$

$$G^d \equiv \exp\left\{-\alpha\left[(\theta_1^d - \bar{\theta})f'(\theta_1^d + z) - f(\theta_1^d + z)\right]\right\}.$$

Eq. (E.2) implies that

$$f(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!}\alpha^{n-1}\theta^n \kappa_n, \tag{E.19}$$

where κ_n is the n'th cumulant of $D - \bar{D}$. Since D is distributed symmetrically around $\bar{D}$, the cumulants for odd n are zero. Since, in addition, $\kappa_2 = \sigma^2$ and $\kappa_4 = \gamma\sigma^4$, (E.19) implies that

$$f(\theta) = \tfrac{1}{2}\alpha\sigma^2\theta^2 + \tfrac{1}{24}\alpha^3\gamma\sigma^4\theta^4 + o(\alpha^3). \tag{E.20}$$

Because the illiquidity discount (derived below) is of order α^3, we need to evaluate (G^s, G^d) up to order α^2 and $f'(\theta_1^d + z)$ up to order α^3. To evaluate (G^s, G^d), we use the Taylor expansion (E.20) up to order α. Noting that $\theta_1^s = \theta_1^d + z = \bar{\theta} + \pi z$ because (7.5) does not bind, we find

$$(1-\pi)G^s + \pi G^d = 1 + \tfrac{1}{2}\alpha^2\sigma^2(\bar{\theta} + \pi z)^2 + o(\alpha^2), \tag{E.21}$$

$$f'(\theta_1^d + z) = \alpha\sigma^2(\bar{\theta} + \pi z) + \tfrac{1}{6}\alpha^3\gamma\sigma^4(\bar{\theta} + \pi z)^3 + o(\alpha^3). \tag{E.22}$$

Substituting (E.21) and (E.22) into (E.18), we find

$$S_0 = \bar{D} - \frac{\mathrm{E}\left\{\left[1 + \frac{1}{2}\alpha^2\sigma^2(\bar{\theta}+\pi z)^2 + o(\alpha^2)\right]\left[\alpha\sigma^2(\bar{\theta}+\pi z) + \frac{1}{6}\alpha^3\gamma\sigma^4(\bar{\theta}+\pi z)^3 + o(\alpha^3)\right]\right\}}{\mathrm{E}\left[1 + \frac{1}{2}\alpha^2\sigma^2(\bar{\theta}+\pi z)^2 + o(\alpha^2)\right]}. \tag{E.23}$$

Since z is distributed symmetrically around zero, $\mathrm{E}(z) = \mathrm{E}(z^3) = 0$ and $\mathrm{E}(z^2) = \sigma_z^2$. Substituting into (E.23), we find

$$S_0 = \bar{D} - \frac{\alpha\sigma^2\bar{\theta} + \alpha^3\sigma^4(\bar{\theta}^3 + 3\bar{\theta}\sigma_z^2\pi^2)\left(\frac{1}{2} + \frac{1}{6}\gamma\right) + o(\alpha^3)}{1 + \frac{1}{2}\alpha^2\sigma^2(\bar{\theta}^2 + \sigma_z^2\pi^2) + o(\alpha^2)}. \tag{E.24}$$

Eq. (E.24) implies (7.10).

We next determine S_0 in the scarce-capital region. Eq. (E.18) holds by redefining G^s as

$$G^s \equiv \exp\left\{-\alpha\left[(\theta_1^s-\bar{\theta})f'(\theta_1^d+z)-f(\theta_1^s)\right]\right\}\left\{1+1_{\{z\in[-b_z,-\bar{z})\cup(\underline{z},b_z]\}}\frac{\left[f'(\theta_1^d+z)-f'(\theta_1^s)\right]\operatorname{sign}(\theta_1^s)}{m(S_1,\theta_1^s)}\right\},$$

where the last term corresponds to the fact that when (7.5) binds, changes in θ_0 affect θ_1^s because they affect W_1. Because the illiquidity discount (derived below) is of order α, we only need to evaluate (G^s, G^d) up to order one and $f'(\theta_1^d+z)$ up to order α. Since

$$G^s = 1 + o(1),$$

$$G^d = 1 + o(1),$$

$$f'(\theta_1^d+z) = \alpha\sigma^2(\theta_1^d+z) + o(\alpha),$$

(E.18) implies that

$$\begin{aligned} S_0 &= \bar{D} - \mathrm{E}\left[\alpha\sigma^2(\theta_1^d+z)+o(\alpha)\right] \\ &= \bar{D} - \mathrm{E}\left[\alpha\sigma^2\left(\bar{\theta}-\frac{1-\pi}{\pi}(\theta_1^s-\bar{\theta})+z\right)+o(\alpha)\right] \\ &= \bar{D} - \alpha\sigma^2\bar{\theta} + \alpha\sigma^2\frac{1-\pi}{\pi}\mathrm{E}\left(\theta_1^s-\bar{\theta}\right)+o(\alpha), \end{aligned} \tag{E.25}$$

where the second step follows from (3.8) and the third because z is distributed symmetrically around zero. When (7.5) does not bind, $\theta_1^s = \bar{\theta}+\pi z$. When $z \in (\underline{z}, b_z]$, (E.10) implies that in order one θ_1^s is independent of z and equal to $\bar{\theta}+\pi\underline{z}$. When $z \in [-b_z, -\bar{z})$, (E.14) implies that in order one θ_1^s is independent of z and equal to $\bar{\theta}-\pi\bar{z}$. Therefore,

$$\theta_1^s - \bar{\theta} = \pi\left[-\bar{z}1_{z\in[-b_z,-\bar{z})} + z1_{z\in[-\bar{z},\underline{z}]} + \underline{z}1_{z\in(\underline{z},b_z]}\right] + o(1). \tag{E.26}$$

Substituting into (E.25), and using the symmetry of the distribution of z around zero, we find

$$S_0 = \bar{D} - \alpha\sigma^2\bar{\theta} - \alpha\sigma^2(1-\pi)\left[\int_{\underline{z}}^{b_z}(z-\underline{z})dF(z) - \int_{\bar{z}}^{b_z}(z-\bar{z})dF(z)\right] + o(\alpha). \tag{E.27}$$

Eq. (E.27) implies (7.11). ∎

Proof of Proposition 7.3: The signed volume of liquidity demanders is

$$\pi(\theta_1^d-\bar{\theta}) = -(1-\pi)(\theta_1^s-\bar{\theta}) \tag{E.28}$$

and the price is

$$S_1 = \bar{D} - \alpha\sigma^2(\theta_1^d + z) + o(\alpha). \tag{E.29}$$

When capital is abundant, $\theta_1^s = \theta_1^d + z = \bar{\theta} + \pi z$. Substituting into (E.28) and (E.29), we find

$$\pi(\theta_1^d - \bar{\theta}) = -\pi(1-\pi)z, \tag{E.30}$$

$$S_1 = \bar{D} - \alpha\sigma^2(\bar{\theta} + \pi z) + o(\alpha). \tag{E.31}$$

Substituting (E.30) and (E.31) into (3.16), we find

$$\lambda = \frac{\alpha\sigma^2}{1-\pi} + o(\alpha). \tag{E.32}$$

When capital is scarce, θ_1^s is given by (E.26). Substituting into (E.28), we find

$$\pi(\theta_1^d - \bar{\theta}) = -\pi(1-\pi)\left[-\overline{z}1_{z\in[-b_z,-\overline{z})} + z1_{z\in[-\overline{z},\underline{z}]} + \underline{z}1_{z\in(\underline{z},b_z]}\right] + o(1). \tag{E.33}$$

Using (E.29) and the symmetry of the distribution of z around zero, we find

$$\mathrm{E}(\theta_1^d - \bar{\theta}) = (1-\pi)\mathrm{E}\left[\int_{\underline{z}}^{b_z}(z-\underline{z})dF(z) - \int_{\overline{z}}^{b_z}(z-\overline{z})dF(z)\right] + o(1), \tag{E.34}$$

$$\mathrm{E}(\theta_1^d - \bar{\theta})^2 = (1-\pi)^2\left[\sigma_z^2 - \int_{\underline{z}}^{b_z}(z^2-\underline{z}^2)dF(z) - \int_{\overline{z}}^{b_z}(z^2-\overline{z}^2)dF(z)\right] + o(1), \tag{E.35}$$

$$\mathrm{E}\left[z(\theta_1^d - \bar{\theta})\right] = -(1-\pi)\left[\sigma_z^2 - \int_{\underline{z}}^{b_z} z(z-\underline{z})dF(z) - \int_{\overline{z}}^{b_z} z(z-\overline{z})dF(z)\right] + o(1). \tag{E.36}$$

Substituting (E.29) into (3.16), we find

$$\begin{aligned}
\lambda &= -\frac{\alpha\sigma^2\left\{\mathrm{Var}\left(\theta_1^d - \bar{\theta}\right) + \mathrm{E}\left[z(\theta_1^d - \bar{\theta})\right]\right\}}{\pi\mathrm{Var}\left(\theta_1^d - \bar{\theta}\right)} + o(\alpha) \\
&= \frac{\alpha\sigma^2}{1-\pi} - \frac{\alpha\sigma^2\left\{\mathrm{Var}\left(\theta_1^d - \bar{\theta}\right) + (1-\pi)\mathrm{E}\left[z(\theta_1^d - \bar{\theta})\right]\right\}}{\pi(1-\pi)\mathrm{Var}\left(\theta_1^d - \bar{\theta}\right)} + o(\alpha) \\
&= \frac{\alpha\sigma^2}{1-\pi} + \frac{\alpha\sigma^2\left\{(1-\pi)^2\left[\int_{\underline{z}}^{b_z}\underline{z}(z-\underline{z})dF(z) + \int_{\overline{z}}^{b_z}\overline{z}(z-\overline{z})dF(z)\right] + \left[\mathrm{E}(\theta_1^d - \bar{\theta})\right]^2\right\}}{\pi(1-\pi)\mathrm{Var}\left(\theta_1^d - \bar{\theta}\right)} + o(\alpha),
\end{aligned} \tag{E.37}$$

where the third step follows from (E.35) and (E.36). Since the term in curly brackets is positive, λ in (E.37) is higher than in (E.32) for small α.

Suppose next that z is drawn from a two-point distribution, i.e., takes the two values $\pm\sigma_z$. Denoting by $(\theta_1^{s+}, \theta_1^{d+}, S_1^+)$ the values of $(\theta_1^s, \theta_1^d, S_1)$ corresponding to σ_z, and by $(\theta_1^{s-}, \theta_1^{d-}, S_1^-)$ those corresponding to $-\sigma_z$,

$$\lambda = \frac{S_1^- - S_1^+}{\pi(\theta_1^{d-} - \theta_1^{d+})} = \frac{S_1^- - S_1^+}{(1-\pi)(\theta_1^{s+} - \theta_1^{s-})}.$$

The first inequality in (E.12) implies that θ_1^{s+} is smaller when capital is scarce than when it is abundant, and (7.3) and (E.13) imply that S_1^+ is lower. Likewise, θ_1^{s-} is larger when capital is scarce than when it is abundant, and S_1^- is higher. Since $S_1^- > S_1^+$ and $\theta_1^{s+} > \theta_1^{s-}$, λ is higher when capital is scarce than when it is abundant. ∎

Proof of Proposition 7.4: Since S_1 and D are independent, $\gamma = \text{Var}(S_1)$. We denote by $S_1(z)$ the price when the liquidity shock is z, and use the superscripts ac for abundant capital and sc for scarce capital. Setting $S_1(\hat{z}) \equiv \text{E}\left[S_1^{sc}(z)\right]$, we can write $\text{Var}\left[S_1^{sc}(z)\right]$ as

$$\text{Var}\left[S_1^{sc}(z)\right] = \int_{-b_z}^{b_z} \left[S_1^{sc}(z) - S_1^{sc}(\hat{z})\right]^2 dF(z). \tag{E.38}$$

Eqs. (7.3) and (E.13) imply that $\frac{dS_1^{sc}(z)}{dz} < \frac{dS_1^{ac}(z)}{dz} < 0$ for $z \in (\underline{z}, b_z]$. The same holds for $z \in [-b_z, -\overline{z})$, while $\frac{dS_1^{sc}(z)}{dz} = \frac{dS_1^{ac}(z)}{dz} < 0$ for $z \in [-\overline{z}, \underline{z}]$. Therefore,

$$\int_{-b_z}^{b_z} \left[S_1^{sc}(z) - S_1^{sc}(\hat{z})\right]^2 dF(z) > \int_{-b_z}^{b_z} \left[S_1^{ac}(z) - S_1^{ac}(\hat{z})\right]^2 dF(z). \tag{E.39}$$

Since

$$\text{Var}\left[S_1^{ac}(z)\right] = \min_x \int_{-b_z}^{b_z} \left[S_1^{ac}(z) - x\right]^2 dF(z),$$

(E.38) and (E.39) imply that $\text{Var}\left[S_1^{sc}(z)\right] > \text{Var}\left[S_1^{ac}(z)\right]$. Therefore, γ is higher when capital is scarce than when it is abundant. ∎

Proof of Proposition 7.5: Since the third term in the right-hand side of (7.11) is positive, S_0 in (7.11) is lower than in (7.10) for small α. ∎

Proof of Proposition 7.6: Eqs. (E.29) and (E.33) imply that for small α and all $z_1 > z_2$,

$$S_1(\omega z_1) - S_1(\omega z_2) < S_1(z_1) - S_1(z_2) < 0.$$

This inequality and the argument in Proposition 7.4 imply that γ is higher under the new distribution. To show the comparisons for (λ, S_0), we distinguish cases. If capital is abundant under the new distribution (and so abundant under both), (E.32) implies that λ is the same under both distributions, and (7.10) implies that S_0 is lower under the new distribution. If capital is scarce under the new distribution and abundant under the old, the comparisons follow from Propositions 7.3 and 7.5. If capital is scarce under both distributions, (E.34), (E.35) and (E.37) imply that λ is higher under the new distribution if

$$\int_{\frac{\underline{z}}{\omega}}^{b_z} \underline{z}(\omega z - \underline{z})dF(z) + \int_{\frac{\overline{z}}{\omega}}^{b_z} \overline{z}(\omega z - \overline{z})dF(z) > \int_{\underline{z}}^{b_z} \underline{z}(z - \underline{z})dF(z) + \int_{\overline{z}}^{b_z} \overline{z}(z - \overline{z})dF(z), \tag{E.40}$$

$$\int_{\frac{\underline{z}}{\omega}}^{b_z} (\omega^2 z^2 - \underline{z}^2)dF(z) + \int_{\frac{\overline{z}}{\omega}}^{b_z} (\omega^2 z^2 - \overline{z}^2)dF(z) > \int_{\underline{z}}^{b_z} (z^2 - \underline{z}^2)dF(z) + \int_{\overline{z}}^{b_z} (z^2 - \overline{z}^2)dF(z), \tag{E.41}$$

$$\int_{\frac{\underline{z}}{\omega}}^{b_z} (\omega z - \underline{z})dF(z) - \int_{\frac{\overline{z}}{\omega}}^{b_z} (\omega z - \overline{z})dF(z) > \int_{\underline{z}}^{b_z} (z - \underline{z})dF(z) - \int_{\overline{z}}^{b_z} (z - \overline{z})dF(z), \tag{E.42}$$

and (7.11) implies that S_0 is lower under the new distribution if

$$\int_{\frac{\underline{z}}{\omega}}^{\frac{\overline{z}}{\omega}} (\omega z - \underline{z})dF(z) + \int_{\frac{\overline{z}}{\omega}}^{b_z} (\overline{z} - \underline{z})dF(z) > \int_{\underline{z}}^{\overline{z}} (z - \underline{z})dF(z) + \int_{\overline{z}}^{b_z} (\overline{z} - \underline{z})dF(z). \tag{E.43}$$

Eqs. (E.40)-(E.43) are satisfied because the left-hand side of each is increasing in $\omega > 1$. ∎

F Non-Competitive Behavior

Proof of Proposition 8.1: Substituting W_1 from (3.10), and using normality, we can write (8.2) as

$$-\mathrm{E}\exp\left\{-\alpha\left[W_0 + \theta_0\left(S_1(\theta_1^d) - S_0\right) + \theta_1^d\left(\mathrm{E}[D|s] - S_1(\theta_1^d)\right) + z\left(\mathrm{E}[D|s] - \bar{D}\right) - \frac{1}{2}\alpha\sigma^2[D|s](\theta_1^d + z)^2\right]\right\}. \tag{F.1}$$

Since in equilibrium $\theta_0 = \bar{\theta}$, the first-order condition with respect to θ_1^d is

$$\mathrm{E}[D|s] - S_1(\theta_1^d) - \hat{\lambda}(\theta_1^d - \bar{\theta}) - \alpha\sigma^2[D|s](\theta_1^d + z) = 0. \tag{F.2}$$

Eq. (8.3) follows by rearranging (F.2). ∎

Proof of Proposition 8.2: The proof is similar to that of Proposition 4.2. Eq. (B.2) is replaced by

$$(1-\pi)\frac{\bar{D}+\frac{\beta_\xi}{b}(S_1-a)-S_1}{\alpha\sigma^2[D|S_1]}+\pi\frac{\bar{D}+\frac{\beta_s}{b}(S_1-a)+\beta_s cz-S_1-\alpha\sigma^2[D|s]z+\hat{\lambda}\bar{\theta}}{\alpha\sigma^2[D|s]+\hat{\lambda}}=\bar{\theta}. \tag{F.3}$$

Eq. (F.3) can be viewed as an affine equation in the variables (S_1-a, z). Setting terms in S_1-a to zero, we find

$$(1-\pi)\frac{\frac{\beta_\xi}{b}-1}{\alpha\sigma^2[D|S_1]}+\pi\frac{\frac{\beta_s}{b}-1}{\alpha\sigma^2[D|s]+\hat{\lambda}}=0. \tag{F.4}$$

Setting terms in z to zero, and using (4.2b), we find (4.6c). Setting constant terms to zero, we find

$$(1-\pi)\frac{\bar{D}-a}{\alpha\sigma^2[D|S_1]}+\pi\frac{\bar{D}-a+\hat{\lambda}\bar{\theta}}{\alpha\sigma^2[D|s]+\hat{\lambda}}=\bar{\theta}$$

$$\Leftrightarrow(1-\pi)\frac{\bar{D}-a}{\alpha\sigma^2[D|S_1]}+\pi\left[\bar{\theta}+\frac{\bar{D}-a-\alpha\sigma^2[D|s]\bar{\theta}}{\alpha\sigma^2[D|s]+\hat{\lambda}}\right]=\bar{\theta}. \tag{F.5}$$

Using (F.4) and the definition of $\hat{\lambda}$, we find (8.4). Using (F.4) and (F.5), and following the same argument as in the proof of Proposition 4.2, we find (4.6a).

A linear equilibrium exists if the liquidity demanders' second-order condition is met. Eq. (F.1) implies that the second-order condition is

$$\alpha\sigma^2[D|s]+2\hat{\lambda}>0$$

$$\Leftrightarrow\sigma^2[D|s]+\frac{2\left[\pi\beta_s\sigma^2[D|S_1]+(1-\pi)\beta_\xi\sigma^2[D|s]\right]}{(\beta_s-2\beta_\xi)(1-\pi)}>0$$

$$\Leftrightarrow\frac{\sigma_\epsilon^2}{\sigma^2+\sigma_\epsilon^2}+\frac{2\left(\sigma_\epsilon^2+\pi c^2\sigma_z^2\right)}{\left(c^2\sigma_z^2-\sigma^2-\sigma_\epsilon^2\right)(1-\pi)}>0, \tag{F.6}$$

where the second step follows from (8.4) and the definition of $\hat{\lambda}$, and the third from (4.2b), (4.3b), and the definitions of (β_s,β_ξ). Eq. (F.6) is satisfied if and only if $c^2\sigma_z^2-\sigma^2-\sigma_\epsilon^2>0$, which from (4.6c) is equivalent to $\sigma_\epsilon^2>\hat{\sigma}_\epsilon^2$. ∎

Proof of Proposition 8.3: The proof is similar to that of Proposition 4.3. The expected utility of a liquidity supplier in Period 1 is (B.6), and the expectation over (s,z) is (B.10) for F_s given by (B.14). Substituting $\mathrm{E}[D|s]-S_1(\theta_1^d)$ from (F.2), we can write the expected utility (F.1) of a liquidity demander as

$$-\exp\left\{-\alpha\left[W_0+\theta_0\left(S_1-S_0\right)+\beta_s\xi z+\tfrac{1}{2}\alpha\sigma^2[D|s]\left[(\theta_1^d)^2+z^2\right]+\hat{\lambda}\theta_1^d(\theta_1^d-\bar{\theta})\right]\right\}. \tag{F.7}$$

(Eq. (F.2) holds for $\theta_0=\bar{\theta}$ even when one agent chooses $\theta_0\neq\bar{\theta}$. This is because agents behave competitively in Period 0, and therefore a non-equilibrium choice $\theta_0\neq\bar{\theta}$ by one agent does not imply non-equilibrium choices by other agents.) Using (4.2a), (4.2b), (4.6c) and the definition of ξ, we can write (8.3) as

$$\theta_1^d=\frac{\bar{D}+\beta_s\xi-S_1+\hat{\lambda}\bar{\theta}}{\alpha\sigma^2[D|s]+\hat{\lambda}}. \tag{F.8}$$

Substituting θ_1^d from (F.8), and S_1 from (4.1), we can write (F.7) as

$$-\exp\left\{-\alpha\left[W_0+\theta_0(a+b\xi-S_0)+\beta_s\xi z+\frac{\alpha\sigma^2[D|s]\left[\bar{D}+\beta_s\xi-(a+b\xi)+\hat{\lambda}\bar{\theta}\right]^2}{2\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}\right.\right.$$
$$\left.\left.+\tfrac{1}{2}\alpha\sigma^2[D|s]z^2+\frac{\hat{\lambda}\left[\bar{D}+\beta_s\xi-(a+b\xi)+\hat{\lambda}\bar{\theta}\right]\left[\bar{D}+\beta_s\xi-(a+b\xi)-\alpha\sigma^2[D|s]\bar{\theta}\right]}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}\right]\right\}. \tag{F.9}$$

To compute the expectation of (F.9) over (s,z), we use Lemma A.1 and set

$$x\equiv\begin{bmatrix}\xi\\ z\end{bmatrix}$$

$$\Sigma\equiv\begin{bmatrix}\sigma^2+\sigma_\epsilon^2+c^2\sigma_z^2 & -c\sigma_z^2\\ -c\sigma_z^2 & \sigma_z^2\end{bmatrix}$$

$$A\equiv W_0+\theta_0(a-S_0)+\frac{\left(\bar{D}-a+\hat{\lambda}\bar{\theta}\right)\left[(\bar{D}-a)\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)-\alpha\sigma^2[D|s]\hat{\lambda}\bar{\theta}\right]}{2\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}$$

$$B\equiv\begin{bmatrix}b\theta_0+\frac{\left[(\bar{D}-a)\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)+\hat{\lambda}^2\bar{\theta}\right](\beta_s-b)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}\\ 0\end{bmatrix}$$

$$C\equiv\begin{bmatrix}\frac{(\beta_s-b)^2\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2} & \beta_s\\ \beta_s & \alpha\sigma^2[D|s]\end{bmatrix}.$$

Proceeding as in the proof of Proposition 4.3, we find

$$U^d=-\exp\left(-\alpha F^d\right)\frac{1}{\sqrt{1+\frac{\alpha(\beta_s-b)^2\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2)-\alpha^2\sigma^2\sigma_z^2}}, \tag{F.10}$$

where

$$F^d = W_0 + \theta_0(a - S_0) + \frac{\left(\bar{D} - a + \hat{\lambda}\bar{\theta}\right)\left[(\bar{D} - a)\left(\alpha\sigma^2[D|s] + 2\hat{\lambda}\right) - \alpha\sigma^2[D|s]\hat{\lambda}\bar{\theta}\right]}{2\left(\alpha\sigma^2[D|s] + \hat{\lambda}\right)^2}$$

$$- \frac{\alpha\left[b\theta_0 + \frac{\left[(\bar{D}-a)\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)+\hat{\lambda}^2\bar{\theta}\right](\beta_s-b)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}\right]^2(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2)}{2\left[1 + \frac{\alpha(\beta_s-b)^2\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2\right]}. \tag{F.11}$$

Substituting $\bar{D} - a$ from (4.6a) into (F.11), we find

$$F^d = W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2$$

$$+ \alpha\left\{b\sigma^2\theta_0\bar{\theta} - \tfrac{1}{2}\sigma^2\bar{\theta}^2 + \frac{\left[\alpha(1-b)\sigma^2 + \hat{\lambda}\right]\left[(1-b)\sigma^2\left(\alpha\sigma^2[D|s] + 2\hat{\lambda}\right) - \sigma^2[D|s]\hat{\lambda}\right]}{2\left(\alpha\sigma^2[D|s] + \hat{\lambda}\right)^2}\bar{\theta}^2\right.$$

$$\left. - \frac{\left[b\theta_0 + \frac{\left[\alpha(1-b)\sigma^2\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)+\hat{\lambda}^2\right](\beta_s-b)\bar{\theta}}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}\right]^2(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2)}{2\left[1 + \frac{\alpha(\beta_s-b)^2\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2\right]}\right\}. \tag{F.12}$$

Using (4.2b) and the definition of β_s, we can write (F.12) as

$$F^d = W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2$$

$$- \frac{\alpha\left\{b^2(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2)\theta_0^2 + 2b(\sigma^2+\sigma_\epsilon^2)\left[\alpha^2\sigma^2\sigma_z^2 - b(1+\alpha^2\sigma_\epsilon^2\sigma_z^2)\right]\theta_0\bar{\theta} + X\bar{\theta}^2\right\}}{2\left[1 + \frac{\alpha(\beta_s-b)^2\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2\right]}, \tag{F.13}$$

where

$$X \equiv \left[\sigma^2 - \frac{(1-b)^2\sigma^4}{\sigma^2[D|s]}\right](1-\alpha^2\sigma^2\sigma_z^2) + \frac{(\beta_s-b)^2\sigma^2}{\sigma^2[D|s]}(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2) - \frac{\alpha^2(\beta_s-b)^2\hat{\lambda}^2\sigma^4(\sigma^2+\sigma_\epsilon^2)\sigma_z^2}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2\sigma^2[D|s]}.$$

Using (B.21) to eliminate the term in $\sigma^2 - \frac{(1-b)^2\sigma^4}{\sigma^2[D|s]}$ in the definition of X, and substituting X into (B.20), we find

$$F^d = W_0 + \theta_0(\bar{D} - S_0) - \alpha\sigma^2\theta_0\bar{\theta} + \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2$$

$$- \frac{\alpha\left\{b^2(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2)(\theta_0-\bar{\theta})^2 + \alpha^2(\sigma^2+\sigma_\epsilon^2)^2\sigma_z^2\left[\frac{2b\sigma^2\theta_0\bar{\theta}}{\sigma^2+\sigma_\epsilon^2} + \hat{X}\bar{\theta}^2\right]\right\}}{2\left[1 + \frac{\alpha(\beta_s-b)^2\left(\alpha\sigma^2[D|s]+2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s]+\hat{\lambda}\right)^2}(\sigma^2+\sigma_\epsilon^2)(1+\alpha^2\sigma_\epsilon^2\sigma_z^2) - \alpha^2\sigma^2\sigma_z^2\right]}, \tag{F.14}$$

where

$$\hat{X} \equiv \frac{(\beta_s - b)^2\sigma^2}{\sigma^2[D|s]}\left[1 - \frac{\hat{\lambda}^2\sigma^2}{\left(\alpha\sigma^2[D|s] + \hat{\lambda}\right)^2(\sigma^2 + \sigma_\epsilon^2)}\right] - b^2.$$

We next note that

$$\begin{aligned}
&\frac{\alpha(\beta_s - b)^2\left(\alpha\sigma^2[D|s] + 2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s] + \hat{\lambda}\right)^2}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) \\
=&\frac{(b - \beta_\xi)^2\left(\alpha\sigma^2[D|s] + 2\hat{\lambda}\right)(1-\pi)^2}{\alpha\sigma^4[D|S_1]\pi^2}(\sigma^2 + \sigma_\epsilon^2)(1 + \alpha^2\sigma_\epsilon^2\sigma_z^2) \\
=&\frac{(b - \beta_\xi)^2\left(\alpha\sigma^2[D|s] + 2\hat{\lambda}\right)(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)(1-\pi)^2}{\alpha\sigma^2[D|s]\sigma^2[D|S_1]\pi^2} \\
=&\Delta_0\left(1 + \frac{\hat{\lambda}}{\alpha\sigma^2[D|s]}\right)(1-\pi)^2, \qquad \text{(F.15)}
\end{aligned}$$

where the first step follows from (F.4), the second from (4.2b), (4.3b), (4.6c) and the definitions of (β_s, β_ξ), and the third from (4.7a). Therefore, (F.10) takes the form (A.5), with Δ_0 replaced by $\Delta_0\left(1 + \frac{2\hat{\lambda}}{\alpha\sigma^2[D|s]}\right)$. Eqs. (B.14), (F.14) and (F.15) imply that when $\theta_0 = \bar{\theta}$, $(dF^s/d\theta_0, F^s, dF^d/d\theta_0, F^d)$ are given by (A.8)-(A.11), with (Δ_1, Δ_2) given by (8.7a) and (8.7b). Since the equations for $(U^s, U^d, dF^s/d\theta_0, F^s, dF^d/d\theta_0, F^d)$ take the same form as in Proposition 3.2, the same applies to S_0. ∎

Proof of Proposition 8.4: The proof that λ is given by (4.8) is the same as for Proposition 4.4. When information is asymmetric ($\sigma_\epsilon^2 < \infty$), (4.6b) and (8.4) imply that b is smaller under non-competitive behavior. Therefore, (4.8) implies that λ is higher. To determine λ when information is symmetric, we consider the limit $\sigma_\epsilon^2 \to \infty$. Eqs. (4.2b), (4.3b), (4.6c), and the definitions of (β_s, β_ξ) imply that $\frac{\beta_\xi}{\beta_s} \to 0$ and $\frac{b}{\beta_s} \to \frac{\pi}{1+\pi}$. Eq. (4.8) then implies that $\lambda \to \frac{\alpha\sigma^2}{1-\pi}$, which coincides with the competitive counterpart (3.19). ∎

Proof of Proposition 8.5: The proof that γ is given by (4.8) is the same as for Proposition 4.5. When information is asymmetric ($\sigma_\epsilon^2 < \infty$), b is smaller under non-competitive behavior, and (4.9) implies that γ is lower. To determine γ when information is symmetric, we consider the limit $\sigma_\epsilon^2 \to \infty$. Since $\frac{\beta_\xi}{\beta_s} \to 0$ and $\frac{b}{\beta_s} \to \frac{\pi}{1+\pi}$, (4.9) implies that $\gamma \to \frac{\alpha^2\sigma^4\sigma_z^2\pi^2}{(1+\pi)^2}$, which is lower than the competitive counterpart (3.21). ∎

Proof of Proposition 8.6: To determine S_0 when information is symmetric, we consider the limit $\sigma_\epsilon^2 \to \infty$. Since $\frac{\beta_\xi}{\beta_s} \to 0$ and $\frac{b}{\beta_s} \to \frac{\pi}{1+\pi}$, (4.7a), (8.7a) and (8.7b) imply that

$$\Delta_0 \to \frac{\alpha^2\sigma^2\sigma_z^2}{(1+\pi)^2} < \Delta_0^c$$

$$\Delta_0\left(1+\frac{2\hat{\lambda}}{\alpha\sigma^2[D|s]}\right) \to \frac{\alpha^2\sigma^2\sigma_z^2}{1-\pi^2} > \Delta_0^c$$

$$\Delta_1 \to \frac{\frac{\alpha^3\sigma^4\sigma_z^2\pi}{1+\pi}}{1+\frac{\alpha^2\sigma^2\sigma_z^2(1-\pi)}{1+\pi}-\alpha^2\sigma^2\sigma_z^2} < \Delta_1^c$$

$$\Delta_2 \to \frac{\alpha^3\sigma^4\sigma_z^2}{1+\frac{\alpha^2\sigma^2\sigma_z^2(1-\pi)}{1+\pi}-\alpha^2\sigma^2\sigma_z^2} < \Delta_2^c,$$

where $\{\Delta_j^c\}_{j=0,1,2}$ denote the competitive counterparts of $\{\Delta_j\}_{j=0,1,2}$, given by (3.15a)-(3.15c). The above inequalities, together with (3.13), (3.14), and (8.6) imply that S_0 is higher under non-competitive behavior.

To show that S_0 can be lower under non-competitive behavior, we consider the limit $\sigma_\epsilon^2 \to \hat{\sigma}_\epsilon^2$ (and $\sigma_\epsilon^2 > \hat{\sigma}_\epsilon^2$ so that the linear equilibrium exists). Eqs. (4.2b), (4.3b), (4.6c), and the definitions of (β_s, β_ξ) imply that $\frac{\beta_\xi}{\beta_s} \to \frac{1}{2}$ and $\frac{b}{\beta_s} \to \frac{1}{2}$. Substituting into (4.7a), (8.7a) and (8.7b), and using the definition of $\hat{\lambda}$, we find

$$\Delta_0 \to 0$$

$$\Delta_0\left(1+\frac{2\hat{\lambda}}{\alpha\sigma^2[D|s]}\right) \to 0$$

$$\Delta_1 \to \frac{\alpha^3\sigma^4\sigma_z^2}{2\left(1-\alpha^2\sigma^2\sigma_z^2\right)}$$

$$\Delta_2 \to \frac{\alpha^3\sigma^4\sigma_z^2}{1-\alpha^2\sigma^2\sigma_z^2}.$$

The competitive counterparts of $\{\Delta_j\}_{j=0,1,2}$ are given by (4.7a)-(4.7c). Since $\Delta_0^c > 0$, the following inequalities hold when $\alpha^2\sigma^2\sigma_z^2 \approx 1$: $\Delta_j > \Delta_j^c$ for $j = 1, 2$, and $M > M^c$, where M^c denotes the competitive counterpart of M. These inequalities, together with (3.13), imply that S_0 is lower under non-competitive behavior. ∎

Proof of Proposition 8.7: Using (4.2b), (4.3b), (4.6c), (8.4) and the definitions of (β_s, β_ξ), we can write (4.8) as

$$\lambda = \frac{\alpha\sigma^2\left(\sigma_\epsilon^2 + c^2\sigma_z^2\pi\right)}{\left(c^2\sigma_z^2 - \sigma^2 - \sigma_\epsilon^2\right)\pi(1-\pi)}, \tag{F.16}$$

(4.9) as

$$\gamma = \frac{\sigma^4(\sigma_\epsilon^2 + c^2\sigma_z^2)(\sigma_\epsilon^2 + c^2\sigma_z^2\pi)(c^2\sigma_z^2 - \sigma^2 - \sigma_\epsilon^2)\pi}{\left[2(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2)\pi + \sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)(1-\pi)\right]^2}, \tag{F.17}$$

(4.7a) as

$$\Delta_0 = \frac{\sigma^2(\sigma_\epsilon^2 + c^2\sigma_z^2)(c^2\sigma_z^2 - \sigma^2 - \sigma_\epsilon^2)^2}{\left[2(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2)\pi + \sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)(1-\pi)\right]^2}, \tag{F.18}$$

and the numerator in (8.7a) as

$$\alpha^3 b\sigma^2(\sigma^2 + \sigma_\epsilon^2)\sigma_z^2 = \frac{\alpha^3\sigma^4(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2\pi)\sigma_z^2}{2(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2)\pi + \sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)(1-\pi)}. \tag{F.19}$$

Using (F.18) and the definition of $\hat{\lambda}$, we find

$$\Delta_0\left(1 + \frac{2\hat{\lambda}}{\alpha\sigma^2[D|s]}\right) = \frac{\sigma^2(\sigma_\epsilon^2 + c^2\sigma_z^2)(c^2\sigma_z^2 - \sigma^2 - \sigma_\epsilon^2)}{\sigma_\epsilon^2\left[2(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2)\pi + \sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)(1-\pi)\right](1-\pi)}. \tag{F.20}$$

Using (F.15) and (F.20), we find

$$\frac{\alpha(\beta_s - b)^2\left(\alpha\sigma^2[D|s] + 2\hat{\lambda}\right)}{\left(\alpha\sigma^2[D|s] + \hat{\lambda}\right)^2}(\sigma^2 + \sigma_\epsilon^2) = \frac{\sigma^2(c^2\sigma_z^2 - \sigma^2 - \sigma_\epsilon^2)(1-\pi)}{2(\sigma^2 + \sigma_\epsilon^2)(\sigma_\epsilon^2 + c^2\sigma_z^2)\pi + \sigma_\epsilon^2(\sigma^2 + \sigma_\epsilon^2 + c^2\sigma_z^2)(1-\pi)}. \tag{F.21}$$

Eq. (F.16) implies that λ is decreasing in σ_z^2. Eq. (F.17) implies that γ is increasing in σ_z^2. Eq. (F.18) implies that Δ_0 is increasing in σ_z^2. Eq. (F.20) implies that

$$1 + \Delta_0\left(1 + \frac{2\hat{\lambda}}{\alpha\sigma^2[D|s]}\right)(1-\pi)^2 - \alpha^2\sigma^2\sigma_z^2$$

is decreasing in σ_z^2. Eq. (F.19) implies that the numerator in (8.7a) is increasing in σ_z^2, and so is Δ_1. Eq. (F.21) implies that the numerator in (8.7b) is increasing in σ_z^2, and so are Δ_2 and M. Therefore, (3.13) implies that S_0 is decreasing in σ_z^2. ∎

G Search

Proof of Proposition 9.1: The proposition follows by substituting the certainty equivalents from (9.1)-(9.4) into (9.5), setting $x = z/2$, and solving for S_1. ∎

Proof of Proposition 9.2: We first compute the interim utility U^s in Period 1/2 of a liquidity supplier who buys $\theta_0 \neq \bar{\theta}$ shares in Period 0. The certainty equivalents (9.1) and (9.2) are replaced by

$$CEQ^{sn} = W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2,$$
$$CEQ^{s} = W_0 + \theta_0(\bar{D} - S_0) + x(\bar{D} - S_1) - \tfrac{1}{2}\alpha\sigma^2(\theta_0 + x)^2,$$

respectively. If the supplier does not meet a demander, he receives expected utility

$$U_1^{sn} = -\exp\left(-\alpha CEQ^{sn}\right) \tag{G.1}$$

in Period 1. If the supplier meets a demander, he buys

$$x = \tfrac{1}{2}(z + \bar{\theta} - \theta_0) \tag{G.2}$$

shares, the maximand of $CEQ^s + CEQ^d$. Expected utility in Period 1 is

$$U_1^{s} = -\exp\left(-\alpha CEQ^{s}\right) = -\exp\left[-\alpha\left(CEQ^{sn} + \phi\alpha\sigma^2x^2\right)\right], \tag{G.3}$$

where the second step follows from (9.5) because the surplus from the transaction is $\alpha\sigma^2x^2$. Expected utility in Period 1/2 is

$$U^s = \pi^s E\left(U_1^s\right) + (1-\pi^s)E\left(U_1^{sn}\right) = \pi^s E\left(U_1^s\right) + (1-\pi^s)U_1^{sn}, \tag{G.4}$$

where the expectation is over z, and the second step follows because U_1^{sn} is independent of z. Lemma A.1 and (G.2) imply that the expectation of (G.3) is

$$E\left(U_1^s\right) = -\exp\left\{-\alpha\left[W_0 + \theta_0(D - S_0) - \tfrac{1}{2}\alpha\sigma^2\theta_0^2 + \frac{\phi\alpha\sigma^2(\theta_0 - \bar{\theta})^2}{4G_1}\right]\right\}\frac{1}{\sqrt{G_1}}. \tag{G.5}$$

Using (G.1), (G.4), and (G.5), we find

$$\left.\frac{dU^s}{d\theta_0}\right|_{\theta_0=\bar{\theta}} = \alpha\exp\left\{-\alpha\left[W_0 + \bar{\theta}(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2\right]\right\}\left(\bar{D} - S_0 - \alpha\sigma^2\bar{\theta}\right)\left(\frac{\pi^s}{\sqrt{G_1}} + 1 - \pi^s\right). \tag{G.6}$$

We next compute the interim utility U^d in Period 1/2 of a liquidity demander who buys $\theta_0 \neq \bar{\theta}$ shares in Period 0. The certainty equivalents (9.3) and (9.4) are replaced by

$$CEQ^{dn} = W_0 + \theta_0(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2(\theta_0 + z)^2,$$

$$CEQ^d = W_0 + \theta_0(\bar{D} - S_0) - x(\bar{D} - S_1) - \tfrac{1}{2}\alpha\sigma^2(\theta_0 + z - x)^2,$$

respectively. If the demander does not meet a supplier, he receives expected utility

$$U_1^{dn} = -\exp\left(-\alpha CEQ^{dn}\right) \tag{G.7}$$

in Period 1. If the supplier meets a demander, he sells

$$x = \tfrac{1}{2}(z + \theta_0 - \bar{\theta}) \tag{G.8}$$

shares, the maximand of $CEQ^s + CEQ^d$. Expected utility in Period 1 is

$$U_1^s = -\exp\left(-\alpha CEQ^d\right) = -\exp\left[-\alpha\left(CEQ^{dn} + (1-\phi)\alpha\sigma^2 x^2\right)\right], \tag{G.9}$$

and expected utility in Period 1/2 is

$$U^d = \pi^d E\left(U_1^d\right) + (1 - \pi^d) E\left(U_1^{dn}\right). \tag{G.10}$$

The expectation of (G.7) is (C.5) because in both cases the demander does not trade in Period 1. Lemma A.1 and (G.8) imply that the expectation of (G.9) is

$$E\left(U_1^d\right) = -\exp\left\{-\alpha\left[W_0 + \theta_0(D - S_0) - \tfrac{1}{2}\alpha\sigma^2 \frac{\frac{1}{2}(1+\phi)\theta_0^2 + (1-\phi)\theta_0\bar{\theta} - \frac{1}{2}(1-\phi)(1-\alpha^2\sigma^2\sigma_z^2)\bar{\theta}^2}{G_2}\right]\right\} \frac{1}{\sqrt{G_2}}. \tag{G.11}$$

Using (C.5), (G.10), and (G.11), we find

$$\left.\frac{dU^d}{d\theta_0}\right|_{\theta_0 = \bar{\theta}} = \alpha \exp\left\{-\alpha\left[W_0 + \bar{\theta}(\bar{D} - S_0) - \tfrac{1}{2}\alpha\sigma^2\bar{\theta}^2\right]\right\}$$
$$\times \left\{\frac{\pi^d}{\sqrt{G_2}} \exp\left(\frac{\bar{\theta}^2\alpha^4\sigma^4\sigma_z^2}{2G_2}\right)\left(\bar{D} - S_0 - \frac{\alpha\sigma^2\bar{\theta}}{G_2}\right) + \frac{1-\pi^d}{\sqrt{G_3}} \exp\left(\frac{\bar{\theta}^2\alpha^4\sigma^4\sigma_z^2}{2G_3}\right)\left(\bar{D} - S_0 - \frac{\alpha\sigma^2\bar{\theta}}{G_3}\right)\right\}. \tag{G.12}$$

Substituting (G.6), (G.12), $\pi^s = N/(1-\pi)$, and $\pi^d = N/\pi$, into (D.7), and solving for S_0, we find (9.7). ∎

Proof of Proposition 9.3: The signed order flow of liquidity demanders is $-Nz/2$ since the measure of meetings is N and in each meeting a demander sells $z/2$ shares. Eq. (9.6) implies that

$$\lambda = \frac{\text{Cov}\left(S_1 - S_0, -\frac{1}{2}Nz\right)}{\text{Var}\left(-\frac{1}{2}Nz\right)} = \frac{\frac{1}{4}\alpha\sigma^2 z(1+2\phi)}{\frac{1}{2}Nz} = \frac{\alpha\sigma^2(1+2\phi)}{2N}.$$

Illiquidity is higher than in the centralized market if

$$\frac{1+2\phi}{2N} \geq \frac{1}{1-\pi}. \tag{G.13}$$

Eq. (G.13) yields the condition in the proposition. ∎

Proof of Proposition 9.4: Eq. (9.9) follows from (3.20) and (9.6). The comparison with the centralized market follows from (3.21) and (9.9). ∎

Proof of Proposition 9.5: The derivative of (9.7) with respect to N has the same sign as

$$\begin{aligned}&(1-\pi)\left[\frac{1+\phi}{2G_3^{\frac{3}{2}}}\exp\left(\frac{\bar{\theta}^2\alpha^4\sigma^4\sigma_z^2}{2G_3}\right) - \frac{1}{G_2^{\frac{3}{2}}}\exp\left(\frac{\bar{\theta}^2\alpha^4\sigma^4\sigma_z^2}{2G_2}\right)\right]\\ &+\left(\frac{1}{\sqrt{G_1}}-1\right)\frac{\pi}{G_3^{\frac{3}{2}}}\exp\left(\frac{\bar{\theta}^2\alpha^4\sigma^4\sigma_z^2}{2G_3}\right) + \frac{\pi(1-\phi)}{2G_2^{\frac{3}{2}}G_3^{\frac{3}{2}}}\exp\left(\frac{\bar{\theta}^2\alpha^4\sigma^4\sigma_z^2}{2G_2}\right)\exp\left(\frac{\bar{\theta}^2\alpha^4\sigma^4\sigma_z^2}{2G_3}\right).\end{aligned} \tag{G.14}$$

The first term in (G.14) is positive because $G_2 > G_3$. A sufficient condition for the sum of the second and third terms to be positive is

$$\begin{aligned}&\frac{1}{\sqrt{G_1}} - 1 + \frac{1-\phi}{2G_2^{\frac{3}{2}}} > 0\\ \Leftrightarrow &\frac{1-\phi}{2G_2^{\frac{3}{2}}} > \frac{\phi\alpha^2\sigma^2\sigma_z^2}{2\sqrt{G_1}\left(1+\sqrt{G_1}\right)}.\end{aligned} \tag{G.15}$$

Eq. (G.15) holds if $\phi \leq 1/2$ because of (2.2) and $G_1 > 1 > G_2$. ∎

Proof of Proposition 9.6: Eq. (9.8) implies that λ is independent of σ_z^2, and (9.9) implies that γ is increasing in σ_z^2. To show that S_0 is decreasing in σ_z^2, we write the illiquidity discount in (9.7) as

$$\frac{\frac{N(1+\phi)}{2G_2^{\frac{3}{2}}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_2}\right)+\frac{\pi-N}{G_3^{\frac{3}{2}}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_3}\right)}{\frac{N}{\sqrt{G_2}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_2}\right)+\frac{\pi-N}{\sqrt{G_3}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_3}\right)}\frac{\alpha^3\sigma^4\sigma_z^2\bar{\theta}}{\frac{\frac{N}{\sqrt{G_1}}+1-\pi-N}{\frac{N}{\sqrt{G_2}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_2}\right)+\frac{\pi-N}{\sqrt{G_3}}\exp\left(\frac{\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2}{2G_3}\right)}+1}.$$

Since G_1 is increasing in σ_z^2, while (G_2, G_3) are decreasing, the second fraction is increasing in σ_z^2. To show that the first fraction is also increasing, we write it as

$$\frac{\frac{1+\phi}{2G_2} + \frac{\pi - N}{NG_3}F}{1 + \frac{\pi - N}{N}F}, \tag{G.16}$$

where

$$F = \sqrt{\frac{G_2}{G_3}} \exp\left[\tfrac{1}{2}\alpha^4\sigma^4\sigma_z^2\bar{\theta}^2\left(\frac{1}{G_3} - \frac{1}{G_2}\right)\right].$$

The derivative of (G.16) with respect to σ_z^2 has the same sign as

$$\left[\tfrac{1}{2}(1+\phi)\frac{d\left(\frac{1}{G_2}\right)}{d\sigma_z^2} + \frac{\pi - N}{N}\frac{d\left(\frac{1}{G_3}\right)}{d\sigma_z^2}F\right]\left(1 + \frac{\pi - N}{N}F\right) + \frac{\pi - N}{N}\frac{dF}{d\sigma_z^2}\left(\frac{1}{G_3} - \frac{1+\phi}{2G_2}\right). \tag{G.17}$$

Eq. (G.17) is positive because (G_2, G_3) are decreasing in σ_z^2, F is increasing, and $G_2 \geq G_3$. ∎