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A PREFERRED-HABITAT MODEL OF THE TERM STRUCTURE OF INTEREST RATES[1]

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We model the term structure of interest rates that results from the interaction between investors with preferences for specific maturities and risk-averse arbitrageurs. Shocks to the short rate are transmitted to long rates through arbitrageurs' carry trades. Arbitrageurs earn rents from transmitting the shocks, through bond risk premia that relate positively to the slope of the term structure. When the short rate is the only risk factor, changes in investor demand have the same relative effect on interest rates across maturities regardless of the maturities where they originate. When investor demand is also stochastic, demand effects become more localized. A calibration indicates that long rates under-react to forward-guidance announcements about short rates. Large-scale asset purchases can be more effective in moving long rates, especially if they are concentrated at long maturities.

Keywords: Interest rates, bond risk premia, limited arbitrage, government debt, monetary policy.

1. INTRODUCTION

What determines the term structure of interest rates? In most macro-finance models, the interest rate for a given maturity depends on the willingness of a representative agent to substitute consumption from today towards that maturity. The consumption-based view of the term structure contrasts with a more informal preferred-habitat view, which has been proposed by Culbertson (1957) and Modigliani and Sutch (1966), and is popular within central banks and the financial industry. According to that view, there are investor clienteles for specific maturity segments, and the interest rate for a given maturity is mainly driven by shocks affecting

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the demand of the corresponding clientele. The term structure thus exhibits a degree of segmentation.

The preferred-habitat view has been used to interpret numerous market episodes. The 2004 U.K. pension reform is one example. The reform required pension funds to evaluate their pension liabilities using the yields of long-maturity bonds. To hedge against drops in long rates, which would raise the value of pension liabilities and trigger regulatory scrutiny, pension funds bought long-maturity bonds in large quantities. This drove long rates to record low levels. A flat term structure in early 2004 became downward-sloping in subsequent years, with the 30-year bond yielding as much as 0.80% (80 basis points, bps) below its 10-year counterpart.[1] More recently, the preferred-habitat view informed decisions by major central banks to engage in Quantitative Easing (QE). A stated goal of QE programmes was that large-scale purchases of long-maturity bonds would drive long rates down, stimulating corporate investment.[2]

The preferred-habitat view cannot be correct in its most extreme form, namely, the interest rate for a given maturity cannot be driven *only* by shocks affecting the demand of the corresponding clientele. Indeed, if that were the case, interest rates for nearby maturities could be very different, generating large profits for term-structure arbitrageurs. At the same time, shocks to clientele demands can affect interest rates. Indeed, because absorbing the shocks exposes arbitrageurs to interest-rate risk, bond prices must change to compensate them for the risk.

How do shocks to clientele demands affect the term structure? What are the effects of large-scale bond purchases by central banks? What are the implications of the preferred-habitat view for the dynamics of interest rates, for bond risk premia, and for the transmission of monetary policy from short to long rates? In this paper we develop a model to answer these questions both qualitatively as well as quantitatively through a calibration exercise. Our model formalizes the preferred-habitat view and embeds it into a modern no-arbitrage term-structure framework.

We describe our model in Section 2. The short rate follows an exogenous mean-reverting process. An exogenous short rate can be interpreted as the return of a linear and instantaneously riskless production technology, or as the instantaneous rate that a (non-modelled) central bank pays on reserves. Bond yields are determined endogenously through trading between preferred-habitat investors and arbitrageurs. Preferred-habitat investors demand zero-coupon bonds with specific maturities, and their demand can be price-elastic.

[1]For accounts of the 2004 U.K. pension reform and other related episodes, see Tzucker and Islam (2005), Garbade and Rutherford (2007), Islam (2007), and Greenwood and Vayanos (2010).

[2]See, for example, the 2011 speeches on large-scale asset purchases by Janet Yellen, the then Vice-Chair of the U.S. Federal Reserve (Yellen (2011)), and John Williams, the then President of the San Francisco Fed (Wiliams (2011)).

We provide an optimizing foundation for that demand in a setting where investors form overlapping generations consuming at the end of their life, are infinitely risk averse, and can invest in bonds and in a private opportunity with exogenous return (e.g., real estate). Arbitrageurs are competitive and maximize a mean-variance objective over instantaneous changes in wealth. We fix their aggregate risk aversion and do not study entry into the arbitrage business.

In Section 3 we solve for equilibrium when the demand of preferred-habitat investors is constant over time and the only risk factor is the short rate. We address three main questions: how shocks to the short rate are transmitted to long rates, how bond risk premia depend on the shape of the term structure, and how changes in preferred-habitat demand affect the term structure. Since demand is constant over time, we take demand changes to be unanticipated and permanent.

Shocks to the short rate are transmitted to bond yields through the trades of arbitrageurs. Suppose that the short rate drops. Since investing in bonds becomes more attractive than investing in the short rate, arbitrageurs buy bonds by borrowing short-term. That trade causes bond prices to rise and yields to drop. Because, however, arbitrageurs become exposed to the risk that the short rate will increase, they do not scale up their trade to the point where it earns zero expected profit. Hence, the drop in bond yields does not fully reflect the drop in the short rate, which means that forward rates under-react to expected future short rates. The under-reaction disappears when arbitrageurs are risk-neutral, or when preferred-habitat demand is price-inelastic since in that case arbitrageurs cause bond prices to rise without actually buying the bonds.

Bond risk premia (expected returns in excess of the short rate) are positively related to the slope of the term structure, consistent with the empirical findings of Fama and Bliss (FB 1987) and Campbell and Shiller (CS 1991). When the short rate is low, the term structure slopes up, and bonds earn positive risk premia so that arbitrageurs are induced to buy them. The risk premia accrue to arbitrageurs as a rent for transmitting short-rate shocks to long rates. Monetary-policy actions by central banks affecting the short rate can hence be viewed as a source of arbitrageur rent.[3] That rent is higher when arbitrageurs are more risk-averse and when preferred-habitat demand is more price-elastic.

When the short rate is the only risk factor, changes in preferred-habitat demand have global effects: the effects depend on how the arbitrageurs' overall exposure to the short rate ("duration risk") changes, and not on the specific maturities where the demand changes originate. To illustrate this result's surprising implications, suppose that the demand for short-maturity bonds increases and the demand for long-maturity

[3]We thank John Cochrane for suggesting this idea (Cochrane (2008)).

bonds decreases by the same amount in present-value terms. Since arbitrageurs buy long-maturity bonds, and these are more sensitive to short-rate changes than short-maturity bonds, all yields rise—including those of short-maturity bonds for which demand *increases*. The same logic implies that all demand changes have the same relative effect across maturities regardless of where they originate. Moreover, the effect is largest at the longest maturity. Indeed, since the longest-maturity bonds are the most sensitive to short-rate changes, their risk premia are also the most sensitive to changes in the arbitrageurs' exposure to the short rate.

In Section 4 we allow the demand of preferred-habitat investors to vary over time. We maintain a stochastic short rate; with a constant short rate, arbitrageur activity would render all yields equal to the short rate. We mainly focus on the case where demand has a one-factor structure and that factor is independent of the short rate, but we also consider multiple demand factors and correlation. Within the two-factor model, we revisit the same three questions as in Section 3.

Demand risk weakens and can even reverse the transmission of short-rate shocks to long rates. Suppose that the short rate drops, in which case arbitrageurs buy bonds. Arbitrageurs become exposed to the risk that the short rate will increase *and* that preferred-habitat demand will decrease. Because demand risk becomes dominant for long-maturity bonds, arbitrageurs buy them in small quantities and may even sell them short to hedge the demand risk of their long positions in intermediate maturities. Long-maturity yields may thus *rise* in response to a short-rate drop.

Demand risk strengthens the positive relationship between bond risk premia and term-structure slope. Indeed, when preferred-habitat demand is low, risk premia are high so that arbitrageurs are induced to buy bonds to make up for the low demand. Because of the high premia, bond yields are high and the term structure slopes up. As a result of the stronger premia-slope relationship, the model-generated coefficients in the FB and CS regressions have properties closer to their empirical counterparts. For example, the FB coefficient can be larger than one and increasing with maturity, rather than only positive and constant as in the one-factor model.

With multiple risk factors, demand effects become more localized. Changes in the demand for short- (long-) maturity bonds have more pronounced effects on short- (long-) maturity yields. As in the one-factor model, the effects arise through the arbitrageurs' exposure to the risk factors. They become more localized because demand changes originating at different maturities affect the exposure to each factor differently, and because changes in each factor exposure have a different relative effect across maturities.

In Section 5 we calibrate the two-factor model and analyze central-bank policies such as forward guidance and QE. We choose the model parameters to match the volatility of U.S. government bond yields and yield changes, the correlation between yield changes at the short and the long end of the term structure, and the composition of bond trading volume across maturities. Since the model can be given both a nominal and a real interpretation, we calibrate it using nominal yields and then again using real yields. The nominal and real calibrations generate remarkably similar results.

Forward guidance about short rates is effective in moving yields of short-maturity bonds, but becomes less effective for long maturities. Lowering the average expected short rate over the next ten years by 100 bps (and holding preferred-habitat demand constant) causes the ten-year yield to drop by 35-50 bps. The same change to the expected short rate over thirty years has almost no effect on the thirty-year yield. QE can be more effective in changing long rates, provided that bond purchases are concentrated at long maturities. Purchases amounting to 12% of GDP and conforming to the maturity distribution used by the Fed during QE1 lower the ten-year yield by 25-30bps and the thirty-year yield by 30-35bps. Tilting purchases towards long maturities, while keeping the fraction of available supply purchased in each maturity bucket within observed ceilings, increases the effects by 10 and 30bps, respectively.

Our model formalizes the preferred-habitat theory of the term structure, proposed by Culbertson (1957) and Modigliani and Sutch (1966). Related to preferred habitat is Tobin's (1958,1969) portfolio-balance theory, in which financial assets are imperfect substitutes, and investors require a rise in interest rates to absorb an increased supply of government bonds. The portfolio-balance channel is present in our model, with Tobin's investors being our arbitrageurs. It is the only channel present in the special case of our model where preferred-habitat demand is price-inelastic.

Andres, Lopez-Salido, and Nelson (2004) study demand effects and the portfolio-balance channel in a calibrated macroeconomic model with trading frictions. Greenwood and Vayanos (2014) use our model's special case with a price-inelastic demand to test for a positive relationship between the maturity of government debt and future bond returns. Other empirical studies of demand effects in the bond market that build on our model include Hamilton and Wu (2012) and Li and Wei (2013) on QE purchases and the zero lower bound (ZLB);[4] Hanson (2014) and Malkhozov, Mueller, Vedolin, and Venter (2016) on mortgage-backed securities; Gorodnichenko and Ray (2018) on Treasury auctions; Kaminska and Zinna (2019) on purchases by foreign

[4]For empirical estimates of the effects of QE, see also Gagnon, Raskin, Remache, and Sack (2011), Joyce, Lasaosa, Stevens, and Tong (2011), Krishnamurthy and Vissing-Jorgensen (2011), Swanson (2011), Christensen and Rudebusch (2012), D'Amico and King (2013), Swanson and Williams (2014), and the survey by Wiliams (2014). Some of these papers emphasize the duration-risk channel. That channel describes demand effects in the one-factor version of our model but not with multiple factors.

central banks; and King (2019) on non-linearities induced by the ZLB. Hayashi (2018) develops numerical algorithms to solve our model with a general number of risk factors.

The notion that demand shocks can drive asset prices away from fundamental values is emphasized in the literature on the limits of arbitrage, surveyed in Gromb and Vayanos (2010). Closest to our paper is the strand of the literature on price distortions across an asset class. See, for example, Barberis and Shleifer (2003) and Vayanos and Woolley (2013) on style investing, momentum and reversal; Greenwood (2005) and Hau (2011) on index redefinitions; Gabaix, Krishnamurthy, and Vigneron (2007) on mortgage-backed securities; Garleanu, Pedersen, and Poteshman (2009) on options; and Gabaix and Maggiori (2015) on foreign exchange.

Preferred habitats in our model concern maturities. They could alternatively concern bonds that differ in liquidity or in the type of issuer, e.g., government versus corporate. Preferences for liquidity have been used to explain the on-the-run phenomenon, whereby just-issued government bonds are more expensive than previously-issued bonds maturing on nearby dates.[5] Preferences for government bonds over corporate bonds could be arising because the former are safer and more widely acceptable as collateral. Krishnamurthy and Vissing-Jorgensen (2012) provide evidence consistent with the existence of an investor clientele pricing those attributes.

Our model belongs to the class of affine no-arbitrage term-structure models (Duffie and Kan (1996)) because yields are affine in the risk factors. Dai and Singleton (2002) and Duffee (2002) develop models within that class that embody the positive relationship between bond risk premia and term-structure slope. We derive such a relationship in an equilibrium model.[6] Our model can address questions that reduced-form models cannot such as how demand shocks affect the term structure and how the effects depend on arbitrageur risk aversion and investor price-elasticity.

2. MODEL

Time is continuous and goes from zero to infinity. The term structure at time t consists of a continuum of zero-coupon government bonds. The maturities of the bonds lie in the interval $(0, \infty)$. Assuming that the interval of bond maturities is infinite is without loss of generality because we can specify preferred-habitat

[5] For evidence on the on-the-run phenomenon, see Amihud and Mendelson (1991), Warga (1992) and Krishnamurthy (2002). For theoretical explanations, see Duffie (1996), Vayanos and Weill (2008) and Banerjee and Graveline (2013).

[6] Other equilibrium models that generate a positive premia-slope relationship include Wachter (2006), Buraschi and Jiltsov (2007) and Lettau and Wachter (2011) who assume habit formation; Xiong and Yan (2010) who assume heterogeneous beliefs; and Gabaix (2012) who assumes rare disasters with time-varying severity.

demand to be zero for bonds with sufficiently long maturities. The bond with maturity τ has face value one, hence paying one unit of the numeraire at time $t+\tau$. We denote by $P_t^{(\tau)}$ and $y_t^{(\tau)}$, respectively the time-t price and yield of the bond with maturity τ. The yield is the spot rate for maturity τ, and is related to the price through

$$y_t^{(\tau)} = -\frac{\log(P_t^{(\tau)})}{\tau}. \tag{1}$$

We denote by $f_t^{(\tau-\Delta\tau,\tau)}$ the time-t forward rate between maturities $\tau - \Delta\tau$ and τ. The forward rate is related to the price through

$$f_t^{(\tau-\Delta\tau,\tau)} = -\frac{\log\left(\frac{P_t^{(\tau)}}{P_t^{(\tau-\Delta\tau)}}\right)}{\Delta\tau}. \tag{2}$$

The short rate r_t is the limit of the yield $y_t^{(\tau)}$ when τ goes to zero. We take r_t as exogenous, and describe its dynamics later in this section (Equation (7)). An exogenous r_t can be interpreted as the return of a linear and instantaneously riskless production technology. Alternatively, r_t can be determined by the central bank in response to exogenous shocks. We sketch the central-bank interpretation in Section 3.3, where we derive some of our model's implications for monetary policy.

Agents are of two types: arbitrageurs and preferred-habitat investors. Arbitrageurs can invest in the bonds and in the short rate. We denote their time-t wealth by W_t and their time-t position, expressed in present-value terms, in the bonds with maturities in $[\tau, \tau + d\tau]$ by $X_t^{(\tau)} d\tau$. The arbitrageurs' budget constraint is

$$dW_t = \left(W_t - \int_0^\infty X_t^{(\tau)} d\tau\right) r_t dt + \int_0^\infty X_t^{(\tau)} \frac{dP_t^{(\tau)}}{P_t^{(\tau)}} d\tau, \tag{3}$$

where the instantaneous change $dP_t^{(\tau)}$ is computed by changing the time subscript t to $t+dt$ and the maturity superscript τ to $\tau - dt$.[7] Arbitrageurs maximize a mean-variance objective over instantaneous changes in

[7]Implicit in our notation is that the arbitrageurs' position in the bonds with maturities in $[\tau, \tau+d\tau]$ is of order $d\tau$. Arbitrageurs hold such a position in equilibrium because preferred-habitat demand for the bonds with maturities in $[\tau, \tau + d\tau]$ is assumed to be of order $d\tau$.

wealth. Their optimization problem is

$$\max_{\{X_t^{(\tau)}\}_{\tau\in(0,\infty)}} \left[\mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\mathrm{ar}_t(dW_t)\right], \tag{4}$$

where $a \geq 0$ is a risk-aversion coefficient that characterizes the trade-off between mean and variance. Arbitrageurs with the objective (4) can be interpreted as overlapping generations living over infinitesimal periods. The generation born at time t is endowed with wealth W, invests from t to $t+dt$, consumes at $t+dt$ and then dies. If preferences over consumption are described by the Von Neumann-Morgenstern (VNM) utility function U, and if all uncertainty is Brownian as is the case in equilibrium, utility maximization yields the objective (4) with the risk-aversion coefficient $a = -\frac{U''(W)}{U'(W)}$.

Preferred-habitat investors have preferences for specific maturities. For example, pension funds prefer long-maturity bonds because their duration matches that of pension liabilities. Insurance companies likewise prefer long- and intermediate-maturity bonds because their duration matches that of liabilities associated to retirement and insurance products that they offer. At the other end of the maturity spectrum, money-market funds are required by their mandates to hold short-maturity bonds. We model the demand of preferred-habitat investors in reduced form and provide an optimizing foundation in Appendix B.

Investors' maturity habitats cover the interval $(0,\infty)$, and investors with habitats in $[\tau, \tau+d\tau]$ are in measure $d\tau$. Investors with habitat τ at time t hold a position

$$Z_t^{(\tau)} = -\alpha(\tau)\log(P_t^{(\tau)}) - \beta_t^{(\tau)}, \tag{5}$$

expressed in present-value terms, in the bond with maturity τ and hold no other bonds. Equation (5) is a demand function linear and decreasing in the logarithm of the bond price. The slope coefficient $\alpha(\tau) \geq 0$ is constant over time but can depend on maturity τ. The intercept coefficient $\beta_t^{(\tau)}$ can depend on both t and τ. For simplicity, we refer to $\alpha(\tau)$ and $\beta_t^{(\tau)}$ as demand slope and demand intercept, respectively. The actual intercept is $-\beta_t^{(\tau)}$. By setting $\alpha(\tau) = \beta_t^{(\tau)} = 0$ for τ larger than a finite threshold T, we can take the interval of bond maturities to be finite and equal to $(0,T)$.

The demand intercept $\beta_t^{(\tau)}$ takes the form

$$\beta_t^{(\tau)} = \theta_0(\tau) + \sum_{k=1}^{K}\theta_k(\tau)\beta_{k,t}, \tag{6}$$

where $\{\theta_k(\tau)\}_{k=0,..,K}$ are constant over time but can depend on maturity τ, and $\{\beta_{k,t}\}_{k=1,..,K}$ are time-varying but independent of τ. We refer to $\{\beta_{k,t}\}_{k=1,..,K}$ as demand risk factors. The functions $\{\theta_k(\tau)\}_{k=1,..,K}$ characterize the maturities where demand changes originate. If, for example, $\theta_k(\tau)$ is independent of τ, then a change in $\beta_{k,t}$ impacts demand for all maturities equally, and can be interpreted as a global demand shock. If instead $\theta_k(\tau)$ peaks at a specific maturity, then a change in $\beta_{k,t}$ impacts demand for that maturity the most, and can be interpreted as a local demand shock. To ensure that integrals involving $(\alpha(\tau), \{\theta_k(\tau)\}_{k=1,..,K})$ are well-defined, we assume that either (i) $(\alpha(\tau), \{\theta_k(\tau)\}_{k=1,..,K})$ become zero for τ larger than a finite threshold T, are are continuous in $(0, T]$, or (ii) $(\alpha(\tau), \{\theta_k(\tau)\}_{k=1,..,K})$ converge to zero at exponential rates when τ goes to infinity, with the rate for $\alpha(\tau)$ not exceeding those for $\{\theta_k(\tau)\}_{k=1,..,K}$, and are continuous in $(0, \infty)$.

The $(K+1) \times 1$ vector $q_t \equiv (r_t, \beta_{1,t}, .., \beta_{K,t})^\top$ follows the process

$$dq_t = -\Gamma(q_t - \overline{r}\mathcal{E})dt + \Sigma dB_t, \tag{7}$$

where $\overline{r}$ is a constant, $\mathcal{E}$ is the $(K+1) \times 1$ vector $(1, 0, .., 0)^\top$, (Γ, Σ) are constant $(K+1) \times (K+1)$ matrices, dB_t is a $(K+1) \times 1$ vector $(dB_{r,t}, dB_{\beta,1,t}, .., dB_{\beta,K,t})^\top$ of independent Brownian motions, and $\top$ denotes transpose. Equation (7) nests the case where the short rate r_t and the K demand factors $\{\beta_{k,t}\}_{k=1,..,K}$ are mutually independent, and the case where they are correlated. Independence arises when the matrices (Γ, Σ) are diagonal. When instead Σ is non-diagonal, shocks to the factors r_t and $\{\beta_{k,t}\}_{k=1,..,K}$ are correlated, and when Γ is non-diagonal, the drift (instantaneous expected change) of each factor depends on all other factors. We assume that the eigenvalues of Γ have positive real parts. Hence, q_t is stationary, and (7) implies that the long-run means of r_t and $\{\beta_{k,t}\}_{k=1,..,K}$ are $\overline{r}$ and zero, respectively. Setting the long-run mean of $\{\beta_{k,t}\}_{k=1,..,K}$ to zero is without loss of generality since we can redefine the function $\theta_0(\tau)$ to include a non-zero long-run mean.

We assume that government bonds are in zero supply. This is without loss of generality because we can redefine the demand function (5) as a net demand: the demand by preferred-habitat investors for the bond with maturity τ, net of the government supply of that bond.

Under the assumed demand function (5), the demand by preferred-habitat investors for the bond with maturity τ depends only on that bond's price and not on the prices of other bonds. This begs the question why rational investors buy the bond with maturity τ if a bond with maturity close to τ is much cheaper. Appendix B shows that the demand function (5), together with the specification (6) and (7) for the demand intercept $\beta_t^{(\tau)}$, can be given an optimizing foundation when bond maturities belong to a finite interval $(0, T)$ and the matrix Σ has full rank. The optimizing foundation requires that the term structure satisfies no-arbitrage, which is the case for the equilibrium derived in Sections 3 and 4.

The preferred-habitat investors in Appendix B form overlapping generations living over a period equal to the maximum bond maturity T. The generation born at time t consumes only at $t + T$ and then dies. Investors are infinitely risk-averse over consumption. They derive consumption by investing in bonds and in a private opportunity whose return at time $t' \geq t$ is exogenous and increasing in $\beta_t^{(T+t-t')}$. Infinite risk aversion ensures that investors' optimal bond portfolio yields a riskless payoff at the time $t + T$ when they consume. That portfolio consists only of the bond maturing at $t + T$. No-arbitrage ensures that investors cannot achieve a higher payoff with certainty by investing in bonds with maturities other than $t + T$: if the payoff is higher with positive probability, then it must also be lower with positive probability.

The elasticity of preferred-habitat demand in Appendix B arises because investors substitute between the bond that matures at the time $t + T$ when they consume, and the private opportunity. When the bond's price decreases, the bond's return from t to $t + T$ increases. Hence, the bond becomes more attractive relative to the private opportunity, and bond demand increases.[8] Conversely, when the return on the private opportunity increases, it becomes more attractive relative to the bond, and bond demand decreases. The private opportunity could represent, for example, an investment in real estate.[9]

Stepping outside of the optimizing foundation in Appendix B, $\beta_t^{(\tau)}$ could vary because of shocks to the supply of bonds issued by the government and shocks to the composition of the preferred-habitat investor pool. The demand specification (5)-(7) can capture these shocks if the maturities affected by the shocks remain fixed as time passes. Suppose, for example, that there is a sudden increase at time t in the demand for the bond with maturity τ. The specification (5)-(7) requires that this increase translates to an increase at time $t' > t$ in the demand for the bond with maturity τ rather than $\tau + t - t'$. That is, the shock does not "roll down" over time in the maturity space.

Some shocks roll down in the maturity space. For example, an increase at time t in the government supply of the bond with maturity τ translates to an increase at time $t' > t$ in the supply of the bond with maturity

[8]Since investors in Appendix B choose their portfolio based on its return at the time $t+T$ when they consume, their demand for the bond that matures at $t + T$ depends on the bond's return to maturity rather than on the return over the next instant.

[9]An example of preferred-habitat investors substituting from government bonds into real estate comes from the UK's pension reform of 2004, mentioned in the Introduction. The drop in long rates induced pension funds to substitute towards non-bond investments, including real estate. For example, Marks & Spencer arranged for their pension fund to receive payments based on the leases of their property portfolio (Islam (2007), p.61).

$\tau+t-t'$ rather than τ. For such shocks, the specification (5)-(7) can be viewed as an approximation. Modifying that specification to allow roll down would render the analysis more complicated because bond demand at time t would depend on the entire history of shocks up to time $t-T$. (The shocks up to time $t-T+\tau$ would affect demand for bonds with maturities up to τ.)

Our model makes a stark distinction between arbitrageurs, who can substitute across maturities, and preferred-habitat investors, who invest only in their maturity habitat. Suppressing this distinction (by making the risk aversion of preferred-habitat investors finite in Appendix B), would complicate the model without changing the basic mechanisms. Preferred-habitat investors would substitute across maturities, acting partly as arbitrageurs, and arbitrage capacity would increase. The analysis would become more complicated because it would involve a continuum of portfolios rather than only the portfolio of arbitrageurs.

An additional distinction between arbitrageurs and preferred-habitat investors, which is implicit in the demand specification (5) and explicit in the optimizing foundation in Appendix B, is that the latter can access investment opportunities outside of the bond market while the former cannot. If arbitrageurs could access investment opportunities outside the government bond market, then shocks to the returns of their opportunities would affect bond prices as well. We suppress that effect by assuming that arbitrageurs specialize in trading only government bonds.

Our model can be given both a nominal and a real interpretation. Under the nominal interpretation, the numeraire is money, arbitrageurs' preferences concern their wealth evaluated in nominal terms, and preferences of preferred-habitat investors (in the optimizing foundation in Appendix B) concern their consumption in nominal terms. Under the real interpretation, the numeraire consists of goods, and preferences concern wealth and consumption in real terms. A short rate determined by the central bank fits better the nominal interpretation, while a short rate determined by a production technology fits better the real interpretation.

The arbitrageurs' optimization problem yields the same solution regardless of whether preferences concern nominal or real wealth. This is because the arbitrageurs' objective involves changes in wealth over an infinitesimal interval, during which inflation is constant.[10] Hence, the assumption under the nominal interpretation that arbitrageurs' preferences concern nominal wealth is innocuous.

Whether preferences concern nominal or real consumption matters for preferred-habitat investors, who have

[10]Denoting by $dW_t = W_{t+dt} - W_t$ the instantaneous change in arbitrageur nominal wealth, the change in real wealth is $dW_t^R = \frac{W_{t+dt}}{1+\pi_t dt} - W_t = dW_t - W_t \pi_t dt$, where π_t is inflation between t and $t+dt$. Since $\mathbb{E}_t(dW_t^R) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t^R) = \mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t) - \pi_t dt$, maximizing $\mathbb{E}_t(dW_t^R) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t^R)$ yields the same solution as maximizing $\mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t)$.

a longer horizon. Preferences over nominal consumption describe, for example, life-insurance companies that offer insurance or retirement products with guaranteed minimum returns typically not indexed to inflation.[11] Preferences over real consumption describe, for example, pension funds that offer pensions rising with, or explicitly indexed to, inflation.[12] Payouts from property and casualty insurance rise with inflation as well. Hence, both nominal and real preferred habitats arise in practice.

Under the nominal interpretation, inflation could affect both the short rate and the intercept $\beta_t^{(\tau)}$ of preferred-habitat demand. Indeed, high inflation could be associated with high nominal returns throughout the economy, and hence with both a high nominal short rate and a high nominal return $\beta_t^{(\tau)}$ on investment opportunities other than government bonds. Inflation could thus generate a positive correlation between the short rate and the demand factors. Because of that correlation, inflation could have only a weak effect on bond demand by preferred-habitat investors: high bond yields raise demand, and high $\beta_t^{(\tau)}$ lowers it.

3. NO DEMAND RISK

In this section we study the case where there are no demand risk factors ($K = 0$). Time-variation in yields arises because of the short rate r_t, which is the only risk factor. For $K = 0$, (7) reduces to

$$dr_t = \kappa_r(\overline{r} - r_t)dt + \sigma_r dB_{r,t}, \tag{8}$$

where $\kappa_r \equiv \Gamma_{1,1} > 0$ and $\sigma_r \equiv \Sigma_{1,1}$.

3.1. *Equilibrium without Arbitrageurs*

We first derive, as a benchmark, the equilibrium that would prevail in the arbitrageurs' absence. We refer to it as the *segmentation equilibrium* because the yield for each maturity is determined solely by the demand of the investors with that maturity habitat. The yield $y_t^{(\tau)}$ for maturity τ is determined by setting the net demand (5) by preferred-habitat investors to zero. Since (1) implies $\log(P_t^{(\tau)}) = -\tau y_t^{(\tau)}$, $y_t^{(\tau)}$ is given by

$$y_t^{(\tau)} = \frac{\beta_t^{(\tau)}}{\alpha(\tau)\tau} = \frac{\theta_0(\tau)}{\alpha(\tau)\tau}, \tag{9}$$

[11] For a description of the products offered by life-insurance companies see, for example, Berends, McMenamin, Plestis, and Rosen (2013) and Sen (2019). Table 1 of Berends, McMenamin, Plestis, and Rosen (2013) indicates that guaranteed minimum returns not indexed to inflation are a common feature of life-insurance products.

[12] Indexation of pensions to inflation was accounted for in the 2004 U.K. pension reform, which required pension funds to evaluate their pension liabilities using the yields of long-maturity *inflation-indexed* bonds.

where the second equality follows by setting $K = 0$ in (6). The yield $y_t^{(\tau)}$ for maturity τ is constant over time and is disconnected from the time-varying short rate r_t. It depends only on the demand intercept $\beta_t^{(\tau)} = \theta_0(\tau)$ and demand slope $\alpha(\tau)$ for maturity τ. An increase in $\theta_0(\tau)$ lowers the demand by preferred-habitat investors for the bond with maturity τ, and hence raises $y_t^{(\tau)}$. The effect is weaker the larger $\alpha(\tau)$ is because the demand by preferred-habitat investors is more price-elastic. The segmentation equilibrium corresponds to an extreme form of the preferred-habitat view (Culbertson (1957), Modigliani and Sutch (1966)).

3.2. *Equilibrium with Arbitrageurs*

We next derive the equilibrium when arbitrageurs are present. We proceed in three steps: (i) conjecture a functional form for equilibrium yields, (ii) derive the arbitrageurs' first-order condition given the conjectured yields, and (iii) combine the arbitrageurs' first-order condition with market clearing, and confirm that yields are as conjectured.

We conjecture that equilibrium yields are affine in the single risk factor r_t. That is, there exist two functions $(A_r(\tau), C(\tau))$ that depend only on τ such that the time-t price of the bond with maturity τ is

$$P_t^{(\tau)} = e^{-[A_r(\tau) r_t + C(\tau)]}. \tag{10}$$

Applying Ito's Lemma to (10), recalling that $dP_t^{(\tau)}$ is computed by changing the time subscript t to $t+dt$ and the maturity superscript τ to $\tau - dt$, and using the dynamics (8) of r_t, we find that the time-t instantaneous return on the bond with maturity τ is

$$\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)} dt - A_r(\tau)\sigma_r dB_{r,t}, \tag{11}$$

where

$$\mu_t^{(\tau)} \equiv A_r'(\tau) r_t + C'(\tau) - A_r(\tau)\kappa_r(\overline{r} - r_t) + \frac{1}{2} A_r(\tau)^2 \sigma_r^2 \tag{12}$$

is the instantaneous expected return.

To derive the arbitrageurs' first-order condition, we substitute the bond return (11) into the the arbitrageurs' budget constraint (3) and optimization problem (4). This yields

$$dW_t = \left[W_t r_t + \int_0^\infty X_t^{(\tau)}(\mu_t^{(\tau)} - r_t)d\tau\right]dt - \left[\int_0^\infty X_t^{(\tau)} A_r(\tau)d\tau\right]\sigma_r dB_{r,t}$$

and

$$\max_{\{X_t^{(\tau)}\}_{\tau\in(0,\infty)}} \left\{\int_0^\infty X_t^{(\tau)}(\mu_t^{(\tau)} - r_t)d\tau - \frac{a\sigma_r^2}{2}\left[\int_0^\infty X_t^{(\tau)} A_r(\tau)d\tau\right]^2\right\}, \tag{13}$$

respectively. Point-wise maximization of (13) yields the arbitrageurs' first-order condition.

LEMMA 1 *The arbitrageurs' first-order condition is*

$$\mu_t^{(\tau)} - r_t = -A_r(\tau)\lambda_{r,t}, \tag{14}$$

where

$$\lambda_{r,t} \equiv -a\sigma_r^2 \int_0^\infty X_t^{(\tau)} A_r(\tau)d\tau. \tag{15}$$

The arbitrageurs' first-order condition (14) balances risk and return. The left-hand side is the increase in the expected return on the arbitrageurs' portfolio if they shift one unit of the numeraire from the short rate r_t to the bond with maturity τ. Portfolio expected return increases by the difference between the bond's expected return $\mu_t^{(\tau)}$ and the short rate r_t. The right-hand side is the increase in the risk of the arbitrageurs' portfolio, times the arbitrageurs' risk-aversion coefficient a. Portfolio risk increases by the covariance between the return on the additional investment in the bond and the return on the portfolio. With one risk factor, the covariance is the product of the sensitivities of the two returns to the factor, times the factor's variance. The risk factor is the short rate, and its variance is σ_r^2. Moreover, (11) implies that the sensitivity of the bond's return to the short rate is $-A_r(\tau)$, and the sensitivity of the portfolio's return is $-\int_0^\infty X_t^{(\tau)} A_r(\tau)d\tau$.

The first-order condition (14) can alternatively be interpreted in the context of no-arbitrage models of the term structure.[13] No-arbitrage in continuous time requires that there exist prices specific to each risk

[13]See, for example, Vasicek (1977) and Cox, Ingersoll, and Ross (1985) for early contributions, and Veronesi (2010) for a textbook treatment.

factor and common across assets, such that the expected return of any asset in excess of the short rate is equal to the sum across factors of the asset's sensitivity to each factor times the factor's price. With one factor, the no-arbitrage condition boils down to requiring that the factor's price is equal to the ratio of any asset's expected excess return to the asset's factor sensitivity. The no-arbitrage condition in our model is the arbitrageurs' first-order condition (14), and the price of the short-rate factor is $\lambda_{r,t}$.

Absence of arbitrage is mute on what the prices of the risk factors are. These prices are instead determined by equilibrium arguments. Equation (15) shows that $\lambda_{r,t}$ is proportional to the factor sensitivity $-\int_0^\infty X_t^{(\tau)} A_r(\tau) d\tau$ of the arbitrageurs' portfolio. To determine that portfolio, we use market clearing.

Market clearing requires that the time-t positions of arbitrageurs and preferred-habitat investors in the bond with maturity τ sum to zero:

$$X_t^{(\tau)} + Z_t^{(\tau)} = 0. \tag{16}$$

Substituting $X_t^{(\tau)}$ from (16) into (15), we find

$$\begin{aligned}
\lambda_{r,t} &= a\sigma_r^2 \int_0^\infty Z_t^{(\tau)} A_r(\tau) d\tau \\
&= a\sigma_r^2 \int_0^\infty \left[-\alpha(\tau)\log(P_t^{(\tau)}) - \beta_t^{(\tau)}\right] A_r(\tau) d\tau \\
&= a\sigma_r^2 \int_0^\infty \left[\alpha(\tau)\left[A_r(\tau) r_t + C(\tau)\right] - \theta_0(\tau)\right] A_r(\tau) d\tau, \qquad (17)
\end{aligned}$$

where the second equality follows by substituting $Z_t^{(\tau)}$ from (5), and the third equality follows by substituting $P_t^{(\tau)}$ from (10) and using $\beta_t^{(\tau)} = \theta_0(\tau)$ (which follows by setting $K = 0$ in (6)). Equation (17) shows that the price $\lambda_{r,t}$ of the short-rate risk factor depends on the short rate r_t and on the demand intercept $\theta_0(\tau)$ and demand slope $\alpha(\tau)$ of preferred-habitat investors. We return to these effects and their economic implications in Sections 3.3-3.5.

Substituting $\lambda_{r,t}$ and $\mu_t^{(\tau)}$ from (17) and (12), respectively, into (14), we find

$$\begin{aligned}
&A_r'(\tau) r_t + C'(\tau) - A_r(\tau)\kappa_r(\overline{r} - r_t) + \frac{1}{2}A_r(\tau)^2\sigma_r^2 - r_t \\
&= a\sigma_r^2 A_r(\tau) \int_0^\infty \left[\theta_0(\tau) - \alpha(\tau)\left[A_r(\tau) r_t + C(\tau)\right]\right] A_r(\tau) d\tau. \qquad (18)
\end{aligned}$$

Equation (18) must hold for all values of r_t. Hence, the linear terms in r_t on both sides must be equal, and the same is true for the terms that are independent of r_t. This yields the two first-order linear ordinary differential equations (ODEs)

$$A_r'(\tau) + \kappa_r A_r(\tau) - 1 = -a\sigma_r^2 A_r(\tau) \int_0^\infty \alpha(\tau) A_r(\tau)^2 d\tau, \tag{19}$$

$$C'(\tau) - \kappa_r \overline{r} A_r(\tau) + \frac{1}{2}\sigma_r^2 A_r(\tau)^2 = a\sigma_r^2 A_r(\tau) \int_0^\infty \left[\theta_0(\tau) - \alpha(\tau)C(\tau)\right] A_r(\tau) d\tau, \tag{20}$$

in the functions $(A_r(\tau), C(\tau))$. Equations (19) and (20) must be solved with the initial conditions $A_r(0) = C(0) = 0$, which follow from (10) because a bond with zero maturity trades at its face value of one. A complicating feature of (19) and (20) is that the coefficient of $A_r(\tau)$ in each equation depends on an integral involving the functions $(A_r(\tau), C(\tau))$. To solve (19) and (20), we proceed in two steps. First, we take the integrals as given and solve (19) and (20) as linear ODEs with constant coefficients. Second, we require that the solution is consistent with the value of the integrals.

The first step yields

$$A_r(\tau) = \frac{1 - e^{-\kappa_r^* \tau}}{\kappa_r^*}, \tag{21}$$

$$C(\tau) = \kappa_r^* \overline{r}^* \int_0^\tau A_r(u) du - \frac{\sigma_r^2}{2} \int_0^\tau A_r(u)^2 du, \tag{22}$$

where the scalars $(\kappa_r^*, \overline{r}^*)$ are defined by

$$\kappa_r^* \equiv \kappa_r + a\sigma_r^2 \int_0^\infty \alpha(\tau) A_r(\tau)^2 d\tau, \tag{23}$$

$$\kappa_r^* \overline{r}^* \equiv \kappa_r \overline{r} + a\sigma_r^2 \int_0^\infty \left[\theta_0(\tau) - \alpha(\tau)C(\tau)\right] A_r(\tau) d\tau. \tag{24}$$

We use the star subscript because $(\kappa_r^*, \overline{r}^*)$ are the counterparts of $(\kappa_r, \overline{r})$ under the risk-neutral measure. The second step requires that $(\kappa_r^*, \overline{r}^*)$ solve (23) and (24) when $(A_r(\tau), C(\tau))$ are substituted in from (21) and (22). Proposition 1 shows that this requirement determines $(\kappa_r^*, \overline{r}^*)$ uniquely.

PROPOSITION 1 *The functions $(A_r(\tau), C(\tau))$ are given by (21) and (22), respectively, where κ_r^* is the unique* *solution to*

$$\kappa_r^* = \kappa_r + a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\kappa_r^* \tau}}{\kappa_r^*} \right)^2 d\tau, \tag{25}$$

and $\overline{r}^*$ is given by

$$\overline{r}^* = \overline{r} + a\sigma_r^2 \frac{\int_0^\infty \left[\theta_0(\tau) - \overline{r}\alpha(\tau)\tau\right] \frac{1-e^{-\kappa_r^*\tau}}{\kappa_r^*} d\tau + \frac{\sigma_r^2}{2} \int_0^\infty \alpha(\tau) \left[\int_0^\tau \left(\frac{1-e^{-\kappa_r^* u}}{\kappa_r^*} \right)^2 du \right] \frac{1-e^{-\kappa_r^*\tau}}{\kappa_r^*} d\tau}{\kappa_r^* \left[1 + a\sigma_r^2 \int_0^\infty \alpha(\tau) \left[\int_0^\tau \frac{1-e^{-\kappa_r^* u}}{\kappa_r^*} du \right] \frac{1-e^{-\kappa_r^*\tau}}{\kappa_r^*} d\tau \right]}. \tag{26}$$

We next explore the economic implications of the equilibrium derived in Proposition 1. Section 3.3 examines how shocks to the short rate are transmitted to longer maturities. Section 3.4 examines how bond expected excess returns depend on the short rate and on the shape of the term structure. Section 3.5 examines how changes in bond demand affect the term structure.

3.3. *Monetary Policy Transmission and Carry Trades*

In the segmentation equilibrium, in which there are no arbitrageurs, bond yields $y_t^{(\tau)}$ are disconnected from the short rate r_t. By contrast, when arbitrageurs are present, they transmit short-rate shocks to bond yields, ensuring that yields are informative about the current and expected future short rates.

Arbitrageurs transmit short-rate shocks to bond yields through their *carry trades*. Suppose that a shock causes the short rate to drop below the value that bond yields would take in the segmentation equilibrium. To benefit from the discrepancy between bond yields and the short rate, arbitrageurs buy bonds and finance their position by borrowing short-term. Their activity causes bond prices to rise and yields to drop, thus reflecting the drop in the short rate. Conversely, following a shock that causes the short rate to exceed the value that bond yields would take under segmentation, arbitrageurs short-sell bonds and invest short-term. Their activity causes bond prices to drop and yields to rise, thus reflecting the rise in the short rate. In both cases, arbitrageurs engage in carry trades—trades that are profitable when prices do not move. For example, buying a bond and financing that position by short-term borrowing is profitable when the short rate remains below the bond's yield until the bond's maturity.

The extent to which arbitrageurs transmit short-rate shocks to bond yields depends on three main parameters of our model: the arbitrageurs' risk-aversion coefficient a, the volatility σ_r of the short rate, and the slope $\alpha(\tau)$ of the demand by preferred-habitat investors. When $a = 0$, arbitrageurs are not averse to the risk that carry trades entail, namely, that the short rate can rise when they borrow short-term to buy bonds, and that the short rate can drop when they short-sell bonds and invest short-term. Hence, arbitrageurs engage in carry trades that are sufficiently large to transmit short-rate shocks fully to bond yields. When $\alpha(\tau) = 0$ for all $\tau \in (0, T)$, shocks are again transmitted fully, but for a different reason. Since the demand of preferred-habitat investors is independent of bond prices, short-rate shocks do not trigger carry trades by arbitrageurs in equilibrium, even though bond yields change. Hence, arbitrageurs impact bond yields without bearing carry-trade risk, in effect having infinite price impact. The transmission of shocks becomes weaker when a, σ_r^2 and $\alpha(\tau)$ increase.

We measure the extent to which arbitrageurs transmit short-rate shocks to bond yields by comparing the reaction of forward rates to that of expected future short rates. We evaluate how a time-t shock to the short rate r_t affects the expected short rate $E_t(r_{t+\tau})$ at time $t+\tau$ and the instantaneous forward rate $f_t^{(\tau)}$ for maturity τ. The latter rate is defined as the limit of the forward rate $f_t^{(\tau-\Delta\tau,\tau)}$ between maturities $\tau - \Delta\tau$ and τ when $\Delta\tau$ goes to zero:

$$f_t^{(\tau)} \equiv \lim_{\Delta\tau\to 0} f_t^{(\tau-\Delta\tau,\tau)} = -\frac{\partial \log(P_t^{(\tau)})}{\partial \tau} = A_r'(\tau) r_t + C'(\tau), \tag{27}$$

where the second step follows from (2), and the third from (10). When the expectations hypothesis (EH) of the term structure holds, forward rates move one-to-one with expected future short rates. Proposition 2 shows that when $a > 0$ and $\alpha(\tau) > 0$, forward rates under-react and hence arbitrageurs transmit short-rate shocks to bond yields only partially.

Formally, a unit shock to r_t raises $E_t(r_{t+\tau})$ by $e^{-\kappa_r \tau}$ because the short rate mean-reverts at rate κ_r. Equation (27) implies that $f_t^{(\tau)}$ rises by $A_r'(\tau) = e^{-\kappa_r^* \tau}$, where the equality follows from (21). Under-reaction occurs because the short rate's mean-reversion parameter κ_r^* under the risk-neutral measure exceeds its counterpart κ_r under the physical measure. Equation (25) implies that the difference $\kappa_r^* - \kappa_r$, and hence the extent of under-reaction, increases in a, σ_r^2 and $\alpha(\tau)$.

Proposition 2 (**Under-Reaction of Forward Rates**) *A unit shock to the short rate r_t:*

- *Raises the expected short rate $E_t(r_{t+\tau})$ at time $t+\tau$ by $\frac{\partial E_t(r_{t+\tau})}{\partial r_t} = e^{-\kappa_r \tau}$.*

- *Raises the instantaneous forward rate* $f_t^{(\tau)}$ *for maturity* τ *by* $\frac{\partial f_t^{(\tau)}}{\partial r_t} = e^{-\kappa_r^* \tau}$.

The forward rate under-reacts ($\kappa_r^* > \kappa_r$*) if arbitrageurs are risk-averse (*$a > 0$*) and the demand by preferred-habitat investors is price-elastic (*$\alpha(\tau) > 0$ *in a positive-measure subset of* $(0, \infty)$*). The extent of under-reaction* $\kappa_r^* - \kappa_r$ *increases in* a*,* σ_r^2 *and* $\alpha(\tau)$*.*

Our results have implications for the transmission of monetary policy. Suppose that the central bank conducts monetary policy by changing the rate that it pays on bank reserves. Suppose also that arbitrageurs are banks, in which case the short rate r_t that they earn on their wealth is the rate paid on reserves. Our model implies that the transmission of monetary-policy shocks to the yields of long-maturity bonds is done by arbitrageurs. Moreover, the transmission mechanism is weaker when arbitrageurs are more risk-averse, central bank actions are more uncertain (the short rate is more volatile), or the demand by preferred-habitat investors is more price-elastic. An additional implication is that in transmitting monetary-policy shocks, arbitrageurs earn a rent. That rent arises from the returns on the carry trades, and reflects bond risk premia, as we explain in Section 3.4. In that section we also show that bond risk premia are larger, resulting in a larger rent for arbitrageurs, under the same conditions that generate a weaker transmission mechanism.

3.4. *Bond Risk Premia*

Under the EH, bond expected returns are equal to the riskless rate. When instead $a > 0$ and $\alpha(\tau) > 0$, they differ from the riskless rate and mirror the carry trades of arbitrageurs. This is because risk-averse arbitrageurs enter into carry trades only if they expect to earn high returns as compensation for the risk they take. Suppose that the short rate drops, in which case bond yields drop and price-elastic preferred-habitat investors sell bonds. Bonds earn then positive expected returns in excess of the riskless rate so that arbitrageurs are induced to buy them. When instead the short rate rises, bonds earn negative expected excess returns so that arbitrageurs are induced to sell them short. We refer to expected excess returns as risk premia because they compensate arbitrageurs for risk.

Since in the absence of demand risk factors, the short rate is the only source of time-variation, bond risk premia are positively related to the slope of the term structure: a low (high) short rate implies both a term structure with slope higher (lower) than average and positive (negative) bond risk premia. The positive premia-slope relationship is a widely documented empirical fact in the term-structure literature, starting with

Fama and Bliss (FB, 1987). FB perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_t^{(\tau)}}\right) - y_t^{(\Delta\tau)} = a_{\text{FB}} + b_{\text{FB}}\left(f_t^{(\tau-\Delta\tau,\tau)} - y_t^{(\Delta\tau)}\right) + e_{t+\Delta\tau}. \tag{28}$$

The dependent variable is the return on a zero-coupon bond with maturity τ held over a period $\Delta\tau$, in excess of the spot rate for maturity $\Delta\tau$. The independent variable is the slope of the term structure as measured by the difference between the forward rate between maturities $\tau - \Delta\tau$ and τ, and the spot rate for maturity $\Delta\tau$. FB find that b_{FB} is positive, larger than one for most τ, and increasing in τ. The implied time-variation of risk premia is economically significant: predicted premia have a standard deviation of about 1-1.5% per year, while average premia are about 0.5% per year.

The behavior of bond risk premia is related to the predictability of changes to long rates. Campbell and Shiller (CS 1991) find that the slope of the term structure predicts changes in long rates, but to a weaker and typically opposite extent than implied by the EH. CS perform the regression

$$y_{t+\Delta\tau}^{(\tau-\Delta\tau)} - y_t^{(\tau)} = a_{\text{CS}} + b_{\text{CS}}\frac{\Delta\tau}{\tau-\Delta\tau}\left(y_t^{(\tau)} - y_t^{(\Delta\tau)}\right) + e_{t+\Delta\tau}. \tag{29}$$

The dependent variable is the change, between times t and $t + \Delta\tau$, in the yield of a zero-coupon bond that has maturity τ at time t. The independent variable is the difference between the spot rates for maturities τ and $\Delta\tau$, normalized so that the regression coefficient b_{CS} is equal to one under the EH. CS find that b_{CS} is smaller than one, negative for most τ, and decreasing in τ. This finding is related to the positive premia-slope relationship. Indeed, suppose that the term structure has slope higher than average. Because bonds earn positive expected excess returns, their yields increase by less than under the EH, implying a regression coefficient b_{CS} smaller than one.[14]

Proposition 3 computes the FB and CS regression coefficients b_{FB} and b_{CS} in the analytically convenient case where $\Delta\tau$ is small. The proposition confirms that when $a > 0$ and $\alpha(\tau) > 0$, b_{FB} is positive and b_{CS} is smaller than one. It also shows that b_{FB} increases in the arbitrageurs' risk-aversion coefficient a, the volatility σ_r of the short rate, and the slope $\alpha(\tau)$ of the demand by preferred-habitat investors.

Additional implications of Proposition 3 are that b_{FB} is independent of τ and is smaller than one, and that b_{CS} increases in τ. In the data, by contrast, b_{FB} increases in τ and exceeds one for most maturities, and b_{CS}

[14]For more material and references on bond return predictability, see the survey by Cochrane (1999). See also Cochrane and Piazzesi (2005) who find that a tent-shaped factor of yields explains bond risk premia even better than the slope of the term structure does.

decreases in τ. Our model can match these empirical properties in the presence of demand risk, as we show in Sections 4 and 5.

PROPOSITION 3 (**Positive Premia-Slope Relationship**) *For $\Delta\tau \to 0$ and for all τ:*

- *The FB regression coefficient in (28) is $b_{FB} = \frac{\kappa_r^* - \kappa_r}{\kappa_r^*}$. It is positive if arbitrageurs are risk-averse ($a > 0$) and the demand by preferred-habitat investors is price-elastic ($\alpha(\tau) > 0$ in a positive-measure subset of $(0, \infty)$). It increases in a, σ_r^2 and $\alpha(\tau)$.*
- *The CS regression coefficient in (29) is $b_{CS} = 1 - \frac{(\kappa_r^* - \kappa_r)A_r(\tau)\tau}{\tau - A_r(\tau)}$. It is smaller than one under the same condition that ensures $b_{FB} > 0$, and it increases in τ.*

3.5. *Demand Effects*

In the segmentation equilibrium, in which there are no arbitrageurs, the yield $y_t^{(\tau)}$ for maturity τ depends only on the demand intercept $\beta_t^{(\tau)} = \theta_0(\tau)$ and demand slope $\alpha(\tau)$ for that maturity. The presence of arbitrageurs changes that aspect of the equilibrium dramatically. The yield $y_t^{(\tau)}$ depends on the demand intercept and slope for all maturities. Moreover, a change in the demand intercept for maturity τ can have its largest effects for maturities other than τ.

Suppose that the demand intercept $\theta_0(\tau)$ changes to $\theta_0(\tau) + \Delta\theta_0(\tau)$, where $\Delta\theta_0(\tau)$ is a general function of τ and represents an unanticipated and permanent change. Maturities for which $\Delta\theta_0(\tau) > 0$ experience a drop in demand because (5) defines the demand intercept with a negative sign. Proposition 1 implies that κ_r^* and $A_r(\tau)$ do not change, that the change $\Delta\bar{r}^*$ in $\bar{r}^*$ has the same sign as $a\sigma_r^2 \int_0^\infty \Delta\theta_0(\tau)A_r(\tau)d\tau$, and that $C(\tau)$ changes by $\kappa_r^*\Delta\bar{r}^* \int_0^\tau A_r(u)du$. Hence, the yield $y_t^{(\tau)}$ for maturity τ changes by $\Delta y_t^{(\tau)} \equiv \kappa_r^*\Delta\bar{r}^* \frac{\int_0^\tau A_r(u)du}{\tau}$. Proposition 4 follows from these observations.

PROPOSITION 4 (**Global Demand Effects**) *A change in the demand intercept from $\theta_0(\tau)$ to $\theta_0(\tau) + \Delta\theta_0(\tau)$ affects yields if arbitrageurs are risk-averse ($a > 0$). Spot rates for all maturities rise if $\int_0^\infty \Delta\theta_0(\tau)A_r(\tau)d\tau > 0$ and drop otherwise. The relative effect across maturities is independent of the maturities where the demand change originates ($\frac{\Delta y_t^{(\tau_2)}}{\Delta y_t^{(\tau_1)}}$ is independent of $\Delta\theta_0(\tau)$). Yields for longer maturities are more affected ($\frac{\Delta y_t^{(\tau_2)}}{\Delta y_t^{(\tau_1)}} > 1$ for $\tau_1 < \tau_2$).*

Proposition 4 shows that the effects of the change $\Delta\theta_0(\tau)$ are characterized fully by the integral $\int_0^\infty \Delta\theta_0(\tau)A_r(\tau)d\tau$. If that integral is positive, then yields for all maturities rise—even for maturities for which demand increases because $\Delta\theta_0(\tau) < 0$. Thus, demand effects are global: demand intercepts across all maturities are aggregated into the one-dimensional index $\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau$, and changes to that index move all yields in the same direction. These global effects are the polar opposite of the local effects derived in the segmentation equilibrium.

Demand effects are represented by a one-dimensional index because there is only one risk factor, the short rate. The index relates to the sensitivity of arbitrageurs' portfolio to that factor. Suppose that following a change in preferred-habitat demand, arbitrageurs are induced to hold a portfolio that realizes more losses when the short rate increases. Arbitrageurs then view bonds as riskier and require higher expected excess returns to hold them, causing yields to increase for all maturities.

The index is derived by multiplying the demand intercept $\theta_0(\tau)$ for maturity τ by the function $A_r(\tau) = \frac{1-e^{-\kappa_r^*\tau}}{\kappa_r^*}$ that characterizes the sensitivity of the τ-maturity bond to the short rate, and integrating across maturities. If a change in the demand intercept raises that integral, then the sensitivity-weighted demand for bonds by preferred-habitat investors declines and the sensitivity of arbitrageurs' portfolio increases. Since $A_r(\tau)$ increases in τ, demand intercepts for longer-maturity bonds receive a larger weight in the index. Hence, changes to the demand for these bonds have a larger effect on the term structure.

While changes to the demand for longer-maturity bonds have a larger effect on yields, the *relative* effect across maturities is the same as when the demand for shorter-maturity bonds changes. Moreover, yields for longer maturities are more affected (by any demand change). Intuitively, a decrease in demand raises the instantaneous expected returns of long-maturity bonds more than of short-maturity bonds. This is because expected excess returns compensate arbitrageurs for risk, and long-maturity bonds are riskier ($A_r(\tau)$ increases in τ). The increase in expected returns causes yields to increase: the yield for maturity τ involves an average of instantaneous expected returns that the bond with maturity τ earns during its life $[t, t+\tau]$. Since demand changes are permanent, the average of instantaneous expected returns increases more for longer-maturity bonds. Hence, yields for longer maturities are more affected by demand changes.

4. DEMAND RISK

In this section we generalize our analysis to the case where demand is time-varying. Since demand affects yields only when arbitrageurs are risk-averse, we assume $a > 0$. Time-variation in yields arises because of the short rate r_t and the K demand factors $\{\beta_{k,t}\}_{k=1,..,K}$.

4.1. *Equilibrium*

We derive the equilibrium following the same three steps as in Section 3.2. We conjecture that there exist $K+2$ functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..,K}, C(\tau))$ that depend only on τ such that the time-t price of the bond with maturity τ is

$$P_t^{(\tau)} = e^{-\left[A(\tau)^\top q_t + C(\tau)\right]}, \tag{30}$$

where $A(\tau)$ is the $(K+1)\times 1$ vector $(A_r(\tau), A_{\beta,1}(\tau), .., A_{\beta,K}(\tau))^\top$. Applying Ito's Lemma to (10), using the dynamics (7) of q_t, and noting that $t+\tau$ stays constant when taking the derivative, we find that the time-t instantaneous return on the bond with maturity τ is

$$\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)} dt - A(\tau)^\top \Sigma dB_t, \tag{31}$$

where

$$\mu_t^{(\tau)} \equiv A'(\tau)^\top q_t + C'(\tau) + A(\tau)^\top \Gamma (q_t - \overline{r}\mathcal{E}) + \frac{1}{2} A(\tau)^\top \Sigma\Sigma^\top A(\tau) \tag{32}$$

is the instantaneous expected return. Substituting the bond return (31) into the the arbitrageurs' optimization problem (4) yields

$$\max_{\{X_t^{(\tau)}\}_{\tau\in(0,T)}} \left\{ \int_0^\infty X_t^{(\tau)} (\mu_t^{(\tau)} - r_t) d\tau - \frac{a}{2} \left[\int_0^\infty X_t^{(\tau)} A(\tau) d\tau\right]^\top \Sigma\Sigma^\top \left[\int_0^\infty X_t^{(\tau)} A(\tau) d\tau\right] \right\}. \tag{33}$$

Point-wise maximization of (33) yields the arbitrageurs' first-order condition.

LEMMA 2 *The arbitrageurs' first-order condition is*

$$\mu_t^{(\tau)} - r_t = a A(\tau)^\top \Sigma\Sigma^\top \left[\int_0^\infty X_t^{(\tau)} A(\tau) d\tau\right]. \tag{34}$$

Equation (34) is the multi-factor counterpart of (14). The left-hand side is the increase in portfolio expected return if arbitrageurs shift one unit of the numeraire from the short rate r_t to the bond with maturity τ. The right-hand side is the increase in portfolio risk, times the arbitrageurs' risk aversion coefficient a. The increase in portfolio risk is equal to the covariance between the return on the additional investment in the bond and the return on the arbitrageurs' portfolio. With multiple risk factors, the covariance is the product of the sensitivity vectors $-A(\tau)$ and $-\int_0^\infty X_t^{(\tau)} A(\tau) d\tau$ of the two returns to the factors, times the factors' covariance matrix $\Sigma\Sigma^\top$. To show the full analogy between (34) and (14), we can write (34) in terms of factor prices. Denoting the $(K+1)\times 1$ vector of factor prices by $\lambda_t \equiv (\lambda_{r,t}, \lambda_{\beta,1,t}, .., \lambda_{\beta,K,t})^\top$, we can write (34) as $\mu_t^{(\tau)} - r_t = -aA(\tau)^\top \lambda_t$ and deduce that factor prices are $\lambda_t = -\Sigma\Sigma^\top \left[\int_0^\infty X_t^{(\tau)} A(\tau) d\tau\right]$.

Substituting $X_t^{(\tau)}$ from the market-clearing equation (16) into (34), using (5), (6), (30) and (32), and denoting by $\Theta(\tau)$ the $1 \times (K+1)$ vector $(0, \theta_1(\tau), .., \theta_K(\tau))$, we find the following counterpart of (18):

$$\begin{aligned} & A'(\tau)^\top q_t + C'(\tau) + A(\tau)^\top \Gamma(q_t - \bar{r}\mathcal{E}) + \frac{1}{2} A(\tau)^\top \Sigma\Sigma^\top A(\tau) - r_t \\ & = aA(\tau)^\top \Sigma\Sigma^\top \int_0^\infty \left[\theta_0(\tau) + \Theta(\tau) q_t - \alpha(\tau)\left(A(\tau)^\top q_t + C(\tau)\right)\right] A(\tau) d\tau. \end{aligned} \tag{35}$$

Setting the linear terms in q_t on both sides of (35) to be equal yields the system of $K+1$ first-order linear ODEs

$$A'(\tau) + MA(\tau) - \mathcal{E} = 0, \tag{36}$$

where M is the $(K+1) \times (K+1)$ matrix

$$M \equiv \Gamma^\top - a \int_0^\infty \left[\Theta(\tau)^\top A(\tau)^\top - \alpha(\tau) A(\tau) A(\tau)^\top\right] d\tau \Sigma\Sigma^\top. \tag{37}$$

Setting the terms that are independent of q_t on both sides of (35) to be equal yields the first-order linear ODE

$$C'(\tau) - \bar{r} A(\tau)^\top \Gamma \mathcal{E} + \frac{1}{2} A(\tau)^\top \Sigma\Sigma^\top A(\tau) = aA(\tau)^\top \Sigma\Sigma^\top \int_0^\infty \left[\theta_0(\tau) - \alpha(\tau) C(\tau)\right] A(\tau) d\tau. \tag{38}$$

Equations (36) and (38) must be solved with the initial conditions $A(0) = C(0) = 0$. To solve (36) and (38), we follow the same two steps as in Section 3. The first step is to take the integrals in (36) and (38) as given and solve these equations as linear ODEs with constant coefficients. The solution is in Lemma 3.

LEMMA 3 *Suppose that the matrix M defined in (37) has $K+1$ distinct eigenvalues $(\nu_1, .., \nu_{K+1})$. The function $A(\tau) = (A_r(\tau), A_{\beta,1}(\tau), .., A_{\beta,K}(\tau))^\top$ is given by*

$$A_r(\tau) = \frac{1-e^{-\nu_1 \tau}}{\nu_1} + \sum_{k'=1}^{K} \phi_{r,k'} \left(\frac{1-e^{-\nu_{k'+1}\tau}}{\nu_{k'+1}} - \frac{1-e^{-\nu_1 \tau}}{\nu_1} \right), \tag{39}$$

$$A_{\beta,k}(\tau) = \sum_{k'=1}^{K} \phi_{\beta,k,k'} \left(\frac{1-e^{-\nu_{k'+1}\tau}}{\nu_{k'+1}} - \frac{1-e^{-\nu_1 \tau}}{\nu_1} \right), \tag{40}$$

where $(\{\phi_{r,k'}\}_{k'=1,..,K}, \{\phi_{\beta,k,k'}\}_{k,k'=1,..,K})$ are scalars derived from the eigenvectors of M. The function $C(\tau)$ is given by

$$C(\tau) = \left[\int_0^\tau A(u)^\top du \right] \chi - \frac{1}{2} \int_0^\tau A(u)^\top \Sigma\Sigma^\top A(u) du, \tag{41}$$

where $\chi \equiv (\chi_r, \chi_{\beta,1}, .., \chi_{\beta,K})^\top$ is the $(K+1) \times 1$ vector

$$\chi \equiv \bar{r}\Gamma\mathcal{E} + a\Sigma\Sigma^\top \int_0^\infty [\theta_0(\tau) - \alpha(\tau)C(\tau)] A(\tau) d\tau. \tag{42}$$

The second step is to ensure that the solution derived in Lemma 3 is consistent with the value of the integrals. There are $(K+1)^2$ integrals in (36). These integrals involve the $K+1$ functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..K})$, and determine the elements of the $(K+1) \times (K+1)$ matrix M defined in (37). In turn, the eigenvalues and eigenvectors of M determine the solution for $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..K})$ in Lemma 3, and that solution determines the value of the integrals. This yields a nonlinear system of $(K+1)^2$ equations in the $(K+1)^2$ integrals. Given a solution to that system, the elements $(\chi_r, \chi_{\beta,1}, .., \chi_{\beta,K})$ of the vector χ in the solution for $C(\tau)$ in Lemma 3 can be derived from a linear system of $K+1$ equations.

In the remainder of this section, we show analytically general properties of the model. We focus on the case where there is one demand factor ($K = 1$, four nonlinear equations) and omit the subscript k from that factor. We additionally assume that the short rate and the demand factor are independent. This corresponds to the matrices (Γ, Σ) being diagonal. We denote their diagonal elements by $(\kappa_r, \kappa_\beta, \sigma_r, \sigma_\beta) \equiv (\Gamma_{1,1}, \Gamma_{2,2}, \Sigma_{1,1}, \Sigma_{2,2})$. The case with one independent demand factor is a natural first case to analyze, and it yields a rich set of results. We analyze the same case numerically in Section 5, where we perform a calibration exercise.[15] We discuss the general case briefly at the end of Section 4.4.

Two useful assumptions for deriving some of our analytical results are that the functions $(\alpha(\tau), \{\theta_k(\tau)\}_{k=1,..,K})$ are exponentials or linear combinations of exponentials. Under these assumptions, the integrals in (36) involve Laplace transforms of the functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..K})$ and of those functions' pairwise products. Moreover, by multiplying the ODE system (36) by the exponentials in $(\alpha(\tau), \{\theta_k(\tau)\}_{k=1,..,K})$ and by the products of these exponentials with the functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..K})$, we find equations that involve the same Laplace transforms. This yields a system of equations in the Laplace transforms, derived in Appendix A for the general case (Lemma A.1). While that system remains nonlinear, a key advantage of the Laplace-transform approach is that we do not need to compute the eigenvalues and eigenvectors of M, which can be real or complex.

We begin our analytical investigation by showing existence of equilibrium. We take the demand elasticity $\alpha(\tau)$ to be the declining exponential $\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}$, where (α, δ_α) are positive constants. We take the impact $\theta(\tau)$ of the single demand factor on the demand intercept to be a difference between two exponentials $\theta(\tau) = \theta\left(e^{-\delta_\alpha \tau} - e^{-\delta_\theta \tau}\right)$, where (θ, δ_θ) are positive constants and $\delta_\alpha < \delta_\theta$. A unit increase in the demand factor β_t raises the spot rate for maturity τ in the segmentation equilibrium by

$$\frac{\theta(\tau)}{\alpha(\tau)\tau} = \frac{\theta\left(1 - \epsilon^{-(\delta_\theta - \delta_\alpha)\tau}\right)}{\alpha\tau}.$$

This function has a positive limit at $\tau = 0$ and decreases in τ.

Theorem 1 (**Equilibrium Existence**) *Suppose that there is one demand factor, the matrices (Γ, Σ) are diagonal, $\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}$ and $\theta(\tau) = \theta\left(e^{-\delta_\alpha \tau} - e^{-\delta_\theta \tau}\right)$, where $(\alpha, \theta, \delta_\alpha, \delta_\theta)$ are positive constants and δ_θ is large. An equilibrium exists under either of the following sufficient conditions:*

- *κ_β is close to zero.*

[15] Hayashi (2018) derives two alternative numerical algorithms for solving our model in the case $\alpha(\tau) = 0$. Both algorithms discretize the functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..K})$, without imposing the structure derived in Lemma 3. They have the advantage of handling large values of K as easily as small values.

- $\delta_\alpha(\delta_\alpha + \kappa_r)(\delta_\alpha + \kappa_\beta) > 2a\theta\sigma_r\sigma_\beta$.

In equilibrium, $M_{1,1} > \kappa_r$*,* $M_{1,2} > 0$*,* $M_{2,1} < 0$ *and* $M_{2,2} > \frac{\kappa_\beta - \delta_\alpha}{2}$.

We complement the existence result in Theorem 1 by computing in Appendix A (Lemma A.2) the equilibrium in closed form when the arbitrageurs' risk-aversion coefficient a is close to zero or to infinity and other parameters can take any values. For our analysis of $a \approx 0$ and $a \approx \infty$, we require that $\alpha(\tau)$ and $\frac{\theta(\tau)}{\tau}$ have a positive and a finite limit, respectively, at $\tau = 0$. That restriction is satisfied by the specification in Theorem 1.[16] We next examine how the results of Sections 3.3-3.5 are modified in the presence of demand risk.

4.2. *Carry Trades and Hedging*

Demand risk weakens the transmission of short-rate shocks to bond yields. This is because the carry trades through which arbitrageurs transmit the shocks become riskier. To hedge against demand risk, arbitrageurs scale down their carry trades or even convert them into *butterfly trades,* reversing the sign of their positions for long maturities. Because of hedging, short-rate shocks can move yields for long maturities in the direction opposite to the shocks.

To explain hedging in our model, suppose as in Section 3.3 that a shock causes the short rate to drop below the value that bond yields would take in the segmentation equilibrium. Arbitrageurs can benefit from the discrepancy between bond yields and the short rate by buying bonds and borrowing short-term. This carry trade leaves them exposed to a rise in the short rate, as in Section 3.3, and to a drop in bond demand by preferred-habitat investors. The importance of demand risk relative to short-rate risk rises with maturity. This is shown in Proposition 5, and can be partly anticipated from the one-factor model, in which short-rate shocks have an effect on yields that declines with maturity, while permanent demand changes have an increasing effect. Because long-maturity bonds are highly exposed to demand risk, arbitrageurs can short-sell them to hedge the demand risk of their aggregate position. Such short-selling occurs when arbitrageurs are sufficiently risk-averse, and causes yields for long maturities to rise despite the drop in current and expected future short rates. Buying intermediate-maturity bonds and short-selling long-maturity ones and very short-maturity ones

[16]For $a \approx 0$, our model becomes approximately a one-factor one, with the factor being the short rate. This is because shocks to the demand factors have small effects on bond yields. The effects of demand shocks are characterized by the one-dimensional index derived in Proposition 4, with $\kappa_r^* = \kappa_r$. The only difference relative to Proposition 4 is that yields for longer maturities may not be the most affected. This is because Proposition 4 assumes permanent demand changes, while shocks to the demand factors mean-revert.

(i.e., borrowing short-term) is a butterfly trade, common in term-structure arbitrage.[17]

Proposition 5 characterizes the response of yields to short-rate and demand shocks. The proposition assumes $M_{2,1} < 0$, a property shown to hold for the equilibrium derived in Theorem 1. The assumptions of Theorem 1 are not needed as long as that property holds.

The characterization is simple when the two eigenvalues of M are real. The function $A_\beta(\tau)$ is positive, which implies that a drop in demand causes yields for all maturities to rise, and increases in τ. The function $A_r(\tau)$ is either positive, or switches sign from positive to negative when τ crosses a threshold $\bar{\tau}$. In the latter case, a drop in the short rate causes yields for maturities $\tau > \bar{\tau}$ to rise. The ratio $\frac{A_r(\tau)}{A_\beta(\tau)}$ decreases in τ, which implies that the effect of demand shocks relative to short-rate shocks rises with maturity.

When the two eigenvalues of M are complex, the functions $(A_r(\tau), A_\beta(\tau))$ exhibit an oscillating pattern driven by the arbitrageurs' hedging activity. Following a rise in the short rate, prices of short-maturity bonds drop. Prices of long-maturity bonds can instead rise because arbitrageurs can buy them to hedge demand risk. Long-maturity bonds can thus hedge the short-rate risk of a portfolio with long positions in bonds, and earn negative expected excess returns when arbitrageurs hold such a portfolio in equilibrium. Since arbitrageurs hold long positions when demand by preferred-habitat investors is low, low demand can cause, through the cumulation of negative expected returns, the prices of bonds of even longer ("very long") maturities to rise. In that case, arbitrageurs do not use the very-long-maturity bonds to hedge demand risk, and those bonds' prices rise following a drop in the short rate. This yields an oscillating pattern of price sensitivity to the short rate as a function of maturity. The properties shown for real eigenvalues carry through to complex ones for the first half-cycle of the oscillation (which can be longer than the maximum maturity T). The functions $(A_r(\tau), A_\beta(\tau))$ begin by being increasing in τ. The function $A_r(\tau)$ eventually reaches a maximum, and the function $A_\beta(\tau)$ does so at a larger value $\hat{\tau}$ which marks the end of the first half-cycle. We set $\hat{\tau} = \infty$ when the

[17] An example of a butterfly trade comes from the 2007-2008 financial crisis. Short-rate cuts triggered by the crisis rendered the US term structure steeply upward sloping. Term structure arbitrageurs took the view that forward rates did not drop enough to reflect the low expected future spot rates—the under-reaction result of Proposition 2. For example, a Barclays Capital report by Pradhan (2009), p.2., points out that while the two-year spot rate was 258 bps lower than the ten-year spot rate, the difference between their two-year forward counterparts was only 93bps. The report goes on to advise lending at the two-year rate two years forward and borrowing at the ten-year rate two years forward. Lending at the two-year rate two years forward is a carry trade: it amounts to shorting two-year bonds and buying four-year bonds. Borrowing at the ten-year rate two years forward amounts to buying two-year bonds and shorting twelve-year bonds. That position is layered to the carry trade to hedge term-structure movements at intermediate maturities, and is for a smaller notional amount since the twelve-year bond is more sensitive to such movements than the four-year bond. The overall trade is a butterfly: a short position in two-year bonds, a long position in four-year bonds, and a short position in twelve-year bonds. It exerts upward pressure on the twelve-year spot rate, even though it is triggered by a drop in the short rate.

two eigenvalues of M are real. We refer to the largest interval of the form $(0,\tau)$ over which a given property holds as a maximal interval.

Proposition 5 (**Effect of Short-Rate and Demand Shocks**) *Suppose that there is one demand factor, the matrices (Γ,Σ) are diagonal, and $M_{2,1}<0$.*

- *If the two eigenvalues of M are real, then $A_\beta(\tau)>0$, $A'_\beta(\tau)>0$ and $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]'<0$. Moreover, $A_r(\tau)>0$ for $\tau\in(0,\bar{\tau})$ and $A_r(\tau)<0$ for $\tau\in(\bar{\tau},\infty)$, where $\bar{\tau}=\infty$ when $a\approx 0$ or $\alpha(\tau)=0$, and $\bar{\tau}<\infty$ when $a\approx\infty$.*
- *If the two eigenvalues of M are complex, then $A_\beta(\tau)>0$ for τ in a maximal interval $(0,\bar{\bar{\tau}})$, $A'_\beta(\tau)>0$ for τ in a maximal interval $(0,\hat{\tau})$, and $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]'<0$ for $\tau\in(0,\hat{\tau})$, where $\bar{\bar{\tau}}>\hat{\tau}>0$. If $\bar{\bar{\tau}}<\infty$, then $A_r(\tau)>0$ for τ in a maximal interval $(0,\bar{\tau})$, where $\bar{\tau}\in(0,\bar{\bar{\tau}})$.*

4.3. *Bond Risk Premia*

Demand risk strengthens the positive premia-slope relationship derived in Section 3.4. Indeed, low demand by preferred-habitat investors implies positive bond risk premia because arbitrageurs must be induced to buy the bonds to make up for the low investor demand. Because of the positive premia, yields are high and the term structure is upward-sloping.

Proposition 6 computes the FB and CS coefficients b_{FB} and b_{CS}. It shows that b_{FB} is positive and b_{CS} is smaller than one for at least all maturities such that the functions $(A_r(\tau),A_\beta(\tau))$ are positive and $A_\beta(\tau)$ increases in τ, and for all maturities when a is close to zero or to infinity. Moreover, when $a\approx\infty$ and the average maturity where demand shocks originate is sufficiently long, b_{FB} exceeds one and increases in τ, while b_{CS} is negative and decreases in τ.

Proposition 6 (**Demand Risk Strengthens Positive Premia-Slope Relationship**) *Suppose that there is one demand factor, the matrices (Γ,Σ) are diagonal, $M_{1,2}\geq 0$, $M_{2,1}<0$ and $\Delta\tau\to 0$.*

- *The FB regression coefficient in (28) is positive for $\tau<\min\{\bar{\tau},\hat{\tau}\}$, and for all τ when $a\approx 0$ or $a\approx\infty$. When $a\approx\infty$ and*

$$\frac{\int_0^\infty \theta(\tau)\tau d\tau}{\int_0^\infty \theta(\tau)d\tau} > \frac{\int_0^\infty \alpha(\tau)\tau^2 d\tau}{\int_0^\infty \alpha(\tau)\tau d\tau}, \tag{43}$$

b_{FB} exceeds one and increases in τ.

- *The CS regression coefficient in (29) is smaller than one for $\tau < \min\{\bar{\tau}, \hat{\tau}\}$, and for all τ when $a \approx 0$ or $a \approx \infty$. When $a \approx 0$, b_{CS} is close to one and increases in τ. When $a \approx \infty$ and (43) holds, b_{CS} is negative and decreases in τ.*

4.4. *Demand Effects*

Suppose, as in Section 3.5, that the demand intercept $\theta_0(\tau)$ changes to $\theta_0(\tau) + \Delta\theta_0(\tau)$, where $\Delta\theta_0(\tau)$ is a general function of τ. The functions $(A_r(\tau), A_\beta(\tau))$ do not change, and the effects on yields are entirely through $C(\tau)$. Because there are two risk factors, the effects are represented by two one-dimensional indices. The indices are $\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau$ and $\int_0^\infty \theta_0(\tau)A_\beta(\tau)d\tau$, and relate to the sensitivity of arbitrageurs' portfolio to the short-rate and the demand factor, respectively.

While demand effects retain a global flavor because they are represented by only two indices across a continuum of maturities, they become more localized relative to the one-factor case. Recall from Section 3.5 that with one factor, demand changes have the same relative effect across maturities regardless of the maturities where they originate. This independence result does not extend to two factors. The maturities where demand shocks originate matter because they influence how the shocks affect one index relative to the other, and because changes to each index have a different relative effect across maturities. Changes to the demand for long-maturity bonds have a large effect on $\int_0^\infty \theta_0(\tau)A_\beta(\tau)d\tau$ relative to $\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau$, and changes to $\int_0^\infty \theta_0(\tau)A_\beta(\tau)d\tau$ have a large effect on long rates relative to short rates. Hence, the effects of long-maturity bond demand are more pronounced at the long end of the term structure. In comparison, changes to the demand for short-maturity bonds have a large relative effect on $\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau$, and changes to that index have a large relative effect on short rates. Hence, the effects of short-maturity bond demand are more pronounced at the short end.

The economic intuition is as follows. Suppose that the demand by preferred-habitat investors for long-maturity bonds declines, in which case arbitrageurs take up the slack by purchasing those bonds. Since bonds' sensitivity to demand shocks relative to short-rate shocks rises with maturity, arbitrageurs' exposure to demand risk increases significantly, while their exposure to short-rate risk increases more mildly. The expected excess returns that arbitrageurs require to bear demand risk increase significantly as well. Since bonds' sensitivity to demand shocks rises faster with maturity than their sensitivity to short-rate shocks, long-maturity bonds experience a sharp increase in their expected excess returns relative to short-maturity bonds. Hence, long rates increase sharply. By contrast, when the demand by preferred-habitat investors for short-maturity bonds declines, long rates increase less than short rates.

To show a formal result on localization, we consider the simple case where the change $\Delta\theta_0(\tau)$ represents a decrease in demand for a specific short maturity τ_1 or a specific long maturity $\tau_2 > \tau_1$. We denote the resulting changes in the yield $y_t^{(\tau)}$ by $\Delta y_{t,\tau_1}^{(\tau)}$ and $\Delta y_{t,\tau_2}^{(\tau)}$, respectively.

PROPOSITION 7 (**Localization of Demand Effects**) *When there is one demand factor, a change in the demand intercept from $\theta_0(\tau)$ to $\theta_0(\tau)+\Delta\theta_0(\tau)$ affects yields only through $\int_0^\infty \Delta\theta_0(\tau)A_r(\tau)d\tau$ and $\int_0^\infty \Delta\theta_0(\tau)A_\beta(\tau)d\tau$. When additionally the matrices (Γ, Σ) are diagonal, $M_{2,1} < 0$, $\alpha(\tau)$ is non-increasing, and the change $\Delta\theta_0(\tau)$ is a Dirac function with point mass at $\tau_1 < \hat{\tau}$ or at $\tau_2 \in (\tau_1, \hat{\tau})$,*

$$\Delta y_{t,\tau_1}^{(\tau_1)} \Delta y_{t,\tau_2}^{(\tau_2)} > \Delta y_{t,\tau_1}^{(\tau_2)} \Delta y_{t,\tau_2}^{(\tau_1)}. \tag{44}$$

Equation (44) states that the product of the "local" effects that the changes have on the maturity where they originate exceeds the product of the "cross" effects on the other maturity. Local effects are thus stronger than cross effects.

We expect full localization when there is a large number of demand factors and arbitrageurs are highly risk-averse. Indeed, suppose that a demand shock originating at maturity τ_1 has its largest effect at maturity $\tau_2 \neq \tau_1$. For this to happen, arbitrageurs must hold non-zero positions in at least the bonds of one of the two maturities. Highly risk-averse arbitrageurs, however, hold non-zero positions only if their exposure to all risk factors is zero, which is infeasible with a large number of factors. Proposition 1 implies a full localization result for the effects of short-rate shocks: since the function $A_r(\tau)$ converges to zero when the arbitrageurs' risk-aversion coefficient a goes to infinity, the effects of short-rate shocks become localized at the zero maturity. We can derive the same localization result with one and two demand factors, using closed-form solutions for the large a limit. Extending the full localization result for the effects of demand shocks requires extending our solutions to a large number of demand factors and is left for future work.

5. CALIBRATION AND POLICY ANALYSIS

In this section we calibrate our model and analyze the effects of different policies by central banks. Since the model can be given both a nominal and a real interpretation, we calibrate it using nominal yields and then again using real yields. In all calibrations we assume that there is one demand factor which is independent of the short rate. We leave the correlated case, which seems more relevant for the nominal calibration, for future work. The independent case is a natural first case to investigate, and it yields a remarkably similar analysis of central-bank policies across the nominal and real calibrations.

5.1. *Calibration*

The equilibrium term structure is determined by the parameters $(\bar{r}, \kappa_r, \sigma_r)$ of the short-rate process, the parameters $(\kappa_\beta, \sigma_\beta)$ of the demand-factor process, the risk-aversion coefficient a of arbitrageurs, and the functions $(\alpha(\tau), \theta_0(\tau), \theta(\tau))$ that describe the demand slope and intercept of preferred-habitat investors.

The values of $(\bar{r}, \theta_0(\tau))$ affect only the long-run averages of yields and of agents' positions. They do not matter for our policy analysis, which concerns how yields and positions respond to shocks. We sketch a calibration of these parameters in Section 5.3, where we compute unconditional moments of bond returns.

We set $\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}$ and $\theta(\tau) = \theta(e^{-\delta_\alpha \tau} - e^{-\delta_\theta \tau})$ for $\tau < T$, and $\alpha(\tau) = \theta(\tau) = 0$ for $\tau > T$. This is the same exponential specification as in Theorem 1, except that we take the maximum bond maturity T to be finite. We set $T = 30$ years, the maximum maturity for U.S. government bonds.

The values of $(\theta(\tau), \sigma_\beta)$ matter only through their product because $(\theta(\tau), \beta_t)$ affect the demand of preferred-habitat investors only through their product as well. We can hence normalize σ_β to an arbitrary value, and we set it equal to σ_r.

We calibrate the remaining eight parameters $(\kappa_r, \sigma_r, \kappa_\beta, a, \alpha, \theta, \delta_\alpha, \delta_\theta)$ using U.S. data on bond yields and trading volume, as well as estimates of demand elasticity from the literature. For bond yields, we use the Gurkaynak, Sack and Wright (GKS) datasets, which report daily spot rates extracted from government bond prices. The dataset on nominal yields goes from June 1961 to the present. We start our main sample of nominal yields in November 1985, because this is the earliest when all maturities from one to 30 years are included, and end it in January 2020. The dataset on real yields goes from January 1999 to the present, and includes all maturities from two to 20 years. We start our sample of real yields in January 1999 and end it in January 2020. In addition to our main sample of nominal yields, we consider a sub-sample covering the same period as the sample of real yields. We source nominal and real yields at the end of each month. For bond trading volume, we use the FR 2004 dataset, which reports daily volume by primary dealers in the Treasury market, split into buckets based on the bonds' remaining time to maturity. Volume on real bonds

(TIPS) is approximately 3% of total volume, and is not split into maturity buckets until March 2020. For that reason, we use the volume split for nominal bonds in all calibrations. We do not include T-bills in our volume calculations because of their special features (e.g., extensive use as collateral). T-bills are also not included in the GKS datasets. The dataset on volume goes from April 2013 to the present. We end it in January 2020, and use averages within that period in all calibrations. For demand elasticity, we use estimates from Krishnamurthy and Vissing-Jorgensen (KVJ 2012).[18]

Table I reports the calibrated parameters and the empirical moments used to determine them, for the main sample of nominal yields. Tables C.I and C.II in Appendix C report the same information for the subsample of nominal yields and the sample of real yields, respectively. We express yields and their volatilities in percentage terms throughout this section, e.g., a yield of 0.02 is expressed as 2.

We determine the first seven parameters in Table I by equating the first seven empirical moments to their model-generated counterparts. This requires solving a seven-equation non-linear system. The formulas for the seven model-generated moments are in Appendix C. The seven moments concern volatilities and correlations of yields and yield changes, and fractions of volume at different maturity buckets. Data on yields and relative volume cannot identify the arbitrageurs' risk-aversion coefficient a separately from the parameters (α, θ) that characterize the slope of preferred-habitat demand and the magnitude of demand shocks, respectively. Only the products $(a\alpha, a\theta)$ can be identified. Intuitively, yields can be volatile because arbitrageurs are highly risk-averse (high a) and demand shocks are small (low θ), or because arbitrageurs are less risk-averse and demand shocks are larger.[19] We determine α, the eighth parameter in Table I, based on KVJ's estimates, and deduce (a, θ) from the products $(a\alpha, a\theta)$.

The empirical moment next to each parameter in Table I is the one identifying that parameter. We address identification formally in Appendix C, where we compute a seven-by-seven table of elasticities of the first seven moments with respect to the first seven parameters. The elasticity table validates the mapping in Table I except for the fourth and fifth moments, for which cross-effects from the fifth and fourth parameter, respectively, are important.

[18]The dataset of nominal yields is available at `https://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html` and is described in Gurkaynak, Sack, and Wright (2007). The dataset of real yields is available at `https://www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html`. The FR 2004 dataset (which reports additional information to volume) is available at `https://www.newyorkfed.org/markets/gsds/search`.

[19]Formally, (37) shows that the matrix M that determines $(A_r(\tau), A_\beta(\tau))$ through the ODE (36) depends on (a, α, θ) only through the products $(a\alpha, a\theta)$. Hence, $(A_r(\tau), A_\beta(\tau))$ have that property as well, and so do the moments of returns and volume computed in Appendix C.

TABLE I

Calibration of model parameters for the main sample of nominal yields.

Parameter	**Value**	**Empirical moment**	**Value**
κ_r Mean-reversion of r_t	0.125	$\sqrt{\mathbb{V}\mathrm{ar}\left(y_t^{(1)}\right)}$ Volatility 1-year yield – Levels	2.62
σ_r Diffusion of r_t	0.0146	$\sqrt{\mathbb{V}\mathrm{ar}\left(y_{t+1}^{(1)} - y_t^{(1)}\right)}$ Volatility 1-year yield – Annual changes	1.27
κ_β Mean-reversion of β_t	0.053	$\frac{1}{30}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\mathrm{ar}\left(y_t^{(\tau)}\right)}$ Volatility τ-year yield – Levels, average over τ	2.20
$a\theta$ Arb. risk-aversion $\times$ PH demand shock	3155	$\frac{1}{30}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\mathrm{ar}\left(y_{t+1}^{(\tau)} - y_t^{(\tau)}\right)}$ Volatility τ-year yield – Annual changes, average over τ	0.796
$a\alpha$ Arb. risk-aversion $\times$ PH demand slope	35.3	$\frac{1}{30}\sum_{\tau=1}^{30}\mathbb{C}\mathrm{orr}\left(y_{t+1}^{(1)} - y_t^{(1)}, y_{t+1}^{(\tau)} - y_t^{(\tau)}\right)$ Correlation 1-year yield with τ-year yield – Annual changes, average over τ	0.504
δ_α PH demand shock – short maturities	0.297	$\frac{\sum_{0<\tau\leq 2}\mathrm{Volume}(\tau)}{\sum_{0<\tau\leq 30}\mathrm{Volume}(\tau)}$ Relative volume for maturities $\tau \in (0, 2]$	0.199
δ_θ PH demand shock – long maturities	0.307	$\frac{\sum_{11<\tau\leq 30}\mathrm{Volume}(\tau)}{\sum_{0<\tau\leq 30}\mathrm{Volume}(\tau)}$ Relative volume for maturities $\tau \in (11, 30]$	0.094
α PH demand slope	5.21	Estimate in KVJ 2012	-0.746

The mean-reversion κ_r and diffusion σ_r of the short rate r_t have their largest effect on the one-year yield $y_t^{(1)}$. An increase in σ_r raises the volatility of that yield and the volatility of yield changes. A decrease in κ_r raises the yield's volatility, but has a weaker effect on the volatility of yield changes because it implies that the short rate mean-reverts more slowly. Since shocks to the demand factor have a weak effect on the one-year yield, the volatility of that yield identifies κ_r, and the volatility of annual changes to that yield identifies σ_r.

The mean-reversion κ_β of the demand factor β_t and the magnitude parameter θ of demand shocks have their largest effect on long-maturity yields. As with (κ_r, σ_r), the volatility of yields identifies κ_β and the volatility of annual changes to yields identifies $a\theta$. We average volatilities across all maturities. Using volatilities at long maturities only does not sharpen the identification.

The slope parameter α of preferred-habitat demand affects how shocks to the short rate are transmitted to longer maturities. An increase in α weakens the transmission (Proposition 2), and this makes yield changes at short and long maturities less correlated. Hence, the correlation between annual changes to the one-year yield and to other yields identifies $a\alpha$. (As we explain in Appendix C, however, there are important cross-effects from $a\theta$ to correlation and from $a\alpha$ to volatility.) As with (κ_β, θ), we average the correlation across all maturities

The parameters $(\delta_\alpha, \delta_\theta)$ control the maturities where demand shocks originate, via the specification $\theta(\tau) = \theta(e^{-\delta_\alpha \tau} - e^{-\delta_\theta \tau})$. Hence, they affect how volume is split across maturities. An increase in δ_α raises the relative volume for short maturities and lowers that for long maturities. An increase in δ_θ has the same effects, with the decline in long-maturity volume being relatively more pronounced. Hence, the relative volume for maturities two years and below identifies δ_α, and the relative volume for maturities eleven years and above identifies δ_θ.

Our moment-matching exercise indicates slow mean-reversion for the short rate ($\kappa_r = 0.125$, half-life of shocks 5.55 years) and even slower mean-reversion for the demand factor ($\kappa_\beta = 0.053$, half-life of shocks 13.1 years). The corresponding parameters for the sub-sample of nominal yields and the sample of real yields are two to three times larger, implying faster mean-reversion. In all samples, demand shocks originate at short and intermediate maturities, consistent with the fact that only 9.4% of volume concerns bonds with remaining time to maturity longer than 11 years.

Figure 1 compares the empirical moments, represented by the black crosses, to the model-generated ones, represented by the red solid lines, for the main sample of nominal yields. Figures C.1 and C.2 in Appendix C show the same comparisons for the sub-sample of nominal yields and the sample of real yields, respectively.

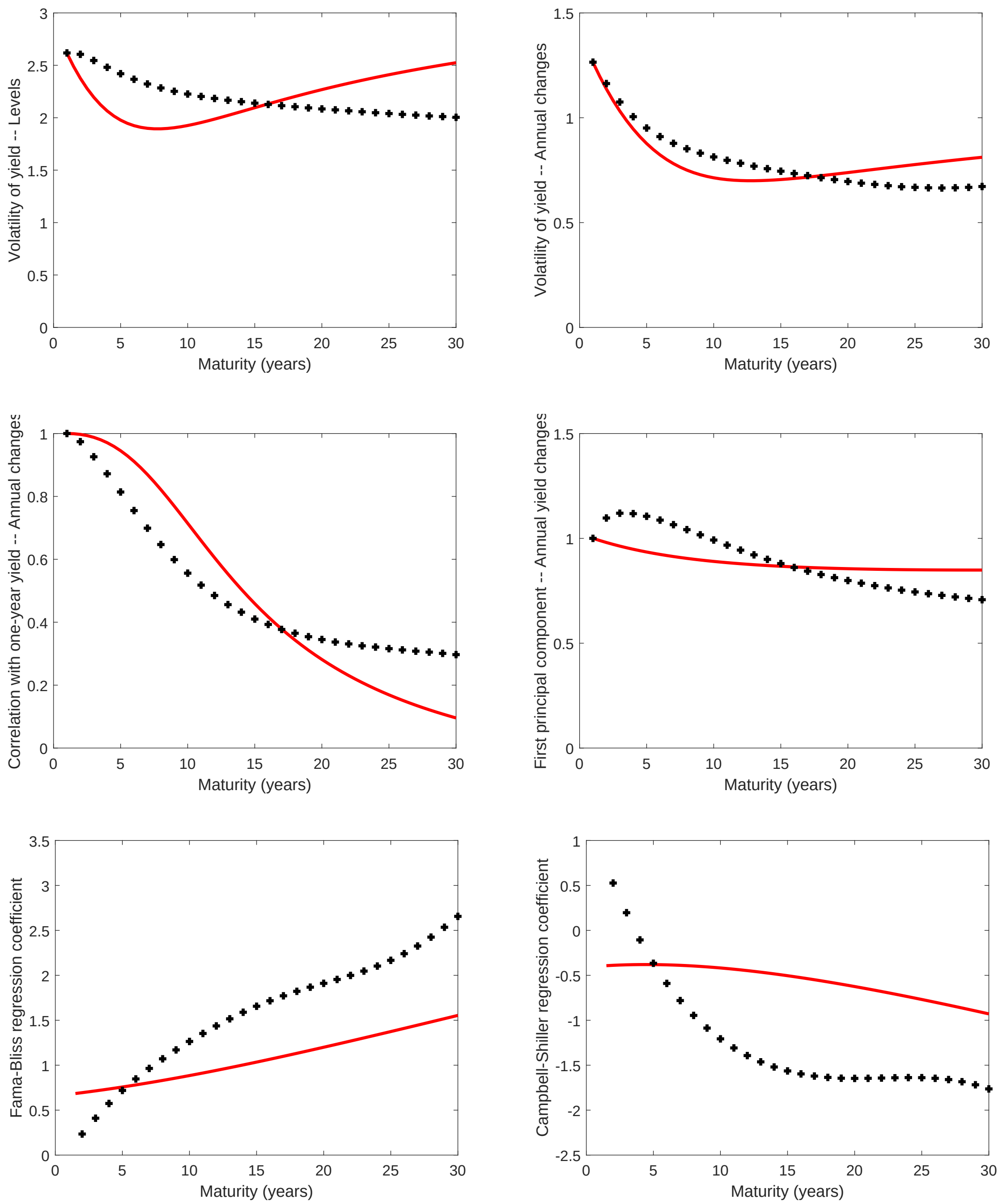

FIGURE 1.— Model-generated and empirical moments for the main sample of nominal yields.

The comparisons are remarkably similar across the three figures. The figures depend only on the first seven parameters in Table I, and not on the separate values of a and (α, θ).

The top two panels in Figure 1 report the volatility of yields and the volatility of annual yield changes, as functions of maturity. The model-generated moments coincide with the empirical ones for the one-year maturity and on average, by construction. While the empirical moments are decreasing functions of maturity, the model-generated ones are inverse hump-shaped. The inverse hump shape seems to be driven by the independence between the short rate and the demand factor, as these factors have their largest effects at different ends of the term structure. The middle-left panel reports the correlation between annual changes to the one-year yield and to other yields, as function of maturity. The model-generated moments coincide with the empirical ones on average, by construction.

The remaining panels in Figure 1 report moments not used in the calibration. The middle-right panel reports the first principal component of annual yield changes as function of maturity, scaled to one for the one-year maturity. The model-generated moments are close to the empirical ones, and so is the fraction of variation explained by the first principal component (76.5% in the model and 81.3% in the data). Hence, our calibration captures closely the empirical factor structure of yields.

The bottom two panels in Figure 1 report the coefficients of the FB and CS regressions (28) and (29), respectively, with $\Delta\tau = 1$ (returns and yield changes are evaluated over one year). The model generates less predictability than is found in the data, especially for long maturities. For those maturities, the model-generated predictability, as measured by the deviation between the FB/CS coefficients and their EH value, is about 60% of its empirical counterpart. The model-generated coefficients have the same monotonicity as in the data. If the model is calibrated to match the FB/CS coefficients instead of the volatility of annual yield changes, then it overshoots that volatility for long maturities, because $a\theta$ must take a larger value.

To determine the slope parameter α of preferred-habitat demand, we use KVJ's estimates of the elasticity of the demand for government debt. KVJ regress the yield spread between long-maturity AAA-rated corporate bonds and government bonds on the logarithm of government debt to GDP, and find a coefficient of -0.746 (Table 1, Panel A). Hence, a 0.01 (1 bp) drop in the yield spread is associated with a 0.0134 ($= \frac{0.01}{0.746}$) increase in the logarithm of debt to GDP. Assuming that debt to GDP takes originally its average value, which is 43.9% in KVJ's sample (1919-2008), it increases by 0.0059 ($= 43.9\% \times (e^{0.0134} - 1)$). To map this estimate into our model, we interpret the increase in debt to GDP as the slope of preferred-habitat demand for government debt. We also assume that the drop in the yield spread results from an increase in government bond yields across all maturities, and use GDP as the unit of account. KVJ's estimate implies $\alpha = 5.21$.

The value $\alpha = 5.21$ is an upper bound for two reasons. First, instrumental-variables estimation of the KVJ regression generates a more negative coefficient and hence a smaller slope for preferred-habitat demand. Second, our model takes as given the returns that preferred-habitat investors earn outside the government bond market (Appendix B). These returns, however, could change in equilibrium when government bond yields change, resulting in a lower effective demand elasticity. In the extreme case where returns outside the government bond market move one-to-one with government bond yields, a change in these yields should not affect preferred-habitat demand, resulting in an effective slope of zero. In the intermediate case where returns outside the government bond market adjust by $x \in (0,1)$, the effective slope is $\alpha(1-x)$.

For $\alpha = 5.21$ and $a\alpha = 35.3$, the coefficient of arbitrageur risk aversion is $a = 6.78$. To map a into a coefficient of relative risk aversion (RRA), we recall that if arbitrageurs have wealth W and a VNM utility function U, then $a = -\frac{U''(W)}{U'(W)}$. Hence, the coefficient of RRA is $\gamma = -\frac{U''(W)W}{U'(W)} = aW$. The macro-finance literature generally assumes that γ is larger than one and does not exceed ten. For $\gamma = 2$ and $a = 6.78$, arbitrageur wealth is $W = 29.5\%$, which is 29.5% of GDP since we are using GDP as the unit of account. Such a value seems large. Suppose that we identify arbitrageurs with hedge funds, which are sophisticated investors with relatively broad mandates. The assets of hedge funds in the fixed-income, macro and balanced categories in the last quarter of 2019 added up to \$1.2 trillion, which was 5.6% ($= \frac{1.2}{21.42}$) of U.S. GDP in that year.[20] Smaller values of W correspond to smaller values of α since W is proportional to α holding $(a\alpha, \gamma)$ fixed. Since smaller values seem plausible for both W and α, for separate reasons for each parameter, we use a parameter range. We use $\alpha = 5.21$ as the upper bound of the range for α, and $\alpha = 1.04$ as the lower bound. The lower bound corresponds to an $x = 80\%$ adjustment of returns outside the government-bond market to government-bond yields.[21] The upper bound $\alpha = 5.21$ corresponds to an upper bound 29.5% for W and a lower bound 6.78 for a. The lower bound $\alpha = 1.04$ corresponds to a lower bound 5.9% for W and an upper bound 33.9 for a.

[20] See `https://www.barclayhedge.com/solutions/assets-under-management/hedge-fund-assets-under-management/`.

[21] Duffee (1998) finds that a unit drop in the Treasury bill rate causes the spread between corporate and government bonds to rise by values ranging from 0.02 for intermediate-term AAA-rated corporate bonds to 0.42 for long-term BBB-rated bonds. An 80% adjustment of corporate bond yields to government bond yields (i.e., a rise in the spread by 0.2) lies within these estimates.

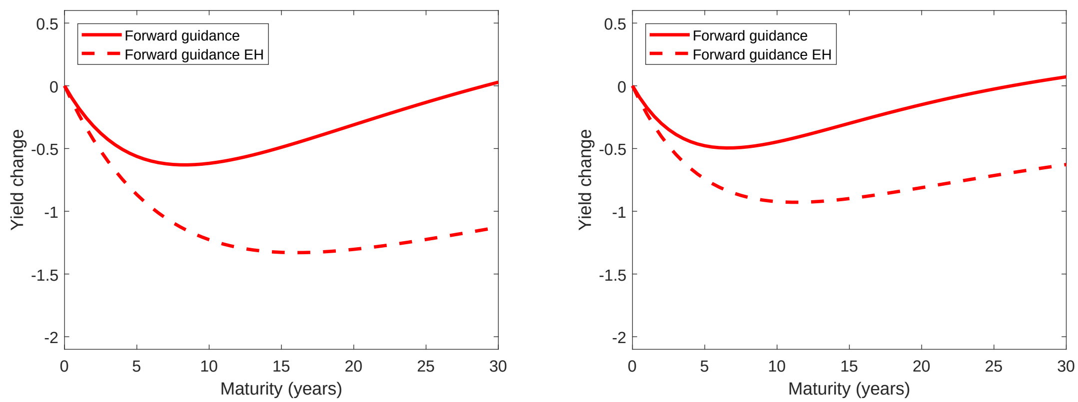

FIGURE 2.— Effect of a forward-guidance announcement about the path of short rates, for the calibration based on the main sample of nominal yields.

5.2. *Policy Analysis*

The first policy that we analyze is a forward-guidance announcement about the path of short rates. We model this announcement as a change $\Delta\bar{r}$ in the long-run mean $\bar{r}$ of the short rate r_t. We assume that the change is unanticipated, takes place at time zero, and reverts deterministically to zero at a rate $\kappa_{\bar{r}}$.

Figure 2 shows the announcement's effect on the term structure at time zero, for the calibration based on the main sample of nominal yields. The figures for the other two calibrations, and the equations describing the announcement's effect, are in Appendix C. In each panel of Figure 2, the red solid line represents the announcement's effect, and the red dashed line represents the same effect when arbitrageurs are risk-neutral and the EH holds. The change $\Delta\bar{r}$ is negative, i.e., the announcement is that future short rates will be lower, and is set to -4 (-400 bps). The change reverts to zero at the rate $\kappa_{\bar{r}} = 0.1$ (half-life 6.93 years) in the left panel and $\kappa_{\bar{r}} = 0.2$ (half-life 3.47 years) in the right panel. When $\kappa_{\bar{r}} = 0.1$, yields are more affected because the same is true for expected future short rates.

For both values of $\kappa_{\bar{r}}$, yields under-react relative to their EH counterparts. This reflects the under-reaction result of Proposition 2. The extent of under-reaction increases with maturity. When $\kappa_{\bar{r}} = 0.1$, under-reaction is 25.6% for the two-year yield, 35.1% for the five-year yield, 49.6% for the ten-year yield, 76.1% for the twenty-year yield, and 102.6% for the thirty-year yield. When $\kappa_{\bar{r}} = 0.2$, these numbers rise to 25.7%, 35.7%, 51.6%, 81.6%, and 111.4%, respectively. Thus, forward guidance is effective in changing yields for short maturities, but less so for longer maturities. To engineer a decline in the ten-year yield by 0.5 (50 bps), for

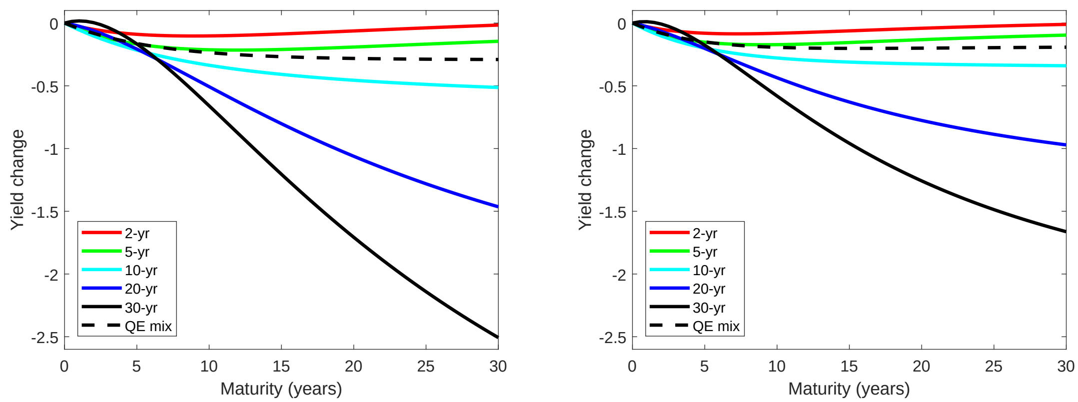

FIGURE 3.— Effect of QE, for the calibration based on the main sample of nominal yields.

example, central banks need to lower the average of expected short rates over the next ten years by about twice as much (100 bps). The calibration based on the sample of real yields generates a similar number. The calibration based on the sub-sample of nominal yields implies instead that the average of expected short rates must drop by about three times as much (150 bps).

The second policy that we analyze is QE. We assume that QE purchases concern government bonds only, and we model them as a decrease $\Delta\theta_0(\tau)$ in the intercept of preferred-habitat demand. (Equation (5) defines the demand intercept with a negative sign.) We assume that the decrease is unanticipated, takes place at time zero, and reverts deterministically to zero at a rate κ_θ.

Figure 3 shows the effect of QE on the term structure at time zero, for the calibration based on the main sample of nominal yields. The figures for the other two calibrations, and the equations describing the effect of QE, are in Appendix C. In each panel of Figure 3, the red, green, light blue (cyan), blue and black solid lines represent the effect of QE purchases of two-, five-, ten-, twenty- and thirty-year bonds, respectively. The black dashed line represents the effect of QE purchases that conform to the maturity distribution used by the Fed during QE1, as reported in D'Amico and King (2013). All lines are drawn for a change $\Delta\theta_0(\tau)$ in the intercept of preferred-habitat demand that satisfies $\int_0^\infty \Delta\theta_0(\tau)d\tau = -0.12$, i.e., QE purchases are 12% of GDP. This is approximately the value of government bonds purchased by the Fed during QE1, QE2 and QE3. The demand change mean-reverts to zero at the rate $\kappa_{\overline{\tau}} = 0.1$ (half-life 6.93 years) in the left panel and $\kappa_{\overline{\tau}} = 0.2$ (half-life 3.47 years) in the right panel.

Figure 3 is the only one in this section that depends on the separate values of a and (α, θ) rather than only on the products $(a\theta, a\alpha)$. An increase in the coefficient of arbitrageur risk aversion a holding $(a\theta, a\alpha)$ constant results in a proportionate increase in the effects of QE. Relative effects across maturities do not change, i.e., Figure 3 looks the same after rescaling the y-axis. We use the value of a that generates the average effect across the lower bound $a = 6.78$ and the upper bound $a = 33.9$.

The effects of QE on the term structure are larger when $\kappa_\theta = 0.1$, i.e., when QE is unwound over a longer period. Intuitively, QE lowers the yield of a bond because it lowers the risk premia that arbitrageurs require to hold the bond. Moreover, the yield depends not only on the risk premium that arbitrageurs require in the current instant but on an average of risk premia during the bond's life. When QE is expected to be unwound more slowly, risk premia in that average are impacted more.

The effects of QE have a global flavor as in Proposition 4, with some localization as in Proposition 7. Consistent with Proposition 4, an increase in demand for bonds with longer maturities generates a larger downward shift in the term structure. For example, the term structure shifts downward more when QE purchases concern thirty-year bonds than when they concern two-year bonds. That downward shift, however, is not larger across all maturities: yields for maturities ranging from one to three years are more sensitive to purchases of two-year bonds than of thirty-year bonds. More generally, and consistent with Proposition 7, an increase in demand for bonds with short (long) maturities has more pronounced effects at the short (long) end of the term structure. For example, purchases of two- and five-year bonds have an effect that peaks at short and intermediate maturities, while purchases of twenty- and thirty-year bonds have an effect that peaks at long maturities. These features are robust to different values of κ_θ.

The effects of QE in Figure 3 are somewhat smaller than in the literature. Wiliams (2014) summarizes a number of QE studies in the U.S. as suggesting that bond purchases of \$600 billion by the Fed reduced the ten-year yield by 0.15-0.25 (15-25 bps). Taking U.S. GDP at that time to be \$15 trillion, the \$600 billion purchases are 4% of GDP. Hence, QE purchases of 12% of GDP should reduce the ten-year yield by 0.45-0.75. The corresponding effect in Figure 3, in the case where the maturities of QE purchases conform to the distribution used by the Fed during QE1, is 0.24 when $\kappa_\theta = 0.1$ and 0.19 when $\kappa_\theta = 0.2$. When $\kappa_\theta = 0.1$, the range of the effect between the upper and lower bound of α is 0.08-0.39. The calibration based on the sub-sample of nominal yields generates the range 0.11-0.54, and that based on the sample of real yields generates 0.09-0.44.

The discrepancy between our calibrations and the estimates from QE studies could arise because some of the observed effect of QE was due to forward guidance about the path of short rates. Additionally, arbitrageur risk aversion during the QE period could have been larger than average because of capital losses and tighter regulation. The latter explanation is consistent with the calibration based on the sub-sample of nominal yields generating larger effects than the one based on the main sample.

Figure 3 suggests that central banks seeking to maximize the effects on QE on yields should concentrate their purchases at long maturities. Moreover, such purchases have particularly large effects on long-maturity yields. In the extreme case where QE purchases of 12% of GDP are concentrated at the thirty-year maturity, and where $\kappa_\theta = 0.1$, the ten-year yield drops by 0.66 (instead of 0.24, under the maturity distribution used by the Fed during QE1) and the thirty-year yield drops by 2.51 (instead of 0.29). Of course, it is not possible to buy 12% of GDP worth of thirty-year bonds because their supply is below that amount.

Even less extreme tilts towards long maturities, in a way consistent with available supply, can generate sizeable effects. The Fed's purchases during QE1 incorporated a mild tilt: the average maturity of purchased bonds was 6.5 years, while that of all available coupon bonds was 5.7 years. To evaluate the effects of a stronger tilt, suppose that the Fed did not change the total value of its purchases during QE1 but bought 15% of all available supply in any given maturity before moving to a shorter maturity (hence not buying at all short maturities). The ceiling of 15% is not overly high: D'Amico and King (2013) report that it was exceeded for the 6-8 and 10-12 maturity buckets. Under the modified maturity distribution, QE purchases of 12% of GDP lower the 10-year yield by 0.33 (instead of 0.24) and the thirty-year yield by 0.59 (instead of 0.29).

5.3. *Unconditional Moments*

To compute unconditional moments of bond returns, we must choose values for $(\overline{r}, \theta_0(\tau))$. We assume that $\theta_0(\tau)$ is proportional to $\theta(\tau)$, thus setting $\theta_0(\tau) = \theta_0(e^{-\delta_\alpha \tau} - e^{-\delta_\theta \tau})$ for $\tau < T$, and $\theta_0(\tau) = 0$ for $\tau > T$. We determine $(\overline{r}, \theta_0)$ by equating empirical averages of yields to their model-generated counterparts. Since the estimation concerns first moments, we use the longest period available in the GKS dataset: we focus on nominal yields and start the sample from June 1961. The empirical average of the one-year yield is 5.01. The empirical average of the seven-year yield, which is the longest maturity covered during the entire sample period, is 5.90. Our model matches these moments when $(\overline{r}, a\theta_0) = (4.80, 289)$.

The model-generated average yield rises with maturity, from 5.01 for the one-year bond to 6.99 for the thirty-year bond. The unconditional expected excess return rises with maturity as well, from 0.40% for the one-year bond to 5.08% for the thirty-year bond. The unconditional Sharpe ratio drops from 0.320 for the one-year bond to 0.206 for the thirty-year bond, but does so non-monotonically, by first rising until the seven-year maturity to 0.365. The rise in expected return with maturity reflects the rise in the yield, and is consistent with the empirical evidence. Empirical Sharpe ratios, by contrast, decline with maturity across the entire maturity range.[22] The increase in the Sharpe ratio that our model generates for short maturities reflects the inverse hump shape of volatility shown in Figure 1, and seems to be driven by the independence between the short rate and the demand factor. The unconditional correlation between bond returns and the stochastic discount factor rises from 0.842 for the one-year bond to one for the seven-year bond, and subsequently drops to 0.563 for the thirty-year bond. The formulas for the model-generated moments are in Appendix C.

6. CONCLUSION

We model the term structure of interest rates that results from the interaction between investors with preferences for specific maturities and risk-averse arbitrageurs. Our model formalizes the preferred-habitat view of the term structure and embeds it into a modern no-arbitrage framework. We use our model to study three main questions: how shocks to the short rate, including monetary-policy actions by central banks, are transmitted to long rates; how bond risk premia depend on the shape of the term structure; and how changes in preferred-habitat demand, including large-scale bond purchases by central banks, affect the term structure. We provide qualitative answers as well as quantitative ones through a calibration exercise.

Our approach can be extended in a number of directions. One direction is to derive optimal debt issuance by governments or corporations when investors have preferences for specific maturities. Work along these lines includes Greenwood, Hanson, and Stein (2010), Guibaud, Nosbusch, and Vayanos (2013) and Bigio, Nuno, and Passadore (2019). Another direction is to broaden the asset-pricing implications by allowing arbitrageurs to trade additional assets. Work along these lines includes Gourinchas, Ray, and Vayanos (2020) and Greenwood, Hanson, Stein, and Sunderam (2020), who study the joint determination of bond prices and exchange rates. A third direction is to analyze broader macro-economic settings, in which term-structure shifts affect investment and output. Work along these lines includes Ray (2019), who embeds our model within a New Keynesian

[22]For evidence on how bond expected returns and Sharpe ratios vary with maturity see, for example, Duffee (2010) and Frazzini and Pedersen (2014).

framework.

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APPENDIX A: PROOFS

Proof of Lemma 1: The proof is in the text. *Q.E.D.*

Proof of Proposition 1: Equations (21) and (22) follow from integrating the linear ODEs (19) and (20) with the initial conditions $A_r(0) = C(0) = 0$. Substituting $A_r(\tau)$ from (21) into (23), we find (25). The left-hand side of (25) is increasing in κ_r^*, is zero for $\kappa_r^* = 0$, and converges to infinity when κ_r^* goes to infinity. The right-hand side of (25) is decreasing in κ_r^*, exceeds $\kappa_r > 0$ for $\kappa_r^* = 0$, and converges to κ_r when κ_r^* goes to infinity. Therefore, (25) has a unique solution for κ_r^*, which is positive.

Substituting $C(\tau)$ from (22) into (24), we find

$$\kappa_r^*\overline{r}^*\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)du\right]A_r(\tau)d\tau\right]$$

$$= \kappa_r\overline{r} + a\sigma_r^2\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau + \frac{a\sigma_r^4}{2}\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)^2du\right]A_r(\tau)d\tau. \tag{A.1}$$

Since

$$\kappa_r\overline{r} = \kappa_r^*\overline{r}\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)du\right]A_r(\tau)d\tau\right]$$
$$+ (\kappa_r-\kappa_r^*)\overline{r} - \kappa_r^*\overline{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)du\right]A_r(\tau)d\tau,$$

and

$$(\kappa_r-\kappa_r^*)\overline{r} - \kappa_r^*\overline{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)du\right]A_r(\tau)d\tau$$
$$= -\overline{r}a\sigma_r^2\int_0^\infty \alpha(\tau)A_r(\tau)^2d\tau - \kappa_r^*\overline{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)du\right]A_r(\tau)d\tau$$
$$= -\overline{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\left[A_r(\tau)+\kappa_r^*\int_0^\tau A_r(u)du\right]A_r(\tau)d\tau$$
$$= -\overline{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\tau A_r(\tau)d\tau,$$

where the first step follows from (21) and (25), and the third step follows from integrating (19) from zero to τ and using (21) and (25), we can write (A.1) as

$$\kappa_r^*\overline{r}^*\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)du\right]A_r(\tau)d\tau\right]$$
$$= \kappa_r^*\overline{r}\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)du\right]A_r(\tau)d\tau\right] - \overline{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\tau A_r(\tau)d\tau$$
$$+ a\sigma_r^2\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau + \frac{a\sigma_r^4}{2}\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)^2du\right]A_r(\tau)d\tau. \tag{A.2}$$

Equations (21) and (A.2) imply (26). *Q.E.D.*

Proof of Proposition 2: Taking expectations conditional on time t in (8), we find

$$dE_t(r_{t+\tau}) = \kappa_r(\overline{r} - E_t(r_{t+\tau}))d\tau$$
$$\Rightarrow E_t(r_{t+\tau}) = (1-e^{-\kappa_r\tau})\overline{r} + e^{-\kappa_r\tau}r_t. \tag{A.3}$$

Equation (A.3) implies

$$\frac{\partial E_t(r_{t+\tau})}{\partial r_t} = e^{-\kappa_r\tau}. \tag{A.4}$$

Equation (27) likewise implies

$$\frac{\partial f_t^{(\tau)}}{\partial r_t} = A_r'(\tau) = e^{-\kappa_r^* \tau}, \tag{A.5}$$

where the second step follows from (21).

Equation (25) implies that if $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$, then $\kappa_r^* > \kappa_r$. Since the right-hand side of (25) increases in a, σ_r^2 and $\alpha(\tau)$, and the difference between the left-hand side and the right-hand side increases in κ_r^*, κ_r^* increases in a, σ_r^2 and $\alpha(\tau)$. *Q.E.D.*

Proof of Proposition 3: Equations (1), (2) and (10) imply that the dependent variable in (28) is

$$\frac{1}{\Delta\tau}\left\{A_r(\tau)r_t + C(\tau) - [A_r(\tau-\Delta\tau)r_{t+\Delta\tau} + C(\tau-\Delta\tau)] - [A_r(\Delta\tau)r_t + C(\Delta\tau)]\right\}$$

and the independent variable is

$$\frac{1}{\Delta\tau}\left\{A_r(\tau)r_t + C(\tau) - [A_r(\tau-\Delta\tau)r_t + C(\tau-\Delta\tau)] - [A_r(\Delta\tau)r_t + C(\Delta\tau)]\right\}.$$

Therefore, the FB regression coefficient is

$$\begin{aligned} b_{\mathrm{FB}} &= \frac{\mathbb{C}\mathrm{ov}\left\{[A_r(\tau) - A_r(\Delta\tau)]r_t - A_r(\tau-\Delta\tau)r_{t+\Delta\tau}, [A_r(\tau) - A_r(\tau-\Delta\tau) - A_r(\Delta\tau)]r_t\right\}}{\mathbb{V}\mathrm{ar}\left\{[A_r(\tau) - A_r(\tau-\Delta\tau) - A_r(\Delta\tau)]r_t\right\}} \\ &= \frac{[A_r(\tau) - A_r(\Delta\tau)]\mathbb{V}\mathrm{ar}(r_t) - A_r(\tau-\Delta\tau)\mathbb{C}\mathrm{ov}(r_{t+\Delta\tau}, r_t)}{[A_r(\tau) - A_r(\tau-\Delta\tau) - A_r(\Delta\tau)]\mathbb{V}\mathrm{ar}(r_t)}. \end{aligned} \tag{A.6}$$

Since (A.3) implies

$$\mathbb{C}\mathrm{ov}(r_{t+\Delta\tau}, r_t) = \mathbb{V}\mathrm{ar}(r_t)e^{-\kappa_r\Delta\tau}, \tag{A.7}$$

we can write (A.6) as

$$b_{\mathrm{FB}} = \frac{A_r(\tau) - A_r(\tau-\Delta\tau)e^{-\kappa_r\Delta\tau} - A_r(\Delta\tau)}{A_r(\tau) - A_r(\tau-\Delta\tau) - A_r(\Delta\tau)}.$$

Taking the limit $\Delta\tau \to 0$ and noting from (21) that $\frac{A_r(\Delta\tau)}{\Delta\tau} \to 1$, we find

$$b_{\mathrm{FB}} \to \frac{A_r'(\tau) + \kappa_r A_r(\tau) - 1}{A_r'(\tau) - 1} = \frac{(\kappa_r^* - \kappa_r)A_r(\tau)}{\kappa_r^* A_r(\tau)} = \frac{\kappa_r^* - \kappa_r}{\kappa_r^*}, \tag{A.8}$$

where the second step follows from (19) and (25). Since $\kappa_r^* > \kappa_r$ when $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$, (A.8) implies $b_{\mathrm{FB}} > 0$. Since κ_r^* increases in a, σ_r^2 and $\alpha(\tau)$, (A.8) implies that b_{FB} increases in the same variables.

Equations (1) and (10) imply that the dependent variable in (29) is

$$\frac{A_r(\tau-\Delta\tau)r_{t+\Delta\tau} + C(\tau-\Delta\tau)}{\tau-\Delta\tau} - \frac{A_r(\tau)r_t + C(\tau)}{\tau}$$

and the independent variable is

$$\frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)r_t+C(\tau)}{\tau}-\frac{A_r(\Delta\tau)r_t+C(\Delta\tau)}{\Delta\tau}\right].$$

Therefore, the CS regression coefficient is

$$b_{\mathrm{CS}}=\frac{\mathbb{C}\mathrm{ov}\left\{\frac{A_r(\tau-\Delta\tau)}{\tau-\Delta\tau}r_{t+\Delta\tau}-\frac{A_r(\tau)}{\tau}r_t,\frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)}{\tau}-\frac{A_r(\Delta\tau)}{\Delta\tau}\right]r_t\right\}}{\mathbb{V}\mathrm{ar}\left\{\frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)}{\tau}-\frac{A_r(\Delta\tau)}{\Delta\tau}\right]r_t\right\}}$$

$$=\frac{\frac{A_r(\tau-\Delta\tau)}{\tau-\Delta\tau}\mathbb{C}\mathrm{ov}(r_{t+\Delta\tau},r_t)-\frac{A_r(\tau)}{\tau}\mathbb{V}\mathrm{ar}(r_t)}{\frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)}{\tau}-\frac{A_r(\Delta\tau)}{\Delta\tau}\right]\mathbb{V}\mathrm{ar}(r_t)}. \tag{A.9}$$

Using (A.7), we can write (A.9) as

$$b_{\mathrm{CS}}=\frac{\frac{A_r(\tau-\Delta\tau)}{\tau-\Delta\tau}e^{-\kappa_r\Delta\tau}-\frac{A_r(\tau)}{\tau}}{\frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)}{\tau}-\frac{A_r(\Delta\tau)}{\Delta\tau}\right]}.$$

Taking the limit $\Delta\tau\to 0$, we find

$$b_{\mathrm{CS}}\to\frac{\frac{A_r(\tau)}{\tau}-[A_r'(\tau)+\kappa_r A_r(\tau)]}{\frac{A_r(\tau)}{\tau}-1}=1-\frac{A_r'(\tau)+\kappa_r A_r(\tau)-1}{\frac{A_r(\tau)}{\tau}-1}=1-\frac{(\kappa_r^*-\kappa_r)A_r(\tau)\tau}{\tau-A_r(\tau)}, \tag{A.10}$$

where the third step follows from (19) and (25). Since $\kappa_r^*>\kappa_r$ when $a>0$ and $\alpha(\tau)>0$ in a positive-measure subset of $(0,T)$, (A.10) implies $b_{\mathrm{CS}}<1$. Since

$$\frac{A_r(\tau)\tau}{\tau-A_r(\tau)}=\frac{1-e^{-\kappa_r^*\tau}}{\kappa_r^*\left(1-\frac{1-e^{-\kappa_r^*\tau}}{\kappa_r^*\tau}\right)},$$

(A.10) implies that b_{CS} increases in τ if the function

$$K(x)\equiv\frac{1-\frac{1-e^{-x}}{x}}{1-e^{-x}}=\frac{1}{1-e^{-x}}-\frac{1}{x}$$

is increasing for $x>0$. The derivative $K'(x)$ has the same sign as the function

$$\hat{K}(x)\equiv 1-e^{-x}-xe^{-\frac{x}{2}}.$$

The function $\hat{K}(x)$ is equal to zero for $x=0$, and its derivative $\hat{K}'(x)$ has the same sign as $e^{-\frac{x}{2}}-1+\frac{x}{2}$ which is positive for all x. Therefore, $\hat{K}(x)>0$ for $x>0$, and $K(x)$ is increasing. *Q.E.D.*

Proof of Proposition 4: The argument in the text shows that $\Delta y_t^{(\tau)} = \kappa_r^* \Delta \bar{r}^* \frac{\int_0^\tau A_r(u)du}{\tau}$ and $\Delta \bar{r}^*$ has the same sign as $a\sigma_r^2 \int_0^\infty \Delta\theta_0(\tau) A_r(\tau) d\tau$. Hence, when $a > 0$, the change $\Delta\theta_0(\tau)$ raises all yields if $\int_0^\infty \Delta\theta_0(\tau) A_r(\tau) d\tau > 0$ and lowers them otherwise. The relative effect across maturities is

$$\frac{\Delta y_t^{(\tau_2)}}{\Delta y_t^{(\tau_1)}} = \frac{\frac{\int_0^{\tau_2} A_r(u)du}{\tau_2}}{\frac{\int_0^{\tau_1} A_r(u)du}{\tau_1}},$$

and is independent of $\Delta\theta_0(\tau)$. Since the function $A_r(\tau)$ increases in τ, the function $\frac{\int_0^\tau A_r(u)du}{\tau}$ also increases, and hence the relative effect across maturities is larger than one for $\tau_1 < \tau_2$. *Q.E.D.*

Proof of Lemma 2: The proof is in the text. *Q.E.D.*

Proof of Lemma 3: Using the diagonalization

$$M = P^{-1} Diag(\nu_1, \nu_2, .., \nu_{K+1}) P,$$

where $Diag(z_1, z_2, .., z_N)$ is the $N \times N$ diagonal matrix with elements $(z_1, z_2, .., z_N)$, and multiplying the ODE system (36) from the left by P, we can write it as

$$PA'(\tau) + Diag(\nu_1, \nu_2, .., \nu_{K+1}) PA(\tau) - P\mathcal{E} = 0. \tag{A.11}$$

Integrating (A.11) with the initial condition $A(0) = 0$ yields

$$PA(\tau) = Diag\left(\frac{1 - e^{-\nu_1\tau}}{\nu_1}, \frac{1 - e^{-\nu_2\tau}}{\nu_2}, .., \frac{1 - e^{-\nu_{K+1}\tau}}{\nu_{K+1}}\right) P\mathcal{E}. \tag{A.12}$$

Using

$$Diag\left(\frac{1 - e^{-\nu_1\tau}}{\nu_1}, \frac{1 - e^{-\nu_2\tau}}{\nu_2}, .., \frac{1 - e^{-\nu_{K+1}\tau}}{\nu_{K+1}}\right)$$
$$= \frac{1 - e^{-\nu_1\tau}}{\nu_1}\mathcal{I}_{K+1} + Diag\left(0, \frac{1 - e^{-\nu_2\tau}}{\nu_2} - \frac{1 - e^{-\nu_1\tau}}{\nu_1}, .., \frac{1 - e^{-\nu_{K+1}\tau}}{\nu_{K+1}} - \frac{1 - e^{-\nu_1\tau}}{\nu_1}\right),$$

where $\mathcal{I}_N$ is the $N \times N$ identity matrix, we can write (A.12) as

$$A(\tau) = \frac{1 - e^{-\nu_1\tau}}{\nu_1}\mathcal{E} + P^{-1} Diag\left(0, \frac{1 - e^{-\nu_2\tau}}{\nu_2} - \frac{1 - e^{-\nu_1\tau}}{\nu_1}, .., \frac{1 - e^{-\nu_{K+1}\tau}}{\nu_{K+1}} - \frac{1 - e^{-\nu_1\tau}}{\nu_1}\right) P\mathcal{E}$$

$$\Rightarrow \begin{bmatrix} A_r(\tau) \\ A_{\beta,1}(\tau) \\ .. \\ A_{\beta,K}(\tau) \end{bmatrix} = \frac{1 - e^{-\nu_1\tau}}{\nu_1} \begin{bmatrix} 1 \\ 0 \\ .. \\ 0 \end{bmatrix}$$

$$+ P^{-1} Diag\left(0, \frac{1 - e^{-\nu_2\tau}}{\nu_2} - \frac{1 - e^{-\nu_1\tau}}{\nu_1}, .., \frac{1 - e^{-\nu_{K+1}\tau}}{\nu_{K+1}} - \frac{1 - e^{-\nu_1\tau}}{\nu_1}\right) P \begin{bmatrix} 1 \\ 0 \\ .. \\ 0 \end{bmatrix}. \tag{A.13}$$

Equation (A.13) implies (39) and (40). Integrating (38) with the initial condition $C(0) = 0$ yields (41). *Q.E.D.*

We next derive the system of equations in the Laplace transforms. We consider the general case where there are K demand factors. We assume $\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}$ and $\theta_k(\tau) = \sum_{n=1}^{N} \theta_{k,n} e^{-\delta_{\theta n} \tau}$, where $N \geq 1$, $(\alpha, \delta_\alpha, \{\theta_{k,n}\}_{k=1,..,K,\ n=1,..,N}, \{\delta_{\theta_n}\}_{n=1,..,N})$ are scalars and $(\alpha, \delta_\alpha, \{\delta_{\theta_n}\}_{n=1,..,N})$ are positive. We set

$$I \equiv \int_0^\infty \alpha(\tau) A(\tau) d\tau,$$

$$J \equiv \int_0^\infty \alpha(\tau) A(\tau) A(\tau)^\top d\tau,$$

For $n = 1, .., N$, we set

$$I_n \equiv \int_0^\infty e^{-\delta_{\theta n} \tau} A(\tau) d\tau,$$

and denote by Θ_n the $1 \times (K+1)$ vector $(0, \theta_{1,n}, .., \theta_{K,n})$. Since the vectors $(I, I_1, .., I_N)$ are $(K+1) \times 1$, and since the matrix J is $(K+1) \times (K+1)$ and symmetric, there are a total of

$$K + 1 + \frac{(K+1)(K+2)}{2} + (K+1)N = (K+1)\left(\frac{K}{2} + N + 2\right)$$

distinct elements. These elements are Laplace transforms of the functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..K})$ and of those functions' pairwise products. Using $(J, \{I_n\}_{n=1,..,N}, \{\Theta_n\}_{n=1,..,N})$, we can write the matrix M defined in (37) as

$$M \equiv \Gamma^\top - a\left(\sum_{n=1}^{N} \Theta_n^\top I_n^\top - J\right) \Sigma\Sigma^\top. \tag{A.14}$$

Lemma A.1 *Suppose that* $\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}$ *and* $\theta_k(\tau) = \sum_{j=1}^{N} \theta_{k,n} e^{-\delta_{\theta n} \tau}$, *where* $N \geq 1$, $(\alpha, \delta_\alpha, \{\theta_{k,n}\}_{k=1,..,K,\ n=1,..,N}, \{\delta_{\theta_n}\}_{n=1,..,N})$ *are scalars and* $(\alpha, \delta_\alpha, \{\delta_{\theta_n}\}_{n=1,..,N})$ *are positive. The* $(K+1)\left(\frac{K}{2} + N + 2\right)$ *elements of* $(I, J, \{I_n\}_{n=1,..,N})$ *solve the system of*

$$(\delta_\alpha I_{K+1} + M)\, I = \frac{\alpha}{\delta_\alpha} \mathcal{E}, \tag{A.15}$$

$$(\delta_{\theta_n} I_{K+1} + M)\, I_n = \frac{1}{\delta_{\theta_n}} \mathcal{E}, \tag{A.16}$$

for $n = 1, .., N$, *and*

$$(\delta_\alpha I_{K+1} + M)\, J + J M^\top = \mathcal{E} I^\top + I \mathcal{E}^\top. \tag{A.17}$$

Proof of Lemma A.1: To derive (A.15), we multiply the ODE system (36) by $\alpha(\tau)$ and integrate from zero to infinity. This yields

$$\int_0^\infty \alpha(\tau)A'(\tau)d\tau + MI - \left[\int_0^\infty \alpha(\tau)d\tau\right]\mathcal{E} = 0. \tag{A.18}$$

Integration by parts implies

$$\begin{aligned}\int_0^\infty \alpha(\tau)A'(\tau)d\tau &= [\alpha(\tau)A(\tau)]_0^\infty - \int_0^\infty \alpha'(\tau)A(\tau)d\tau \\ &= \lim_{\tau\to\infty}\alpha(\tau)A(\tau) - \alpha(0)A(0) + \delta_\alpha\int_0^\infty \alpha(\tau)A(\tau)d\tau \\ &= \lim_{\tau\to\infty}\alpha(\tau)A(\tau) + \delta_\alpha\int_0^\infty \alpha(\tau)A(\tau)d\tau,\end{aligned}$$

where the second step follows from $\alpha'(\tau) = -\delta_\alpha\alpha(\tau)$ and the third step from $A(0) = 0$. Assuming $\lim_{\tau\to\infty}\alpha(\tau)A(\tau) = 0$, a property that is required for the matrix M to be finite (and that holds for the solution in Theorem 1, as we show at the end of that theorem's proof), we find

$$\int_0^\infty \alpha(\tau)A'(\tau)d\tau = \delta_\alpha\int_0^\infty \alpha(\tau)A(\tau)d\tau = \delta_\alpha I. \tag{A.19}$$

Using (A.18), (A.19) and $a(\tau) = \alpha e^{-\delta_\alpha\tau}$, we find (A.15).

To derive (A.16), we likewise multiply the ODE system (36) by $e^{-\delta_{\theta_n}\tau}$ and integrate from zero to infinity. This yields

$$\int_0^\infty e^{-\delta_{\theta_n}\tau}A'(\tau)d\tau + MI_n - \left[\int_0^\infty e^{-\delta_{\theta_n}\tau}d\tau\right]\mathcal{E} = 0. \tag{A.20}$$

Integration by parts and a zero limit at infinity imply

$$\int_0^\infty e^{-\delta_{\theta_n}\tau}A'(\tau)d\tau = \delta_{\theta_n}\int_0^\infty e^{-\delta_{\theta_n}\tau}A(\tau)d\tau = \delta_{\theta_n}I_n. \tag{A.21}$$

Using (A.20) and (A.21), we find (A.16).

To derive (A.17), we multiply the ODE system (36) from the left by $\alpha(\tau)A(\tau)^\top$, add to the resulting $(K+1)\times(K+1)$ matrix its transpose, and integrate from zero to infinity. This yields

$$\int_0^\infty \alpha(\tau)\left[A'(\tau)A(\tau)^\top + A(\tau)A'(\tau)^\top\right]d\tau + MJ + JM^\top - \mathcal{E}I^\top - I\mathcal{E}^\top = 0. \tag{A.22}$$

Integration by parts and a zero limit at infinity imply

$$\int_0^\infty \alpha(\tau)\left[A'(\tau)A(\tau)^\top + A(\tau)A'(\tau)^\top\right]d\tau = \delta_\alpha\int_0^\infty \alpha(\tau)A(\tau)A(\tau)^\top d\tau = \delta_\alpha J. \tag{A.23}$$

Using (A.22) and (A.23), we find (A.17).

The total number of equations is $(K+1)\left(\frac{K}{2}+N+2\right)$, same as the number of unknown Laplace transforms: the vector equation (A.15) yields $K+1$ scalar equations, the vector equations (A.16) for $n=1,..,N$ yield $(K+1)N$ scalar equations, and the matrix equation (A.17) yields $\frac{(K+1)(K+2)}{2}$ scalar equations because the matrices in it are symmetric. *Q.E.D.*

Proof of Theorem 1: The theorem specializes Lemma A.1 to the case $K=1$, $N=2$, $\theta_{11}=-\theta_{12}=\theta$, $\delta_{\theta_1}=\delta_\alpha$, $\delta_{\theta_2}=\delta_\theta$, $\Gamma=Diag(\kappa_r,\kappa_\beta)$ and $\Sigma=Diag(\sigma_r,\sigma_\beta)$. Since $K=1$ and $N=2$, there are nine unknown Laplace transforms, which reduce to seven because $\delta_{\theta_1}=\delta_\alpha$ implies $I_1=\frac{I}{\alpha}$. Setting $I\equiv(I_r,I_\beta)^\top$, $I_2\equiv(I_{r,2},I_{\beta,2})^\top$ and

$$J\equiv\left[\begin{array}{cc} I_{r,r} & I_{r,\beta}\\ I_{r,\beta} & I_{\beta,\beta}\end{array}\right],$$

the seven unknown Laplace transforms are $(I_r,I_\beta,I_{r,2},I_{\beta,2},I_{r,r},I_{r,\beta},I_{\beta,\beta})$. Setting

$$\Delta I_{r,\theta}\equiv\theta\left(\frac{I_r}{\alpha}-I_{r,2}\right)-I_{r,\beta}, \tag{A.24}$$

$$\Delta I_{\beta,\theta}\equiv\theta\left(\frac{I_\beta}{\alpha}-I_{\beta,2}\right)-I_{\beta,\beta}, \tag{A.25}$$

we can write the matrix M given by (A.14) as

$$\left[\begin{array}{cc} \kappa_r+a\sigma_r^2 I_{r,r} & a\sigma_\beta^2 I_{r,\beta}\\ -a\sigma_r^2\Delta I_{r,\theta} & \kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\end{array}\right]. \tag{A.26}$$

The vector equation (A.15) yields the two scalar equations

$$\left(\delta_\alpha+\kappa_r+a\sigma_r^2 I_{r,r}\right)I_r+a\sigma_\beta^2 I_{r,\beta}I_\beta=\frac{\alpha}{\delta_\alpha}, \tag{A.27}$$

$$-a\sigma_r^2\Delta I_{r,\theta}I_r+\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)I_\beta=0. \tag{A.28}$$

The vector equation (A.16) yields the two scalar equations

$$\left(\delta_\theta+\kappa_r+a\sigma_r^2 I_{r,r}\right)I_{r,2}+a\sigma_\beta^2 I_{r,\beta}I_{\beta,2}=\frac{1}{\delta_\theta}, \tag{A.29}$$

$$-a\sigma_r^2\Delta I_{r,\theta}I_{r,2}+\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)I_{\beta,2}=0. \tag{A.30}$$

The matrix equation (A.17) yields the three scalar equations

$$\left(\frac{\delta_\alpha}{2}+\kappa_r+a\sigma_r^2 I_{r,r}\right)I_{r,r}+a\sigma_\beta^2 I_{r,\beta}^2=I_r, \tag{A.31}$$

$$\left(\delta_\alpha+\kappa_r+\kappa_\beta+a\sigma_r^2 I_{r,r}-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)I_{r,\beta}+a\sigma_\beta^2 I_{r,\beta}I_{\beta,\beta}-a\sigma_r^2\Delta I_{r,\theta}I_{r,r}=I_\beta, \tag{A.32}$$

$$-a\sigma_r^2\Delta I_{r,\theta}I_{r,\beta}+\left(\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)I_{\beta,\beta}=0. \tag{A.33}$$

Equations (A.27)-(A.32) constitute a system of seven equations in the seven unknowns $(I_r, I_\beta, I_{r,r}, I_{r,\beta}, I_{\beta,\beta}, I_{r,2}, I_{\beta,2})$. We next reduce this system into one of four equations in the four unknowns $(I_{r,r}, I_{r,\beta}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$.

The system of (A.27) and (A.28) is linear in (I_r, I_β) and its solution is

$$I_r = \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}, \tag{A.34}$$

$$I_\beta = \frac{\frac{\alpha}{\delta_\alpha} a\sigma_r^2 \Delta I_{r,\theta}}{(\delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}. \tag{A.35}$$

Likewise, the system of (A.29) and (A.30) is linear in $(I_{r,2}, I_{\beta,2})$ and its solution is

$$I_{r,2} = \frac{\frac{1}{\delta_\theta}\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\theta + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}, \tag{A.36}$$

$$I_{\beta,2} = \frac{\frac{1}{\delta_\theta} a\sigma_r^2 \Delta I_{r,\theta}}{(\delta_\theta + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}. \tag{A.37}$$

Equation (A.33) is linear in $I_{\beta,\beta}$ and its solution is

$$I_{\beta,\beta} = \frac{a\sigma_r^2 I_{r,\beta}\Delta I_{r,\theta}}{\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}}. \tag{A.38}$$

Substituting I_r from (A.34), we can write (A.31) as

$$\begin{aligned}&\left(\frac{\delta_\alpha}{2} + \kappa_r + a\sigma_r^2 I_{r,r}\right) I_{r,r} + a\sigma_\beta^2 I_{r,\beta}^2 \\ &- \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}} = 0.\end{aligned} \tag{A.39}$$

Substituting $(I_r, I_{r,2})$ from (A.34) and (A.36), respectively, into the definition (A.24) of $\Delta I_{r,\theta}$, we find

$$\begin{aligned}&\Delta I_{r,\theta} - \frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}} \\ &+ \frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\theta + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}} + I_{r,\beta} = 0.\end{aligned} \tag{A.40}$$

Substituting $(I_\beta, I_{\beta,2}, I_{\beta,\beta})$ from (A.34), (A.36) and (A.38), respectively, into the definition (A.25) of $\Delta I_{\beta,\theta}$, we find

$$\Delta I_{\beta,\theta} - \frac{\frac{\theta}{\delta_\alpha} a\sigma_r^2 \Delta I_{r,\theta}}{(\delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}$$

$$+ \frac{\frac{\theta}{\delta_\theta} a\sigma_r^2 \Delta I_{r,\theta}}{(\delta_\theta + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}} + \frac{a\sigma_r^2 I_{r,\beta}\Delta I_{r,\theta}}{\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}} = 0. \tag{A.41}$$

Substituting $(I_\beta, I_{\beta,\beta})$ from (A.34) and (A.38), respectively, we can write (A.31) as

$$\left(\delta_\alpha + \kappa_r + \kappa_\beta + a\sigma_r^2 I_{r,r} - a\sigma_\beta^2 \Delta I_{\beta,\theta} + a\sigma_\beta^2 \frac{a\sigma_r^2 I_{r,\beta}\Delta I_{r,\theta}}{\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}}\right) I_{r,\beta} - a\sigma_r^2 I_{r,r}\Delta I_{r,\theta}$$

$$- \frac{\frac{\alpha}{\delta_\alpha} a\sigma_r^2 \Delta I_{r,\theta}}{(\delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r})\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}} = 0. \tag{A.42}$$

Equations (A.39)-(A.42) form the system of four equations in the four unknowns $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta})$. Given a solution to that system, we can determine $(I_r, I_\beta, I_{r,2}, I_{\beta,2}, I_{\beta,\beta})$ from (A.34)-(A.38).

To show that the system (A.39)-(A.42) has a solution, we proceed in two steps. In Step 1 we take $I_{r,\beta} > 0$ as given, and construct $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$ and $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$ uniquely from (A.39)-(A.41). In Step 2 we treat $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$ as implicit functions of $I_{r,\beta}$, and show that (A.42) has a solution $I_{r,\beta} > 0$. We denote the left-hand sides of (A.39), (A.40), (A.41) and (A.42) by $L_{r,r}$, $L_{r,\theta}$, $L_{\beta,\theta}$ and $L_{r,\beta}$, respectively, and set

$$D_j \equiv \left(\delta_j + \kappa_r + a\sigma_r^2 I_{r,r}\right)\left(\delta_j + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}$$

for $j = \alpha, \theta$. For $I_{r,r} \geq 0$, $\Delta I_{r,\theta} \geq 0$, $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$ and $I_{r,\beta} > 0$, $D_\theta > D_\alpha > 0$, and hence $(L_{r,r}, L_{r,\theta}, L_{\beta,\theta}, L_{r,\beta})$ are continuous functions of $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta})$.

Step 1: We first take $\Delta I_{r,\theta} \geq 0$, $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$ and $I_{r,\beta} > 0$ as given, and construct $I_{r,r} > 0$ from (A.39). Equation (A.39) implies

$$\frac{\partial L_{r,r}}{\partial I_{r,r}} = \frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma_r^2 I_{r,r} + \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)^2}{D_\alpha^2} a\sigma_r^2, \tag{A.43}$$

which in turn implies $\frac{\partial L_{r,r}}{\partial I_{r,r}} > 0$ for $I_{r,r} \geq 0$. Hence, if $L_{r,r} < 0$ for $I_{r,r} = 0$, and $L_{r,r} > 0$ for $I_{r,r}$ large enough, then (A.39) has a unique positive solution for $I_{r,r}$. Equation (A.39) implies that $L_{r,r}$ converges to infinity when $I_{r,r}$ goes to infinity. We assume that $(\Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta})$ are such that $L_{r,r} < 0$ for $I_{r,r} = 0$, and return to this issue in Step 2.

We next take $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ and $I_{r,\beta} > 0$ as given, treat $I_{r,r} > 0$ as an implicit function of $(\Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta})$, and construct $\Delta I_{r,\theta} > 0$ from (A.40). Equation (A.40) implies that the partial derivative of $L_{r,\theta}$ with respect to $\Delta I_{r,\theta}$ when the variation of $I_{r,r}$ is taken into account is

$$\hat{L}_{r,\theta} \equiv \frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}}.$$

We show that if $L_{r,\theta} = 0$ for a value $\Delta I_{r,\theta} > 0$, then $\hat{L}_{r,\theta} > 0$ for the same value. Equation (A.40) implies

$$\frac{\partial L_{r,\theta}}{\partial I_{r,r}} = \left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)^2}{D_\alpha^2} - \frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)^2}{D_\theta^2} \right) a\sigma_r^2, \tag{A.44}$$

$$\frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} = 1 + \left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{D_\alpha^2} - \frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{D_\theta^2} \right) a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}. \tag{A.45}$$

Equation (A.39) implies

$$\frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} = \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{D_\alpha^2} a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}. \tag{A.46}$$

Since $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ and $I_{r,\beta} > 0$, (A.46) implies $\frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} > 0$ and hence

$$\frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} = -\frac{\frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}}}{\frac{\partial L_{r,r}}{\partial I_{r,r}}} < 0. \tag{A.47}$$

Combining (A.44) and (A.45) with

$$\begin{aligned}
&\frac{\frac{\theta}{\delta_j}\left(\delta_j + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)^2}{D_j^2} a\sigma_r^2 \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + \frac{\frac{\theta}{\delta_j}\left(\delta_j + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{D_j^2} a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta} \\
&= \frac{\frac{\theta}{\delta_j}\left(\delta_j + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{D_j^2} \left(\left(\delta_j + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) a\sigma_r^2 \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta} \right),
\end{aligned}$$

for $j = \alpha, \theta$,

$$\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) a\sigma_r^2 \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta} > \left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) a\sigma_r^2 \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta},$$

which follows from $\frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} < 0$, $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ and $\delta_\theta > \delta_\alpha$,

$$\begin{aligned}
&\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) a\sigma_r^2 \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta} \\
&= -\frac{\frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2} a^3\sigma_r^4\sigma_\beta^2 I_{r,\beta}}{\frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma_r^2 I_{r,r} + \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2} a\sigma_r^2} + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta} \\
&= a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\left(1 - \frac{\frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2} a\sigma_r^2}{\frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma_r^2 I_{r,r} + \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2} a\sigma_r^2}\right) > 0,
\end{aligned}$$

which follows from (A.43), (A.46) and (A.47), $D_\theta > D_\alpha > 0$, and

$$\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\alpha} - \frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\theta} = \Delta I_{r,\theta} + I_{r,\beta} > 0,$$

which follows from $L_{r,\theta} = 0$ (i.e., (A.40)), we find

$$\hat{L}_{r,\theta} = \frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} > 1 > 0. \tag{A.48}$$

Since $\hat{L}_{r,\theta} > 0$ at any point where $L_{r,\theta} = 0$, $L_{r,\theta}$ can be equal to zero only once. Hence, if $L_{r,\theta} < 0$ for $\Delta I_{r,\theta} = 0$, and $L_{r,\theta} > 0$ for $\Delta I_{r,\theta} = \overline{\Delta I}_{r,\theta}$ sufficiently large, and if all values of $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$ yield $I_{r,r} > 0$, then (A.40) yields a unique solution for $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$. We assume that $(\Delta I_{\beta,\theta}, I_{r,\beta})$ are such that these conditions hold, and return to this issue in Step 2.

We finally take $I_{r,\beta} > 0$ as given, treat $I_{r,r} > 0$ and $\Delta I_{r,\theta} > 0$ as implicit functions of $(\Delta I_{\beta,\theta}, I_{r,\beta})$, and construct $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ from (A.41). Equation (A.41) implies that the partial derivative of $L_{\beta,\theta}$ with respect to $\Delta I_{\beta,\theta}$ when the variation of $(I_{r,r}, \Delta I_{r,\theta})$ is taken into account is

$$\hat{L}_{\beta,\theta} \equiv \frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} \frac{\partial \Delta I_{r,\theta}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{\beta,\theta}}. \tag{A.49}$$

We show that if $L_{\beta,\theta} = 0$ for a value $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$, then $\hat{L}_{\beta,\theta} > 0$ for the same value. Differentiating (A.39) and (A.40) at the values of $(I_{r,r}, \Delta I_{r,\theta})$ that render $(L_{r,r}, L_{r,\theta})$ equal to zero, we find

$$\frac{\partial L_{r,r}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} \frac{\partial \Delta I_{r,\theta}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{r,r}}{\partial \Delta I_{\beta,\theta}} = 0, \tag{A.50}$$

$$\frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} \frac{\partial \Delta I_{r,\theta}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{r,\theta}}{\partial \Delta I_{\beta,\theta}} = 0, \tag{A.51}$$

respectively. Equations (A.50) and (A.51) form a linear system in the unknowns $\left(\frac{\partial I_{r,r}}{\partial \Delta I_{\beta,\theta}}, \frac{\partial \Delta I_{r,\theta}}{\partial \Delta I_{\beta,\theta}}\right)$. The determinant of that system is

$$\frac{\partial L_{r,r}}{\partial I_{r,r}}\frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} - \frac{\partial L_{r,\theta}}{\partial I_{r,r}}\frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} = \frac{\partial L_{r,r}}{\partial I_{r,r}}\left(\frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} - \frac{\partial L_{r,\theta}}{\partial I_{r,r}}\frac{\frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}}}{\frac{\partial L_{r,r}}{\partial I_{r,r}}}\right)$$

$$= \frac{\partial L_{r,r}}{\partial I_{r,r}}\left(\frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} + \frac{\partial L_{r,\theta}}{\partial I_{r,r}}\frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}}\right)$$

$$= \frac{\partial L_{r,r}}{\partial I_{r,r}}\hat{L}_{r,\theta},$$

and is positive because $\frac{\partial L_{r,r}}{\partial I_{r,r}} > 0$ and $\hat{L}_{r,\theta} > 0$. Substituting the solution of the system (A.50)-(A.51) into (A.49), we find that (A.49) has the same sign as the Jacobian determinant

$$\begin{vmatrix} \frac{\partial L_{r,r}}{\partial I_{r,r}} & \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} & \frac{\partial L_{r,r}}{\partial \Delta I_{\beta,\theta}} \\ \frac{\partial L_{r,\theta}}{\partial I_{r,r}} & \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} & \frac{\partial L_{r,\theta}}{\partial \Delta I_{\beta,\theta}} \\ \frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} & \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} & \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{\beta,\theta}} \end{vmatrix}. \tag{A.52}$$

The partial derivatives $(\frac{\partial L_{r,r}}{\partial I_{r,r}}, \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}}, \frac{\partial L_{r,\theta}}{\partial I_{r,r}}, \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}})$ are given by (A.43), (A.46), (A.44) and (A.45), respectively. Equations (A.39), (A.40) and (A.41) imply that the remaining partial derivatives are

$$\frac{\partial L_{r,r}}{\partial \Delta I_{\beta,\theta}} = \frac{\frac{\alpha}{\delta_\alpha}}{D_\alpha^2} a^3\sigma_r^2\sigma_\beta^4 I_{r,\beta}\Delta I_{r,\theta}, \tag{A.53}$$

$$\frac{\partial L_{r,\theta}}{\partial \Delta I_{\beta,\theta}} = \left(\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha^2} - \frac{\frac{\theta}{\delta_\theta}}{D_\theta^2}\right) a^3\sigma_r^2\sigma_\beta^4 I_{r,\beta}\Delta I_{r,\theta}, \tag{A.54}$$

$$\frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} = \left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\alpha^2} - \frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\theta^2}\right) a^2\sigma_r^4\Delta I_{r,\theta}, \tag{A.55}$$

$$\frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} = \left(-\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha} + \frac{\frac{\theta}{\delta_\theta}}{D_\theta} + \frac{I_{r,\beta}}{\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}}\right) a\sigma_r^2 + \left(\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha^2} - \frac{\frac{\theta}{\delta_\theta}}{D_\theta^2}\right) a^3\sigma_r^4\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}, \tag{A.56}$$

$$\begin{aligned}\frac{\partial L_{\beta,\theta}}{\partial \Delta I_{\beta,\theta}} = 1 &- \left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r}\right)}{D_\alpha^2} - \frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta + \kappa_r + a\sigma_r^2 I_{r,r}\right)}{D_\theta^2}\right) a^2\sigma_r^2\sigma_\beta^2\Delta I_{r,\theta} \\ &+ \frac{a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}{\left(\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}.\end{aligned} \tag{A.57}$$

The sign of the Jacobian determinant (A.52) does not change if we multiply the last row by $\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}\right)$. The resulting determinant does not change if we subtract the middle row times $a\sigma_r^2\Delta I_{r,\theta}$ from the last row, and then the first row times $\frac{\theta}{\alpha}\left(1-\frac{\delta_\alpha D_\alpha^2}{\delta_\theta D_\theta^2}\right)$ from the middle row. In the resulting determinant, the elements (1,1), (1,2) and (1,3) are given by (A.43), (A.46) and (A.53), respectively, the element (2,1) is given by

$$\left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\theta^2}\right)a\sigma_r^2$$

$$-\frac{\theta}{\alpha}\left(1-\frac{\delta_\alpha D_\alpha^2}{\delta_\theta D_\theta^2}\right)\left(\frac{\delta_\alpha}{2}+\kappa_r+2a\sigma_r^2 I_{r,r}+\frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2}a\sigma_r^2\right)$$

$$=-\frac{\frac{\theta}{\delta_\theta}(\delta_\theta-\delta_\alpha)\left(\delta_\theta+\delta_\alpha+2\kappa_\beta-2a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\theta^2}a\sigma_r^2-\frac{\theta}{\alpha}\left(1-\frac{\delta_\alpha D_\alpha^2}{\delta_\theta D_\theta^2}\right)\left(\frac{\delta_\alpha}{2}+\kappa_r+2a\sigma_r^2 I_{r,r}\right),$$

the element (2,2) by

$$1+\left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\theta^2}\right)a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}$$

$$-\frac{\theta}{\alpha}\left(1-\frac{\delta_\alpha D_\alpha^2}{\delta_\theta D_\theta^2}\right)\frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\alpha^2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}=1-\frac{\frac{\theta}{\delta_\theta}(\delta_\theta-\delta_\alpha)}{D_\theta^2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta},$$

the element (2,3) by

$$\left(\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}}{D_\theta^2}\right)a^3\sigma_r^2\sigma_\beta^4 I_{r,\beta}\Delta I_{r,\theta}-\frac{\theta}{\alpha}\left(1-\frac{\delta_\alpha D_\alpha^2}{\delta_\theta D_\theta^2}\right)\frac{\frac{\alpha}{\delta_\alpha}}{D_\alpha^2}a^3\sigma_r^2\sigma_\beta^4 I_{r,\beta}\Delta I_{r,\theta}=0,$$

the element (3,1) by

$$\left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\theta^2}\right)a^2\sigma_r^4\Delta I_{r,\theta}$$

$$-\left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\theta^2}\right)a^2\sigma_r^4\Delta I_{r,\theta}$$

$$=\frac{\frac{\theta}{\delta_\theta}(\delta_\theta-\delta_\alpha)\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\theta^2}a^2\sigma_r^4\Delta I_{r,\theta},$$

the element (3,2) by

$$\left[\left(-\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha}+\frac{\frac{\theta}{\delta_\theta}}{D_\theta}+\frac{I_{r,\beta}}{\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}}\right)a\sigma_r^2+\left(\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}}{D_\theta^2}\right)a^3\sigma_r^4\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}\right]\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)$$

$$-\left[1+\left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\theta^2}\right)a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\right]a\sigma_r^2\Delta I_{r,\theta}$$

$$=-a\sigma_r^2\Delta I_{r,\theta}+\frac{\frac{\theta}{\delta_\theta}(\delta_\theta-\delta_\alpha)}{D_\theta^2}a^3\sigma_r^4\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}-\frac{\Delta I_{\beta,\theta}}{\Delta I_{r,\theta}}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right),$$

where we use $L_{\beta,\theta}=0$ (i.e., (A.41)), and the element (3,3) by

$$\left[1-\left(\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha+\kappa_r+a\sigma_r^2 I_{r,r}\right)}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta+\kappa_r+a\sigma_r^2 I_{r,r}\right)}{D_\theta^2}\right)a^2\sigma_r^2\sigma_\beta^2\Delta I_{r,\theta}\right]\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)$$

$$+\frac{\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}{\left(\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}-\left(\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha^2}-\frac{\frac{\theta}{\delta_\theta}}{D_\theta^2}\right)a^4\sigma_r^4\sigma_\beta^4 I_{r,\beta}\Delta I_{r,\theta}^2$$

$$=\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}-\left(\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha}-\frac{\frac{\theta}{\delta_\theta}}{D_\theta}\right)a^2\sigma_r^2\sigma_\beta^2\Delta I_{r,\theta}-\frac{\frac{\theta}{\delta_\theta}(\delta_\theta-\delta_\alpha)\left(\delta_\theta+\kappa_r+a\sigma_r^2 I_{r,r}\right)}{D_\theta^2}a^2\sigma_r^2\sigma_\beta^2\Delta I_{r,\theta}$$

$$+\frac{\frac{\delta_\alpha}{2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}{\left(\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}+\frac{a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}{\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}}$$

$$=\delta_\alpha+\kappa_\beta-2a\sigma_\beta^2\Delta I_{\beta,\theta}-\frac{\frac{\theta}{\delta_\theta}(\delta_\theta-\delta_\alpha)\left(\delta_\theta+\kappa_r+a\sigma_r^2 I_{r,r}\right)}{D_\theta^2}a^2\sigma_r^2\sigma_\beta^2\Delta I_{r,\theta}+\frac{\frac{\delta_\alpha}{2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}{\left(\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2},$$

where the last step follows from $L_{\beta,\theta}=0$.

For large δ_θ, all the terms with D_θ in the denominator are close to zero, and the determinant obtained by multiplying (A.52) by $\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)$ becomes

$$\begin{vmatrix} \begin{array}{c}\frac{\delta_\alpha}{2}+\kappa_r+2a\sigma_r^2 I_{r,r}\\ +\frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2}a\sigma_r^2\end{array} & \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\alpha^2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta} & \frac{\frac{\alpha}{\delta_\alpha}}{D_\alpha^2}a^3\sigma_r^2\sigma_\beta^4 I_{r,\beta}\Delta I_{r,\theta} \\ -\frac{\theta}{\alpha}\left(\frac{\delta_\alpha}{2}+\kappa_r+2a\sigma_r^2 I_{r,r}\right) & 1 & 0 \\ 0 & -a\sigma_r^2\Delta I_{r,\theta}-\frac{\Delta I_{\beta,\theta}}{\Delta I_{r,\theta}}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right) & \begin{array}{c}\delta_\alpha+\kappa_\beta-2a\sigma_\beta^2\Delta I_{\beta,\theta}\\ +\frac{\frac{\delta_\alpha}{2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}{\left(\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}\end{array}\end{vmatrix}$$

$$\begin{aligned}&=\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha^2}\left(\frac{\delta_\alpha}{2}+\kappa_r+2a\sigma_r^2 I_{r,r}\right)\left(a\sigma_r^2\Delta I_{r,\theta}+\frac{\Delta I_{\beta,\theta}}{\Delta I_{r,\theta}}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)\right)a^3\sigma_r^2\sigma_\beta^4 I_{r,\beta}\Delta I_{r,\theta}\\ &\quad+\left(\delta_\alpha+\kappa_\beta-2a\sigma_\beta^2\Delta I_{\beta,\theta}+\frac{\frac{\delta_\alpha}{2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}{\left(\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}\right)\left[\frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2}a\sigma_r^2\right.\\ &\quad\left.+\left(\frac{\delta_\alpha}{2}+\kappa_r+2a\sigma_r^2 I_{r,r}\right)\left(1+\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\alpha^2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\right)\right]. \end{aligned}\tag{A.58}$$

To show that (A.58) is positive, and hence $\hat{L}_{\beta,\theta}>0$, we distinguish cases. When $\Delta I_{\beta,\theta}<0$, the only negative term in (A.58) is the one generated by $\frac{\Delta I_{\beta,\theta}}{\Delta I_{r,\theta}}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)$. We group it together with the term generated by one of the two $-a\sigma_\beta^2\Delta I_{\beta,\theta}$ in $\left(\delta_\alpha+\kappa_\beta-2a\sigma_\beta^2\Delta I_{\beta,\theta}\right)$ and note that (A.58) exceeds

$$\begin{aligned}&\frac{\frac{\theta}{\delta_\alpha}}{D_\alpha^2}\left(\frac{\delta_\alpha}{2}+\kappa_r+2a\sigma_r^2 I_{r,r}\right)a^4\sigma_r^4\sigma_\beta^4 I_{r,\beta}\Delta I_{r,\theta}^2\\ &+\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}+\frac{\frac{\delta_\alpha}{2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}}{\left(\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}\right)\left[\frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)^2}{D_\alpha^2}a\sigma_r^2\right.\\ &\left.+\left(\frac{\delta_\alpha}{2}+\kappa_r+2a\sigma_r^2 I_{r,r}\right)\left(1+\frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{D_\alpha^2}a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\right)\right],\end{aligned}$$

which is positive. When instead $\Delta I_{\beta,\theta}\in\left(0,\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}\right)$, all the terms in (A.58), with $\left(\delta_\alpha+\kappa_\beta-2a\sigma_\beta^2\Delta I_{\beta,\theta}\right)$ counted as a single term, are positive. Hence, $\hat{L}_{\beta,\theta}>0$ at any point $\Delta I_{\beta,\theta}<\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ where $L_{\beta,\theta}=0$, which implies that $L_{\beta,\theta}$ can be equal to zero only once. Moreover, if $L_{\beta,\theta}<0$ for $\Delta I_{\beta,\theta}=\underline{\Delta I}_{\beta,\theta}$ sufficiently negative, and $L_{\beta,\theta}>0$ for $\Delta I_{\beta,\theta}=\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$, and if all values of $\Delta I_{\beta,\theta}\in\left(\underline{\Delta I}_{\beta,\theta},\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}\right)$ yield $I_{r,r}>0$ and $\Delta I_{r,\theta}>0$, then (A.40) yields a unique solution for $\Delta I_{\beta,\theta}\in\left(\underline{\Delta I}_{\beta,\theta},\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}\right)$. We assume that $I_{r,\beta}$ is such that these conditions hold, and return to this issue in Step 2.

Step 2: Suppose that $I_{r,\beta} > 0$ satisfies

$$a\sigma_\beta^2 I_{r,\beta}^2 < \frac{\frac{\alpha}{\delta_\alpha}}{\delta_\alpha + \kappa_r} \tag{A.59}$$

and define $\bar{I}_{r,r} > 0$ by

$$\left(\frac{\delta_\alpha}{2} + \kappa_r + a\sigma_r \bar{I}_{r,r}\right)\bar{I}_{r,r} + a\sigma_\beta^2 I_{r,\beta}^2 - \frac{\frac{\alpha}{\delta_\alpha}}{\delta_\alpha + \kappa_r + a\sigma_r^2 \bar{I}_{r,r}} = 0. \tag{A.60}$$

Equation (A.60) defines $\bar{I}_{r,r} > 0$ uniquely because the left-hand side increases for $I_{r,r} \geq 0$, converges to infinity when $\bar{I}_{r,r}$ goes to infinity, and is negative for $\bar{I}_{r,r} = 0$ because of (A.59). Suppose that $I_{r,\beta}$ satisfies additionally

$$I_{r,\beta} < \frac{\frac{\theta}{\delta_\alpha}}{\delta_\alpha + \kappa_r + a\sigma_r^2 \bar{I}_{r,r}} - \frac{\frac{\theta}{\delta_\theta}}{\delta_\theta + \kappa_r + a\sigma_r^2 \bar{I}_{r,r}}, \tag{A.61}$$

$$a\sigma_\beta^2 I_{r,\beta} < \frac{\alpha}{\theta}. \tag{A.62}$$

We can then construct $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$ and $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$ uniquely, following the procedure in Step 1. That procedure assumes some of the boundary conditions, which we next prove using (A.59), (A.61) and (A.62).

Take first $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$, $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$ and $I_{r,\beta} > 0$ as given, where $\overline{\Delta I}_{r,\theta} > 0$ is defined by

$$a\sigma_\beta^2 I_{r,\beta}^2 = \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\alpha + \kappa_r)\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\overline{\Delta I}_{r,\theta}} \tag{A.63}$$

and is positive because of (A.59). Equation (A.39) implies that for $I_{r,r} = 0$,

$$L_{r,r} = a\sigma_\beta^2 I_{r,\beta}^2 - \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\alpha + \kappa_r)\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}} < 0,$$

where the inequality follows from $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$ and (A.63). Equation (A.39) and (A.60) imply that for $I_{r,r} = \bar{I}_{r,r}$,

$$\begin{aligned} L_{r,r} &= \left(\frac{\delta_\alpha}{2} + \kappa_r + a\sigma_r \bar{I}_{r,r}\right)\bar{I}_{r,r} + a\sigma_\beta^2 I_{r,\beta}^2 \\ &\quad - \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{\left(\delta_\alpha + \kappa_r + a\sigma_r^2 \bar{I}_{r,r}\right)\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}} \\ &= \frac{\frac{\alpha}{\delta_\alpha}}{\delta_\alpha + \kappa_r + a\sigma_r^2 \bar{I}_{r,r}} - \frac{\frac{\alpha}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{\left(\delta_\alpha + \kappa_r + a\sigma_r^2 \bar{I}_{r,r}\right)\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}} > 0. \end{aligned}$$

Hence (A.39) has a unique positive solution for $I_{r,r} \in (0, \bar{I}_{r,r})$.

Take next $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ and $I_{r,\beta} > 0$ as given, and treat $I_{r,r} \in (0, \bar{I}_{r,r})$ as an implicit function of $(\Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta})$. For $\Delta I_{r,\theta} = 0$, (A.39) and (A.60) imply $I_{r,r} = \bar{I}_{r,r}$, and (A.40) implies

$$L_{r,\theta} = -\frac{\frac{\theta}{\delta_\alpha}}{\delta_\alpha + \kappa_r + a\sigma_r^2 \bar{I}_{r,r}} + \frac{\frac{\theta}{\delta_\theta}}{\delta_\theta + \kappa_r + a\sigma_r^2 \bar{I}_{r,r}} + I_{r,\beta} < 0,$$

where the inequality follows from (A.61). For $\Delta I_{r,\theta} = \overline{\Delta I}_{r,\theta}$, (A.39) and (A.63) imply $I_{r,r} = 0$, and (A.40) implies

$$\begin{aligned}
L_{r,\theta} &= \overline{\Delta I}_{r,\theta} - \frac{\frac{\theta}{\delta_\alpha}\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\alpha + \kappa_r)\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\overline{\Delta I}_{r,\theta}} \\
&\quad + \frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\theta + \kappa_r)\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\overline{\Delta I}_{r,\theta}} + I_{r,\beta} \\
&= \overline{\Delta I}_{r,\theta} - \frac{\theta}{\alpha} a\sigma_\beta^2 I_{r,\beta}^2 + \frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{(\delta_\theta + \kappa_r)\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\overline{\Delta I}_{r,\theta}} + I_{r,\beta} \\
&> I_{r,\beta}\left(1 - \frac{\theta}{\alpha} a\sigma_\beta^2 I_{r,\beta}\right) > 0, \qquad \text{(A.64)}
\end{aligned}$$

where the second step follows from (A.63) and the fourth from (A.62). Hence, (A.40) has a unique solution for $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$.

Take finally $I_{r,\beta} > 0$ as given, and treat $I_{r,r} \in (0, \bar{I}_{r,r})$ and $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$ as implicit functions of $(\Delta I_{\beta,\theta}, I_{r,\beta})$. When $\Delta I_{\beta,\theta}$ goes to minus infinity, (A.63) implies that $\frac{\overline{\Delta I}_{r,\theta}}{\delta_\alpha+\kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}}$ converges to a positive limit. Since, in addition, $\bar{I}_{r,r}$ is independent of $\Delta I_{\beta,\theta}$, $I_{r,r} \in (0, \bar{I}_{r,r})$ and $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$, (A.41) implies that $L_{\beta,\theta}$ converges to minus infinity. We next determine conditions so that $L_{\beta,\theta} > 0$ for $\Delta I_{\beta,\theta} = \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$. Equations (A.40) and (A.41) imply

$$\begin{aligned}
L_{\beta,\theta} &= \Delta I_{\beta,\theta} - \frac{a\sigma_r^2(\Delta I_{r,\theta} + I_{r,\beta})\Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}} \\
&\quad - \frac{\frac{\theta}{\delta_\theta}(\delta_\theta - \delta_\alpha)\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) D_\theta} + \frac{a\sigma_r^2 I_{r,\beta}\Delta I_{r,\theta}}{\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}} \\
&= \Delta I_{\beta,\theta} - \frac{a\sigma_r^2 \Delta I_{r,\theta}^2}{\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}} \\
&\quad - \frac{\frac{\theta}{\delta_\theta}(\delta_\theta - \delta_\alpha)\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}{\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right) D_\theta} + \frac{\frac{\delta_\alpha}{2} a\sigma_r^2 I_{r,\beta}\Delta I_{r,\theta}}{\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)\left(\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)}.
\end{aligned}$$

Hence, $L_{\beta,\theta} > 0$ for large δ_θ if

$$\Delta I_{\beta,\theta} - \frac{a\sigma_r^2 \Delta I_{r,\theta}^2}{\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}} + \frac{\frac{\delta_\alpha}{2} a\sigma_r^2 I_{r,\beta} \Delta I_{r,\theta}}{\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)\left(\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)} > 0. \tag{A.65}$$

Setting $\Delta I_{\beta,\theta} = \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$ in (A.65), we can write it as

$$\frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2} - \frac{a\sigma_r^2 \Delta I_{r,\theta}^2}{\frac{\delta_\alpha + \kappa_\beta}{2}} + \frac{\frac{\delta_\alpha}{2} a\sigma_r^2 I_{r,\beta} \Delta I_{r,\theta}}{\frac{\kappa_\beta(\delta_\alpha + \kappa_\beta)}{4}} > 0. \tag{A.66}$$

Equation (A.66) is satisfied for $\kappa_\beta \approx 0$. It is also satisfied for a general value of κ_β if

$$\frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2} - \frac{a\sigma_r^2 \left(\frac{\frac{\theta}{\delta_\theta}}{\delta_\theta + \kappa_r}\right)^2}{\frac{\delta_\alpha + \kappa_\beta}{2}} > 0 \Leftrightarrow \delta_\alpha(\delta_\alpha + \kappa_r)(\delta_\alpha + \kappa_\beta) > 2a\theta\sigma_r\sigma_\beta, \tag{A.67}$$

which follows from (A.66) by noting that (A.40) implies $\Delta I_{r,\theta} < \frac{\frac{\theta}{\delta_\theta}}{\delta_\theta + \kappa_r}$. Under either $\kappa_\beta \approx 0$ or $\delta_\alpha(\delta_\alpha + \kappa_r)(\delta_\alpha + \kappa_\beta) > 2a\theta\sigma_r\sigma_\beta$, $L_{\beta,\theta} > 0$ for $\Delta I_{\beta,\theta} = \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$, and hence (A.41) has a unique solution for $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$.

Inequalities (A.59), (A.61) and (A.62) hold for $I_{r,\beta}$ close to zero. Consider the largest value $\bar{I}_{r,\beta}$ such that (A.59), (A.61) and (A.62) hold for all $I_{r,\beta} < \bar{I}_{r,\beta}$. The implicit function theorem ensures that the functions $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$ are continuous in $I_{r,\beta} \le \bar{I}_{r,\beta}$. For $I_{r,\beta}$ close to zero, (A.39) and (A.40) imply that $I_{r,r}$ and $\Delta I_{r,\theta}$ are bounded away from zero. Since, in addition, $\Delta I_{\beta,\theta}$ is bounded above by $\frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta^2}$, (A.42) implies $L_{r,\beta} < 0$. We next determine a value $I_{r,\beta}^* \ge \bar{I}_{r,\beta}$ such that $L_{r,\beta} > 0$ (and such that $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$ are well-defined and continuous in $I_{r,\beta} \in (\bar{I}_{r,\beta}, I_{r,\beta}^*]$). Continuity then ensures that a solution $I_{r,\beta} < I_{r,\beta}^*$ to (A.42) exists, and hence a solution $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta})$ to the system (A.39)-(A.42) also exists.

The inequality among (A.59), (A.61) and (A.62) that switches to an equality at $\bar{I}_{r,\beta}$ cannot be (A.59). Indeed, if (A.59) switches to an equality at $\bar{I}_{r,\beta}$, then (A.60) implies $\bar{I}_{r,r} = 0$, and (A.61) becomes

$$\bar{I}_{r,\beta} < \frac{\frac{\theta}{\delta_\alpha}}{\delta_\alpha + \kappa_r} - \frac{\frac{\theta}{\delta_\theta}}{\delta_\theta + \kappa_r}. \tag{A.68}$$

Multiplying (A.62) by (A.68), we find

$$a\sigma_\beta^2 \bar{I}_{r,\beta}^2 < \frac{\frac{\alpha}{\delta_\alpha}}{\delta_\alpha + \kappa_r} - \frac{\frac{\alpha}{\delta_\theta}}{\delta_\theta + \kappa_r} < \frac{\frac{\alpha}{\delta_\alpha}}{\delta_\alpha + \kappa_r},$$

which implies that (A.59) holds, a contradiction.

If (A.61) switches to an equality at $\bar{I}_{r,\beta}$, then $L_{r,\theta}=0$ for $\Delta I_{r,\theta}=0$, and hence the solution to (A.40) is $\Delta I_{r,\theta}=0$. Equation (A.42) then implies $L_{r,\beta}>0$ for $I_{r,\beta}=\bar{I}_{r,\beta}=\bar{I}^*_{r,\beta}$.

Suppose instead that (A.62) switches to an equality at $\bar{I}_{r,\beta}$. Consider a value of $I_{r,\beta}>\bar{I}_{r,\beta}=\frac{\alpha}{\theta a\sigma_\beta^2}$ such that (A.59) and (A.61) hold. Define $\overline{\Delta I}_{r,\theta}>0$ by (A.63) and consider the set of $\Delta I_{\beta,\theta}<\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ such that $L_{r,\theta}>0$ for $\Delta I_{r,\theta}=\overline{\Delta I}_{r,\theta}$. Proceeding as in (A.64) and substituting $\overline{\Delta I}_{r,\theta}$ from (A.63), we can write the condition defining that set as

$$\frac{\frac{\alpha}{\delta_\alpha}-(\delta_\alpha+\kappa_r)a\sigma_\beta^2 I_{r,\beta}^2}{a^3\sigma_r^2\sigma_\beta^4 I_{r,\beta}^3}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)+I_{r,\beta}\left(1-\frac{\theta}{\alpha}a\sigma_\beta^2 I_{r,\beta}\right)$$

$$+\frac{\frac{\theta}{\delta_\theta}\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}{(\delta_\theta+\kappa_\beta)\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)+\frac{\frac{\alpha}{\delta_\alpha}-(\delta_\alpha+\kappa_r)a\sigma_\beta^2 I_{r,\beta}^2}{a\sigma_\beta^2 I_{r,\beta}^2}\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)}>0. \tag{A.69}$$

If (A.69) holds for all $\Delta I_{\beta,\theta}<\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$, then we can proceed as in the case where (A.59), (A.61) and (A.62) hold, and construct $I_{r,r}>0$, $\Delta I_{r,\theta}>0$ and $\Delta I_{\beta,\theta}<\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ uniquely. Denote by $\bar{I}'_{r,\beta}>\bar{I}_{r,\beta}$ the maximum value of $I_{r,\beta}$ such that (A.69) holds for all $\Delta I_{\beta,\theta}<\frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ and for all $I_{r,\beta}\in[\bar{I}_{r,\beta},\bar{I}'_{r,\beta})$.

If (A.61) switches to an equality at $\bar{I}''_{r,\beta}\in(\bar{I}_{r,\beta},\bar{I}'_{r,\beta}]$ and (A.59) holds for all $I_{r,\beta}\in[\bar{I}_{r,\beta},\bar{I}''_{r,\beta}]$, then $(I_{r,r},\Delta I_{r,\theta},\Delta I_{\beta,\theta})$ are well-defined and continuous in $I_{r,\beta}\in[\bar{I}_{r,\beta},\bar{I}''_{r,\beta}]$ and $L_{r,\beta}>0$ for $I_{r,\beta}=\bar{I}''_{r,\beta}=I^*_{r,\beta}$.

Suppose instead that (A.61) holds for all $I_{r,\beta}\in[\bar{I}_{r,\beta},\bar{I}'_{r,\beta}]$. Then (A.59) also holds for all $I_{r,\beta}\in[\bar{I}_{r,\beta},\bar{I}'_{r,\beta}]$. Indeed, if (A.59) switches to an equality at $\bar{I}''_{r,\beta}\in(\bar{I}_{r,\beta},\bar{I}'_{r,\beta}]$, then (A.60) implies $\bar{I}_{r,r}=0$, and (A.64) implies

$$\bar{I}''_{r,\beta}\left(1-\frac{\theta}{\alpha}a\sigma_\beta^2\bar{I}''_{r,\beta}\right)+\frac{\frac{\theta}{\delta_\theta}}{\delta_\theta+\kappa_\beta}>0$$

$$\Rightarrow \bar{I}''_{r,\beta}-\frac{\theta}{\alpha}a\sigma_\beta^2\left(\bar{I}'''_{r,\beta}\right)^2+\frac{\frac{\theta}{\delta_\theta}}{\delta_\theta+\kappa_\beta}>0$$

$$\Rightarrow \bar{I}''_{r,\beta}-\frac{\frac{\theta}{\delta_\alpha}}{\delta_\alpha+\kappa_\beta}+\frac{\frac{\theta}{\delta_\theta}}{\delta_\theta+\kappa_\beta}>0, \tag{A.70}$$

where the first and third steps follow from (A.59) switching to an equality at $\bar{I}''_{r,\beta}$. Hence, (A.61) holds in the opposite direction, a contradiction. Since (A.59) and (A.61) hold for all $I_{r,\beta}\in[\bar{I}_{r,\beta},\bar{I}'_{r,\beta}]$, $(I_{r,r},\Delta I_{r,\theta},\Delta I_{\beta,\theta})$ are well-defined and continuous in $I_{r,\beta}\in[\bar{I}_{r,\beta},\bar{I}'_{r,\beta}]$. For $I_{r,\beta}=\bar{I}'_{r,\beta}$, (A.64) switches to an equality for a single value $\underline{\Delta I}_{\beta,\theta}$. (Since the left-hand side is convex in $\Delta I_{\beta,\theta}$, if (A.64) switches to an equality for two values of $\Delta I_{\beta,\theta}$, then it switches to an inequality in the opposite direction for values of $\Delta I_{\beta,\theta}$ in-between, which contradicts the definition of $\bar{I}'_{r,\beta}$.) Suppose without loss of generality that the solution $\Delta I_{\beta,\theta}$

is to the right of $\underline{\Delta I}_{\beta,\theta}$, in which case $L_{\beta,\theta} < 0$ for $\Delta I_{\beta,\theta} = \underline{\Delta I}_{\beta,\theta}$. Consider a value of $I_{r,\beta} > \bar{I}'_{r,\beta}$ such that (A.59) and (A.61) hold, and denote by $\underline{\Delta I}_{\beta,\theta}$ the minimum value of $\Delta I_{\beta,\theta}$ such that (A.69) holds for all $\Delta I_{\beta,\theta} \in \left(\underline{\Delta I}_{\beta,\theta}, \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}\right)$. Proceeding as in the case where (A.59), (A.61) and (A.62) hold, we can construct $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$ and $\Delta I_{\beta,\theta} \in \left(\underline{\Delta I}_{\beta,\theta}, \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}\right)$ uniquely. Consider the largest value $\bar{I}'''_{r,\beta} > \bar{I}'_{r,\beta}$ such that for all $I_{r,\beta} \in [\bar{I}'_{r,\beta}, \bar{I}'''_{r,\beta})$, (A.59) and (A.61) hold and $L_{\beta,\theta} < 0$ for $\Delta I_{\beta,\theta} = \underline{\Delta I}_{\beta,\theta}$. The functions $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$ are well-defined and continuous in $I_{r,\beta} \in (\bar{I}'_{r,\beta}, \bar{I}'''_{r,\beta}]$. The same argument as in (A.70) implies that the inequality among (A.59), (A.61) and $L_{\beta,\theta} < 0$ for $\Delta I_{\beta,\theta} = \underline{\Delta I}_{\beta,\theta}$ that switches to an equality at $\bar{I}'''_{r,\beta}$ cannot be (A.59). If (A.61) switches to an equality at $\bar{I}'''_{r,\beta}$, then $L_{r,\beta} > 0$ for $I_{r,\beta} = \bar{I}'''_{r,\beta} = I^*_{r,\beta}$. If instead, $L_{\beta,\theta} = 0$ for $\Delta I_{\beta,\theta} = \underline{\Delta I}_{\beta,\theta}$, then $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}) = (0, \overline{\Delta I}_{r,\theta}, \underline{\Delta I}_{\beta,\theta})$. Hence,

$$\begin{aligned}
L_{r,\beta} &= \left(\delta_\alpha + \kappa_r + \kappa_\beta - a\sigma_\beta^2\underline{\Delta I}_{\beta,\theta} + a\sigma_\beta^2 \frac{a\sigma_r^2 \bar{I}'''_{r,\beta}\underline{\Delta I}_{r,\theta}}{\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2\overline{\Delta I}_{\beta,\theta}}\right) I'''_{r,\beta} \\
&\quad - \frac{\frac{\alpha}{\delta_\alpha} a\sigma_r^2\overline{\Delta I}_{r,\theta}}{(\delta_\alpha+\kappa_r)\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2\overline{\Delta I}_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I'''_{r,\beta}\underline{\Delta I}_{r,\theta}} \\
&> \left(\delta_\alpha + \kappa_r + \kappa_\beta - a\sigma_\beta^2\underline{\Delta I}_{\beta,\theta} + a\sigma_\beta^2 \frac{a\sigma_r^2 \bar{I}'''_{r,\beta}\underline{\Delta I}_{r,\theta}}{\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2\overline{\Delta I}_{\beta,\theta}}\right) I'''_{r,\beta} \\
&\quad - \frac{\frac{\theta}{\delta_\alpha} a^2\sigma_r^2\sigma_\beta^2\overline{\Delta I}_{r,\theta} I'''_{r,\beta}}{(\delta_\alpha+\kappa_r)\left(\delta_\alpha + \kappa_\beta - a\sigma_\beta^2\overline{\Delta I}_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I'''_{r,\beta}\underline{\Delta I}_{r,\theta}} \\
&= (\delta_\alpha + \kappa_r + \kappa_\beta - 2a\sigma_\beta^2\underline{\Delta I}_{\beta,\theta}) I'''_{r,\beta} - \frac{\frac{\theta}{\delta_\theta} a^2\sigma_r^2\sigma_\beta^2 I'''_{r,\beta}\overline{\Delta I}_{r,\theta}}{(\delta_\theta+\kappa_r)\left(\delta_\theta + \kappa_\beta - a\sigma_\beta^2\overline{\Delta I}_{\beta,\theta}\right) + a^2\sigma_r^2\sigma_\beta^2 I'''_{r,\beta}\overline{\Delta I}_{r,\theta}},
\end{aligned}$$

where the first step follows from $I'''_{r,\beta} > \bar{I}_{r,\beta} = \frac{\alpha}{\theta a\sigma_\beta^2}$ and the second step from (A.41). For large δ_θ, $L_{r,\beta} > 0$ if

$$\delta_\alpha + \kappa_r + \kappa_\beta - 2a\sigma_\beta^2\underline{\Delta I}_{\beta,\theta} > 0, \tag{A.71}$$

which holds because $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$. Hence, $I_{r,\beta} = \bar{I}'''_{r,\beta} = I^*_{r,\beta}$. The solution satisfies $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$, $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$ and $I_{r,\beta} > 0$. Combining these inequalities with (A.26), we find $M_{1,1} > \kappa_r$, $M_{1,2} > 0$, $M_{2,1} < 0$ and $M_{2,2} > \frac{\kappa_\beta - \delta_\alpha}{2}$.

To complete the existence proof, we show that the integrals in the Laplace transforms $(I_r, I_\beta, I_{r,r}, I_{r,\beta}, I_{\beta,\beta}, I_{r,2}, I_{\beta,2})$ converge. That property is assumed when performing the integration by parts in Lemma A.1. Since $\delta_\theta > \delta_\alpha$, the Laplace-transform integrals converge if the real parts of the eigenvalues of M exceed $-\frac{\delta_\alpha}{2}$. Using (A.26), we find that the characteristic polynomial of M is

$$P(\lambda) = \left(\kappa_r + a\sigma_r^2 I_{r,r} - \lambda\right)\left(\kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} - \lambda\right) + a^2\sigma_r^2\sigma_\beta^2 I_{r,\beta}\Delta I_{r,\theta}. \tag{A.72}$$

Since $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$, $\Delta I_{\beta,\theta} < \frac{\delta_a+\kappa_\beta}{2a\sigma_\beta^2}$ and $I_{r,\beta} > 0$, $P(\lambda) > 0$ for all $\lambda < -\frac{\delta_\alpha}{2}$. Hence, if the eigenvalues are real, they must exceed $-\frac{\delta_\alpha}{2}$. If the eigenvalues are complex, their real part is

$$\frac{\kappa_r + a\sigma_r^2 I_{r,r} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}}{2}$$

and exceeds $-\frac{\delta_\alpha}{2}$ because $I_{r,r} > 0$ and $\Delta I_{\beta,\theta} < \frac{\delta_\alpha+\kappa_\beta}{2a\sigma_\beta^2}$. *Q.E.D.*

Proof of Proposition 5: Using $K = 1$ and (A.26), we can write the system (36) as

$$A_r'(\tau) + \left(\kappa_r + a\sigma_r^2 I_{r,r}\right)A_r(\tau) + a\sigma_\beta^2 I_{r,\beta}A_\beta(\tau) - 1 = 0, \tag{A.73}$$

$$A_\beta'(\tau) - a\sigma_r^2\Delta I_{r,\theta}A_r(\tau) + \left(\kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}\right)A_\beta(\tau) = 0, \tag{A.74}$$

and the solution to that system, given in Lemma 3, as

$$A_r(\tau) = \frac{1-e^{-\nu_1\tau}}{\nu_1} + \phi_r\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{A.75}$$

$$A_\beta(\tau) = \phi_\beta\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right). \tag{A.76}$$

Equations (A.73) and (A.74), together with the initial conditions $A_r(0) = A_\beta(0) = 0$, imply $A_r'(0) = 1$ and $A_\beta'(0) = 0$. Differentiating (A.74) at zero and using $\Delta I_{r,\theta} > 0$, which follows from $M_{2,1} < 0$ and (A.26), we find $A_\beta''(0) > 0$. Hence, $A_r(\tau) > 0$, $A_\beta'(\tau) > 0$ and $A_\beta(\tau) > 0$ for small τ.

Suppose that the two eigenvalues of M are real, and without loss of generality set $\nu_1 > \nu_2$. Since the function $(\nu,\tau) \longrightarrow \frac{1-e^{-\nu\tau}}{\nu}$ decreases in ν, the term in parenthesis in (A.76) is positive. Since, in addition, $A_\beta(\tau) > 0$ for small τ, $\phi_\beta > 0$ and hence $A_\beta(\tau) > 0$ for all τ. Since

$$A_\beta'(\tau) = \phi_\beta\left(e^{-\nu_2\tau} - e^{-\nu_1\tau}\right)$$

and $\phi_\beta > 0$, $A_\beta'(\tau) > 0$. Since

$$\frac{A_r(\tau)}{A_\beta(\tau)} = \frac{\frac{1-e^{-\nu_1\tau}}{\nu_1}}{\phi_\beta\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right)} + \frac{\phi_r}{\phi_\beta} = \frac{1}{\phi_\beta\left(\frac{\nu_1}{\nu_2}\frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}} - 1\right)} + \frac{\phi_r}{\phi_\beta},$$

and the function $(\nu_1,\nu_2,\tau) \longrightarrow \frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}}$ increases in τ because its derivative has the same sign as $\frac{e^{\nu_1\tau}-1}{\nu_1} - \frac{e^{\nu_2\tau}-1}{\nu_2}$, $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]' < 0$. Since

$$A_r'(\tau) = e^{-\nu_1\tau} + \phi_r\left(e^{-\nu_2\tau} - e^{-\nu_1\tau}\right),$$

the sign of $A_r'(\tau)$ can change at most once. Hence, $A_r'(\tau) > 0$ for $\tau \in (0, \bar{\tau}')$ and $A_r'(\tau) < 0$ for $\tau \in (\bar{\tau}', \infty)$, where $\bar{\tau}'$ is a threshold in $(0, \infty]$. The function $A_r(\tau)$ has the same behavior for a different threshold $\bar{\tau}$.

When $a \approx 0$, $A_r(\tau) > 0$ because Lemma A.2 implies $\phi_r \approx 0$, $\nu_1 \approx \kappa_r > 0$ and $\nu_2 \approx \kappa_\beta > 0$. When $\alpha(\tau) = 0$, $I_{r,r} = I_{r,\beta} = 0$, and hence (A.73) implies $A_r(\tau) = \frac{1-e^{-\kappa_r \tau}}{\kappa_r} > 0$. In both cases, $\bar{\tau} = \infty$. When $a \approx \infty$, Lemma A.2 implies that for τ bounded away from zero

$$
\begin{aligned}
A_r(\tau) &\approx \frac{1}{a^{\frac{1}{3}}}\left(\frac{1}{\overline{n}_1} + \overline{c}_r \frac{1-e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2}\right) \\
&= \frac{1}{a^{\frac{1}{3}}\overline{n}_1}\left(1 - \frac{\int_0^\infty \alpha(\tau')\frac{1-e^{-\overline{\nu}_2\tau'}}{\overline{\nu}_2}d\tau'}{\int_0^\infty \alpha(\tau')\left(\frac{1-e^{-\overline{\nu}_2\tau'}}{\overline{\nu}_2}\right)^2 d\tau'} \frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\right) \\
&= \frac{1}{a^{\frac{1}{3}}\overline{n}_1} \frac{\int_0^\infty \alpha(\tau')(1-e^{-\overline{\nu}_2\tau'})(e^{-\overline{\nu}_2\tau} - e^{-\overline{\nu}_2\tau'})d\tau'}{\int_0^\infty \alpha(\tau')(1-e^{-\overline{\nu}_2\tau'})^2 d\tau'}.
\end{aligned}
$$

Since this is negative for τ close to ∞, $\bar{\tau} < \infty$.

Suppose that the two eigenvalues of M are complex. Since they are conjugates, we set $\nu_1 = \mu + i\xi$ and $\nu_2 = \mu - i\xi$ for real numbers (μ, ξ). Equations (A.75) and (A.76) imply that $(A_r(\tau), A_\beta(\tau))$ take the form

$$A_r(\tau) = \phi_{r,0} + \phi_{r,1}e^{-\mu\tau}\cos(\xi\tau) + \phi_{r,2}e^{-\mu\tau}\sin(\xi\tau), \tag{A.77}$$

$$A_\beta(\tau) = \phi_{\beta,0} + \phi_{\beta,1}e^{-\mu\tau}\cos(\xi\tau) + \phi_{\beta,2}e^{-\mu\tau}\sin(\xi\tau), \tag{A.78}$$

for real numbers $\{\phi_{j,n}\}_{j=r,\beta,\ n=0,1,2}$. Since the initial conditions $A_r(0) = A_\beta(0) = 0$ imply $\phi_{j,0} + \phi_{j,1} = 0$ for $j = r, \beta$, condition $A_r'(0) = 1$ implies $-\phi_{r,1}\mu + \phi_{r,2}\xi = 1$, and condition $A_\beta'(0) = 0$ implies $-\phi_{\beta,1}\mu + \phi_{\beta,2}\xi = 0$, we can write (A.77) and (A.78) as

$$A_r(\tau) = \phi_{r,0}\left[1 - \frac{\mu}{\xi}e^{-\mu\tau}\sin(\xi\tau) - e^{-\mu\tau}\cos(\xi\tau)\right] + \frac{1}{\xi}e^{-\mu\tau}\sin(\xi\tau), \tag{A.79}$$

$$A_\beta(\tau) = \phi_{\beta,0}\left[1 - \frac{\mu}{\xi}e^{-\mu\tau}\sin(\xi\tau) - e^{-\mu\tau}\cos(\xi\tau)\right]. \tag{A.80}$$

Differentiating (A.79) and (A.80), we find

$$A_r'(\tau) = \phi_{r,0}\frac{\mu^2+\xi^2}{\xi}e^{-\mu\tau}\sin(\xi\tau) + e^{-\mu\tau}\left[\cos(\xi\tau) - \frac{\mu}{\xi}\sin(\xi\tau)\right], \tag{A.81}$$

$$A_\beta'(\tau) = \phi_{\beta,0}\frac{\mu^2+\xi^2}{\xi}e^{-\mu\tau}\sin(\xi\tau). \tag{A.82}$$

Since $A'_\beta(\tau) > 0$ for small τ, $\phi_{\beta,0} > 0$, and hence $A'_\beta(\tau) > 0$ for $\tau \in (0, \frac{\pi}{|\xi|})$. The derivative $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]'$ has the same sign as

$$
\begin{aligned}
&A'_r(\tau)A_\beta(\tau) - A_r(\tau)A'_\beta(\tau)\\
&= e^{-\mu\tau}\left[\cos(\xi\tau) - \frac{\mu}{\xi}\sin(\xi\tau)\right]\phi_{\beta,0}\left[1 - \frac{\mu}{\xi}e^{-\mu\tau}\sin(\xi\tau) - e^{-\mu\tau}\cos(\xi\tau)\right]\\
&\quad - \frac{1}{\xi}e^{-\mu\tau}\sin(\xi\tau)\phi_{\beta,0}\frac{\mu^2+\xi^2}{\xi}e^{-\mu\tau}\sin(\xi\tau)\\
&= \phi_{\beta,0}e^{-\mu\tau}\left[\cos(\xi\tau) - \frac{\mu}{\xi}\sin(\xi\tau) - e^{-\mu\tau}\right], \qquad \text{(A.83)}
\end{aligned}
$$

where the second step follows from (A.79)-(A.82) and the third by rearranging. Since $\phi_{\beta,0} > 0$, $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]'$ is negative if the term in brackets in (A.83) is negative. That term is concave in μ and is maximized for μ given by

$$
-\frac{1}{\xi}\sin(\xi\tau) + \tau e^{-\mu\tau} = 0 \Leftrightarrow e^{-\mu\tau} = \frac{\sin(\xi\tau)}{\xi\tau}.
$$

The maximum is

$$
\cos(\xi\tau) - \frac{\sin(\xi\tau)}{\xi\tau}\left[1 - \log\left(\frac{\sin(\xi\tau)}{\xi\tau}\right)\right] = H(\xi\tau)\frac{\sin(\xi\tau)}{\xi\tau}, \qquad \text{(A.84)}
$$

where

$$
H(x) \equiv \frac{x\cos(x)}{\sin(x)} - 1 + \log\left(\frac{\sin(x)}{x}\right).
$$

The function $H(x)$ is equal to zero for $x = 0$, and its derivative is

$$
H'(x) = -\frac{x}{\sin^2(x)} + \frac{\cos(x)}{\sin(x)} + \frac{\frac{x\cos(x)-\sin(x)}{x^2}}{\frac{\sin(x)}{x}} = -\frac{x^2 - 2x\cos(x)\sin(x) + \sin^2(x)}{x\sin^2(x)}.
$$

Since

$$
x^2 - 2x\cos(x)\sin(x) + \sin^2(x) > x^2 - 2|x\sin(x)| + \sin^2(x) = (|x| - |\sin(x)|)^2 > 0
$$

for $x \neq 0$, $H'(x) > 0$ for $x < 0$, and $H'(x) < 0$ for $x > 0$. Since, in addition, $H(0) = 0$, $H(x) < 0$. Hence, the maximum (A.84) is negative for $\tau \in (0, \frac{\pi}{|\xi|})$, and so is $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]'$. This establishes the results in the proposition for $A'_\beta(\tau)$ and $\frac{A_r(\tau)}{A_\beta(\tau)}$ and for the threshold $\hat{\tau} = \frac{\pi}{|\xi|}$. The result for $A_\beta(\tau)$ and for a threshold $\bar{\bar{\tau}} > \hat{\tau}$ follows because $A_\beta(0) = 0$ and $A'_\beta(\tau) > 0$ for $\tau \in (0, \hat{\tau})$ imply $A_\beta(\tau) > 0$ for $\tau \in (0, \hat{\tau}]$.

If $\bar{\bar{\tau}} < \infty$, then $A_\beta(\bar{\bar{\tau}}) = 0$ and $A'_\beta(\bar{\bar{\tau}}) \leq 0$. If $A'_\beta(\bar{\bar{\tau}}) < 0$, then $\Delta I_{r,\theta} > 0$ and (A.74) imply $A_r(\bar{\bar{\tau}}) < 0$. If $A'_\beta(\bar{\bar{\tau}}) = 0$, then $\Delta I_{r,\theta} > 0$ and (A.74) imply $A_r(\bar{\bar{\tau}}) = 0$, and (A.74) implies $A'_r(\bar{\bar{\tau}}) = 1$. Hence, in both cases, $A_r(\tau) < 0$ for τ smaller than and close to $\bar{\bar{\tau}}$. This yields the result in the proposition for $A_r(\tau)$ and for a threshold $\bar{\tau} < \bar{\bar{\tau}}$. *Q.E.D.*

Lemma A.2 derives the asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ when $a \approx 0$ and $a \approx \infty$. To state and prove the lemma, we define the functions

$$F(\nu, \nu') \equiv \int_0^\infty \alpha(\tau) \frac{1 - e^{-\nu\tau}}{\nu} \frac{1 - e^{-\nu'\tau}}{\nu'} d\tau,$$

$$\hat{F}(\nu, \nu') \equiv F(\nu, \nu') - F(\nu, \nu),$$

$$\hat{\hat{F}}(\nu, \nu') \equiv F(\nu, \nu) + F(\nu', \nu') - 2F(\nu, \nu'),$$

$$G(\nu) \equiv \int_0^\infty \theta(\tau) \frac{1 - e^{-\nu\tau}}{\nu} d\tau,$$

$$\hat{G}(\nu, \nu') \equiv G(\nu') - G(\nu).$$

We also note that the definitions of $(J, I_{r,r}, I_{r,\beta})$ imply

$$I_{r,r} = \int_0^\infty \alpha(\tau) A_r(\tau)^2 d\tau, \tag{A.85}$$

$$I_{r,\beta} = \int_0^\infty \alpha(\tau) A_r(\tau) A_\beta(\tau) d\tau. \tag{A.86}$$

Lemma A.2 *Suppose that there is one demand factor, the matrices (Γ, Σ) are diagonal, and $\alpha(\tau)$ and $\frac{\theta(\tau)}{\tau}$ have a positive and a finite limit, respectively, at $\tau = 0$. The asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ when $a \approx 0$ and $a \approx \infty$ is as follows:*

- *When $a \approx 0$, $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (\kappa_r, \kappa_\beta, a^3 \underline{c}_r, a\underline{c}_\beta)$, where*

$$\underline{c}_r = -\frac{\underline{c}_\beta^2 \sigma_\beta^2 \hat{F}(\kappa_r, \kappa_\beta)}{\kappa_r - \kappa_\beta}, \tag{A.87}$$

$$\underline{c}_\beta = \frac{\sigma_r^2 G(\kappa_r)}{\kappa_r - \kappa_\beta}. \tag{A.88}$$

- *When $a \approx \infty$, $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (a^{\frac{1}{3}} \overline{n}_1, \overline{\nu}_2, a^{-\frac{1}{3}} \overline{c}_r, \overline{\phi}_\beta)$, where*

$$\overline{n}_1 = \sigma_r^{\frac{2}{3}} \left[\int_0^\infty \alpha(\tau) d\tau - \frac{\left[\int_0^\infty \alpha(\tau) \frac{1 - e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2} d\tau \right]^2}{\int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2} \right)^2 d\tau} \right]^{\frac{1}{3}} > 0. \tag{A.89}$$

$$\overline{c}_r = -\frac{1}{\overline{n}_1} \frac{\int_0^\infty \alpha(\tau) \frac{1 - e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2} d\tau}{\int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2} \right)^2 d\tau} < 0, \tag{A.90}$$

$$\overline{\phi}_\beta = \frac{\int_0^\infty \theta(\tau) \frac{1 - e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2} d\tau}{\int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2} \right)^2 d\tau}, \tag{A.91}$$

and $\overline{\nu}_2$ *solves*

$$\frac{\int_0^\infty \theta(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}d\tau}{\int_0^\infty \theta(\tau)d\tau}=\frac{\int_0^\infty \alpha(\tau)\left(\frac{1-e^{-\nu_2\tau}}{\nu_2}\right)^2 d\tau}{\int_0^\infty \alpha(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}d\tau}. \tag{A.92}$$

Proof: Substituting (A.75) and (A.76) into (A.73) and identifying terms in $\frac{1-e^{-\nu_1\tau}}{\nu_1}$ and $\left(\frac{1-e^{-\nu_2\tau}}{\nu_2}-\frac{1-e^{-\nu_1\tau}}{\nu_1}\right)$, we find

$$\phi_r(\nu_1-\nu_2)-\nu_1+\kappa_r+a\sigma_r^2 I_{r,r}=0, \tag{A.93}$$

$$-\phi_r\nu_2+\phi_r\left(\kappa_r+a\sigma_r^2 I_{r,r}\right)+\phi_\beta a\sigma_\beta^2 I_{r,\beta}=0, \tag{A.94}$$

respectively. Using (A.93), we can write (A.94) as

$$\phi_r(1-\phi_r)(\nu_1-\nu_2)+\phi_\beta a\sigma_\beta^2 I_{r,\beta}=0. \tag{A.95}$$

Substituting (A.75) and (A.76) into (A.74) and identifying terms, we find

$$\phi_\beta(\nu_1-\nu_2)-a\sigma_r^2\Delta I_{r,\theta}=0, \tag{A.96}$$

$$-\phi_\beta\nu_2-\phi_r\Delta I_{r,\theta}+\phi_\beta\left(\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)=0, \tag{A.97}$$

respectively. Using (A.96), we can write (A.97) as

$$-\nu_2-\phi_r(\nu_1-\nu_2)+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}=0. \tag{A.98}$$

Equations (A.93), (A.95), (A.96) and (A.98) constitute a system of four equations in the four unknowns $(\nu_1,\nu_2,\phi_r,\phi_\beta)$. Substituting (A.75) and (A.76) into the definitions (A.85), (A.86), (A.24) and (A.25) of $(I_{r,r}, I_{r,\beta}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$, we can write that system as

$$\phi_r(\nu_1-\nu_2)-\nu_1+\kappa_r+a\sigma_r^2\left[F(\nu_1,\nu_1)+2\phi_r\hat{F}(\nu_1,\nu_2)+\phi_r^2\hat{\hat{F}}(\nu_1,\nu_2)\right]=0, \tag{A.99}$$

$$\phi_r(1-\phi_r)(\nu_1-\nu_2)+\phi_\beta^2 a\sigma_\beta^2\left[\hat{F}(\nu_1,\nu_2)+\phi_r\hat{\hat{F}}(\nu_1,\nu_2)\right]=0, \tag{A.100}$$

$$\phi_\beta(\nu_1-\nu_2)-a\sigma_r^2\left[G(\nu_1)+\phi_r\hat{G}(\nu_1,\nu_2)-\phi_\beta\left[\hat{F}(\nu_1,\nu_2)+\gamma_r\hat{\hat{F}}(\nu_1,\nu_2)\right]\right]=0, \tag{A.101}$$

$$-\nu_2-\phi_r(\nu_1-\nu_2)+\kappa_\beta-\phi_\beta a\sigma_\beta^2\left[\hat{G}(\nu_1,\nu_2)-\phi_\beta\hat{\hat{F}}(\nu_1,\nu_2)\right]=0. \tag{A.102}$$

Suppose that $a\approx 0$. Setting $(\phi_r,\phi_\beta)=(a^3c_r,ac_\beta)$, we can write (A.99)-(A.102) as

$$a^3c_r(\nu_1-\nu_2)-\nu_1+\kappa_r+a\sigma_r^2\left[F(\nu_1,\nu_1)+2a^3c_r\hat{F}(\nu_1,\nu_2)+a^6c_r^2\hat{\hat{F}}(\nu_1,\nu_2)\right]=0, \tag{A.103}$$

$$c_r(1-a^3c_r)(\nu_1-\nu_2)+c_\beta^2\sigma_\beta^2\left[\hat{F}(\nu_1,\nu_2)+a^3c_r\hat{\hat{F}}(\nu_1,\nu_2)\right]=0, \tag{A.104}$$

$$c_\beta(\nu_1-\nu_2)-\sigma_r^2\left[G(\nu_1)+a^3c_r\hat{G}(\nu_1,\nu_2)-ac_\beta\left[\hat{F}(\nu_1,\nu_2)+a^3c_r\hat{\hat{F}}(\nu_1,\nu_2)\right]\right]=0, \tag{A.105}$$

$$-\nu_2-a^3c_r(\nu_1-\nu_2)+\kappa_\beta-a^2c_\beta\sigma_\beta^2\left[\hat{G}(\nu_1,\nu_2)-ac_\beta\hat{\hat{F}}(\nu_1,\nu_2)\right]=0. \tag{A.106}$$

The asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ is as in the lemma if (A.103)-(A.106) has a non-zero solution $(\nu_1, \nu_2, c_r, c_\beta)$ for $a = 0$. For $a = 0$, (A.103) implies $\nu_1 = \kappa_r$, (A.106) implies $\nu_2 = \kappa_\beta$, (A.105) implies $c_\beta = \underline{c}_\beta$ and (A.104) implies $c_r = \underline{c}_r$.

Suppose that $a \approx \infty$. Setting $(\nu_1, \phi_r) = (a^{\frac{1}{3}} n_1, a^{-\frac{1}{3}} c_r)$, we can write (A.99)-(A.102) as

$$a^{-\frac{2}{3}} c_r \left(a^{\frac{1}{3}} n_1 - \nu_2\right) - n_1 + a^{-\frac{1}{3}} \kappa_r + a^{\frac{2}{3}} \sigma_r^2 \left[F\left(a^{\frac{1}{3}} n_1, a^{\frac{1}{3}} n_1\right) + 2a^{-\frac{1}{3}} c_r \hat{F}\left(a^{\frac{1}{3}} n_1, \nu_2\right) + a^{-\frac{2}{3}} c_r^2 \hat{\hat{F}}\left(a^{\frac{1}{3}} n_1, \nu_2\right)\right] = 0, \tag{A.107}$$

$$a^{-1} c_r (1 - a^{-\frac{1}{3}} c_r) \left(a^{\frac{1}{3}} n_1 - \nu_2\right) + a^{\frac{1}{3}} \phi_\beta^2 \sigma_\beta^2 \left[\hat{F}\left(a^{\frac{1}{3}} n_1, \nu_2\right) + a^{-\frac{1}{3}} c_r \hat{\hat{F}}\left(a^{\frac{1}{3}} n_1, \nu_2\right)\right] = 0, \tag{A.108}$$

$$a^{-\frac{2}{3}} \phi_\beta \left(a^{\frac{1}{3}} n_1 - \nu_2\right) - a^{\frac{1}{3}} \sigma_r^2 \left[G\left(a^{\frac{1}{3}} n_1\right) + a^{-\frac{1}{3}} c_r \hat{G}\left(a^{\frac{1}{3}} n_1, \nu_2\right) - \phi_\beta \left[\hat{F}\left(a^{\frac{1}{3}} n_1, \nu_2\right) + a^{-\frac{1}{3}} c_r \hat{\hat{F}}\left(a^{\frac{1}{3}} n_1, \nu_2\right)\right]\right] = 0, \tag{A.109}$$

$$a^{-1} \left[-\nu_2 - a^{-\frac{1}{3}} c_r \left(a^{\frac{1}{3}} n_1 - \nu_2\right) + \kappa_\beta\right] - \phi_\beta \sigma_\beta^2 \left[\hat{G}\left(a^{\frac{1}{3}} n_1, \nu_2\right) - \phi_\beta \hat{\hat{F}}\left(a^{\frac{1}{3}} n_1, \nu_2\right)\right] = 0. \tag{A.110}$$

The asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ is as in the lemma if (A.107)-(A.110) has a non-zero solution $(n_1, \nu_2, c_r, \phi_\beta)$ for $a = \infty$. Noting that

$$\lim_{a \to \infty} a^{\frac{2}{3}} F\left(a^{\frac{1}{3}} n_1, a^{\frac{1}{3}} n_1\right) = \frac{1}{n_1^2} \int_0^\infty \alpha(\tau) d\tau,$$

$$\lim_{a \to \infty} a^{\frac{1}{3}} F\left(a^{\frac{1}{3}} n_1, \nu_2\right) = \frac{1}{n_1} \int_0^\infty \alpha(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau,$$

$$\lim_{a \to \infty} a^{\frac{1}{3}} G\left(a^{\frac{1}{3}} n_1\right) = \frac{1}{n_1} \int_0^\infty \theta(\tau) d\tau,$$

we can write (A.107)-(A.110) for $a = \infty$ as

$$n_1 - \sigma_r^2 \left[\frac{1}{n_1^2} \int_0^\infty \alpha(\tau) d\tau + 2c_r \frac{1}{n_1} \int_0^\infty \alpha(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau + c_r^2 \int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau\right] = 0, \tag{A.111}$$

$$\frac{1}{n_1} \int_0^\infty \alpha(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau + c_r \int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau = 0, \tag{A.112}$$

$$\frac{1}{n_1} \int_0^\infty \theta(\tau) d\tau + c_r \int_0^\infty \theta(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau - \phi_\beta \left[\frac{1}{n_1} \int_0^\infty \alpha(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau + c_r \int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau\right] = 0, \tag{A.113}$$

$$\int_0^\infty \theta(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau - \phi_\beta \int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau = 0. \tag{A.114}$$

Equations (A.112) and (A.113) imply (A.92). Equation (A.92) has a solution $\overline{\nu}_2$. Indeed, when ν_2 goes to infinity, the left-hand side is

$$\frac{1}{\nu_2} \left[1 - \frac{\int_0^\infty \theta(\tau) e^{-\nu_2 \tau} d\tau}{\int_0^\infty \theta(\tau) d\tau}\right] = \frac{1}{\nu_2} \left[1 + o\left(\frac{1}{\nu_2}\right)\right]$$

because $\frac{\theta(\tau)}{\tau}$ has a finite limit at zero, and the right-hand side is

$$\frac{1}{\nu_2}\left[1-\frac{\int_0^\infty \alpha(\tau)(1-e^{-\nu_2\tau})e^{-\nu_2\tau}d\tau}{\int_0^\infty \alpha(\tau)(1-e^{-\nu_2\tau})d\tau}\right]=\frac{1}{\nu_2}\left[1-\frac{\alpha(0)}{\nu_2\int_0^\infty \alpha(\tau)d\tau}+o\left(\frac{1}{\nu_2}\right)\right]$$

because $\alpha(\tau)$ has a positive limit at zero. Hence, the left-hand side exceeds the right-hand side. When $(\alpha(\tau),\{\theta_k(\tau)\}_{k=1,..,K})$ become zero for τ larger than a finite threshold T, and ν_2 goes to minus infinity, the left-hand side is

$$\frac{e^{-\nu_2 T}}{\nu_2}\frac{\int_0^\infty \theta(\tau)\left[e^{\nu_2 T}-e^{\nu_2(T-\tau)}\right]d\tau}{\int_0^\infty \theta(\tau)d\tau}=\frac{e^{-\nu_2 T}}{\nu_2^2}\frac{\theta(T)}{\int_0^\infty \theta(\tau)d\tau}+o\left(\frac{1}{\nu_2^2}\right),$$

and is smaller than the right-hand side, which is

$$\frac{e^{-\nu_2 T}}{\nu_2}\frac{\int_0^\infty \alpha(\tau)\left[e^{\nu_2 T}-e^{\nu_2(T-\tau)}\right]^2d\tau}{\int_0^\infty \alpha(\tau)\left[e^{\nu_2 T}-e^{\nu_2(T-\tau)}\right]d\tau}=\frac{e^{-\nu_2 T}}{-2\nu_2}+o\left(\frac{1}{\nu_2}\right).$$

Hence, a solution $\overline{\nu}_2\in(-\infty,\infty)$ to (A.92) exists. When $T=\infty$, $(\alpha(\tau),\theta(\tau))\approx(\alpha e^{-\delta_\alpha\tau},\theta e^{-\delta'_\alpha\tau})$ for τ large and for $0<\delta_\alpha\le\delta'_\alpha$. When ν_2 goes to $-\frac{\delta_\alpha}{2}$, the right-hand side goes to infinity, while the left-hand side remains finite. Hence, a solution $\overline{\nu}_2\in\left(-\frac{\delta_\alpha}{2},\infty\right)$ to (A.92) exists.

Using (A.112) to eliminate c_r in (A.111), we find $n_1=\overline{n}_1$. Equations (A.112) and (A.114) imply $c_r=\overline{c}_r$ and $\phi_\beta=\overline{\phi}_\beta$, respectively. The Cauchy-Schwarz inequality implies $\overline{n}_1>0$, and hence $\overline{c}_r<0$. *Q.E.D.*

Proof of Proposition 6: Proceeding as in the proof of Proposition 3, we find that the FB regression coefficient is

$$b_{\text{FB}}=\frac{N_{\text{FB},r}\text{Var}(r_t)+N_{\text{FB},\beta}\text{Var}(\beta_t)}{\left[A_r(\tau)-A_r(\tau-\Delta\tau)-A_r(\Delta\tau)\right]^2\text{Var}(r_t)+\left[A_\beta(\tau)-A_\beta(\tau-\Delta\tau)\right]^2\text{Var}(\beta_t)}$$

$$=\frac{N_{\text{FB},r}\frac{\sigma_r^2}{\kappa_r}+N_{\text{FB},\beta}\frac{\sigma_\beta^2}{\kappa_\beta}}{\left[A_r(\tau)-A_r(\tau-\Delta\tau)-A_r(\Delta\tau)\right]^2\frac{\sigma_r^2}{\kappa_r}+\left[A_\beta(\tau)-A_\beta(\tau-\Delta\tau)\right]^2\frac{\sigma_\beta^2}{\kappa_\beta}},\tag{A.115}$$

where

$$N_{\text{FB},j}=\left[A_j(\tau)-A_j(\tau-\Delta\tau)e^{-\kappa_j\Delta\tau}-A_j(\Delta\tau)\right]\left[A_j(\tau)-A_j(\tau-\Delta\tau)-A_j(\Delta\tau)\right]$$

for $j=r,\beta$. Taking the limit in (A.115) when $\Delta\tau\to 0$, and noting from (A.75) and (A.76) that $\frac{A_r(\Delta\tau)}{\Delta\tau}\to 1$ and $\frac{A_\beta(\Delta\tau)}{\Delta\tau}\to 0$, we find

$$b_{\text{FB}}=\frac{\left[A'_r(\tau)+\kappa_r A_r(\tau)-1\right]\left[A'_r(\tau)-1\right]\frac{\sigma_r^2}{\kappa_r}+\left[A'_\beta(\tau)+\kappa_\beta A_\beta(\tau)\right]A'_\beta(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}{\left[A'_r(\tau)-1\right]^2\frac{\sigma_r^2}{\kappa_r}+A'_\beta(\tau)^2\frac{\sigma_\beta^2}{\kappa_\beta}}.\tag{A.116}$$

For $\tau < \min\{\bar{\tau}, \hat{\tau}\}$, $A_r(\tau) > 0$, $A_\beta(\tau) > 0$ and $A'_\beta(\tau) > 0$. Moreover, (A.73) implies

$$A'_r(\tau) + \kappa_r A_r(\tau) - 1 = -a\sigma_r^2 I_{r,r} A_r(\tau) - a\sigma_\beta^2 I_{r,\beta} A_\beta(\tau) \leq 0, \tag{A.117}$$

$$A'_r(\tau) - 1 = -\left(\kappa_r + a\sigma_r^2 I_{r,r}\right) A_r(\tau) - a\sigma_\beta^2 I_{r,\beta} A_\beta(\tau) < 0, \tag{A.118}$$

where the inequalities follow from $A_r(\tau) > 0$, $A_\beta(\tau) > 0$, $I_{r,r} \geq 0$ and $I_{r,\beta} \geq 0$, which in turn follows from $M_{1,2} \geq 0$ and (A.26). Equations (A.116), $A_\beta(\tau) > 0$, $A'_\beta(\tau) > 0$, (A.117) and (A.118) imply $b_{\mathrm{FB}} > 0$.

When $a \approx 0$, (A.75), (A.76) and $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (\kappa_r, \kappa_\beta, a^3 \underline{c}_r, a\underline{c}_\beta)$ (Lemma A.2) imply

$$b_{\mathrm{FB}} = \frac{\frac{\nu_1 - \kappa_r}{\nu_1}(1 - e^{-\kappa_r \tau})^2 \frac{\sigma_r^2}{\kappa_r} + a^2 \underline{c}_\beta^2 \left[L'_\beta(\tau) + \kappa_\beta L_\beta(\tau)\right] L'_\beta(\tau) \frac{\sigma_\beta^2}{\kappa_\beta}}{(1 - e^{-\kappa_r \tau})^2 \frac{\sigma_r^2}{\kappa_r} + a^2 \underline{c}_\beta^2 L'_\beta(\tau)^2 \frac{\sigma_\beta^2}{\kappa_\beta}} + o(a^2),$$

where

$$L_\beta(\tau) \equiv \frac{1 - e^{-\kappa_\beta \tau}}{\kappa_\beta} - \frac{1 - e^{-\kappa_r \tau}}{\kappa_r}.$$

Since $L_\beta(\tau) L'_\beta(\tau) > 0$, and (A.85) and (A.93) imply

$$\nu_1 - \kappa_r = a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\frac{1 - e^{-\kappa_r \tau}}{\kappa_r}\right)^2 d\tau + o(a^2), \tag{A.119}$$

$b_{\mathrm{FB}} > 0$.

When $a \approx \infty$, (A.75), (A.76) and $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (a^{\frac{1}{3}} \overline{n}_1, \overline{\nu}_2, a^{-\frac{1}{3}} \overline{c}_r, \overline{\phi}_\beta)$ (Lemma A.2) imply that for τ bounded away from zero

$$b_{\mathrm{FB}} = \frac{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 \left(e^{-\overline{\nu}_2 \tau} + \kappa_\beta \frac{1 - e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2}\right) e^{-\overline{\nu}_2 \tau} \frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 e^{-2\overline{\nu}_2 \tau} \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1) = 1 + \frac{\overline{\phi}_\beta^2 \frac{1 - e^{-\overline{\nu}_2 \tau}}{\overline{\nu}_2} e^{-\overline{\nu}_2 \tau} \sigma_\beta^2}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 e^{-2\overline{\nu}_2 \tau} \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1). \tag{A.120}$$

Hence, $b_{\mathrm{FB}} > 1$. We next show that b_{FB} increases in τ if (43) holds. Equation (43) implies that the left-hand side of (A.92) exceeds the right-hand side for $\nu_2 = 0$, and hence (A.92) has a solution $\overline{\nu}_2 < 0$. We write (A.120) as

$$b_{\mathrm{FB}} = 1 + \frac{\overline{\phi}_\beta^2 N_{\mathrm{FB}}(\tau) \sigma_\beta^2}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{FB}}(\tau) \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1), \tag{A.121}$$

where

$$N_{\mathrm{FB}}(\tau) \equiv \frac{e^{2z\tau} - e^{z\tau}}{z},$$

$$D_{\mathrm{FB}}(\tau) \equiv e^{2z\tau},$$

and $z \equiv -\overline{\nu}_2 > 0$, and consider the derivative

$$\left[\frac{\overline{\phi}_\beta^2 N_{\mathrm{FB}}(\tau)\sigma_\beta^2}{\frac{\sigma_r^2}{\kappa_r}+\overline{\phi}_\beta^2 D_{\mathrm{FB}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}\right]' = \frac{\frac{\sigma_r^2}{\kappa_r}\overline{\phi}_\beta^2\sigma_\beta^2 N'_{\mathrm{FB}}(\tau)+\overline{\phi}_\beta^4\frac{\sigma_\beta^4}{\kappa_\beta}\left[N'_{\mathrm{FB}}(\tau)D_{\mathrm{FB}}(\tau)-N_{\mathrm{FB}}(\tau)D'_{\mathrm{FB}}(\tau)\right]}{\left[\frac{\sigma_r^2}{\kappa_r}+\overline{\phi}_\beta^2 D_{\mathrm{FB}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}\right]^2}.$$

Since

$$\left[\frac{N_{\mathrm{FB}}(\tau)}{D_{\mathrm{FB}}(\tau)}\right]' = \left[\frac{1-e^{-z\tau}}{z}\right]' = e^{-z\tau} > 0,$$

$N'_{\mathrm{FB}}(\tau)D_{\mathrm{FB}}(\tau)-N_{\mathrm{FB}}(\tau)D'_{\mathrm{FB}}(\tau) > 0$. Since, in addition,

$$N'_{\mathrm{FB}}(\tau) = 2e^{2z\tau}-e^{z\tau} > 0,$$

b_{FB} increases in τ.

Proceeding as in the proof of Proposition 3, we find that the CS regression coefficient is

$$b_{\mathrm{CS}} = \frac{N_{\mathrm{CS},r}\mathrm{Var}(r_t)+N_{\mathrm{CS},\beta}\mathrm{Var}(\beta_t)}{\frac{\Delta\tau}{\tau-\Delta\tau}\left\{\left[\frac{A_r(\tau)}{\tau}-\frac{A_r(\Delta\tau)}{\Delta\tau}\right]^2\mathrm{Var}(r_t)+\left[\frac{A_\beta(\tau)}{\tau}-\frac{A_\beta(\Delta\tau)}{\Delta\tau}\right]^2\mathrm{Var}(\beta_t)\right\}}$$

$$= \frac{N_{\mathrm{CS},r}\frac{\sigma_r^2}{\kappa_r}+N_{\mathrm{CS},\beta}\frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\Delta\tau}{\tau-\Delta\tau}\left\{\left[\frac{A_r(\tau)}{\tau}-\frac{A_r(\Delta\tau)}{\Delta\tau}\right]^2\frac{\sigma_r^2}{\kappa_r}+\left[\frac{A_\beta(\tau)}{\tau}-\frac{A_\beta(\Delta\tau)}{\Delta\tau}\right]^2\frac{\sigma_\beta^2}{\kappa_\beta}\right\}}, \tag{A.122}$$

where

$$N_{\mathrm{CS},j} = \left[\frac{A_j(\tau-\Delta\tau)}{\tau-\Delta\tau}e^{-\kappa_j\Delta\tau}-\frac{A_j(\tau)}{\tau}\right]\left[\frac{A_j(\tau)}{\tau}-\frac{A_j(\Delta\tau)}{\Delta\tau}\right]$$

for $j = r,\beta$. Taking the limit in (A.122) when $\Delta\tau \to 0$, we find

$$b_{\mathrm{CS}} \to \frac{\left[\frac{A_r(\tau)}{\tau}-\left[A'_r(\tau)+\kappa_r A_r(\tau)\right]\right]\left[\frac{A_r(\tau)}{\tau}-1\right]\frac{\sigma_r^2}{\kappa_r}+\left[\frac{A_\beta(\tau)}{\tau}-\left[A'_\beta(\tau)+\kappa_\beta A_\beta(\tau)\right]\right]\frac{A_\beta(\tau)}{\tau}\frac{\sigma_\beta^2}{\kappa_\beta}}{\left[\frac{A_r(\tau)}{\tau}-1\right]^2\frac{\sigma_r^2}{\kappa_r}+\left[\frac{A_\beta(\tau)}{\tau}\right]^2\frac{\sigma_\beta^2}{\kappa_\beta}}$$

$$= 1-\frac{\left[A'_r(\tau)+\kappa_r A_r(\tau)-1\right]\left[\frac{A_r(\tau)}{\tau}-1\right]\frac{\sigma_r^2}{\kappa_r}+\left[A'_\beta(\tau)+\kappa_\beta A_\beta(\tau)\right]\frac{A_\beta(\tau)}{\tau}\frac{\sigma_\beta^2}{\kappa_\beta}}{\left[\frac{A_r(\tau)}{\tau}-1\right]^2\frac{\sigma_r^2}{\kappa_r}+\left[\frac{A_\beta(\tau)}{\tau}\right]^2\frac{\sigma_\beta^2}{\kappa_\beta}}. \tag{A.123}$$

For $\tau < \min\{\bar{\tau},\hat{\tau}\}$, $A_\beta(\tau) > 0$, $A'_\beta(\tau) > 0$, and (A.117) and (A.118) hold. Equation (A.118) and the initial condition $A_r(0) = 0$ imply $A_r(\tau)-\tau < 0$. Equations (A.9), $A_\beta(\tau) > 0$, $A'_\beta(\tau) > 0$, (A.117) and $A_r(\tau)-\tau < 0$ imply $b_{\mathrm{CS}} < 1$.

When $a \approx 0$, (A.75), (A.76), $(\nu_1,\nu_2,\phi_r,\phi_\beta) \approx (\kappa_r,\kappa_\beta,a^3\underline{c}_r,a\underline{c}_\beta)$ (Lemma A.2) and (A.119) imply

$$b_{\mathrm{CS}} = 1-a\frac{\sigma_r^2\left(1-e^{-\kappa_r\tau}\right)}{\kappa_r\left(1-\frac{1-e^{-\kappa_r\tau}}{\kappa_r\tau}\right)}\int_0^\infty \alpha(\tau)\left(\frac{1-e^{-\kappa_r\tau}}{\kappa_r}\right)^2 d\tau + o(a).$$

Hence, b_{CS} is smaller than and close to one. Moreover, b_{CS} increases in τ because the function $K(x)$ defined in Proposition 3 is increasing for $x > 0$.

When $a \approx \infty$, (A.75), (A.76) and $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (a^{\frac{1}{3}}\overline{n}_1, \overline{\nu}_2, a^{-\frac{1}{3}}\overline{c}_r, \overline{\phi}_\beta)$ (Lemma A.2) imply that for τ bounded away from zero

$$b_{\mathrm{CS}} = 1 - \frac{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 \left(e^{-\overline{\nu}_2\tau} + \kappa_\beta \frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\right) \frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2\tau} \frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 \left(\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2\tau}\right)^2 \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1). \tag{A.124}$$

Hence, $b_{\mathrm{CS}} < 1$. We next show that b_{CS} is negative and decreasing in τ if (43) holds. We write (A.124) as

$$b_{\mathrm{CS}} = 1 - \frac{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 N_{\mathrm{CS}}(\tau) \frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{CS}}(\tau) \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1), \tag{A.125}$$

where

$$N_{\mathrm{CS}}(\tau) \equiv \left(e^{z\tau} + \kappa_\beta \frac{e^{z\tau}-1}{z}\right) \frac{e^{z\tau}-1}{z\tau},$$

$$D_{\mathrm{CS}}(\tau) \equiv \left(\frac{e^{z\tau}-1}{z\tau}\right)^2,$$

and $z \equiv -\overline{\nu}_2 > 0$. Equation (A.125) implies

$$b_{\mathrm{CS}} = -\frac{\overline{\phi}_\beta^2 \left[N_{\mathrm{CS}}(\tau) - D_{\mathrm{CS}}(\tau)\right] \frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{CS}}(\tau) \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1). \tag{A.126}$$

Since

$$\begin{aligned} N_{\mathrm{CS}}(\tau) - D_{\mathrm{CS}}(\tau) &= \left[e^{z\tau} + \left(\kappa_\beta - \frac{1}{\tau}\right)\frac{e^{z\tau}-1}{z}\right]\frac{e^{z\tau}-1}{z\tau} \\ &> \left[e^{z\tau} - \frac{e^{z\tau}-1}{z\tau}\right]\frac{e^{z\tau}-1}{z\tau} = \frac{z\tau e^{z\tau} - e^{z\tau} + 1}{z\tau}\frac{e^{z\tau}-1}{z\tau} \end{aligned}$$

and $xe^x - e^x + 1 > 0$ for all x, (A.126) implies $b_{\mathrm{CS}} < 0$. Consider next the derivative

$$\left[\frac{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 N_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}\right]' = \frac{\frac{\sigma_r^2}{\kappa_r}\overline{\phi}_\beta^2\frac{\sigma_\beta^2}{\kappa_\beta}\left[N'_{\mathrm{CS}}(\tau) - D'_{\mathrm{CS}}(\tau)\right] + \overline{\phi}_\beta^4\frac{\sigma_\beta^4}{\kappa_\beta^2}\left[N'_{\mathrm{CS}}(\tau)D_{\mathrm{CS}}(\tau) - N_{\mathrm{CS}}(\tau)D'_{\mathrm{CS}}(\tau)\right]}{\left[\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}\right]^2}.$$

Since

$$N'_{\mathrm{CS}}(\tau) - D'_{\mathrm{CS}}(\tau) = \left[ze^{z\tau} + \left(\kappa_\beta - \frac{1}{\tau}\right)e^{z\tau} + \frac{e^{z\tau}-1}{z\tau^2}\right]\frac{e^{z\tau}-1}{z\tau}$$

$$+ \left[e^{z\tau} + \left(\kappa_\beta - \frac{1}{\tau}\right)\frac{e^{z\tau}-1}{z}\right]\frac{z^2\tau e^{z\tau} - z\left(e^{z\tau}-1\right)}{z^2\tau^2}$$

$$> \frac{z^2\tau^2 e^{z\tau} - z\tau e^{z\tau} + e^{z\tau} - 1}{z\tau^2}\frac{e^{z\tau}-1}{z\tau} + \frac{z\tau e^{z\tau} - e^{z\tau} + 1}{z\tau}\frac{z\tau e^{z\tau} - e^{z\tau} + 1}{z\tau^2}$$

and $x^2e^x - xe^x + e^x - 1 > 0$ for all x, $N'_{\mathrm{CS}}(\tau) - D'_{\mathrm{CS}}(\tau) > 0$. Since

$$\left[\frac{N_{\mathrm{CS}}(\tau)}{D_{\mathrm{CS}}(\tau)}\right]' = \left[\frac{z\tau e^{z\tau}}{e^{z\tau}-1}\right]' = ze^{z\tau}\frac{(1+z\tau)\left(e^{z\tau}-1\right) - z\tau e^{z\tau}}{\left(e^{z\tau}-1\right)^2} = ze^{z\tau}\frac{e^{z\tau}-1-z\tau}{\left(e^{z\tau}-1\right)^2}$$

and $e^x - 1 - x > 0$ for all x, $N'_{\mathrm{CS}}(\tau)D_{\mathrm{CS}}(\tau) - N_{\mathrm{CS}}(\tau)D'_{\mathrm{CS}}(\tau) > 0$. Hence, b_{CS} decreases in τ. *Q.E.D.*

Proof of Proposition 7: Substituting $C(\tau)$ from (41) into (42), using $\Gamma = Diag(\kappa_r, \kappa_\beta)$ and $\Sigma = Diag(\sigma_r^2, \sigma_\beta^2)$, and dropping the subscript 1 from functions of the single demand factor, we find

$$\chi_r = \kappa_r\bar{r} + a\sigma_r^2\left[\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau\right.$$

$$- \chi_r\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)du\right)A_r(\tau)d\tau - \chi_\beta\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)du\right)A_r(\tau)d\tau$$

$$\left.+\frac{\sigma_r^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)^2du\right)A_r(\tau)d\tau + \frac{\sigma_\beta^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)^2du\right)A_r(\tau)d\tau\right], \tag{A.127}$$

$$\chi_\beta = a\sigma_\beta^2\left[\int_0^\infty \theta_0(\tau)A_\beta(\tau)d\tau\right.$$

$$- \chi_r\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)du\right)A_\beta(\tau)d\tau - \chi_\beta\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)du\right)A_\beta(\tau)d\tau$$

$$\left.+\frac{\sigma_r^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)^2du\right)A_\beta(\tau)d\tau + \frac{\sigma_\beta^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)^2du\right)A_\beta(\tau)d\tau\right]. \tag{A.128}$$

The system of (A.127) and (A.128) is linear in (χ_r, χ_β) and its solution is

$$\chi_r = \frac{1}{D}\left\{\left[\kappa_r\bar{r} + a\sigma_r^2\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau + C_r\right]\left[1 + a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)du\right)A_\beta(\tau)d\tau\right]\right.$$

$$\left.-\left[a\sigma_\beta^2\int_0^\infty \theta_0(\tau)A_\beta(\tau)d\tau + C_\beta\right]\left[a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)du\right)A_r(\tau)d\tau\right]\right\}, \tag{A.129}$$

$$\chi_\beta = \frac{1}{D}\left\{\left[a\sigma_\beta^2\int_0^\infty \theta_0(\tau)A_\beta(\tau)d\tau + C_\beta\right]\left[1 + a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)du\right)A_r(\tau)d\tau\right]\right.$$

$$\left.-\left[\kappa_r\bar{r} + a\sigma_r^2\int_0^\infty \theta_0(\tau)A_r(\tau)d\tau + C_r\right]\left[a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)du\right)A_\beta(\tau)d\tau\right]\right\}, \tag{A.130}$$

where

$$D \equiv \left[1 + a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u)du\right) A_r(\tau)d\tau\right] \left[1 + a\sigma_\beta^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u)du\right) A_\beta(\tau)d\tau\right]$$
$$- \left[a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u)du\right) A_r(\tau)d\tau\right] \left[a\sigma_\beta^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u)du\right) A_\beta(\tau)d\tau\right]$$

and

$$C_j \equiv \frac{a\sigma_j^2\sigma_r^2}{2} \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u)^2du\right) A_j(\tau)d\tau + \frac{a\sigma_j^2\sigma_\beta^2}{2} \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u)^2du\right) A_j(\tau)d\tau$$

for $j = r, \beta$. The effect of a change in the demand intercept from $\theta_0(\tau)$ to $\theta_0(\tau) + \Delta\theta_0(\tau)$ on the yield $y_t^{(\tau)}$ for maturity τ is $\Delta y_t^{(\tau)} \equiv \frac{\Delta C(\tau)}{\tau}$, which from (41), (A.129) and (A.130) is

$$\Delta y_t^{(\tau)} = \frac{1}{D}\left\{\left[a\sigma_r^2 \int_0^\infty \Delta\theta_0(\tau)A_r(\tau)d\tau\right] \left[1 + a\sigma_\beta^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u)du\right) A_\beta(\tau)d\tau\right]\right.$$
$$\left.- \left[a\sigma_\beta^2 \int_0^\infty \Delta\theta_0(\tau)A_\beta(\tau)d\tau\right] \left[a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u)du\right) A_r(\tau)d\tau\right]\right\} \frac{\int_0^\tau A_r(u)du}{\tau}$$
$$+ \frac{1}{D}\left\{\left[a\sigma_\beta^2 \int_0^\infty \Delta\theta_0(\tau)A_\beta(\tau)d\tau\right] \left[1 + a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u)du\right) A_r(\tau)d\tau\right]\right.$$
$$\left.- \left[a\sigma_r^2 \int_0^\infty \Delta\theta_0(\tau)A_r(\tau)d\tau\right] \left[a\sigma_\beta^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u)du\right) A_\beta(\tau)d\tau\right]\right\} \frac{\int_0^\tau A_\beta(u)du}{\tau}. \tag{A.131}$$

Hence, the change $\Delta\theta_0(\tau)$ affects yields only through $\int_0^\infty \Delta\theta_0(\tau)A_r(\tau)d\tau$ and $\int_0^\infty \Delta\theta_0(\tau)A_\beta(\tau)d\tau$.

When the change $\Delta\theta_0(\tau)$ is a Dirac function with point mass at τ^*,

$$\int_0^\infty \Delta\theta_0(\tau)A_j(\tau)d\tau = A_j(\tau^*)$$

for $j = r, \beta$, and (A.131) becomes

$$\Delta y_{t,\tau^*}^{(\tau)} = \frac{1}{D}\left[\Lambda_r(\tau^*)\frac{\int_0^\tau A_r(u)du}{\tau} + \Lambda_\beta(\tau^*)\frac{\int_0^\tau A_\beta(u)du}{\tau}\right], \tag{A.132}$$

where

$$\Lambda_r(\tau^*) \equiv a\sigma_r^2 A_r(\tau^*)\left[1 + a\sigma_\beta^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u)du\right) A_\beta(\tau)d\tau\right]$$
$$- a\sigma_\beta^2 A_\beta(\tau^*)\left[a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u)du\right) A_r(\tau)d\tau\right],$$
$$\Lambda_\beta(\tau^*) \equiv a\sigma_\beta^2 A_\beta(\tau^*)\left[1 + a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u)du\right) A_r(\tau)d\tau\right]$$
$$- a\sigma_r^2 A_r(\tau^*)\left[a\sigma_\beta^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u)du\right) A_\beta(\tau)d\tau\right].$$

Using (A.132), we can write (44) in the equivalent form

$$\left[\Lambda_r(\tau_1)\frac{\int_0^{\tau_1} A_r(u)du}{\tau_1} + \Lambda_\beta(\tau_1)\frac{\int_0^{\tau_1} A_\beta(u)du}{\tau_1}\right]\left[\Lambda_r(\tau_2)\frac{\int_0^{\tau_2} A_r(u)du}{\tau_2} + \Lambda_\beta(\tau_2)\frac{\int_0^{\tau_2} A_\beta(u)du}{\tau_2}\right]$$

$$> \left[\Lambda_r(\tau_1)\frac{\int_0^{\tau_2} A_r(u)du}{\tau_2} + \Lambda_\beta(\tau_1)\frac{\int_0^{\tau_2} A_\beta(u)du}{\tau_2}\right]\left[\Lambda_r(\tau_2)\frac{\int_0^{\tau_1} A_r(u)du}{\tau_1} + \Lambda_\beta(\tau_2)\frac{\int_0^{\tau_1} A_\beta(u)du}{\tau_1}\right]$$

$$\Leftrightarrow \left[\Lambda_r(\tau_1)\Lambda_\beta(\tau_2) - \Lambda_r(\tau_2)\Lambda_\beta(\tau_1)\right]\left[\frac{\int_0^{\tau_1} A_r(u)du}{\tau_1}\frac{\int_0^{\tau_2} A_\beta(u)du}{\tau_2} - \frac{\int_0^{\tau_2} A_r(u)du}{\tau_2}\frac{\int_0^{\tau_1} A_\beta(u)du}{\tau_1}\right] > 0. \tag{A.133}$$

To show that (A.133) holds, we show that each of the two terms in brackets is positive. The second term is positive because it has the same sign as

$$\int_0^{\tau_1} A_r(u)du \int_0^{\tau_2} A_\beta(u)du - \int_0^{\tau_2} A_r(u)du \int_0^{\tau_1} A_\beta(u)du$$

$$= \int_0^{\tau_1} A_r(u)du \int_{\tau_1}^{\tau_2} A_\beta(u)du - \int_{\tau_1}^{\tau_2} A_r(u)du \int_0^{\tau_1} A_\beta(u)du$$

$$> \int_0^{\tau_1}\left[A_\beta(u)\frac{A_r(\tau_1)}{A_\beta(\tau_1)}\right]du \int_{\tau_1}^{\tau_2} A_\beta(u)du - \int_{\tau_1}^{\tau_2}\left[A_\beta(u)\frac{A_r(\tau_1)}{A_\beta(\tau_1)}\right]du \int_0^{\tau_1} A_\beta(u)du = 0,$$

where the second step follows because $A_\beta(\tau) > 0$ and $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]' < 0$ for $\tau \in (0, \hat{\tau})$. The first term is equal to

$$\left[A_r(\tau_1)A_\beta(\tau_2) - A_r(\tau_2)A_\beta(\tau_1)\right] D,$$

and is positive if $D > 0$ since $A_\beta(\tau) > 0$ and $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]' < 0$ for $\tau \in (0, \hat{\tau})$. Integration by parts implies that for $j = r, \beta$,

$$\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_j(u)du\right)A_j(\tau)d\tau$$

$$= \left[\alpha(\tau)\left(\int_0^\tau A_j(u)du\right)^2\right]_0^\infty + \int_0^\infty \left(\int_0^\tau A_j(u)du\right)^2 d\hat{\alpha}(\tau) - \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_j(u)du\right)A_j(\tau)d\tau, \tag{A.134}$$

where $d\hat{\alpha}(\tau)$ denotes the measure generated by the non-decreasing function $-\alpha(\tau)$ (which is possibly discontinuous at a finite threshold T). Since

$$\left[\alpha(\tau)\left(\int_0^\tau A_j(u)du\right)^2\right]_0^\infty = \lim_{\tau\to\infty}\left[\alpha(\tau)\left(\int_0^\tau A_j(u)du\right)^2\right] = 0,$$

where the second step follows because M is finite, (A.134) implies

$$\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_j(u)du\right)A_j(\tau)d\tau = \frac{\int_0^\infty \left(\int_0^\tau A_j(u)du\right)^2 d\hat{\alpha}(\tau)}{2} \geq 0, \tag{A.135}$$

Likewise,

$$\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)du\right)A_\beta(\tau)d\tau + \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)du\right)A_r(\tau)d\tau$$

$$= 2\left[\alpha(\tau)\left(\int_0^\tau A_r(u)du\right)\left(\int_0^\tau A_\beta(u)du\right)\right]_0^\infty + 2\int_0^\infty \left(\int_0^\tau A_r(u)du\right)\left(\int_0^\tau A_\beta(u)du\right)d\hat{\alpha}(\tau)$$

$$- \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)du\right)A_\beta(\tau)d\tau - \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)du\right)A_r(\tau)d\tau$$

$$\Rightarrow \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)du\right)A_\beta(\tau)d\tau + \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)du\right)A_r(\tau)d\tau$$

$$= \int_0^\infty \left(\int_0^\tau A_r(u)du\right)\left(\int_0^\tau A_\beta(u)du\right)d\hat{\alpha}(\tau),$$

and hence

$$\left[\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)du\right)A_\beta(\tau)d\tau\right]\left[\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)du\right)A_r(\tau)d\tau\right]$$

$$\le \frac{\left[\int_0^\infty \left(\int_0^\tau A_r(u)du\right)\left(\int_0^\tau A_\beta(u)du\right)d\hat{\alpha}(\tau)\right]^2}{4}. \tag{A.136}$$

Equations (A.135) and (A.136) imply that $D > 0$ if

$$\left[\int_0^\infty \left(\int_0^\tau A_r(u)du\right)^2 d\hat{\alpha}(\tau)\right]\left[\int_0^\infty \left(\int_0^\tau A_\beta(u)du\right)^2 d\hat{\alpha}(\tau)\right]$$

$$\ge \left[\int_0^\infty \left(\int_0^\tau A_r(u)du\right)\left(\int_0^\tau A_\beta(u)du\right)d\hat{\alpha}(\tau)\right]^2,$$

which holds because of the Cauchy-Schwarz inequality. *Q.E.D.*

APPENDIX B: DEMAND OF PREFERRED-HABITAT INVESTORS

There are overlapping generations of preferred-habitat investors living for a period of length $T < \infty$, and of arbitrageurs living for a period of length dt. Thus, at each point in time there is a continuum of investor generations and one arbitrageur generation. Arbitrageurs and investors receive endowment W at the beginning of their life and consume at the end of their life. Arbitrageurs use their endowment to buy bonds. Investors use their endowment to buy bonds and to invest in a private opportunity ("real estate") that pays at the end of their life. To ensure that the slope of the investors' demand for bonds is finite, we require that substitution between bonds and the private opportunity is imperfect. We model imperfect substitution by assuming that bonds pay in a good 1 ("money") and the private opportunity pays in a different good 2 ("real estate services"). The endowment W is in good 1. Arbitrageurs and investors can use good 1 to invest in bonds and in the private opportunity.

Consider the optimization problem of an investor n born at time 0. We denote by $\hat{Z}_{n,t}^{(\tau)}$ the number of units of the bond with maturity τ that the investor holds at time $t \in [0, T]$, where one unit of the bond is an investment in the bond with face value one. We denote by $W_{n,t}$ the value of the investor's bond portfolio at time t and by $dc_{n,t}$ the investment in the private opportunity between t and $t+dt$, both expressed in units of good 1. We denote by $(\hat{W}_{n,t}, d\hat{c}_{n,t})$ the counterparts of $(W_{n,t}, dc_{n,t})$ when expressed in units of the bond maturing at time T:

$$\hat{W}_{n,t} \equiv \frac{W_{n,t}}{P_t^{(T-t)}},$$

$$d\hat{c}_{n,t} \equiv \frac{dc_{n,t}}{P_t^{(T-t)}}.$$

We finally denote by $\hat{\beta}_{n,t}^{(T-t)} > 0$ the number of units of good 2 that an investment of one unit of good 1 at time t yields at time T. The investor's budget constraint is

$$d\hat{W}_{n,t} = \int_0^T \hat{Z}_{n,t}^{(\tau)} d\left(\frac{P_t^{(\tau)}}{P_t^{(T-t)}}\right) d\tau - d\hat{c}_{n,t}. \tag{B.1}$$

The investor's utility at time T is

$$u(C_{n,T}) + \int_0^T \hat{\beta}_{n,t}^{(T-t)} P_t^{(T-t)} d\hat{c}_{n,t}, \tag{B.2}$$

and consists of two parts: a utility $u(C_{n,T})$ that is an increasing and concave function of the consumption $C_{n,T}$ of good 1 at time T, and a utility $\int_0^T \hat{\beta}_{n,t}^{(T-t)} P_t^{(T-t)} d\hat{c}_{n,t}$ that is equal to the consumption of good 2 at time T and is derived from the accumulated investment in the private opportunity between times 0 and T. The marginal utility $u'(C_{n,T})$ converges to infinity when $C_{n,T}$ goes to a lower bound $\underline{C}$ and to zero when $C_{n,T}$ goes to infinity. The investor has max-min preferences. At each time $t \in [0, T]$, the investor chooses $(\hat{Z}_{n,t}^{(\tau)}, \hat{c}_{n,t})$ to maximize the minimum of (B.2) over sample paths of $q_t = (r_t, \beta_{1,t}, .., \beta_{K,t})^\top$ and $\hat{\beta}_{n,t}^{(T-t)}$, subject to the budget constraint (B.1) and the terminal condition $C_{n,T} = \hat{W}_{n,T}$.

Proposition B.1 *Assume that Σ has full rank, $K \geq 1$, $\hat{\beta}_{n,t}^{(T-t)}$ is a function of $(\beta_{1,t}, .., \beta_{K,t})$, and the term structure involves no arbitrage, i.e., (34) holds. At time t, the investor holds only the bond maturing at time T and no other bonds. The number $\hat{Z}_{n,t}^{(T-t)}$ of units of the bond held by the investor solves*

$$u'\left(\hat{Z}_{n,t}^{(T-t)}\right) = P_t^{(T-t)} \hat{\beta}_{n,t}^{(T-t)}. \tag{B.3}$$

Proof: Defining $(\mu_{\hat{Z},n,t}, \sigma_{\hat{Z},n,t})$ by

$$\int_0^T \hat{Z}_{n,t}^{(\tau)} d\left(\frac{P_t^{(\tau)}}{P_t^{(T-t)}}\right) d\tau \equiv \mu_{\hat{Z},n,t} dt + \sigma_{\hat{Z},n,t} dB_t,$$

where $dB_t = (dB_{r,t}, dB_{\beta,1,t}, .., dB_{\beta,K,t})^\top$, we write the budget constraint (B.1) as

$$d\hat{W}_{n,t} = \mu_{\hat{Z},n,t}dt + \sigma_{\hat{Z},n,t}dB_t - d\hat{c}_{n,t}. \tag{B.4}$$

Integrating (B.4) from 0 to T and using the terminal condition $C_{n,T} = \hat{W}_{n,T}$, we write the investor's optimization problem at $t = 0$ as

$$\begin{aligned}\max_{(\hat{Z}^{(\tau)}_{n,t}, \hat{c}_{n,t}, \Delta\hat{c}_{n,0})} \min_{(q_t, \hat{\beta}^{(T-t)}_{n,t})} & \left[u\left(\hat{W}_{n,0} + \int_0^T \mu_{\hat{Z},n,t}dt + \int_0^T \sigma_{\hat{Z},n,t}dB_t - \Delta\hat{c}_{n,0} - \int_0^T d\hat{c}_{n,t}\right)\right. \\ & \left. + \hat{\beta}^{(T)}_{n,0}P^{(T)}_0 \Delta\hat{c}_{n,0} + \int_0^T \hat{\beta}^{(T-t)}_{n,t} P^{(T-t)}_t d\hat{c}_{n,t}\right],\end{aligned} \tag{B.5}$$

where we allow for the possibility that $\hat{c}_{n,t}$ has a discrete change $\Delta\hat{c}_{n,0}$ at $t = 0$. Since Σ has full rank and $K \geq 1$, r_t is not perfectly correlated with $(\beta_{1,t}, .., \beta_{K,t})$. Since, in addition, $\hat{\beta}^{(T-t)}_{n,t}$ is a function of $(\beta_{1,t}, .., \beta_{K,t})$, sample paths of q_t and $\hat{\beta}^{(T-t)}_{n,t}$ exist such that $\hat{\beta}^{(T-t)}_{n,t} P^{(T-t)}_t = u'\left(\hat{W}_{n,0} - \Delta\hat{c}_{n,0}\right)$ for $t > \epsilon$ and for any $\epsilon > 0$. Hence, the minimum in (B.5) is smaller than

$$\begin{aligned}\min_{q_t, \hat{\beta}^{(T-t)}_{n,t}} & \left[u\left(\hat{W}_{n,0} + \int_0^T \mu_{\hat{Z},n,t}dt + \int_0^T \sigma_{\hat{Z},n,t}dB_t - \Delta\hat{c}_{n,0} - \int_0^T d\hat{c}_{n,t}\right)\right. \\ & \left. + \hat{\beta}^{(T)}_{n,0}P^{(T)}_0 \Delta\hat{c}_{n,0} + u'\left(\hat{W}_{n,0} - \Delta\hat{c}_{n,0}\right) \int_0^T d\hat{c}_{n,t}\right],\end{aligned}$$

which in turn is smaller than

$$\min_{q_t, \hat{\beta}^{(T-t)}_{n,t}} \left[u\left(\hat{W}_{n,0} - \Delta\hat{c}_{n,0}\right) + u'\left(\hat{W}_{n,0} - \Delta\hat{c}_{n,0}\right)\left(\int_0^T \mu_{\hat{Z},n,t}dt + \int_0^T \sigma_{\hat{Z},n,t}dB_t\right) + \hat{\beta}^{(T)}_{n,0}P^{(T)}_0 \Delta\hat{c}_{n,0}\right] \tag{B.6}$$

because u is concave. If $\sigma_{\hat{Z},n,t} \neq 0$ for any interval in $(0, T)$, then the minimum in (B.6) is minus infinity because the Brownian motion has infinite variation. Therefore, $\sigma_{\hat{Z},n,t} = 0$, i.e., the investor holds the bond maturing at time T and zero units of all other bonds. Since absence of arbitrage requires $\mu_{\hat{Z},n,t} = 0$, (B.6) is smaller than

$$u\left(\hat{W}_{n,0} - \Delta\hat{c}_{n,0}\right) + \hat{\beta}^{(T)}_{n,0}P^{(T)}_0 \Delta\hat{c}_{n,0},$$

and hence

$$\begin{aligned}\max_{(\hat{Z}^{(\tau)}_{n,t}, \hat{c}_{n,t}, \Delta\hat{c}_{n,0})} \min_{(q_t, \hat{\beta}^{(T-t)}_{n,t})} & \left[u\left(\hat{W}_{n,0} + \int_0^T \mu_{\hat{Z},n,t}dt + \int_0^T \sigma_{\hat{Z},n,t}dB_t - \Delta\hat{c}_{n,0} - \int_0^T d\hat{c}_{n,t}\right)\right. \\ & \left. + \hat{\beta}^{(T)}_{n,0}P^{(T)}_0 \Delta\hat{c}_{n,t} + \int_0^T \hat{\beta}^{(T-t)}_{n,t} P^{(T-t)}_t d\hat{c}_{n,t}\right] \\ \leq & \max_{\Delta\hat{c}_{n,0}} \left[u\left(\hat{W}_{n,0} - \Delta\hat{c}_{n,0}\right) + \hat{\beta}^{(T)}_{n,0}P^{(T)}_0 \Delta\hat{c}_{n,0}\right].\end{aligned} \tag{B.7}$$

Setting $\hat{Z}_{n,t}^{(\tau)} = 0$ for $t \geq 0$ and $\tau \neq T - t$, and $d\hat{c}_{n,t} = 0$ for $t > 0$, in (B.5), we find that (B.7) holds also in the reverse sense, and is therefore an equality. The optimal $\Delta\hat{c}_{n,0}$ thus satisfies

$$u'\left(\hat{W}_{n,0} - \Delta\hat{c}_{n,0}\right) = \hat{\beta}_{n,0}^{(T)} P_0^{(T)}. \tag{B.8}$$

Since $\hat{W}_{n,0} - \Delta\hat{c}_{n,0}$ represents units of the bond maturing at time T that the investor holds at time 0, (B.8) yields (B.3) for $t = 0$. The same argument yields (B.3) for $t > 0$. *Q.E.D.*

Proposition B.1 implies that preferred-habitat investors demand only the bond whose maturity coincides with the time when they consume. To ensure that the demand by preferred-habitat investors takes the specific functional form (5)-(7), we assume specific functions for the utility u and the return $\hat{\beta}_{n,t}^{(\tau)}$ on the private opportunity.

Suppose $\underline{C} = -\infty$, $u(C_{n,T}) = -e^{-C_{n,T}}$ and $\hat{\beta}_{n,t}^{(\tau)} = e^{\beta_t^{(\tau)}}$, where $\beta_t^{(\tau)}$ is given by (6) and (7). Proposition B.1 implies that the number $\hat{Z}_{n,t}^{(T-t)}$ of units of the bond maturing at time T and held at time t by an investor n born at time 0 is given by

$$e^{-\hat{Z}_{n,t}^{(T-t)}} = P_t^{(T-t)} \hat{\beta}_{n,t}^{(T-t)} \Leftrightarrow \hat{Z}_{n,t}^{(T-t)} = -\log\left(P_t^{(T-t)}\right) - \beta_t^{(T-t)}.$$

This coincides with the demand (5)-(7) with $\alpha(\tau) = 1$, except that (5)-(7) concern the present value of the bond rather than its face value, i.e., the units of the bond. To derive the demand (5)-(7) expressed in present-value terms, we modify the assumed functions for u and $\hat{\beta}_{n,t}^{(\tau)}$. We can obtain the demand (5)-(7) for a set of values of q_t whose probability can be made arbitrarily close to one.

Suppose that there are two types of preferred-habitat investors born at each time t, in equal measure. For type 1 investors, $\underline{C} = 0$, $u(C_{n,t+T}) = \log(C_{n,t+T})$ and $\hat{\beta}_{n,t'}^{(T+t-t')} = -\frac{1}{\min\{\beta_{t'}^{(T+t-t')}, -\epsilon\}}$, where $\beta_t^{(\tau)}$ is given by (6) and (7), and ϵ is positive and small. For type 2 investors, $\underline{C} = -\infty$ and $\hat{\beta}_{n,t'}^{(T+t-t')} = 1$. To define $u(C_{n,t+T})$ for type 2 investors, we start with the function

$$N(x) \equiv -\frac{\log(x)}{x},$$

defined for $x > 0$. The function $N(x)$ converges to infinity when x goes to zero, and to zero when x goes to infinity. It decreases for $x \in (0, e)$, and increases for $x \in (e, T)$. Its minimum value, obtained for $x = e$, is $-\frac{1}{e}$. We take x to represent marginal utility $u'(C_{n,t+T})$, and $N(x)$ to represent $C_{n,t+T}$. This defines $u(C_{n,t+T})$ for $C_{n,t+T} > -\frac{1}{e}$ and $u'(C_{n,t+T}) \in (0, e)$. To define $u(C_{n,t+T})$ for $C_{n,t+T} < -\frac{1}{e}$ and $u'(C_{n,t+T}) > e$, we extend $u'(C_{n,t+T})$ as a linear function of $C_{n,T}$. (Other extensions are possible as well.) We set the derivative of the linear function so that $u'(C_{n,t+T})$ is continuously differentiable at the extension point, and take the extension point to be $u'(C_{n,t+T}) = e(1-\epsilon)$ (rather than $u'(C_{n,t+T}) = e$) so that the derivative is finite.

We thus set

$$u'(C_{n,t+T}) = N^{-1}(C_{n,t+T}) \quad \text{for } C_{n,t+T} \geq N[e(1-\epsilon)],$$

$$u'(C_{n,t+T}) = e(1-\epsilon) - \frac{e^2(1-\epsilon)^2}{\log(1-\epsilon)}[C_{n,t+T} - N[e(1-\epsilon)]] \quad \text{for } C_{n,t+T} < N[e(1-\epsilon)].$$

Since $u'(C_{n,t+T})$ is positive and decreasing, $u(C_{n,t+T})$ is increasing and concave.

Proposition B.1 implies that the number $\hat{Z}_{n,t}^{(T-t)}$ of units of the bond maturing at time T and held at time t by a type 1 investor born at time 0 is given by

$$\frac{1}{\hat{Z}_{n,t}^{(T-t)}} = P_t^{(T-t)} \hat{\beta}_{n,t}^{(T-t)}.$$

This yields the demand

$$P_t^{(T-t)} \hat{Z}_{n,t}^{(T-t)} = \frac{1}{\hat{\beta}_{n,t}^{(T-t)}} = -\beta_t^{(T-t)},$$

expressed in present-value terms, when $\beta_t^{(T-t)} < -\epsilon$. Proposition B.1 implies that the number $\hat{Z}_{n,t}^{(T-t)}$ of units of the bond maturing at time T and held at time t by a type 2 investor born at time 0 is given by

$$N^{-1}\left(\hat{Z}_{n,t}^{(T-t)}\right) = P_t^{(T-t)}$$

when $P_t^{(T-t)} < e(1-\epsilon)$. This yields the demand

$$P_t^{(T-t)} \hat{Z}_{n,t}^{(T-t)} = P_t^{(T-t)} N\left(P_t^{(T-t)}\right) = -\log\left(P_t^{(T-t)}\right),$$

expressed in present-value terms. The aggregate demand, expressed in present-value terms, across type 1 and type 2 investors when $\beta_t^{(T-t)} < -\epsilon$ and $P_t^{(T-t)} < e(1-\epsilon)$ is

$$-\log\left(P_t^{(T-t)}\right) - \beta_t^{(T-t)}$$

and coincides with the demand (5)-(7) with $\alpha(\tau) = 1$. Condition $\beta_t^{(T-t)} < -\epsilon$ requires that the demand intercept in (5) is negative (smaller than $-\epsilon$). Condition $P_t^{(T-t)} < e(1-\epsilon)$ requires that zero-coupon bonds trade below $e(1-\epsilon)$ and hence below par value. The probability of the set of values of q_t such that the two conditions hold simultaneously can be made arbitrarily close to one if $\overline{r}$ is sufficiently large and $\theta_0(\tau)$ sufficiently small.

Proposition B.1 and the subsequent analysis require $K \geq 1$. To extend them to $K = 0$, we assume that $\hat{\beta}_{n,t}^{(T-t)}$ is equal to a deterministic function of $T-t$ plus random noise that is independent across investors n in the same generation. Because of the random noise, $\hat{\beta}_{n,t}^{(T-t)}$ is not perfectly correlated with r_t, and the proof of Proposition B.1 goes through. Because the random noise is independent across investors in the same generation, $\hat{\beta}_{n,t}^{(T-t)}$ averages to a deterministic function of $T-t$.

APPENDIX C: CALIBRATION

C.1. *Model-Generated Moments*

Equations (1) and (30) imply that when there is one demand factor, the yield for maturity τ is

$$y_t^{(\tau)} = \frac{A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)}{\tau}.$$

When, in addition, the demand factor is independent of the short rate, the volatility of the yield is

$$\sqrt{\mathbb{V}\text{ar}\left(y_t^{(\tau)}\right)} = \frac{\sqrt{A_r(\tau)^2\text{Var}(r_t) + A_\beta(\tau)^2\text{Var}(\beta_t)}}{\tau} = \frac{\sqrt{A_r(\tau)^2\frac{\sigma_r^2}{2\kappa_r} + A_\beta(\tau)^2\frac{\sigma_\beta^2}{2\kappa_\beta}}}{\tau}. \tag{C.1}$$

The volatility of yield changes during an interval of length $\Delta\tau$ is

$$\begin{aligned}\sqrt{\mathbb{V}\text{ar}\left(y_{t+\Delta\tau}^{(\tau)} - y_t^{(\tau)}\right)} &= \frac{\sqrt{A_r(\tau)^2\text{Var}(r_{t+\Delta\tau} - r_t) + A_\beta(\tau)^2\text{Var}(\beta_{t+\Delta\tau} - \beta_t)}}{\tau} \\ &= \frac{\sqrt{A_r(\tau)^2\frac{\sigma_r^2}{\kappa_r}(1 - e^{-\kappa_r\Delta\tau}) + A_\beta(\tau)^2\frac{\sigma_\beta^2}{\kappa_\beta}(1 - e^{-\kappa_\beta\Delta\tau})}}{\tau}, \qquad \text{(C.2)}\end{aligned}$$

where the second step follows from (A.7) and its counterpart equation for β_t. The covariance of yield changes is

$$\begin{aligned}&\mathbb{C}\text{ov}\left(y_{t+\Delta\tau}^{(\tau_1)} - y_t^{(\tau_1)}, y_{t+\Delta\tau}^{(\tau_2)} - y_t^{(\tau_2)}\right) \\ &= \frac{A_r(\tau_1)A_r(\tau_2)\text{Var}(r_{t+\Delta\tau} - r_t) + A_\beta(\tau_1)A_\beta(\tau_2)\text{Var}(\beta_{t+\Delta\tau} - \beta_t)}{\tau} \\ &= \frac{A_r(\tau_1)A_r(\tau_2)\frac{\sigma_r^2}{\kappa_r}(1 - e^{-\kappa_r\Delta\tau}) + A_\beta(\tau_1)A_\beta(\tau_2)\frac{\sigma_\beta^2}{\kappa_\beta}(1 - e^{-\kappa_\beta\Delta\tau})}{\tau}. \qquad \text{(C.3)}\end{aligned}$$

The correlation of yield changes can be computed from (C.2) and (C.3). The principal components can be computed from the covariance matrix of yield changes, with element (τ_1, τ_2) given by (C.3). The FB and CS regression coefficients are given by (A.115) and (A.122), respectively.

The volume during an infinitesimal interval $[t, t + dt]$ for the bond with maturity $\tau \in (0, T)$ is the absolute value of the change $dZ_t^{(\tau)}$ in the demand of preferred-habitat investors. The change $dZ_t^{(\tau)}$ is

$$\begin{aligned}dZ_t^{(\tau)} &= -d\left\{\alpha(\tau)\log(P_t^{(\tau)}) + \beta_t^{(\tau)}\right\} \\ &= d\left\{\alpha(\tau)\left[A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)\right] - \left[\theta_0(\tau) + \theta(\tau)\beta_t\right]\right\}, \qquad \text{(C.4)}\end{aligned}$$

where the first step follows from (5), and the second from (6) and (30) written for one demand factor. Equation (C.4) implies that expected volume is

$$\mathbb{E}\left(\left|dZ_t^{(\tau)}\right|\right) = \mathbb{E}\left[\mathbb{E}_t\left(\left|dZ_t^{(\tau)}\right|\right)\right] = \mathbb{E}\left[\sqrt{\frac{2}{\pi}\mathbb{V}\text{ar}_t\left(dZ_t^{(\tau)}\right)}\right] = \mathbb{E}\left[\sqrt{\frac{2}{\pi}V(\tau)dt}\right] = \sqrt{\frac{2}{\pi}V(\tau)dt},$$

where independence between the short rate and demand implies

$$\begin{aligned} V(\tau) &\equiv \alpha(\tau)^2 A_r(\tau)^2\sigma_r^2 + \left[\alpha(\tau)A_\beta(\tau) - \theta(\tau)\right]^2 \sigma_\beta^2 \\ &= \alpha e^{-2\delta_\alpha\tau} A_r(\tau)^2\sigma_r^2 + \left[\alpha e^{-\delta_\alpha\tau} A_\beta(\tau) - \theta\left(e^{-\delta_\alpha\tau} - e^{-\delta_\theta\tau}\right)\right]^2 \sigma_\beta^2. \end{aligned}$$

In our calculations of relative volume we use $\sqrt{V(\tau)}$, which is proportional to expected volume.

When yields across all maturities change by Δy, (1) and (5) imply that the demand of preferred-habitat investors changes by

$$\Delta y \int_0^\infty \alpha(\tau)\tau d\tau = \alpha\Delta y \int_0^\infty e^{-\delta_\alpha\tau}\tau d\tau = \alpha\Delta y \frac{1 - e^{-\delta_\alpha T} - \delta_\alpha T e^{-\delta_\alpha T}}{\delta_\alpha^2}.$$

Setting $(T, \delta_\alpha, \Delta y) = (30, 0.297, 0.0001)$ and the demand change to 0.0059, we find $\alpha = 5.21$.

C.2. *Calibrated Parameters*

Tables C.I and C.II report the calibrated parameters and the empirical moments used to determine them, for the sub-sample of nominal yields and the sample of real yields, respectively.

C.3. *Elasticities*

The matrix in the top panel of Table C.III reports the elasticities of the first seven model-generated moments in Table I with respect to the first seven parameters, for the main sample of nominal yields. The elasticities are computed by varying each parameter from its value in Table I times 1.001 to its value in Table I times 0.999, computing the change in the corresponding model-generated moment, dividing by the value of that moment in the base case, and multiplying by 500.

The elasticities involving $(\delta_\alpha, \delta_\theta)$ are hard to interpret because they combine multiple effects. For example, an increase in δ_θ lowers the relative volume for long maturities. It also strengthens the effect of demand shocks on yields, since the shocks' magnitude is $\theta(\tau) = \theta\left(e^{-\delta_\alpha\tau} - e^{-\delta_\theta\tau}\right)$, which increases in δ_θ. This raises the volatility of yields and lowers the correlation between yield changes at short and long maturities.

To disentangle the effects and facilitate the interpretation of the elasticities, we modify the matrix in the top panel of Table C.III by subtracting columns $i = 4, 5$ from columns $j = 6, 7$, after multiplying each time column i by the scalar needed to make element (i, j) equal to zero. For $i = 4$, this amounts to keeping the volatility of annual yield changes constant when changing $(\delta_\alpha, \delta_\theta)$, through a compensating change in $a\theta$. For $i = 5$, this amounts to keeping the correlation between annual changes to the one-year yield and to other yields constant when changing $(\delta_\alpha, \delta_\theta)$, through a compensating change in $a\alpha$. Eliminating the effects of $(\delta_\alpha, \delta_\theta)$ on the volatility of yields and on the correlation between them results in the simpler matrix of modified elasticities in the bottom panel of Table C.III. We focus on that matrix from now on.

The parameter κ_r has its strongest, negative, effect on the volatility of the one-year yield. The parameter σ_r has its strongest, positive, effect on the volatility of the one-year yield and on the volatility of annual changes to that yield. Other parameters

TABLE C.I

CALIBRATION OF MODEL PARAMETERS FOR THE SUB-SAMPLE OF NOMINAL YIELDS.

Parameter	**Value**	**Empirical moment**	**Value**
κ_r Mean-reversion of r_t	0.240	$\sqrt{\mathbb{V}\mathrm{ar}\left(y_t^{(1)}\right)}$ Volatility 1-year yield – Levels	1.89
σ_r Diffusion of r_t	0.0159	$\sqrt{\mathbb{V}\mathrm{ar}\left(y_{t+1}^{(1)}-y_t^{(1)}\right)}$ Volatility 1-year yield – Annual changes	1.24
κ_β Mean-reversion of β_t	0.127	$\frac{1}{30}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\mathrm{ar}\left(y_t^{(\tau)}\right)}$ Volatility τ-year yield – Levels, average over τ	1.36
$a\theta$ Arb. risk-aversion $\times$ PH demand shock	5305	$\frac{1}{30}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\mathrm{ar}\left(y_{t+1}^{(\tau)}-y_t^{(\tau)}\right)}$ Volatility τ-year yield – Annual changes, average over τ	0.705
$a\alpha$ Arb. risk-aversion $\times$ PH demand slope	80.3	$\frac{1}{30}\sum_{\tau=1}^{30}\mathbb{C}\mathrm{orr}\left(y_{t+1}^{(1)}-y_t^{(1)},y_{t+1}^{(\tau)}-y_t^{(\tau)}\right)$ Correlation 1-year yield with τ-year yield – Annual changes, average over τ	0.369
δ_α PH demand shock – short maturities	0.269	$\frac{\sum_{0<\tau\leq 2}\mathrm{Volume}(\tau)}{\sum_{0<\tau\leq 30}\mathrm{Volume}(\tau)}$ Relative volume for maturities $\tau\in(0,2]$	0.199
δ_θ PH demand shock – long maturities	0.279	$\frac{\sum_{11<\tau\leq 30}\mathrm{Volume}(\tau)}{\sum_{0<\tau\leq 30}\mathrm{Volume}(\tau)}$ Relative volume for maturities $\tau\in(11,30]$	0.094
α PH demand slope	4.28	Estimate in KVJ 2012	-0.746

TABLE C.II

CALIBRATION OF MODEL PARAMETERS FOR THE SAMPLE OF REAL YIELDS.

Parameter	**Value**	**Empirical moment**	**Value**
κ_r Mean-reversion of r_t	0.395	$\sqrt{\mathbb{V}\text{ar}\left(y_t^{(2)}\right)}$ Volatility 2-year yield – Levels	1.59
σ_r Diffusion of r_t	0.0216	$\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(2)} - y_t^{(2)}\right)}$ Volatility 2-year yield – Annual changes	1.23
κ_β Mean-reversion of β_t	0.098	$\frac{1}{19}\sum_{\tau=2}^{20}\sqrt{\mathbb{V}\text{ar}\left(y_t^{(\tau)}\right)}$ Volatility τ-year yield – Levels, average over τ	1.30
$a\theta$ Arb. risk-aversion $\times$ PH demand shock	643	$\frac{1}{19}\sum_{\tau=2}^{20}\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(\tau)} - y_t^{(\tau)}\right)}$ Volatility τ-year yield – Annual changes, average over τ	0.674
$a\alpha$ Arb. risk-aversion $\times$ PH demand slope	44.5	$\frac{1}{19}\sum_{\tau=2}^{20}\mathbb{C}\text{orr}\left(y_{t+1}^{(2)} - y_t^{(2)}, y_{t+1}^{(\tau)} - y_t^{(\tau)}\right)$ Correlation 2-year yield with τ-year yield – Annual changes, average over τ	0.660
δ_α PH demand shock – short maturities	0.265	$\frac{\sum_{0<\tau\leq 2}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$ Relative volume for maturities $\tau \in (0,2]$	0.199
δ_θ PH demand shock – long maturities	0.308	$\frac{\sum_{11<\tau\leq 30}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$ Relative volume for maturities $\tau \in (11,30]$	0.094
α PH demand slope	4.16	Estimate in KVJ 2012	-0.746

TABLE C.III

Elasticities and modified elasticities of model-generated moments with respect to model parameters for the main sample of nominal yields.

Parameter	κ_r	σ_r	κ_β	$a\theta$	$a\alpha$	δ_α	δ_θ
$\sqrt{\mathbb{V}\text{ar}\left(y_t^{(1)}\right)}$	-0.538	0.468	-0.006	0.017	-0.041	-0.448	0.500
$\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(1)}-y_t^{(1)}\right)}$	-0.074	0.467	-0.001	0.012	-0.039	-0.315	0.009
$\frac{1}{19}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\text{ar}\left(y_t^{(\tau)}\right)}$	-0.318	0.344	-0.672	1.493	-0.903	-43.873	43.661
$\frac{1}{19}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(\tau)}-y_t^{(\tau)}\right)}$	-0.202	0.330	-0.256	1.243	-0.791	-36.446	36.340
$\frac{1}{19}\sum_{\tau=1}^{30}\mathbb{C}\text{orr}\left(y_{t+1}^{(2)}-y_t^{(2)},y_{t+1}^{(\tau)}-y_t^{(\tau)}\right)$	0.083	-0.207	0.225	-1.443	0.514	43.209	-42.273
$\frac{\sum_{0<\tau\leq 2}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$	-0.019	0.179	-0.036	0.028	0.165	-0.379	1.192
$\frac{\sum_{11<\tau\leq 30}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$	0.189	-0.365	0.236	-0.519	-0.106	15.304	-16.877

Parameter	κ_r	σ_r	κ_β	$a\theta$	$a\alpha$	δ_α	δ_θ
$\sqrt{\mathbb{V}\text{ar}\left(y_t^{(1)}\right)}$	-0.538	0.468	-0.006	0.017	-0.041	-0.017	0.011
$\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(1)}-y_t^{(1)}\right)}$	-0.074	0.467	-0.001	0.012	-0.039	-0.018	0.009
$\frac{1}{19}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\text{ar}\left(y_t^{(\tau)}\right)}$	-0.318	0.344	-0.672	1.493	-0.903	0.008	0.003
$\frac{1}{19}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(\tau)}-y_t^{(\tau)}\right)}$	-0.202	0.330	-0.256	1.243	-0.791	0	0
$\frac{1}{19}\sum_{\tau=1}^{30}\mathbb{C}\text{orr}\left(y_{t+1}^{(2)}-y_t^{(2)},y_{t+1}^{(\tau)}-y_t^{(\tau)}\right)$	0.083	-0.207	0.225	-1.443	0.514	0	0
$\frac{\sum_{0<\tau\leq 2}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$	-0.019	0.179	-0.036	0.028	0.165	0.856	0.327
$\frac{\sum_{11<\tau\leq 30}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$	0.189	-0.365	0.236	-0.519	-0.106	-0.875	-1.617

have much weaker effects on these volatilities. Hence, the volatility of the one-year yield identifies κ_r, and the volatility of annual changes to that yield identifies σ_r.

The parameter κ_β has its strongest, negative, effect on the average volatility of yields. The parameters $(a\theta, a\alpha)$ have their strongest effects on the average volatility of yields, on the average volatility of annual yield changes, and on the average correlation between annual changes to the one-year yield and to other yields. Hence, the average volatility of yields identifies κ_β, and the other two moments identify $(a\theta, a\alpha)$.[23]

The parameters $(\delta_\alpha, \delta_\theta)$ have their strongest effect on relative volume, positive for short maturities and negative for long maturities. The effect of δ_θ on short-maturity volume is weaker. Hence, the relative volume for maturities two years and below identifies δ_α, and the relative volume for maturities eleven years and above identifies δ_θ.

Tables C.IV and C.V provide counterpart matrices to that in the bottom panel of Table C.III, for the sub-sample of nominal yields and the sample of real yields, respectively. The modified elasticities for these samples have similar magnitudes and signs to those for the main sample of nominal yields.

C.4. *Figures*

Figures C.1 and C.2 compare the empirical moments to the model-generated ones, for the sub-sample of nominal yields and the sample of real yields, respectively. For the sub-sample of nominal yields, the fraction of variation of annual yield changes explained by the first principal component is 74% in the model and 73.8% in the data. For the sample of real yields, maturities range from two to twenty. The one-year yield needed to compute the empirical FB and CS coefficients is obtained by spline interpolation. The first principal component of annual yield changes is scaled to one for the two-year maturity. The fraction of variation of annual yield changes explained by the first principal component is 83.6% in the model and 85.2% in the data.

C.5. *Policy Analysis*

Consider an unanticipated change $\Delta\overline{r}$ in the long-run mean $\overline{r}$ of the short rate r_t at time zero that reverts deterministically to zero at the rate $\kappa_{\overline{r}}$. Writing bond prices at time t as

$$P_t^{(\tau)} = e^{-[A_r(\tau)r_t + A_\beta(\tau)\beta_t + A_{\overline{r}}(\tau)\Delta\overline{r}e^{-\kappa_{\overline{r}}t} + C(\tau)]}$$

[23]Table C.III shows that an increase in $a\alpha$ *raises* the correlation between yield changes at short and long maturities (element (5,5) is positive). Intuitively, an increase in α, holding (a, θ) constant, weakens the transmission of short-rate shocks to longer maturities, and this lowers correlation. At the same time, demand shocks have weaker effects on yields because shocks are better absorbed when preferred-habitat demand has higher slope. The latter effect lowers volatility and raises correlation. The latter effect also makes the mapping between $a\theta$ and volatility, and between $a\alpha$ and correlation, less clear-cut. To isolate the former effect, we consider an increase in $a\alpha$ accompanied by an increase in $a\theta$ such that the volatility of annual yield changes remains constant. (This amounts to subtracting column 4 from column 5, after multiplying column 4 by the scalar needed to make element $(4, 5)$ equal to zero.) Element $(5, 5)$ then turns negative, capturing only the former effect.

TABLE C.IV

Modified elasticities of model-generated moments with respect to model parameters for the sub-sample of nominal yields.

Parameter	κ_r	σ_r	κ_β	$a\theta$	$a\alpha$	δ_α	δ_θ
$\sqrt{\mathbb{V}\text{ar}\left(y_t^{(1)}\right)}$	-0.572	0.454	-0.007	0.025	-0.058	-0.063	0.065
$\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(1)} - y_t^{(1)}\right)}$	-0.137	0.454	-0.001	0.021	-0.057	-0.062	0.061
$\frac{1}{19}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\text{ar}\left(y_t^{(\tau)}\right)}$	-0.195	0.191	-0.924	2.054	-1.335	0.009	-0.001
$\frac{1}{19}\sum_{\tau=1}^{30}\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(\tau)} - y_t^{(\tau)}\right)}$	-0.122	0.198	-0.507	1.898	-1.251	0	0
$\frac{1}{19}\sum_{\tau=1}^{30}\mathbb{C}\text{orr}\left(y_{t+1}^{(2)} - y_t^{(2)}, y_{t+1}^{(\tau)} - y_t^{(\tau)}\right)$	0.157	-0.142	0.406	-2.145	0.930	0	0
$\frac{\sum_{0<\tau\leq 2}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$	-0.045	0.225	-0.107	0.134	0.158	0.938	-0.008
$\frac{\sum_{11<\tau\leq 30}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$	0.269	-0.362	0.767	-1.626	0.451	-0.724	-1.143

TABLE C.V

Modified elasticities of model-generated moments with respect to model parameters for the sample of real yields.

Parameter	κ_r	σ_r	κ_β	$a\theta$	$a\alpha$	δ_α	δ_θ
$\sqrt{\mathbb{V}\text{ar}\left(y_t^{(2)}\right)}$	-0.627	0.460	-0.012	0.028	-0.054	-0.027	0.026
$\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(2)} - y_t^{(2)}\right)}$	-0.230	0.459	-0.001	0.016	-0.049	-0.028	0.020
$\frac{1}{19}\sum_{\tau=2}^{20}\sqrt{\mathbb{V}\text{ar}\left(y_t^{(\tau)}\right)}$	-0.455	0.362	-0.911	1.903	-1.090	0.012	0.003
$\frac{1}{19}\sum_{\tau=2}^{20}\sqrt{\mathbb{V}\text{ar}\left(y_{t+1}^{(\tau)} - y_t^{(\tau)}\right)}$	-0.386	0.352	-0.427	1.536	-0.916	0	0
$\frac{1}{19}\sum_{\tau=2}^{20}\mathbb{C}\text{orr}\left(y_{t+1}^{(2)} - y_t^{(2)}, y_{t+1}^{(\tau)} - y_t^{(\tau)}\right)$	0.129	-0.153	0.377	-1.671	0.683	0	0
$\frac{\sum_{0<\tau\leq 2}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$	-0.094	0.241	-0.084	0.022	0.230	0.838	0.176
$\frac{\sum_{11<\tau\leq 30}\text{Volume}(\tau)}{\sum_{0<\tau\leq 30}\text{Volume}(\tau)}$	0.457	-0.422	0.514	-0.945	0.051	-0.823	-1.458

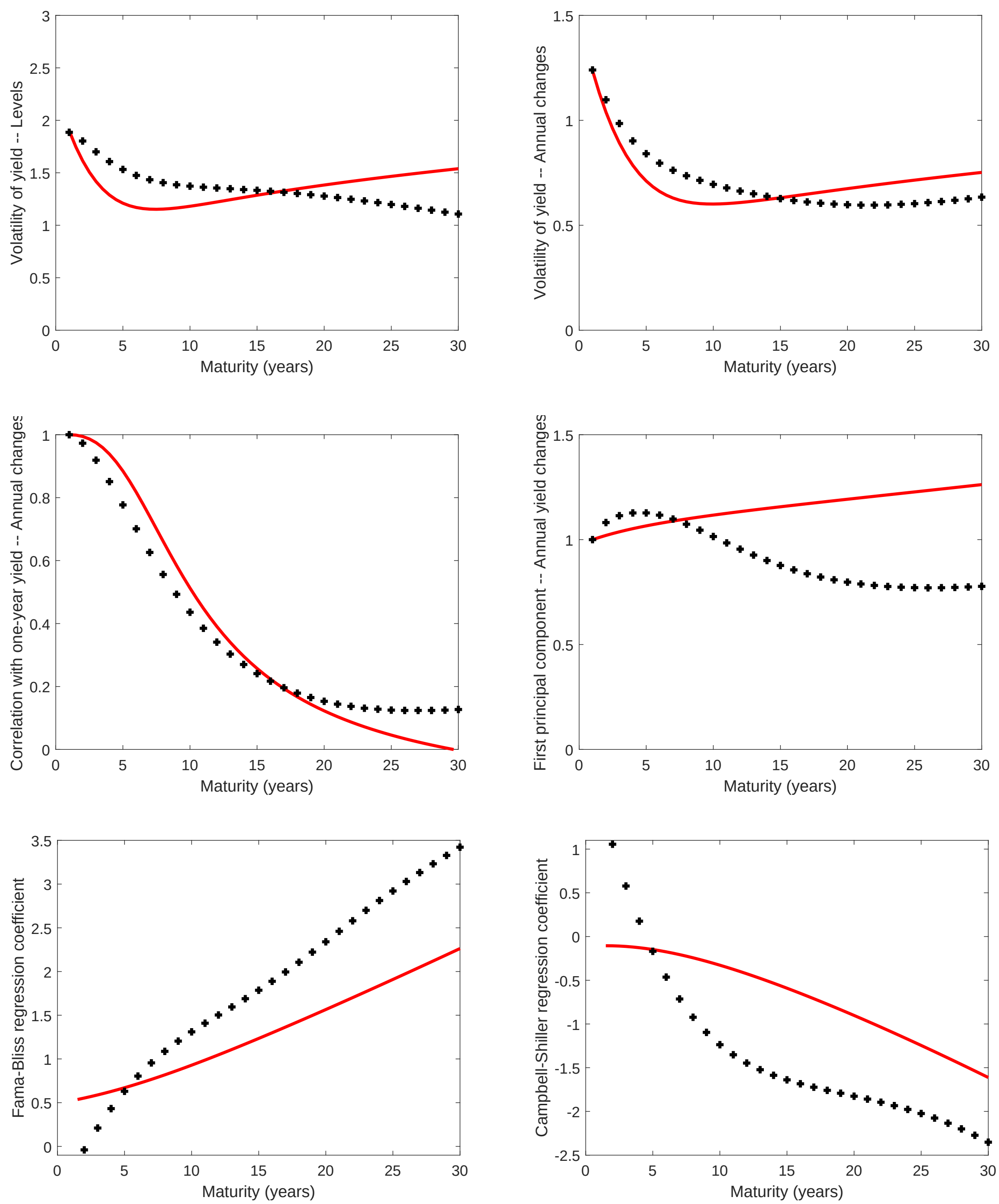

FIGURE C.1.— Model-generated and empirical moments for the sub-sample of nominal yields.

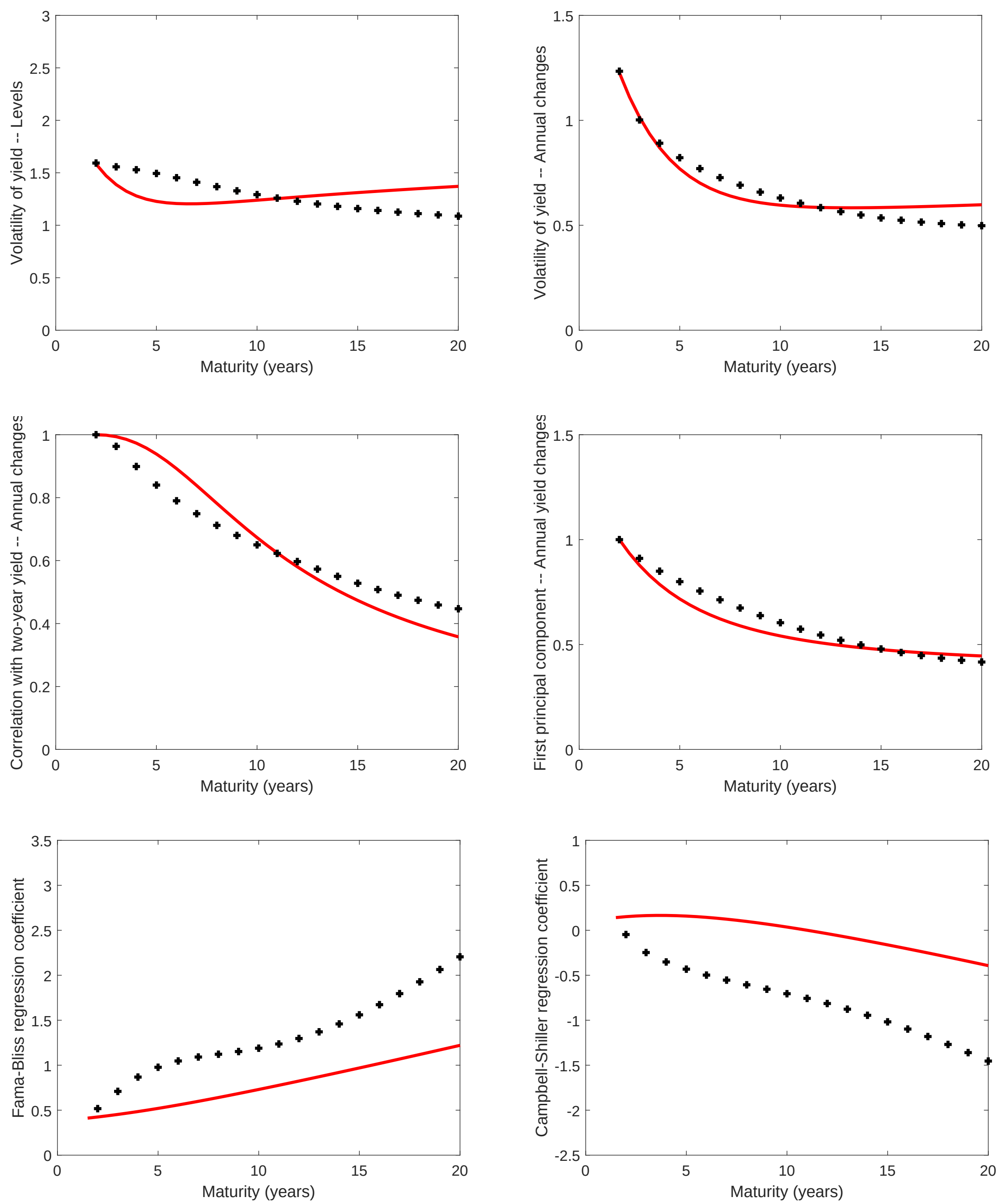

FIGURE C.2.— Model-generated and empirical moments for the sample of real yields.

and proceeding as in Sections 3 and 4, we find that $A_{\overline{r}}(\tau)$ solves the ODE

$$A'_{\overline{r}}(\tau)+\kappa_{\overline{r}}A_{\overline{r}}(\tau)-\kappa_r A_r(\tau)=-a\sigma_r^2 A_r(\tau)\int_0^\infty \alpha(\tau)A_{\overline{r}}(\tau)A_r(\tau)d\tau - a\sigma_\beta^2 A_\beta(\tau)\int_0^\infty \alpha(\tau)A_{\overline{r}}(\tau)A_\beta(\tau)d\tau.$$

Proceeding as in the proofs of Lemma 3 and Proposition 7, we find that the solution to the ODE is

$$A_{\overline{r}}(\tau)=\chi_r\int_0^\tau A_r(u)e^{-\kappa_{\overline{r}}(\tau-u)}du+\chi_\beta\int_0^\tau A_\beta(u)e^{-\kappa_{\overline{r}}(\tau-u)}du,$$

where

$$\chi_r\equiv\frac{\kappa_r}{D}\left[1+a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)e^{-\kappa_{\overline{r}}(\tau-u)}du\right)A_\beta(\tau)d\tau\right],$$

$$\chi_\beta\equiv-\frac{\kappa_r}{D}a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)e^{-\kappa_{\overline{r}}(\tau-u)}du\right)A_\beta(\tau)d\tau,$$

and

$$\begin{aligned}D\equiv&\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)e^{-\kappa_{\overline{r}}(\tau-u)}du\right)A_r(\tau)d\tau\right]\\&\times\left[1+a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)e^{-\kappa_{\overline{r}}(\tau-u)}du\right)A_\beta(\tau)d\tau\right]\\&-\left[a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)e^{-\kappa_{\overline{r}}(\tau-u)}du\right)A_r(\tau)d\tau\right]\\&\times\left[a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)e^{-\kappa_{\overline{r}}(\tau-u)}du\right)A_\beta(\tau)d\tau\right].\end{aligned}$$

When $a=0$, $(\chi_r,\chi_\beta,A_r(\tau),A_\beta(\tau))=(\kappa_r,0,\frac{1-e^{-\kappa_r\tau}}{\kappa_r},0)$ and

$$A_{\overline{r}}(\tau)=\int_0^\tau\left(1-e^{-\kappa_r u}\right)e^{-\kappa_{\overline{r}}(\tau-u)}du.$$

Figures C.3 and C.4 show the effects of a forward-guidance announcement about the path of short rates, for the calibrations based on the sub-sample of nominal yields and the sample of real yields, respectively. In each panel, the red solid line represents the announcement's effect on the term structure, and the red dashed line represents the same effect when arbitrageurs are risk-neutral and the EH holds. The change $\Delta\overline{r}$ in the long-run mean $\overline{r}$ of the short rate r_t is set to -4 (-400 bps). It reverts to zero at the rate $\kappa_{\overline{r}}=0.1$ in the left panel and $\kappa_{\overline{r}}=0.2$ in the right panel.

Consider next an unanticipated change $\Delta\theta_0(\tau)$ in the intercept of preferred-habitat demand at time zero that reverts deterministically to zero at the rate κ_θ. Writing bond prices at time t as

$$P_t^{(\tau)}=e^{-[A_r(\tau)r_t+A_\beta(\tau)\beta_t+A_\theta(\tau)\Delta\theta_0(\tau)e^{-\kappa_\theta t}+C(\tau)]}$$

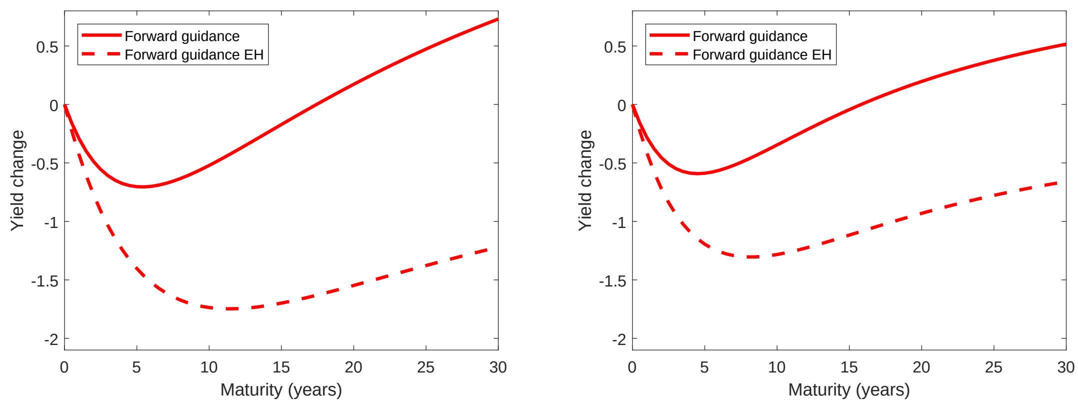

FIGURE C.3.— Effect of a forward-guidance announcement about the path of short rates, for the calibration based on the sub-sample of nominal yields.

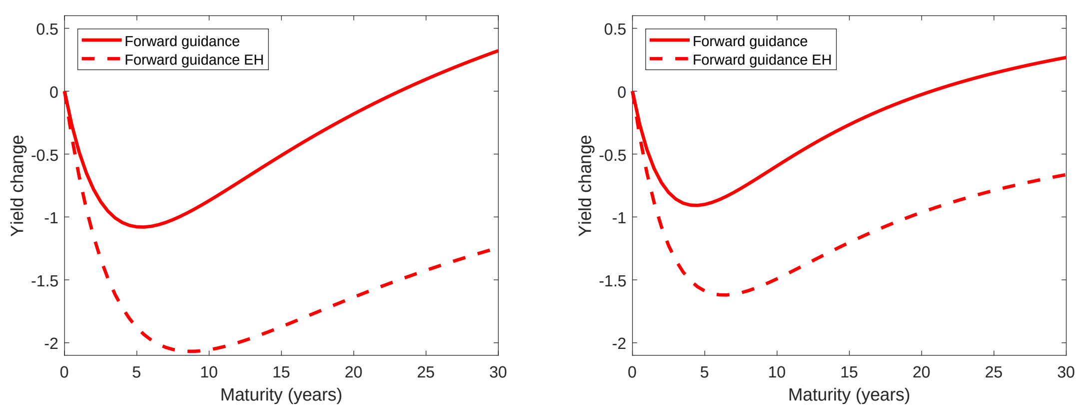

FIGURE C.4.— Effect of a forward-guidance announcement about the path of short rates, for the calibration based on the sample of real yields.

and proceeding as in Sections 3 and 4, we find that $A_\theta(\tau)$ solves the ODE

$$A'_\theta(\tau)+\kappa_\theta A_\theta(\tau)=a\sigma_r^2 A_r(\tau)\int_0^\infty[\Delta\theta_0(\tau)-\alpha(\tau)A_\theta(\tau)]A_r(\tau)d\tau+a\sigma_\beta^2 A_\beta(\tau)\int_0^\infty[\Delta\theta_0(\tau)-\alpha(\tau)A_\theta(\tau)]A_\beta(\tau)d\tau.$$

Proceeding as in the proofs of Lemma 3 and Proposition 7, we find that the solution to the ODE is

$$A_\theta(\tau)=\chi_r\int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du+\chi_\beta\int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du,$$

where

$$\begin{aligned}\chi_r\equiv\frac{1}{D}\Bigg\{&a\sigma_r^2\left[\int_0^\infty\Delta\theta_0(\tau)A_r(\tau)d\tau\right]\left[1+a\sigma_\beta^2\int_0^\infty\alpha(\tau)\left(\int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du\right)A_\beta(\tau)d\tau\right]\\&-a\sigma_\beta^2\left[\int_0^\infty\Delta\theta_0(\tau)A_\beta(\tau)d\tau\right]\left[a\sigma_r^2\int_0^\infty\alpha(\tau)\left(\int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du\right)A_r(\tau)d\tau\right]\Bigg\},\\ \chi_\beta\equiv\frac{1}{D}\Bigg\{&a\sigma_\beta^2\left[\int_0^\infty\Delta\theta_0(\tau)A_\beta(\tau)d\tau\right]\left[1+a\sigma_r^2\int_0^\infty\alpha(\tau)\left(\int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du\right)A_r(\tau)d\tau\right]\\&-a\sigma_r^2\left[\int_0^\infty\Delta\theta_0(\tau)A_r(\tau)d\tau\right]\left[a\sigma_\beta^2\int_0^\infty\alpha(\tau)\left(\int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du\right)A_\beta(\tau)d\tau\right]\Bigg\},\end{aligned}$$

and

$$\begin{aligned}D\equiv&\left[1+a\sigma_r^2\int_0^\infty\alpha(\tau)\left(\int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du\right)A_r(\tau)d\tau\right]\\&\times\left[1+a\sigma_\beta^2\int_0^\infty\alpha(\tau)\left(\int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du\right)A_\beta(\tau)d\tau\right]\\&-\left[a\sigma_r^2\int_0^\infty\alpha(\tau)\left(\int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du\right)A_r(\tau)d\tau\right]\\&\times\left[a\sigma_\beta^2\int_0^\infty\alpha(\tau)\left(\int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du\right)A_\beta(\tau)d\tau\right].\end{aligned}$$

When the change $\Delta\theta_0(\tau)$ is a Dirac function with point mass at τ^*,

$$\int_0^\infty\Delta\theta_0(\tau)A_j(\tau)d\tau=A_j(\tau^*)$$

for $j=r,\beta$. Hence, the time-zero change in the yield for maturity τ is

$$\Delta y_{t,\tau^*}^{(\tau)}=\frac{1}{D}\left[\Lambda_r(\tau^*)\frac{\int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du}{\tau}+\Lambda_\beta(\tau^*)\frac{\int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du}{\tau}\right], \tag{C.5}$$

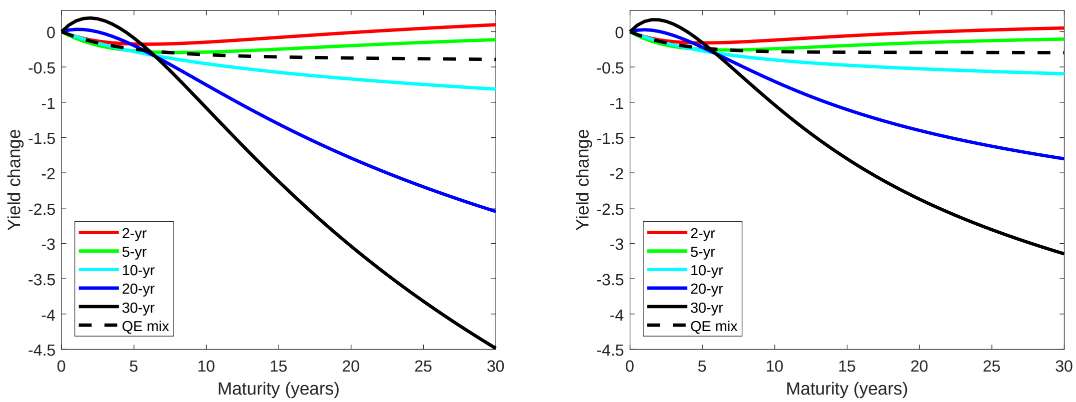

FIGURE C.5.— Effect of QE, for the calibration based on the sub-sample of nominal yields.

where

$$\Lambda_r(\tau^*) \equiv a\sigma_r^2 A_r(\tau^*) \left[1 + a\sigma_\beta^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u) e^{-\kappa_\theta(\tau-u)} du\right) A_\beta(\tau) d\tau\right]$$

$$- a\sigma_\beta^2 A_\beta(\tau^*) \left[a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_\beta(u) e^{-\kappa_\theta(\tau-u)} du\right) A_r(\tau) d\tau\right],$$

$$\Lambda_\beta(\tau^*) \equiv a\sigma_\beta^2 A_\beta(\tau^*) \left[1 + a\sigma_r^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u) e^{-\kappa_\theta(\tau-u)} du\right) A_r(\tau) d\tau\right]$$

$$- a\sigma_r^2 A_r(\tau^*) \left[a\sigma_\beta^2 \int_0^\infty \alpha(\tau) \left(\int_0^\tau A_r(u) e^{-\kappa_\theta(\tau-u)} du\right) A_\beta(\tau) d\tau\right].$$

Figures C.5 and C.6 show the effects of QE for the calibrations based on the sub-sample of nominal yields and the sample of real yields, respectively. In each panel, the red, green, light blue (cyan), blue and black solid lines represent the effect of QE purchases of two-, five-, ten-, twenty- and thirty-year bonds, respectively. The black dashed line represents the effect of QE purchases that conform to the maturity distribution used by the Fed during QE1, as reported in D'Amico and King (2013). In all cases, the change $\Delta\theta_0(\tau)$ in the intercept of preferred-habitat demand is such that $\int_0^\infty \Delta\theta_0(\tau)d\tau = -0.12$, i.e., QE purchases are 12% of GDP. QE is unwound at the rate $\kappa_{\overline{r}} = 0.1$ in the left panel and $\kappa_{\overline{r}} = 0.2$ in the right panel. We use the value of a that generates the average effect across the lower and the upper bound. These bounds are $a = 18.8$ and $a = 93.8$, respectively, in Figure C.5, and $a = 10.7$ and $a = 53.5$, respectively, in Figure C.6.

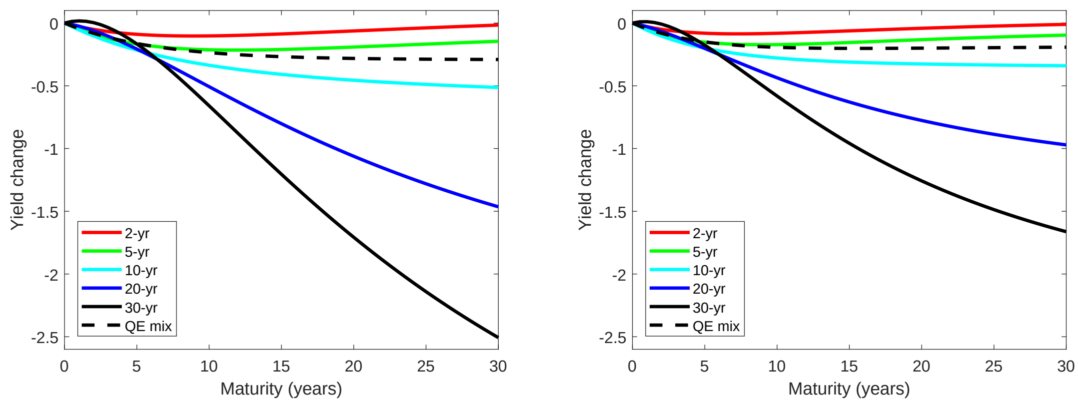

FIGURE C.6.— Effect of QE, for the calibration based on the sample of real yields.

C.6. *Unconditional Moments*

The expected excess return of the bond with maturity τ is equal to the right-hand side of (35). When there is one demand factor which is independent of the short rate, the right-hand side of (35) becomes

$$a\sigma_r^2 A_r(\tau)\int_0^\infty \left[\theta_0(\tau)+\theta(\tau)\beta_t-\alpha(\tau)\left(A_r(\tau)r_t+A_\beta(\tau)\beta_t+C(\tau)\right)\right]A_r(\tau)d\tau$$

$$+\,a\sigma_r^2 A_\beta(\tau)\int_0^\infty \left[\theta_0(\tau)+\theta(\tau)\beta_t-\alpha(\tau)\left(A_r(\tau)r_t+A_\beta(\tau)\beta_t+C(\tau)\right)\right]A_\beta(\tau)d\tau.$$

Taking expectations with respect to (r_t,β_t), we find that the unconditional expected excess return is

$$a\sigma_r^2 A_r(\tau)\mathcal{M}_r + a\sigma_\beta^2 A_\beta(\tau)\mathcal{M}_\beta, \tag{C.6}$$

where

$$\mathcal{M}_r \equiv \int_0^\infty \left[\theta_0(\tau)-\alpha(\tau)\left(A_r(\tau)\overline{r}+C(\tau)\right)\right]A_r(\tau)d\tau,$$

$$\mathcal{M}_\beta \equiv \int_0^\infty \left[\theta_0(\tau)-\alpha(\tau)\left(A_r(\tau)\overline{r}+C(\tau)\right)\right]A_\beta(\tau)d\tau.$$

The Sharpe ratio of the bond with maturity τ is

$$\frac{a\sigma_r^2 A_r(\tau)\mathcal{M}_r + a\sigma_\beta^2 A_\beta(\tau)\mathcal{M}_\beta}{\sqrt{\sigma_r^2 A_r(\tau)^2+\sigma_\beta^2 A_\beta(\tau)^2}}.$$

The correlation between the return on the bond with maturity τ and the stochastic discount factor is

$$\frac{\sigma_r^2 A_r(\tau)\mathcal{M}_r + \sigma_\beta^2 A_\beta(\tau)\mathcal{M}_\beta}{\sqrt{\sigma_r^2 A_r(\tau)^2+\sigma_\beta^2 A_\beta(\tau)^2}\sqrt{\sigma_r^2\mathcal{M}_r^2+\sigma_\beta^2\mathcal{M}_\beta^2}}.$$

The stochastic discount factor parameters $(\mathcal{M}_r, \mathcal{M}_\beta)$ depend on $C(\tau)$. When there is one demand factor which is independent of the short rate, (41) becomes

$$C(\tau) = \chi_r \int_0^\tau A_r(u)du + \chi_\beta \int_0^\tau A_\beta(u)du - \frac{1}{2}\left(\sigma_r^2 \int_0^\tau A_r(u)^2 du + \sigma_\beta^2 \int_0^\tau A_\beta(u)^2 du\right).$$

The constants (χ_r, χ_β) are given by (A.129) and (A.130).