Search and Endogenous Concentration of Liquidity in Asset Markets ⋆

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Abstract

We develop a search-based model of asset trading, in which investors of different horizons can invest in two assets with identical payoffs. The asset markets are partially segmented: buyers can search for only one asset, but can decide which one. We show the existence of a "clientele" equilibrium where all short-horizon investors search for the same asset. This asset has more buyers and sellers, lower search times, and trades at a higher price relative to its identical-payoff counterpart. The clientele equilibrium dominates the one where all investor types split equally across assets, implying that the concentration of liquidity is socially desirable.

Key words: Liquidity, Search, Asset pricing

JEL classification numbers: G1, D8

Running title: Endogenous Liquidity Concentration

⋆ We thank an anonymous referee, Peter DeMarzo, Darrell Duffie, Nicholas Economides, Simon Gervais, Arvind Krishnamurthy, Anna Pavlova, Lasse Pedersen, Ken Singleton, Pierre-Olivier Weill, seminar participants at Alberta, Athens, Tsinghua, UCLA, UT Austin, and participants at the SITE 2003 and WFA 2003 conferences for helpful comments. Jiro Kondo provided excellent research assistance.

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[1] Supported by the Social Sciences and Humanities Research Council of Canada.

1 Introduction

Financial assets differ in their liquidity, defined as the ease of trading them. For example, government bonds are more liquid than stocks or corporate bonds. A large body of research has attempted to measure liquidity and relate it to asset-price differentials. An important and complementary question is why liquidity differs across assets.

A leading theory of liquidity is based on asymmetric information. For example, [15], [21] show that market makers can widen their bid-ask spread to compensate for the risk of trading against informed agents. This increases trading costs for all agents, including the uninformed. In many cases, however, asymmetric information cannot be the explanation for liquidity differences. For example, AAA-rated bonds of US corporations are essentially default-free, but are significantly less liquid than Treasury bonds. Since both sets of bonds have essentially riskless cash flows, their value should depend only on interest rates. But information about the latter is generally symmetric, and in any event, possible asymmetries should be common across bonds. An even starker example comes from within the Treasury market: just-issued ("on-the-run") bonds are significantly more liquid than previously issued ("off-the-run") bonds maturing on nearby dates. [1]

In this paper we explore an alternative theory of liquidity based on the notion that asset trading can involve search, i.e., locating counterparties takes time. Search is a fundamental feature of over-the-counter markets, where trade is conducted through bilateral negotiations rather than a Walrasian auction. [2] We show that liquidity, measured by search costs, can differ across otherwise identical assets, and this translates into equilibrium price differentials. We also perform a welfare analysis, showing that the concentration of liquidity in one asset dominates equal liquidity in all assets.

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[1] Evidence on the default risk of corporate bonds is in [25], on the trading costs of corporate bonds in [5], on the trading costs of government bonds in [12], and on the on-the-run phenomenon in [13], [34].

[2] Examples of over-the-counter markets are for government, corporate, and municipal bonds, and for many derivatives. We elaborate on the role of search in these markets in Section 2. See also the discussion in [10].

We assume that a constant flow of investors enter into a market, seeking to buy one of two infinitely-lived assets with identical payoffs. After buying an asset, investors become "inactive" owners, until the time they seek to sell. That event occurs when the investors' valuation of asset payoffs switches to a lower level. The switching rate is inversely related to investors' horizons, and we assume that horizons are heterogeneous across investors. To model search, we adopt the standard framework (e.g., [8]) where investors are matched randomly over time in pairs. We also assume that markets are partially segmented in that buyers must decide which of the two assets to search for, and then search for that asset only.

We show that there exists an asymmetric ("clientele") equilibrium, where assets differ in liquidity despite having identical payoffs. The market of the more liquid asset has more buyers and sellers. This results in short search times, i.e., high liquidity, and high trading volume. Moreover, prices are higher in that market, reflecting a liquidity premium that investors are willing to pay for the short search times. The tradeoff between prices and search times gives rise to a clientele effect: buyers with high switching rates, who have a stronger preference for short search times, search for the liquid asset, while the opposite holds for the more patient, low-switching-rate buyers. The clientele effect is, in turn, what generates the higher trading volume in the liquid asset: high-switching-rate buyers turn faster into sellers, thus generating more turnover. Critical to the clientele equilibrium is the assumption that buyers cannot search for both assets simultaneously. Indeed, we show that under simultaneous search, investors would buy the first asset they find, and assets would have the same liquidity and price. [3]

The liquidity premium increases as the distribution of investors' switching rates becomes more dispersed around its median, and is equal to zero when investors are homogenous. One might expect the premium to increase with an

[3] Additionally, the clientele equilibrium might not exist if buyers' bargaining power, defined as the probability that they get to make the take-it-or-leave-it offer in a match, is increasing in the switching rate. Intuitively, if high-switching-rate buyers can extract most of the surplus, sellers in the liquid market have a low reservation value. This encourages buyer entry into the liquid market, and can possibly reduce the measure of sellers below that in the illiquid market, contradicting the existence of clientele equilibrium.

upward shift in the switching-rate distribution, consistent with the notion that short-horizon investors value liquidity more highly. Surprisingly, however, the premium can decrease because shorter horizons generate more trading, and this reduces search times and trading costs.

In addition to the clientele equilibrium, there exist symmetric ones, where the two markets are identical in terms of prices and search times. Comparing the two types of equilibria reveals, in the context of our model, whether the concentration of liquidity in one asset is socially desirable. As a benchmark for this comparison, we determine the socially optimal allocation of entering buyers across the two markets. Under this allocation, the measure of sellers differs across markets, and so do the buyers' search times (which are decreasing in the measure of sellers). Such a dispersion is optimal so that markets can cater to different clienteles: buyers with high switching rates go to the market with the short search times, while the opposite holds for low-switching-rate buyers.

In the symmetric equilibria the buyers' search times are identical across markets, while in the clientele equilibrium some dispersion exists. A sufficient condition for the clientele equilibrium to dominate the symmetric ones is that this dispersion does not exceed the socially optimal level. To examine whether this is the case, we consider the social optimality of buyers' entry decisions in the clientele equilibrium. We show that despite the higher prices, buyers do not fully internalize the relatively short supply of sellers in the liquid market, and enter excessively in that market. This pushes the measure of sellers in the liquid market below the socially optimal level, and has the same effect on the dispersion in buyers' search times. Thus, the clientele equilibrium dominates the symmetric ones.

This paper is related to [28], which studies the concentration of liquidity across two markets. [28] shows that the markets can coexist, but the equilibrium is generally dominated by shutting one market and concentrating all trade in the other. The main difference with [28] is that we consider the concentration of liquidity across assets, rather than across market venues for the same asset. In particular, when one asset is traded in different venues, sellers have the choice of venue. By contrast, when venues correspond to physically different assets (e.g., Treasury vs. corporate bonds), sellers do not have such choice because they can only sell the asset they own. For example, in the clientele equilibrium, sellers of the less liquid asset cannot convert it to the liquid asset and sell it at the higher price. If such conversion were possible, we would effectively be back to the one-asset case.

[1] studies the concentration of liquidity under asymmetric information. It shows that if uninformed traders have discretion over the timing of their trades, they will all trade when the market is the most liquid. This reduces the informational content of order flow, feeding back into market liquidity. [6] shows that uninformed traders can all choose to trade in one of multiple locations for similar reasons. As [28], these papers concern the concentration of liquidity across market venues (defined by time or location) rather than assets.

Search-theoretic approaches to liquidity have been explored in the monetary literature following [20], [29]. [4] [2] shows the coexistence of currencies that differ in liquidity and price, and [33] analyzes the relative liquidity of currency and dividend-paying assets. In our model there is no room for currency, and the focus is on the relative liquidity of dividend-paying assets.

[9], [10], [11] integrate search in models of asset market equilibrium. This paper builds on their framework, extending it to multiple assets and heterogeneous investors. Independent work in [35] also considers multiple assets. Investors are homogeneous, however, and differences in liquidity arise because of exogenous differences in assets' issue sizes. Work subsequent to this paper in [32] shows that differences in liquidity can arise even with identical horizons and issue sizes, provided that there are short-sellers.

Finally, our welfare analysis is related to [8]. [8] shows that search can drive a wedge between workers' wages and marginal products, and this can distort the choice between different labor markets. In our model a similar distortion applies to the choice between the markets of different assets. [5]

[4] See also [22] which links liquidity to search in a partial equilibrium setting.
[5] For search models where agents choose between sub-markets, see also [19], [24],

The rest of this paper is organized as follows. Section 2 presents the model. Section 3 determines investor populations, expected utilities, and prices, taking the allocation of investors across markets as given. Section 4 endogenizes this allocation and determines the set of market equilibria. The welfare analysis is in Section 5. Section 6 considers several extensions, and Section 7 concludes. All proofs are in the Appendix.

2 Model

Time is continuous and goes from 0 to ∞. There are two assets, 1 and 2, traded in markets 1 and 2, respectively. Both assets pay a constant flow δ of dividends and are in supply S.

Investors are risk-neutral and have a discount rate equal to r. Upon entering the economy, they seek to buy one unit of either asset 1 or 2. After buying the asset, they become "inactive" owners, until the time when they seek to sell. Thus, there are three groups of investors: buyers, inactive owners, and sellers. To model trading motives, we assume that upon entering the economy investors enjoy the full value δ of the dividend flow, but their valuation can switch to a lower level $\delta - x$ with Poisson rate κ. The parameter $x > 0$ can capture, in reduced form, the effect of a liquidity shock or a hedging need arising from a position in another market. Buyers and inactive owners enjoy the full value δ of the dividend flow. Buyers experiencing a switch to low valuation simply exit the economy. Inactive owners experiencing the switch become sellers, and upon selling the asset, they also exit the economy.

There is a flow of investors entering the economy. We assume that investors are heterogeneous in their horizons, i.e., some have a long horizon and some a shorter one. In our model, horizons are inversely related to the switching rates κ to low valuation. Thus, we can describe the investor heterogeneity by a function $f(\kappa)$ such that the flow of investors with switching rates in $[\kappa, \kappa + d\kappa]$ is $f(\kappa)d\kappa$. The total flow is $\int_{\underline{\kappa}}^{\overline{\kappa}} f(\kappa)d\kappa$, where $[\underline{\kappa}, \overline{\kappa}]$ denotes the support of

[26], [27].

$f(\kappa)$. To avoid technicalities, we assume that $f(\kappa)$ is continuous and strictly positive.

The main feature of our model is that the market operates through search. Search is a fundamental feature of over-the-counter markets, such as those for government, corporate, and municipal bonds, and for many derivatives. Indeed, trades in these markets are negotiated bilaterally between dealers and their customers. And while a customer can easily contact a dealer, dealers often need to engage in search to rebalance their inventories. For example, after acquiring a large inventory from a customer, a dealer needs to unload the inventory to a new customer. This can involve search, and the dealers' ability to search efficiently, by knowing which customers are likely to be interested in a specific transaction, affects the prices they quote in the market. [6]

To model search, we adopt the standard framework (e.g., [8]) where buyers and sellers are matched randomly over time in pairs. This framework is, of course, a stylized representation of over-the-counter markets because it abstracts away from the role of dealers. In some fundamental sense, however, dealers come into existence precisely because customers need to search for counterparties. The existence of dealers cannot eliminate the search cost, but only can reduce it and express it in a different form, e.g. bid-ask spread. Thus, modelling over-the-counter markets in a "pure" search framework allows us to study the effects of the search friction in a more fundamental manner. Of course, incorporating dealers could be an interesting extension of our research. [7]

We assume that markets are partially segmented in that buyers must decide which of the two assets to search for, and then search for that asset only.

[6] According to [14], pp.436-437: "Liquidity in the corporate bond market is not derived by knowing what is available and what is being sought in the form of active bids and offerings... Instead, it is derived by knowing what may be available from, or what may be sold to, public investors.... A corporate bond dealer will quote some bid price if a customer wants to sell an issue, but he is likely to quote a better price if he thinks he knows of the existence of another buyer to whom he can quickly resell the same issue."

[7] It could also relate our approach to the inventory literature in market microstructure (e.g., [3], [18]). That literature assumes that buyers and sellers arrive randomly in the market and can trade with dealers who face costs to holding inventory. [11] consider a search-based model of asset trading with a continuum of competitive dealers.

This assumption is critical. Indeed, Section 6.1 shows that if investors can search simultaneously for both assets, they buy the first asset they find, and assets have the same liquidity and price. One interpretation of our assumption is that investors are mutual-fund managers who are constrained to hold specific types of assets. (For example, government-bond funds are restricted from investing in corporate bonds.) Managers can, however, decide between asset types when the fund is incorporated. An alternative interpretation is that dealers/brokers specialize in different asset types. Market segmentation could then follow from the costs of employing multiple dealers. One such cost is complexity: an investor who wants to buy one unit of an asset through multiple dealers would have to give each dealer an order contingent on the other dealers' search outcomes.[8]

Summarizing, we can describe the economy by the flow diagram in Figure 1. To each asset, are associated three groups of investors: buyers, inactive owners, and sellers. Investors entering the economy come from the pool of outside investors, and investors exiting the economy return to that pool.

To describe the search process, we need to specify the rate at which buyers meet sellers. We assume that an investor seeking to trade meets investors from the overall population according to a Poisson process with a fixed arrival rate. Consequently, meetings with investors seeking the opposite side of the trade occur at a rate proportional to the measure of that investor group. Denoting the coefficient of proportionality by λ, and the measures of buyers and sellers of asset i by μ_b^i and μ_s^i, respectively, a buyer of asset i meets sellers at the rate $\lambda\mu_s^i$, and a seller meets buyers at the rate $\lambda\mu_b^i$. Moreover, the overall flow of meetings for asset i is $\lambda\mu_b^i\mu_s^i$.

The function $M(\mu_b^i, \mu_s^i) \equiv \lambda\mu_b^i\mu_s^i$ describes the search technology in our model. While the assumed form of M is partly motivated from tractability, it also embodies a notion of increasing returns to scale: doubling the measures of

[8] The two interpretations are somewhat related: dealers could specialize to better serve the investors who are constrained to hold specific asset types. We should add that our assumption does not preclude investors from searching for one asset, and then switching and searching for the other. It restricts investors from searching simultaneously for both assets at a given point in time.

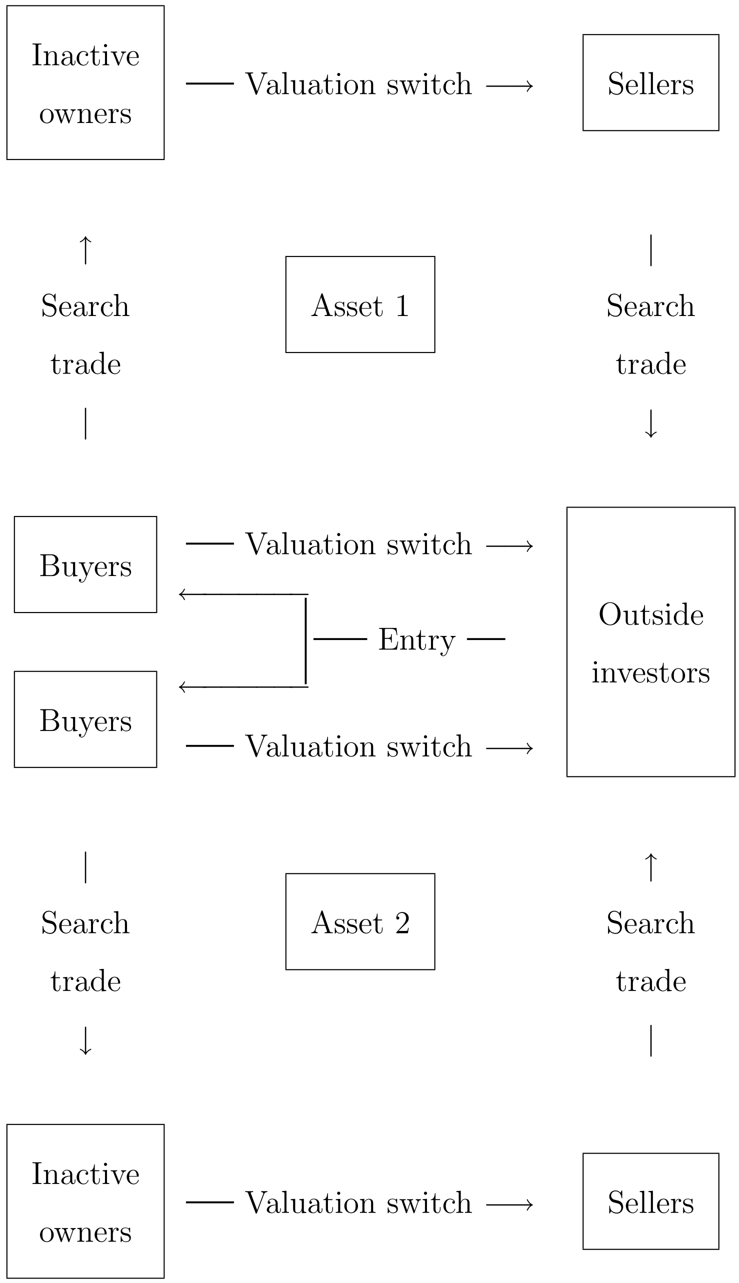

Fig. 1. Flow Diagram for the Two Markets

buyers and sellers more than doubles the flow of meetings. Increasing returns seem realistic for financial-market search because they imply that an increase in market size reduces search times of both buyers and sellers. This fits with the well-documented notion that trading costs are decreasing with trading volume.

When a buyer meets a seller, the price is determined through bilateral bargaining. We assume that the bargaining game takes a simple form, where one party is randomly selected to make a take-it-or-leave-it offer. The probability of the buyer being selected is $z/(1+z)$, where the parameter $z \in (0, \infty)$ measures the buyer's bargaining power.

Because buyers differ in their switching rates κ, they have different reservation values in the bargaining game, and this can introduce asymmetric information. We mainly focus on the symmetric-information case, where switching rates are publicly observable. For example, switching rates could correspond to buyers' observable institutional characteristics (e.g., insurance companies have a long horizon, while hedge funds a shorter one). When κ is publicly observable, the bargaining-power parameter z could in principle depend on κ. We mainly focus on the case where z is constant, but allow it to depend on κ in Section 6.2. Finally, in Section 6.3 we consider the asymmetric-information case, where switching rates are observable only to buyers.

3 Analysis

In this section we take as given the investors' decisions about which asset to search for, i.e., which market to enter. We then determine the measures of buyers, inactive owners, and sellers in each market, the expected utilities of investors in each group, and the market prices. Throughout, we focus on steady states, where all of the above are constant over time.

3.1 Demographics

We denote by $\nu^i(\kappa)$ the fraction of investors with switching rate κ who decide to enter into market i. We also denote by μ_o^i the measure of inactive owners in market i, and recall that the measures of buyers and sellers are denoted by μ_b^i and μ_s^i, respectively.

Because buyers and inactive owners are heterogeneous in their switching rates κ, we need to consider the distribution of switching rates within each population. This distribution is not the same as for the investors entering the market, because investors with different switching rates exit the market at different speeds. To describe the distribution of switching rates within the population of buyers in market i, we introduce the function $\mu_b^i(\kappa)$ such that the measure of buyers with switching rates in $[\kappa, \kappa + d\kappa]$ is $\mu_b^i(\kappa)d\kappa$. We similarly describe the distribution of switching rates within the population of inactive owners in market i by the function $\mu_o^i(\kappa)$. These functions satisfy the accounting identities

$$\int_{\underline{\kappa}}^{\overline{\kappa}} \mu_b^i(\kappa)d\kappa = \mu_b^i, \tag{1}$$

$$\int_{\underline{\kappa}}^{\overline{\kappa}} \mu_o^i(\kappa)d\kappa = \mu_o^i. \tag{2}$$

To determine $\mu_b^i(\kappa)$, we consider the flows in and out of the population of buyers with switching rates in $[\kappa, \kappa + d\kappa]$. The inflow is $f(\kappa)\nu^i(\kappa)d\kappa$, coming from the outside investors. The outflow consists of those buyers whose valuation switches to low and who exit the economy ($\kappa\mu_b^i(\kappa)d\kappa$), and of those who meet sellers and trade ($\lambda\mu_b^i(\kappa)\mu_s^i d\kappa$). (We are implicitly assuming that all buyer-seller matches result in a trade, a result we show in Proposition 1.) Since in steady state inflow equals outflow, it follows that

$$\mu_b^i(\kappa) = \frac{f(\kappa)\nu^i(\kappa)}{\kappa + \lambda\mu_s^i}. \tag{3}$$

To determine $\mu_o^i(\kappa)$, we similarly consider the flows in and out of the population of inactive owners with switching rates in $[\kappa, \kappa + d\kappa]$. The inflow is $\lambda\mu_b^i(\kappa)\mu_s^i d\kappa$, coming from the buyers who meet sellers, and the outflow is $\kappa\mu_o^i(\kappa)d\kappa$, coming from the inactive owners whose valuation switches to low and who become sellers. Writing that inflow equals outflow, and using (3), we find

$$\mu_o^i(\kappa) = \frac{\lambda \mu_s^i f(\kappa) \nu^i(\kappa)}{\kappa \left(\kappa + \lambda \mu_s^i\right)}. \tag{4}$$

Market equilibrium requires that the measure of asset owners in each market is equal to the asset supply. Since asset owners are either inactive owners or sellers, we have

$$\mu_o^i + \mu_s^i = S. \tag{5}$$

Combining (2), (4), and (5), we find

$$\int_{\underline{\kappa}}^{\overline{\kappa}} \frac{\lambda \mu_s^i f(\kappa) \nu^i(\kappa)}{\kappa \left(\kappa + \lambda \mu_s^i\right)} d\kappa + \mu_s^i = S. \tag{6}$$

Eq. (6) determines μ_s^i. Eqs. (1) and (3) then determine μ_b^i, and (2) and (4) determine μ_o^i.

3.2 Expected Utilities and Prices

We denote by $v_b^i(\kappa)$ and $v_o^i(\kappa)$, respectively, the expected utilities of a buyer and an inactive owner with switching rate κ in market i. We also denote by v_s^i the expected utility of a seller, and by $p^i(\kappa)$ the expected price when a buyer with switching rate κ meets a seller. (The actual price is stochastic, depending on which party makes the take-it-or-leave-it offer.)

To determine $v_b^i(\kappa)$, we note that in a small time interval $[t, t+dt]$, a buyer can either switch to low valuation and exit the economy (probability κdt, utility 0), or meet a seller and trade (probability $\lambda \mu_s^i dt$, utility $v_o^i(\kappa) - p^i(\kappa)$), or remain a buyer (utility $v_b^i(\kappa)$). The buyer's expected utility at time t is the expectation of the above utilities, discounted at the rate r:

$$v_b^i(\kappa)(1-rdt)\left[\kappa dt 0+\lambda\mu_s^i dt(v_o^i(\kappa)-p^i(\kappa))+(1-\lambda\mu_s^i dt-\kappa dt)v_b^i(\kappa)\right].(7)$$

Rearranging, we find that $v_b^i(\kappa)$ is given by

$$rv_b^i(\kappa)=-\kappa v_b^i(\kappa)+\lambda\mu_s^i(v_o^i(\kappa)-p^i(\kappa)-v_b^i(\kappa)). \tag{8}$$

The term $rv_b^i(\kappa)$ can be interpreted as the flow utility of being a buyer. According to (8), this flow utility is equal to the expected flow cost of switching to low valuation and exiting the economy, plus the expected flow benefit of meeting a seller and trading.

Proceeding similarly, we find that $v_o^i(\kappa)$ and v_s^i are given by

$$rv_o^i(\kappa)=\delta+\kappa(v_s^i-v_o^i(\kappa)), \tag{9}$$

and

$$rv_s^i=\delta-x+\lambda\mu_b^i(E_b^i(p^i(\kappa))-v_s^i), \tag{10}$$

respectively, where E_b^i denotes expectation under the probability distribution of κ in the population of buyers in market i. According to (9), the flow utility of being an inactive owner is equal to the dividend flow from owning the asset, plus the expected flow cost of switching to a low valuation and becoming a seller. Likewise, the flow utility of being a seller is equal to the seller's valuation of the dividend flow, plus the expected flow benefit of meeting a buyer and trading.

The price $p^i(\kappa)$ is the expectation of the buyer's and the seller's take-it-or-leave-it offers. The buyer is selected to make the offer with probability $z/(1+z)$, and offers the seller's revervation value, v_s^i. The seller is selected with probability $1/(1+z)$, and offers the buyer's reservation value, $v_o(\kappa)-v_b(\kappa)$.

Therefore,

$$p^i(\kappa) = \frac{z}{1+z} v_s^i + \frac{1}{1+z}(v_o^i(\kappa) - v_b^i(\kappa)). \tag{11}$$

Proposition 1 *Eqs. (8)-(11) have a unique solution* $(v_b^i(\kappa), v_o^i(\kappa), v_s^i, p^i(\kappa))$*. This solution satisfies, in particular,* $v_o^i(\kappa) - v_b^i(\kappa) - v_s^i > 0$ *for all* κ*.*

Since $v_o^i(\kappa) - v_b^i(\kappa) - v_s^i > 0$ for all κ, any buyer's reservation value exceeds a seller's. Thus, all buyer-seller matches result in a trade, a result that we have implicitly assumed so far. The intuition is simply that any buyer is a more efficient asset holder than a seller: the buyer values the dividend flow more highly than the seller, and upon switching to low valuation, faces the same rate of meeting new buyers as the seller.

4 Equilibrium

In this section, we endogenize investors' entry decisions, and determine the set of market equilibria. An investor will enter into the market where the expected utility of being a buyer is highest. Thus, the fraction $\nu^1(\kappa)$ of investors with switching rate κ who enter into market 1 is given by

$$\nu^1(\kappa) = 1 \quad \text{if} \quad v_b^1(\kappa) > v_b^2(\kappa) \tag{12}$$
$$0 \leq \nu^1(\kappa) \leq 1 \quad \text{if} \quad v_b^1(\kappa) = v_b^2(\kappa) \tag{13}$$
$$\nu^1(\kappa) = 0 \quad \text{if} \quad v_b^1(\kappa) < v_b^2(\kappa). \tag{14}$$

Definition 1 *A market equilibrium consists of fractions* $\{\nu^i(\kappa)\}_{i=1,2}$ *of investors entering in each market, measures* $\{(\mu_b^i, \mu_o^i, \mu_s^i)\}_{i=1,2}$ *of each group of investors, and expected utilities and prices* $\{(v_b^i(\kappa), v_o^i(\kappa), v_s^i, p^i(\kappa))\}_{i=1,2}$*, such that*

(a) $\{(\mu_b^i, \mu_o^i, \mu_s^i)\}_{i=1,2}$ *are given by (1)-(4) and (6).*

(b) $\{(v_b^i(\kappa), v_o^i(\kappa), v_s^i, p^i(\kappa))\}_{i=1,2}$ *are given by (8)-(11).*

(c) $\nu^1(\kappa)$ *is given by (12)-(14), and* $\nu^2(\kappa) = 1 - \nu^1(\kappa)$*.*

To determine the set of market equilibria, we establish a sorting condition. We consider an investor who is indifferent between the two markets, i.e., κ^* such that $v_b^1(\kappa^*) = v_b^2(\kappa^*)$, and examine which market other investors prefer.

Lemma 1 *Suppose that $v_b^1(\kappa^*) = v_b^2(\kappa^*)$. Then, $v_b^1(\kappa) - v_b^2(\kappa)$ has the same sign as $(\mu_s^1 - \mu_s^2)(\kappa - \kappa^*)$.*

According to Lemma 1, the measure of sellers serves as a sorting device. If, for example, market 1 has the most sellers, then investors with high switching rates will have a stronger preference for that market than investors with low switching rates. The intuition is that high-switching-rate investors have a stronger preference for short search times, and buyers' search times are short in a market with more sellers.

Lemma 1 implies that there can only be two types of equilibria. First, one market can have more sellers than the other, in which case it attracts the investors with high switching rates. We refer to such equilibria as *clientele* equilibria, to emphasize that each market attracts a different clientele of investors. Alternatively, both markets can have the same measure of sellers, in which case all investors are indifferent between the two markets. We refer to such equilibria as *symmetric* equilibria, to emphasize that markets are symmetric from the viewpoint of all investors.

4.1 Clientele Equilibria

We focus on the case where market 1 is the one with the most sellers. This is without loss of generality as any equilibrium derived in this case has a symmetric counterpart derived by switching the indices of the two markets.

Theorem 1 *There exists a unique clientele equilibrium in which market 1 is the one with the most sellers.*

A clientele equilibrium is characterized by the switching rate κ^* of the investor who is indifferent between the two markets. Investors with $\kappa > \kappa^*$ enter into market 1, and investors with $\kappa < \kappa^*$ enter into market 2. According to Theorem

1, such a cutoff κ^* exists and is unique.

Theorem 2 *The clientele equilibrium where market 1 is the one with the most sellers, has the following properties:*

(a) More buyers and sellers in market 1: $\mu_b^1 > \mu_b^2$ *and* $\mu_s^1 > \mu_s^2$*.*

(b) Higher buyer-seller ratio in market 1: $\mu_b^1/\mu_s^1 > \mu_b^2/\mu_s^2$*.*

(c) Higher prices in market 1: $p^1(\kappa) > p^2(\kappa)$ *for all* κ*.*

According to Theorem 2, market 1 has not only more sellers than market 2, but also more buyers, and a higher buyer-seller ratio. Moreover, the price that any given buyer expects to pay is higher in market 1. The intuition is as follows. Since there are more sellers in market 1, buyers' search times are shorter. Therefore, holding all else constant, buyers prefer entering into market 1. To restore equilibrium, prices in market 1 must be higher than in market 2. This is accomplished by higher buying pressure in market 1, i.e., higher buyer-seller ratio.

In the resulting equilibrium, there is a clientele effect. Investors with high switching rates, who have a stronger preference for short search times, prefer market 1 despite the higher prices. On the other hand, low-switching-rate investors, who are more patient, value more the lower prices in market 2. The clientele effect is, in turn, what accounts for the larger measure of sellers in market 1 since the high-switching-rate buyers turn faster into sellers.

Our model of search provides a natural measure of liquidity. Since investors cannot trade immediately, they incur a cost of delay. A measure of this cost is the expected time it takes to find a counterparty, and conversely, a measure of liquidity is the inverse of this expected time. Since a buyer in market i meets sellers at the rate $\lambda\mu_s^i$, the expected time it takes to meet a seller is $\tau_b^i \equiv 1/(\lambda\mu_s^i)$. Likewise, the expected time it takes for a seller to meet a buyer is $\tau_s^i \equiv 1/(\lambda\mu_b^i)$. Since the measures of buyers and sellers are higher in market 1, the expected times τ_b^i and τ_s^i are lower in that market, and thus market 1 is more liquid. Note that because there are more buyers and sellers in market 1, the trading volume, defined as the flow $\lambda\mu_b^i\mu_s^i$ at which matches occur, is higher in that market.

Since market 1 is more liquid than market 2, the price difference between the two markets can be interpreted as a liquidity premium: buyers are willing to pay a higher price for asset 1 because of its greater liquidity. In generating a liquidity premium, our model is analogous to the literature on asset pricing with transaction costs (e.g., [4], [7], [16], [17], [23], [30], [31]). The main difference with that literature is that we endogenize transaction costs. In particular, we do not assume that these differ exogenously across assets, but show that differences can arise endogenously in equilibrium, even when assets are otherwise identical.

To gain more intuition into the liquidity premium, we compute the equilibrium in closed form when search frictions are small, i.e., the parameter λ characterizing the rate of meetings is large. For small frictions, the market converges to Walrasian equilibrium (WE). In the WE both assets trade at the same price, determined by demand and supply. If the measure D_h of high-valuation agents exceeds the total asset supply $2S$, there is "excess demand": high-valuation agents are marginal and the WE price is equal to their valuation δ/r. If instead D_h is lower than $2S$, there is "excess supply": low-valuation agents are marginal and the WE price is equal to their valuation $(\delta - x)/r$. In what follows, we focus on the case $D_h = 2S$, where there is no excess demand or supply. This symmetric case has the advantage that calculations are the simplest.[9] We denote the population density of high-valuation agents by $g(\kappa)$, so that these agents' measure is

$$D_h \equiv \int_{\underline{\kappa}}^{\overline{\kappa}} g(\kappa) d\kappa.$$

[9] One simplifying feature of the case $D_h = 2S$ is that when λ goes to ∞, the measures of buyers and sellers are of order $1/\sqrt{\lambda}$. Thus, the rates of meeting buyers and sellers are of order $\lambda \times (1/\sqrt{\lambda})$, and converge to ∞. When $D_h > 2S$, sellers are the short side of the market and their measure is of order $1/\lambda$, while the measure of buyers is of order 1. Thus, the rate of meeting buyers converges to ∞ but that for sellers remains finite. When $D_h < 2S$, the opposite is true.

Since the inflow into the group of high-valuation agents with switching rates in $[\kappa, \kappa + d\kappa]$ is $f(\kappa)d\kappa$, and the outflow generated by switching to low valuation is $\kappa g(\kappa)d\kappa$, we have $g(\kappa) = f(\kappa)/\kappa$.

Proposition 2 *Suppose that $D_h = 2S$. When λ goes to infinity, $p^1(\kappa)$ and $p^2(\kappa)$ converge to the common limit $\frac{\delta}{r} - \frac{x}{r}\frac{z}{1+z}$. Moreover, the following asymptotics hold:*

$$\mu_s^i = \frac{\alpha^i}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right) \tag{15}$$

$$\kappa^* = \hat{\kappa} + o\left(\frac{1}{\sqrt{\lambda}}\right) \tag{16}$$

$$p^1(\kappa) - p^2(\kappa) = \frac{x(r + \hat{\kappa}(1+z))}{\sqrt{\lambda}r(1+z)}\left(\frac{1}{\alpha^2} - \frac{1}{\alpha^1}\right) + o\left(\frac{1}{\sqrt{\lambda}}\right), \tag{17}$$

where $o(1/\sqrt{\lambda})$ denotes terms of order smaller than $1/\sqrt{\lambda}$, and $(\alpha^1, \alpha^2, \hat{\kappa})$ are defined by

$$\alpha^1 = \sqrt{\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)\kappa d\kappa} \tag{18}$$

$$\alpha^2 = \sqrt{\int_{\underline{\kappa}}^{\hat{\kappa}} g(\kappa)\kappa d\kappa} \tag{19}$$

$$\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)d\kappa = \int_{\underline{\kappa}}^{\hat{\kappa}} g(\kappa)d\kappa. \tag{20}$$

When search frictions are small, the measures of sellers in the two markets, $\{\mu_s^i\}_{i=1,2}$, are of order $1/\sqrt{\lambda}$, and the same can be shown for the measures of buyers. The switching rate κ^* of the agent who is indifferent between markets converges to the median $\hat{\kappa}$ of the distribution $g(\kappa)$, meaning that the measures of high-valuation agents are equal across markets. Intuitively, since the measures of buyers and sellers converge to zero, the set of high-valuation agents in each market coincides in the limit with the set of owners. Moreover, the measures of owners are equal across markets because assets are in identical supply.

The liquidity premium $p^1(\kappa)-p^2(\kappa)$ is of order $1/\sqrt{\lambda}$. Corollary 1 explores how the premium depends on the distribution $g(\kappa)$ of high-valuation investors, and on the bargaining-power parameter z. To state the corollary, we consider the set $\Phi_{a,b}$ of real functions ϕ such that (i) ϕ has support $[a,b]$, (ii) $\int_a^b \phi(y)dy = 0$, and (iii) there exists $c \in (a,b)$ such that $\phi(y) < 0$ for $y \in (a,c)$ and $\phi(y) > 0$ for $y \in (c,b)$. Adding a function $\phi \in \Phi_{a,b}$ to a distribution shifts weight to the right, while keeping total weight constant.

Corollary 1 *Suppose that $D_h = 2S$ and λ is large.*

(a) The liquidity premium decreases when $g(\kappa)$ is replaced by $g(\kappa) + \underline{\phi}(\kappa) - \overline{\phi}(\kappa)$, for $\underline{\phi} \in \Phi_{\underline{\kappa},\hat{\kappa}}$ and $\overline{\phi} \in \Phi_{\hat{\kappa},\overline{\kappa}}$.

(b) The liquidity premium can increase or decrease when $g(\kappa)$ is replaced by $g(\kappa) + \phi(\kappa)$, for $\phi \in \Phi_{\underline{\kappa},\overline{\kappa}}$.

(c) The liquidity premium decreases when z increases.

According to Property (a), the liquidity premium decreases when the distribution $g(\kappa)$ becomes more concentrated around its median. In the extreme case of a point distribution, the liquidity premium is zero because investors are homogeneous. As heterogeneity increases, holding the median constant, the measure of sellers increases in market 1 and decreases in market 2. This increases the gap between the buyers' search times across markets, raising the liquidity premium.

Property (b) concerns a shift in weight towards larger values of κ. One might expect the liquidity premium to increase since with shorter horizons investors should value liquidity more highly. The premium can decrease, however, since shorter horizons imply more trading volume and lower search costs. Property (b) highlights the importance of endogenizing transaction costs: with exogenous costs, a decrease in horizons generally leads to an increase in the liquidity premium.

Property (c) shows that the liquidity premium decreases in the buyers' bargaining power. The intuition is that buyers' utility from a transaction is more sensitive to liquidity than sellers' utility. Indeed, sellers exit the market after a transaction, while buyers benefit from the market's future liquidity when turning into sellers. When buyers' have more bargaining power, the price is driven more by sellers' utility, and is thus less dependent on liquidity.

Note finally that in order $1/\sqrt{\lambda}$, the liquidity premium does not depend on κ, and the same can be shown for the prices $(p^1(\kappa), p^2(\kappa))$. Thus, when frictions are small, prices are almost independent of buyers' switching rates, and asymmetric information on switching rates has no effect. We return to this point when studying the asymmetric-information case in Section 6.3.

4.2 Symmetric Equilibria

In a symmetric equilibrium the measure of sellers is the same across the two markets. For investors to be indifferent between markets, the prices must also be the same. These requirements, however, do not determine a unique symmetric equilibrium.

Proposition 3 *There exist a continuum of symmetric equilibria. In any such equilibrium,* $p^1(\kappa) = p^2(\kappa)$ *for all* κ.

The intuition for the indeterminacy is that there are infinitely many ways to allocate investors in the two markets so that the measure of sellers, and an index of buying pressure that determines prices, are the same across markets. One trivial example is that for any switching rate, half of the investors go to each market, i.e., $\nu^i(\kappa) = 1/2$ for all κ.

5 Welfare Analysis

In this section we perform a welfare analysis of the allocation of liquidity across assets. We examine, in particular, whether it is socially desirable that liquidity is concentrated in one asset, possibly at the expense of others. In the context of our model, this amounts to comparing the clientele equilibrium, where concentration occurs, to the symmetric equilibria.

We use a simple welfare measure which gives the utilities of all investors present in the market equal weight, and discounts those of the future entrants at the common discount rate r. Discounting is consistent with equal weighting since future entrants can be viewed as outside investors whose utility is the discounted value of entering the market. Our welfare measure thus is

$$\mathcal{W} \equiv \sum_{i=1,2} \left[\int_{\underline{\kappa}}^{\overline{\kappa}} [v_b^i(\kappa)\mu_b^i(\kappa) + v_o^i(\kappa)\mu_o^i(\kappa)] d\kappa + v_s^i \mu_s^i + \frac{1}{r} \int_{\underline{\kappa}}^{\overline{\kappa}} v_b^i(\kappa) f(\kappa) \nu^i(\kappa) d\kappa \right]$$

where the last term reflects the welfare of the stream of future entrants. Lemma 2 shows that welfare takes a simple and intuitive form.

Lemma 2 *Welfare is*

$$\mathcal{W} = \frac{2\delta}{r} S - \frac{x}{r}(\mu_s^1 + \mu_s^2). \tag{21}$$

The first term in (21) is the present value of the dividends paid by the two assets. Welfare would coincide with this present value if all asset owners enjoyed the full value δ of the dividends. Some owners, however, enjoy only the value $\delta - x$. These are the sellers in the two markets, and welfare needs to be adjusted downwards by their total measure.

5.1 Entry in the Clientele Equilibrium

We start by examining the social optimality of investors' entry decisions in the clientele equilibrium. This serves as a useful first step for comparing the clientele equilibrium to the symmetric ones. Investors' entry decisions are characterized by a cutoff κ^* such that investors above κ^* enter into market 1, and those below κ^* enter into market 2. To examine whether private decisions are socially optimal, we consider the change in welfare if some investors close to κ^* enter into a different market than the one prescribed in equilibrium. More specifically, we assume that at time 0, some buyers with switching rates in $[\kappa^*, \kappa^* + d\kappa]$ are reallocated from market 1 to market 2, but from then on entry is according to κ^*. This reallocation causes the markets to be temporarily out of steady state and to converge over time to the original steady state.

To compute the change in welfare, we need to evaluate welfare out of steady state. We first consider the non-steady state that results when the measure of buyers in market i with switching rates in $[\kappa, \kappa + d\kappa]$ is increased by ϵ relative to the steady state. Denoting welfare in the non-steady state by $\mathcal{W}(\epsilon)$, we set

$$V_b^i(\kappa) \equiv \left. \frac{d\mathcal{W}(\epsilon)}{d\epsilon} \right|_{\epsilon=0}.$$

The variable $V_b^i(\kappa)$ measures the increase in social welfare by adding buyers with switching rate κ in market i. It thus represents the social value of these buyers. Proceeding similarly, we can define the social value $V_o^i(\kappa)$ of owners with switching rate κ, and the social value V_s^i of sellers.

Proposition 4 *The social values* $(V_b^i(\kappa), V_o^i(\kappa), V_s^i)$ *are given by*

$$rV_b^i(\kappa) = -\kappa V_b^i(\kappa) + \lambda\mu_s^i(V_o^i(\kappa) - V_b^i(\kappa) - V_s^i), \tag{22}$$

$$rV_o^i(\kappa) = \delta + \kappa(V_s^i - V_o^i(\kappa)), \tag{23}$$

$$rV_s^i = \delta - x + \lambda\mu_b^i(E_b^i(V_o^i(\kappa) - V_b^i(\kappa)) - V_s^i). \tag{24}$$

Eqs. (22)-(24) are analogous to (8)-(10) that determine investors' expected utilities. To compare the two sets of equations, we reproduce (8)-(10) below, using (11) to eliminate the price:

$$rv_b^i(\kappa) = -\kappa v_b^i(\kappa) + \lambda\mu_s^i\frac{z}{1+z}(v_o^i(\kappa) - v_b^i(\kappa) - v_s^i), \tag{25}$$

$$rv_o^i(\kappa) = \delta + \kappa(v_s^i - v_o^i(\kappa)), \tag{26}$$

$$rv_s^i = \delta - x + \lambda\mu_b^i \frac{1}{1+z}(E_b^i(v_o^i(\kappa) - v_b^i(\kappa)) - v_s^i). \tag{27}$$

The key difference between expected utilities and social values concerns the flow benefit of meeting a counterparty. Consider, for example, the flow benefit associated to a buyer. In computing the buyer's expected utility, we multiply the buyer's rate of meeting a seller, $\lambda\mu_s^i$, times the surplus realized by the buyer-seller pair, $v_o^i(\kappa) - v_b^i(\kappa) - v_s^i$, times the fraction of that surplus that the buyer appropriates, $z/(1+z)$. In computing the buyer's social value, however, we need to attribute the full surplus to the buyer. This is because the social value measures an investor's marginal contribution to social welfare. Since a trade involving a specific buyer is realized only because that buyer is added to the market, the buyer's marginal contribution is the full surplus associated to the trade. The same is obviously true for the seller. [10] [11]

Proposition 5 *In the clientele equilibrium where market 1 is the one with the most sellers, the social value of buyer κ^* is higher in market 2, i.e., $V_b^1(\kappa^*) < V_b^2(\kappa^*)$.*

Since the social value of buyer κ^* is higher in market 2, welfare can be improved by reallocating some buyers close to κ^* from market 1 to market 2. Thus, in the clientele equilibrium, there is excessive entry into market 1, i.e., the more liquid market. The intuition is as follows. Since buyer κ^* is indifferent between the two markets, the buyer's flow benefit of meeting a seller is the same across markets. A seller's flow benefit of meeting a buyer, however, is higher in market 1. This is because the seller's rate of meeting a buyer involves the measure of buyers rather than that of sellers, and the buyer-seller ratio is higher in

[10] Additionally, in computing the buyer's social value, we need to consider not the buyer's rate of meeting a seller, but the marginal increase in the rate of buyer-seller meetings achieved by adding the buyer in the market. The two coincide, however, because the search technology is linear in the measures of buyers and sellers.

[11] It is worth explaining why our search model generates discrepancies between expected utilities and social values, while the standard Walrasian model does not. In the Walrasian model, the surplus that a buyer-seller pair bargain over is zero, since either party can leave the pair and obtain immediately the market price from another counterparty. In the search model, by contrast, the surplus is non-zero, since finding another counterparty is costly. It is because each party gets only a fraction of this non-zero surplus that discrepancies between expected utilities and social values arise.

market 1. Since a seller's flow benefit is higher in market 1, the discrepancy between the seller's social value and expected utility is larger in that market. (Recall that social value attributes the full benefit of a meeting to each party, while expected utility attributes only a fraction.) Conversely, since buyers bargain on the basis of a seller's expected utility rather than social value, the discrepancy between their own social value and expected utility is smaller in market 1. Given that for the indifferent buyer, expected utility is the same across the two markets, social value is greater in market 2. Intuitively, sellers are more socially valuable in market 1 because they are in relatively short supply in that market. Buyers internalize this through the higher prices, but only partially, and thus they enter excessively into market 1.

5.2 Clientele vs. Symmetric Equilibria

We start with a methodological observation. Both the clientele and the symmetric equilibria are dynamic steady states, and comparing these can be misleading. Indeed, an action aiming to take the market from an inferior steady state to a superior one, must involve non-steady-state dynamics. For such an action to be evaluated based only on a comparison between steady states, these dynamics must be unimportant relative to the long-run limit. This is the case when the discount rate r is small, which we assume below.

Both the clientele and the symmetric equilibria are fully characterized by the decisions of investors as to which market to enter. We next determine, and use as a benchmark, the socially optimal entry decisions in steady state. These are the solution to the problem

$$\max_{\nu^1(\kappa)} \mathcal{W},$$

where $\mathcal{W}$ is given by Lemma 2, μ_s^i by (6), and $\nu^2(\kappa) = 1 - \nu^1(\kappa)$. We solve this problem, $(\mathcal{P})$, in Proposition 6.

Proposition 6 *The problem $(\mathcal{P})$ has two symmetric solutions. The first sat-*

isfies $\mu_s^1 > \mu_s^2$, $\nu^1(\kappa) = 1$ for $\kappa > \kappa_w^*$, and $\nu^1(\kappa) = 0$ for $\kappa < \kappa_w^*$, for a cutoff κ_w^*. The second is derived from the first by switching the indices of the two markets.

Proposition 6 implies that it is socially optimal to create two markets with different measures of sellers. This is because the two markets can cater to different clienteles of investors: buyers with switching rates above a cutoff κ_w^*, who have a greater preference for lower search times, are allocated to the market with the most sellers, while the opposite holds for buyers below κ_w^*.

The cutoff κ_w^* determines the heterogeneity of the two markets. Increasing κ_w^*, reduces the entry into the more liquid market, say market 1. This increases the ratio of sellers μ_s^1/μ_s^2, and makes the markets more heterogeneous from a buyer's viewpoint.

We next treat the cutoff above which buyers enter into market 1 as a free variable, and denote it by ℓ. Social welfare is maximized for $\ell = \kappa_w^*$. As ℓ decreases below κ_w^*, the two markets become more homogenous from a buyer's viewpoint, and welfare decreases. Consider now two values of ℓ: the cutoff κ^* corresponding to the clientele equilibrium, and the cutoff κ' for which the measure of sellers is the same across the two markets. Since in the clientele equilibrium there is excessive entry into market 1, markets are not heterogeneous enough from a buyer's viewpoint, and thus $\kappa^* < \kappa_w^*$. At the same time, since there is some heterogeneity, $\kappa^* > \kappa'$. Therefore, welfare under the clientele equilibrium exceeds that under the allocation corresponding to κ'.

Interestingly, welfare under the latter allocation is the same as under any of the symmetric equilibria. To see why, note that both types of allocations have the property that the measure of sellers is the same across the two markets. Consider now an arbitrary allocation with this property, and denote by $\mu_s \equiv \mu_s^1 = \mu_s^2$ the common measure of sellers. The aggregate measure of inactive owners (i.e., the sum across both markets) depends on this allocation only through μ_s, since μ_s is the only determinant of the buyers' matching rate. Since the aggregate measure of inactive owners plus sellers must equal the aggregate asset supply, μ_s is uniquely determined regardless of the specific allocation. [12] Since, in addition, welfare depends only on μ_s, it is also independent of the specific allocation. Summarizing, we can show the following theorem:

Theorem 3 *All symmetric equilibria achieve the same welfare. Moreover, for small r, they are dominated by the clientele equilibrium.*

6 Extensions

6.1 Market Integration

Our analysis assumes that markets are partially segmented in that buyers must decide which of the two assets to search for, and then search for that asset only. For example, a buyer deciding to search for asset 1 is precluded from meeting sellers of asset 2. In Proposition 7 we show that this assumption is critical for the existence of equilibria where assets differ in liquidity and price.

Proposition 7 *If buyers can search simultaneously for both assets, then they buy the first asset they find. Moreover, prices and sellers' search times are identical across assets.*

Proposition 7 shows that under simultaneous search, each asset's buyer pool consists of the entire buyer population. In particular, there cannot be equilibria where some buyers decline to buy one asset because they prefer to wait for the other. Indeed, waiting for one asset could be optimal if sellers sell that asset cheaply. But then, the asset would attract a large buyer population, and sellers' reservation value would be greater than for the other asset.

[12] To show this formally, we add (6) for market 1 to the same equation for market 2, and find

$$\int_{\underline{\kappa}}^{\overline{\kappa}} \frac{\lambda \mu_s f(\kappa)}{\kappa\left(\lambda \mu_s+\kappa\right)} d\kappa + 2\mu_s = 2S.$$

This equation determines μ_s uniquely, regardless of the specific allocation.

A broad implication of Proposition 7 is that search can explain differences in liquidity across otherwise identical assets, but only when combined with some notion of segmentation. In this paper, segmentation takes the form that buyers are constrained to search for a specific asset (but can choose which one). Work subsequent to this paper in [32] considers two types of buyers: agents who establish long positions and can search for both assets, and agents who need to cover previously established short positions. A short position is established by borrowing an asset and selling it in the market. Segmentation arises because of the institutional constraint that short-sellers can deliver to their lender only the exact same asset they borrowed. Thus, in line with this paper, short-sellers can only buy a specific asset, but can choose which one at the borrowing stage.

In addition to assuming that buyers can only search in one market, we are implicitly assuming that sellers can only sell in the market where they originally bought. In some sense, this captures the difference between multiple market venues for the same asset (e.g., [28]) and multiple assets. When one asset is traded in different venues, sellers can sell in any venue and not necessarily where they bought. By contrast, when venues correspond to different assets, sellers must sell in the venue where they bought because they can only sell the asset they own. For example, in the clientele equilibrium, a seller of asset 2 would be better off converting it into asset 1: this would enable him to access the buyers searching for asset 1, and to sell faster at the higher price. Such conversion, however, is not feasible because the assets are physically different (e.g., Treasury and corporate bonds are different certificates).

6.2 Type-Dependent Bargaining Power

In this section we extend our analysis to the case where the bargaining-power parameter z is a function of κ, rather than a constant.

Proposition 8 *Suppose that $z(\kappa)$ is decreasing. Then, there exists a unique clientele equilibrium in which market 1 is the one with the most sellers. In this* *equilibrium,* $p^1(\kappa) > p^2(\kappa)$ *for all* κ*, if* $\frac{z(\underline{\kappa})}{1+z(\underline{\kappa})} < z(\overline{\kappa})$*.*

Proposition 8 shows that a clientele equilibrium can exist when z is a function of κ, provided that it is a decreasing function. Indeed, suppose instead that it is increasing, i.e., buyers with high switching rates have high bargaining power. Then, sellers in the more liquid market 1 have low utility relative to sellers in market 2 because they receive a small fraction of the surplus. This induces more buyer entry into market 1 (relative to the case where z is constant). Due to this entry, the measure of sellers in market 1 can become lower than in market 2, contradicting the existence of clientele equilibrium.

When z is a function of κ, the clientele equilibrium can have different properties than in Theorem 2. Implicit in the existence result, is the property that market 1 has the most sellers. For prices, however, results are less clearcut. We can show that regardless of the form of $z(\kappa)$, the indifferent buyer κ^* pays a higher price in market 1 than in market 2, reflecting the shorter search time. The same holds for buyers $\kappa < \kappa^*$: they would pay a higher price if they enter into market 1 (rather than market 2 as they do in equilibrium). Buyers $\kappa > \kappa^*$, however, might end up paying more if they enter into market 2. Intuitively, these buyers' low bargaining power can hurt them more in a market with few sellers. Our numerical solutions suggest that this phenomenon occurs only for a small set of parameters, and Proposition 8 rules it out if z does not decrease too quickly with κ. An additional property in Theorem 2 that does not always extend is that the buyer-seller ratio is higher in market 1. The intuition is analogous to that in the previous paragraph: if z is decreasing in κ, buyer entry in market 1 is limited.

6.3 Asymmetric Information

In this section we extend our analysis to the case where buyers' switching rates are not publicly observable. We start by examining whether a clientele equilibrium exists. Assuming that market 1 is the most liquid, and denoting by $\overline{\kappa}^i$ the maximum switching rate of an investor in market i, we have $\overline{\kappa}^1 = \overline{\kappa}$

and $\overline{\kappa}^2 = \kappa^*$. The buyer with switching rate $\overline{\kappa}^i$ has the lowest reservation value in market i. Indeed, reservation values decrease in switching rates since high-switching-rate buyers turn faster into sellers and have to re-incur the search costs.

For simplicity, we restrict the clientele equilibrium to be in pure strategies, i.e., all sellers in a given market make the same offer. In a pure-strategy equilibrium, the sellers' offer must be accepted by all buyers entering a market. Indeed, suppose that buyers with switching rates above a cutoff $\ell^i < \overline{\kappa}^i$ reject the sellers' offer in market i. Then, the density function $\mu_b^i(\kappa)$ of buyers in market i would increase discontinuously at ℓ^i, as buyers above ℓ^i would exit the buyer pool at lower rates.[13] This discontinuity would induce the sellers to slightly lower their offer, to trade with buyers above ℓ^i.

Since all buyer-seller matches result in a trade, the equations for the measures of buyers, inactive owners, and sellers are as in Section 3.1. The equations for the expected utilities and prices are, however, different, because the price is the same for all buyers entering a market. More specifically, the sellers' offer in market i is $v_o^i(\overline{\kappa}^i) - v_b^i(\overline{\kappa}^i)$, i.e., the reservation value of the highest-switching-rate buyer, and the buyers' offer is v_s^i, i.e., the reservation value of a seller. Since buyers make the offer with probability $z/(1+z)$, and sellers with probability $1/(1+z)$, the expected price in market i is

$$p^i = \frac{z}{1+z} v_s^i + \frac{1}{1+z} (v_o^i(\overline{\kappa}^i) - v_b^i(\overline{\kappa}^i)). \tag{28}$$

The expected utility of a buyer in market i is given by

$$r v_b^i(\kappa) = -\kappa v_b^i(\kappa) + \lambda \mu_s^i (v_o^i(\kappa) - p^i - v_b^i(\kappa)), \tag{29}$$

the expected utility of a seller by

$$r v_s^i = \delta - x + \lambda \mu_b^i (p^i - v_s^i), \tag{30}$$

[13] More specifically, $\mu_b^i(\kappa)$ would be given by (3) for $\kappa < \ell^i$, and $\mu_b^i(\kappa) = f(\kappa)\nu^i(\kappa)/\kappa$ for $\kappa > \ell^i$, as buyers above ℓ^i would exit the buyer pool only because of a switch to low valuation.

and the expected utility of an inactive owner by (9).

For a clientele equilibrium to exist, each seller must find it optimal to make an offer which is accepted by all buyers. Suppose that upon meeting a buyer, a seller decides to make an offer which is accepted only when the buyer's switching rate is up to κ. Then, the offer is $v_o^i(\kappa) - v_b^i(\kappa)$, and if it is rejected the seller re-enters the search process with expected utility v_s^i. Thus, the seller finds it optimal to trade with all buyers if

$$\overline{\kappa}^i \in \text{argmax}_\kappa \left[P_b^i(\kappa)(v_o^i(\kappa) - v_b^i(\kappa)) + (1 - P_b^i(\kappa))v_s^i \right], \tag{31}$$

where $P_b^i(\kappa)$ denotes the probability that a buyer in market i has switching rate up to κ.

Additionally, in a clientele equilibrium, buyer κ^* must be indifferent between the two markets. In the asymmetric-information case, an indifferent buyer might not exist. Indeed, suppose that the seller has all the bargaining power ($z = 0$). Then, buyer κ^* receives zero surplus in market 2 (because the price is equal to his reservation value), but positive surplus in market 1. To formulate a sufficient condition for the existence of an indifferent buyer, we treat the cutoff above which investors enter into market 1 as a free variable, and consider population measures and expected utilities as functions of that variable. We also consider the value κ' of the cutoff for which the measures of sellers are equal in the two markets. Then, the sufficient condition is that when the cutoff takes the value κ', buyer κ' prefers entering into market 2. We refer to this condition as Condition (C). Proposition 9 confirms that a clientele equilibrium exists under Conditions (31) and (C), and has the properties in Theorem 2.[14]

Proposition 9 *If Conditions (31) and (C) hold, a clientele equilibrium exists and has the following properties:*

(a) More buyers and sellers in market 1: $\mu_b^1 > \mu_b^2$ *and* $\mu_s^1 > \mu_s^2$.

[14] In fact, some properties of a clientele equilibrium can be proven more generally, without using Conditions (31) and (C). These are that market 1 has more buyers and higher trading volume, and has a higher buyer-seller ratio and higher prices if it has more sellers.

(b) *Higher buyer-seller ratio in market 1:* $\mu_b^1/\mu_s^1 > \mu_b^2/\mu_s^2$.

(c) *Higher prices in market 1:* $p^1 > p^2$.

Conditions (31) and (C) hold, for example, when search frictions are small and $D_h = 2S$, i.e., the parameters in the asymptotic analysis of Section 4.1. This is not surprising: the asymptotic analysis shows that for small frictions, prices are almost independent of buyers' switching rates. Therefore, when switching rates are observable only to buyers, the outcome should be the same as under symmetric information: a clientele equilibrium should exist and have the properties in Theorem 2.

Proposition 10 *If $D_h = 2S$ and λ is large, then Conditions (31) and (C) hold.*

Having established the existence of a clientele equilibrium, we next examine its welfare properties. As shown in Section 5, a sufficient condition for the clientele equilibrium to dominate the symmetric ones is that entry into market 1 is at or above the socially optimal level. To examine whether this condition holds in the asymmetric-information case, we compare entry decisions with the symmetric-information case. Under asymmetric information, buyer κ^* receives positive surplus from the seller's offer when entering into market 1, because the same offer must also be accepted by buyer $\overline{\kappa}$. This induces more entry into market 1. At the same time, a seller's outside option is reduced by his inability to price-discriminate, and this lowers the offer a buyer can make, thus raising the buyer's utility. Whether this induces more or less entry into market 1 depends on the relative heterogeneity of investors in the two markets. When, for example, κ^* is close to $\overline{\kappa}$, market 1 is more homogeneous. Thus, the inability to price-discriminate hurts more the sellers in market 2, inducing more entry into that market. The overall effect is ambiguous. Suppose, for example, that $f(\kappa) = c\kappa^\alpha$, where $\alpha \in \mathbb{R}$ measures the tilt of the distribution towards high switching rates, c is a normalizing constant (so that $D_h = 2S$), and $\overline{\kappa}/\underline{\kappa} = 2$. Then, entry into market 1 is greater in the asymmetric-information case as long as α is smaller than 0.51.

Even when entry into market 1 is lower in the asymmetric-information case, it can still be socially excessive, because it is so under symmetric information. For example, when $f(\kappa) = c\kappa^{\alpha}$, entry into market 1 is socially excessive for all values of α and $\overline{\kappa}/\underline{\kappa}$.[15]

7 Conclusion

In this paper we explore a theory of asset liquidity based on the notion that trading involves search. We assume that investors of different horizons can invest in two identical assets. The asset markets are partially segmented in that buyers must decide which of the two assets to search for, and then search for that asset only. We show that there exists a "clientele" equilibrium where all short-horizon investors search for the same asset. This asset has more buyers and sellers, lower search times, and trades at a higher price relative to its identical-payoff counterpart. Thus, our model can provide an explanation for why assets with similar cash flows can differ substantially in their liquidity and price (e.g., AAA-rated corporate bonds vs. Treasury bonds, and on- vs. off-the-run Treasury bonds). This phenomenon cannot be readily explained with theories based on asymmetric information. Our model also allows for a welfare analysis of the allocation of liquidity across assets. We show that the clientele equilibrium dominates the ones where the two markets are identical, implying that the concentration of liquidity in one asset is socially desirable.

[15] There might, however, be counterexamples for more complicated distributions.

A Appendix

Proof of Proposition 1: Using (11), we can write (8) and (10) as

$$rv_b^i(\kappa) = -\kappa v_b^i(\kappa) + \lambda\mu_s^i \frac{z}{1+z}(v_o^i(\kappa) - v_b^i(\kappa) - v_s^i) \tag{A.1}$$

$$rv_s^i = \delta - x + \lambda\mu_b^i \frac{1}{1+z}(E_b^i(v_o^i(\kappa) - v_b^i(\kappa)) - v_s^i). \tag{A.2}$$

Subtracting (A.1) from (9), we find

$$\begin{aligned} r(v_o^i(\kappa) - v_b^i(\kappa)) &= \delta + \kappa(v_s^i - v_o^i(\kappa) + v_b^i(\kappa)) - \lambda\mu_s^i \frac{z}{1+z}(v_o^i(\kappa) - v_b^i(\kappa) - v_s^i) \\ \Rightarrow v_o^i(\kappa) - v_b^i(\kappa) &= \frac{\delta + \left(\kappa + \lambda\mu_s^i \frac{z}{1+z}\right) v_s^i}{r + \kappa + \lambda\mu_s^i \frac{z}{1+z}}. \end{aligned} \tag{A.3}$$

Plugging (A.3) into (A.2), we find

$$rv_s^i = \delta - x + \lambda\mu_b^i \frac{1}{1+z}(\delta - rv_s^i) E_b^i \left[\frac{1}{r + \kappa + \lambda\mu_s^i \frac{z}{1+z}}\right] \Rightarrow v_s^i = \frac{\delta}{r} - \frac{x}{rQ^i}, \tag{A.4}$$

where

$$Q^i \equiv 1 + \lambda\mu_b^i \frac{1}{1+z} E_b^i \left[\frac{1}{r + \kappa + \lambda\mu_s^i \frac{z}{1+z}}\right].$$

Given v_s^i, the variables $v_o^i(\kappa)$, $v_b^i(\kappa)$, and $p^i(\kappa)$ are uniquely determined from (9), (A.1), and (11), respectively. In the rest of the proof, we compute $v_b^i(\kappa)$ and $p^i(\kappa)$ for use in subsequent proofs. Plugging (A.4) into (A.3), we find

$$v_o^i(\kappa) - v_b^i(\kappa) = \frac{\delta}{r} - \frac{x}{r} \frac{\kappa + \lambda\mu_s^i \frac{z}{1+z}}{\left(r + \kappa + \lambda\mu_s^i \frac{z}{1+z}\right) Q^i}. \tag{A.5}$$

Subtracting (A.4) from (A.5), we find

$$v_o^i(\kappa) - v_b^i(\kappa) - v_s^i = \frac{x}{\left(r + \kappa + \lambda\mu_s^i \frac{z}{1+z}\right) Q^i} > 0. \tag{A.6}$$

Plugging (A.6) into (A.1), we can compute $v_b^i(\kappa)$:

$$\begin{aligned} r v_b^i(\kappa) &= -\kappa v_b^i(\kappa) + \lambda\mu_s^i \frac{z}{1+z} \frac{x}{\left(r + \kappa + \lambda\mu_s^i \frac{z}{1+z}\right) Q^i} \\ \Rightarrow v_b^i(\kappa) &= \frac{\lambda\mu_s^i \frac{z}{1+z} x}{(r+\kappa)\left(r + \kappa + \lambda\mu_s^i \frac{z}{1+z}\right) Q^i}. \end{aligned} \tag{A.7}$$

Plugging (A.4) and (A.5) into (11), we can compute $p^i(\kappa)$:

$$p^i(\kappa) = \frac{\delta}{r} - \frac{x}{r} \frac{1 - \frac{r}{1+z} \frac{1}{r+\kappa+\lambda\mu_s^i \frac{z}{1+z}}}{Q^i}. \tag{A.8}$$

∎

Proof of Lemma 1: Since $v_b^1(\kappa^*) = v_b^2(\kappa^*) > 0$, the difference $v_b^1(\kappa) - v_b^2(\kappa)$ has the same sign as

$$\frac{v_b^1(\kappa)}{v_b^1(\kappa^*)} - \frac{v_b^2(\kappa)}{v_b^2(\kappa^*)}.$$

(A.7) implies that

$$\begin{aligned} \frac{v_b^1(\kappa)}{v_b^1(\kappa^*)} - \frac{v_b^2(\kappa)}{v_b^2(\kappa^*)} &= \frac{r+\kappa^*}{r+\kappa} \left[\frac{r + \kappa^* + \lambda\mu_s^1 \frac{z}{1+z}}{r + \kappa + \lambda\mu_s^1 \frac{z}{1+z}} - \frac{r + \kappa^* + \lambda\mu_s^2 \frac{z}{1+z}}{r + \kappa + \lambda\mu_s^2 \frac{z}{1+z}} \right] \\ &= \frac{r+\kappa^*}{r+\kappa} \frac{\lambda(\mu_s^1 - \mu_s^2)\frac{z}{1+z}(\kappa - \kappa^*)}{\left(r + \kappa + \lambda\mu_s^1 \frac{z}{1+z}\right)\left(r + \kappa + \lambda\mu_s^2 \frac{z}{1+z}\right)}, \end{aligned} \tag{A.9}$$

which proves the lemma. ∎

To prove Theorem 1, we first prove the following lemma:

Lemma 3 *Suppose that investors' entry decisions are given by* $\nu^1(\kappa) = 1$ *for* $\kappa > \kappa^*$, *and* $\nu^1(\kappa) = 0$ *for* $\kappa < \kappa^*$, *for some cutoff* κ^*. *Then,* μ_s^1 *and* μ_s^2 *are uniquely determined,* μ_s^1 *is increasing in* κ^*, *and* μ_s^2 *is decreasing in* κ^*.

Proof: Using (6), and setting $i = 1$, $\nu^1(\kappa) = 1$ for $\kappa > \kappa^*$, and $\nu^1(\kappa) = 0$ for $\kappa < \kappa^*$, we find

$$\int_{\kappa^*}^{\overline{\kappa}} \frac{\lambda \mu_s^1 f(\kappa)}{\kappa\left(\kappa + \lambda \mu_s^1\right)} d\kappa + \mu_s^1 = S. \tag{A.10}$$

The LHS of this equation is strictly increasing in μ_s^1, is zero for $\mu_s^1 = 0$, and is infinite for $\mu_s^1 = \infty$. Therefore, (A.10) has a unique solution μ_s^1. Moreover, differentiating implicitly w.r.t. κ^*, we find

$$\frac{d\mu_s^1}{d\kappa^*} = \frac{\frac{\lambda \mu_s^1 f(\kappa^*)}{\kappa^*(\kappa^* + \lambda \mu_s^1)}}{1 + \int_{\kappa^*}^{\overline{\kappa}} \frac{\lambda f(\kappa)}{(\kappa + \lambda \mu_s^1)^2} d\kappa} > 0.$$

Proceeding similarly, we find that μ_s^2 is uniquely determined by

$$\int_{\underline{\kappa}}^{\kappa^*} \frac{\lambda \mu_s^2 f(\kappa)}{\kappa\left(\kappa + \lambda \mu_s^2\right)} d\kappa + \mu_s^2 = S. \tag{A.11}$$

Differentiating implicitly w.r.t. κ^*, we find

$$\frac{d\mu_s^2}{d\kappa^*} = -\frac{\frac{\lambda \mu_s^2 f(\kappa^*)}{\kappa^*(\kappa^* + \lambda \mu_s^2)}}{1 + \int_{\underline{\kappa}}^{\kappa^*} \frac{\lambda f(\kappa)}{(\kappa + \lambda \mu_s^2)^2} d\kappa} < 0.$$

∎

Proof of Theorem 1: The cutoff κ^* is determined by the indifference condition $v_b^1(\kappa^*) = v_b^2(\kappa^*)$. Using (A.7), we can write this condition as

$$\frac{\mu_s^1}{\left(r+\kappa+\lambda\mu_s^1\frac{z}{1+z}\right)Q^1} = \frac{\mu_s^2}{\left(r+\kappa+\lambda\mu_s^2\frac{z}{1+z}\right)Q^2} \tag{A.12}$$

$$\Leftrightarrow \frac{\mu_s^1}{r+\kappa^*+\lambda\mu_s^1\frac{z}{1+z}+\lambda\mu_b^1\frac{1}{1+z}E^1} = \frac{\mu_s^2}{r+\kappa^*+\lambda\mu_s^2\frac{z}{1+z}+\lambda\mu_b^2\frac{1}{1+z}E^2}, \tag{A.13}$$

where

$$E^i \equiv E_b^i\left[\frac{r+\kappa^*+\lambda\mu_s^i\frac{z}{1+z}}{r+\kappa+\lambda\mu_s^i\frac{z}{1+z}}\right].$$

Multiplying by the denominators in (A.13), we find

$$(r+\kappa^*)(\mu_s^1-\mu_s^2)+\lambda\frac{1}{1+z}(\mu_s^1\mu_b^2E^2-\mu_s^2\mu_b^1E^1)=0. \tag{A.14}$$

Since

$$E^1=\frac{1}{\mu_b^1}\int_{\underline{\kappa}}^{\overline{\kappa}}\frac{r+\kappa^*+\lambda\mu_s^1\frac{z}{1+z}}{r+\kappa+\lambda\mu_s^1\frac{z}{1+z}}\mu_b^1(\kappa)d\kappa=\frac{1}{\mu_b^1}\int_{\kappa^*}^{\overline{\kappa}}\frac{f(\kappa)\left(r+\kappa^*+\lambda\mu_s^1\frac{z}{1+z}\right)}{(\kappa+\lambda\mu_s^1)\left(r+\kappa+\lambda\mu_s^1\frac{z}{1+z}\right)}d\kappa$$

and

$$E^2=\frac{1}{\mu_b^2}\int_{\underline{\kappa}}^{\kappa^*}\frac{f(\kappa)\left(r+\kappa^*+\lambda\mu_s^2\frac{z}{1+z}\right)}{(\kappa+\lambda\mu_s^2)\left(r+\kappa+\lambda\mu_s^2\frac{z}{1+z}\right)}d\kappa,$$

(A.14) can be written as

$$\begin{aligned}\mu_s^1-\mu_s^2+\mu_s^1\frac{1}{(r+\kappa^*)(1+z)}\int_{\underline{\kappa}}^{\kappa^*}\frac{\lambda f(\kappa)\left(r+\kappa^*+\lambda\mu_s^2\frac{z}{1+z}\right)}{(\kappa+\lambda\mu_s^2)\left(r+\kappa+\lambda\mu_s^2\frac{z}{1+z}\right)}d\kappa&\\ -\mu_s^2\frac{1}{(r+\kappa^*)(1+z)}\int_{\kappa^*}^{\overline{\kappa}}\frac{\lambda f(\kappa)\left(r+\kappa^*+\lambda\mu_s^1\frac{z}{1+z}\right)}{(\kappa+\lambda\mu_s^1)\left(r+\kappa+\lambda\mu_s^1\frac{z}{1+z}\right)}d\kappa&=0.\end{aligned} \tag{A.15}$$

To prove the proposition, we consider (A.15) as a function of the single unknown κ^*, i.e., treat μ_s^1 and μ_s^2 as implicit functions of κ^* (Lemma 3). To show that an equilibrium exists, it suffices to show that (A.15) has a solution κ^* satisfying $\mu_s^1 > \mu_s^2$. For $\kappa^* = \underline{\kappa}$, the LHS is negative, since (A.11) implies that $\mu_s^2 = S > \mu_s^1$. Conversely, for $\kappa^* = \overline{\kappa}$, the LHS is positive. Therefore, (A.15) has a solution $\kappa^* \in (\underline{\kappa}, \overline{\kappa})$. To show that $\mu_s^1 > \mu_s^2$, we note that

$$\begin{aligned}
&\int_{\kappa^*}^{\overline{\kappa}} \frac{\lambda f(\kappa)\left(r+\kappa^*+\lambda\mu_s^1\frac{z}{1+z}\right)}{(\kappa+\lambda\mu_s^1)\left(r+\kappa+\lambda\mu_s^1\frac{z}{1+z}\right)}d\kappa - \int_{\kappa^*}^{\overline{\kappa}} \frac{\lambda f(\kappa)\kappa^*}{(\kappa+\lambda\mu_s^1)\kappa}d\kappa \\
&= \int_{\kappa^*}^{\overline{\kappa}} \frac{\lambda f(\kappa)\left(r+\lambda\mu_s^1\frac{z}{1+z}\right)(\kappa-\kappa^*)}{(\kappa+\lambda\mu_s^1)\kappa\left(r+\kappa+\lambda\mu_s^1\frac{z}{1+z}\right)}d\kappa > 0,
\end{aligned}$$

and

$$\int_{\underline{\kappa}}^{\kappa^*} \frac{\lambda f(\kappa)\left(r+\kappa^*+\lambda\mu_s^2\frac{z}{1+z}\right)}{(\kappa+\lambda\mu_s^2)\left(r+\kappa+\lambda\mu_s^2\frac{z}{1+z}\right)}d\kappa - \int_{\underline{\kappa}}^{\kappa^*} \frac{\lambda f(\kappa)\kappa^*}{(\kappa+\lambda\mu_s^2)\kappa}d\kappa < 0.$$

Plugging into (A.15), we find

$$\mu_s^1 - \mu_s^2 + \frac{\kappa^*}{(r+\kappa^*)(1+z)}\left[\mu_s^1\int_{\underline{\kappa}}^{\kappa^*}\frac{\lambda f(\kappa)}{(\kappa+\lambda\mu_s^2)\kappa}d\kappa - \mu_s^2\int_{\kappa^*}^{\overline{\kappa}}\frac{\lambda f(\kappa)}{(\kappa+\lambda\mu_s^1)\kappa}d\kappa\right] > 0.$$

Combining with (A.10) and (A.11), we find

$$\begin{aligned}
&\mu_s^1 - \mu_s^2 + \frac{\kappa^*}{(r+\kappa^*)(1+z)}\left[\mu_s^1\left(\frac{S}{\mu_s^2}-1\right) - \mu_s^2\left(\frac{S}{\mu_s^1}-1\right)\right] > 0 \\
\Rightarrow\ &(\mu_s^1-\mu_s^2)\left[1+\frac{\kappa^*}{(r+\kappa^*)(1+z)}\left[\frac{S(\mu_s^1+\mu_s^2)}{\mu_s^1\mu_s^2}-1\right]\right] > 0.
\end{aligned}$$

Since the term in brackets is positive, we have $\mu_s^1 > \mu_s^2$.

To show that the equilibrium is unique, it suffices to show that for any κ^* that solves (A.15), the derivative of the LHS w.r.t. κ^* is strictly positive. Denoting the LHS by $F(\kappa^*, \mu_s^1, \mu_s^2)$, we have

$$\frac{dF(\kappa^*,\mu_s^1,\mu_s^2)}{d\kappa^*} = \frac{\partial F(\kappa^*,\mu_s^1,\mu_s^2)}{\partial\kappa^*} + \frac{\partial F(\kappa^*,\mu_s^1,\mu_s^2)}{\partial\mu_s^1}\frac{d\mu_s^1}{d\kappa^*} + \frac{\partial F(\kappa^*,\mu_s^1,\mu_s^2)}{\partial\mu_s^2}\frac{d\mu_s^2}{d\kappa^*}.$$

We will show that the partial derivatives w.r.t. κ^* and μ_s^1 are strictly positive, while that w.r.t. μ_s^2 is strictly negative. Since $d\mu_s^1/d\kappa^* > 0$ and $d\mu_s^2/d\kappa^* < 0$, this will imply that $dF(\kappa^*, \mu_s^1, \mu_s^2)/d\kappa^* > 0$. Setting

$$h^i(\kappa) \equiv \frac{\lambda f(\kappa)}{(\kappa + \lambda\mu_s^i)\left(r + \kappa + \lambda\mu_s^i \frac{z}{1+z}\right)},$$

we have

$$\begin{aligned}\frac{\partial F(\kappa^*, \mu_s^1, \mu_s^2)}{\partial \kappa^*} =& \frac{\lambda\mu_s^1 f(\kappa^*)}{(r+\kappa^*)(1+z)(\kappa^* + \lambda\mu_s^2)} + \frac{\lambda\mu_s^2 f(\kappa^*)}{(r+\kappa^*)(1+z)(\kappa^* + \lambda\mu_s^1)} \\ &+ \frac{\lambda\mu_s^1\mu_s^2 \frac{z}{1+z}}{(r+\kappa^*)^2(1+z)}\left[\int_{\kappa^*}^{\overline{\kappa}} h^1(\kappa)d\kappa - \int_{\underline{\kappa}}^{\kappa^*} h^2(\kappa)d\kappa\right].\end{aligned}$$

To show that the RHS is positive, it suffices to show that the term in brackets is positive. The latter follows by writing (A.15) as

$$\mu_s^1 - \mu_s^2 + \frac{\mu_s^1 - \mu_s^2}{1+z}\int_{\underline{\kappa}}^{\kappa^*} h^2(\kappa)d\kappa - \mu_s^2 \frac{r + \kappa^* + \lambda\mu_s^1 \frac{z}{1+z}}{(r+\kappa^*)(1+z)}\left[\int_{\kappa^*}^{\overline{\kappa}} h^1(\kappa)d\kappa - \int_{\underline{\kappa}}^{\kappa^*} h^2(\kappa)d\kappa\right] = 0,$$

and recalling that $\mu_s^1 > \mu_s^2$. We next have

$$\begin{aligned}\frac{\partial F(\kappa^*, \mu_s^1, \mu_s^2)}{\partial \mu_s^1} =& 1 + \frac{r + \kappa^* + \lambda\mu_s^2 \frac{z}{1+z}}{(r+\kappa^*)(1+z)}\int_{\underline{\kappa}}^{\kappa^*} h^2(\kappa)d\kappa - \mu_s^2 \frac{\lambda \frac{z}{1+z}}{(r+\kappa^*)(1+z)}\int_{\kappa^*}^{\overline{\kappa}} h^1(\kappa)d\kappa \\ &+ \mu_s^2 \frac{r + \kappa^* + \lambda\mu_s^1 \frac{z}{1+z}}{(r+\kappa^*)(1+z)}\int_{\kappa^*}^{\overline{\kappa}} h^1(\kappa)\left[\frac{\lambda}{\kappa + \lambda\mu_s^1} + \frac{\lambda \frac{z}{1+z}}{r + \kappa + \lambda\mu_s^1 \frac{z}{1+z}}\right]d\kappa.\end{aligned}$$

To show that the RHS is positive, it suffices to show that the sum of the first three terms is positive. The latter follows by writing (A.15) as

$$\mu_s^1\left[1 + \frac{r + \kappa^* + \lambda\mu_s^2 \frac{z}{1+z}}{(r+\kappa^*)(1+z)}\int_{\underline{\kappa}}^{\kappa^*} h^2(\kappa)d\kappa - \mu_s^2 \frac{\lambda \frac{z}{1+z}}{(r+\kappa^*)(1+z)}\int_{\kappa^*}^{\overline{\kappa}} h^1(\kappa)d\kappa\right]$$

$$-\mu_s^2\left[1+\frac{1}{1+z}\int_{\kappa^*}^{\overline{\kappa}} h^1(\kappa)d\kappa\right]=0.$$

An analogous argument establishes that $\partial F(\kappa^*,\mu_s^1,\mu_s^2)/\partial\mu_s^2<0$. ∎

Proof of Theorem 2: Property (a) follows from $\mu_s^1>\mu_s^2$ and Property (b). To prove Property (b), we note that since $\kappa>\kappa^*$ in market 1 and $\kappa<\kappa^*$ in market 2, $E^1<1$ and $E^2>1$. (A.14) then implies that

$$\begin{aligned}&(r+\kappa^*)(\mu_s^1-\mu_s^2)+\lambda\frac{1}{1+z}(\mu_s^1\mu_b^2-\mu_s^2\mu_b^1)<0\\ \Rightarrow &\lambda\frac{1}{1+z}(\mu_s^2\mu_b^1-\mu_s^1\mu_b^2)>(r+\kappa^*)(\mu_s^1-\mu_s^2)>0,\end{aligned}$$

which, in turn, implies Property (b).

We finally prove Property (c). Substituting the price from (A.8), we have to prove that

$$\frac{1-\frac{r}{1+z}\frac{1}{r+\kappa+\lambda\mu_s^1\frac{z}{1+z}}}{Q^1}<\frac{1-\frac{r}{1+z}\frac{1}{r+\kappa+\lambda\mu_s^2\frac{z}{1+z}}}{Q^2}.$$

Dividing both sides by (A.12), we can write this inequality as $G(\mu_s^1)<G(\mu_s^2)$, where

$$G(\mu)\equiv\frac{\left[1-\frac{r}{1+z}\frac{1}{r+\kappa+\lambda\mu\frac{z}{1+z}}\right]\left[r+\kappa^*+\lambda\mu\frac{z}{1+z}\right]}{\mu}.$$

Given that $\mu_s^1>\mu_s^2$, the inequality $G(\mu_s^1)<G(\mu_s^2)$ will follow if we show that $G(\mu)$ is decreasing. Simple calculations show that

$$G'(\mu)=-\frac{r+\kappa^*}{\mu^2}\left[1-\frac{r}{(1+z)\left(r+\kappa+\lambda\mu\frac{z}{1+z}\right)}-\frac{r\lambda\mu\frac{z}{1+z}\left(r+\kappa^*+\lambda\mu\frac{z}{1+z}\right)}{(1+z)\left(r+\kappa+\lambda\mu\frac{z}{1+z}\right)^2(r+\kappa^*)}\right].\quad\text{(A.16)}$$

The term in brackets in increasing in both κ and κ^*, and is equal to $z/(1+z) > 0$ for $\kappa = \kappa^* = 0$. Therefore, it is positive, and $G(\mu)$ is decreasing. ∎

Proof of Proposition 2: To determine $(\alpha^1, \alpha^2, \hat{\kappa})$, we use (A.10)-(A.12). Recalling that $g(\kappa) = f(\kappa)/\kappa$, we can write (A.10) and (A.11) as

$$\int_{\kappa^*}^{\overline{\kappa}} \frac{g(\kappa)}{1 + \frac{\kappa}{\lambda \mu_s^1}} d\kappa + \mu_s^1 = S, \tag{A.17}$$

$$\int_{\underline{\kappa}}^{\kappa^*} \frac{g(\kappa)}{1 + \frac{\kappa}{\lambda \mu_s^2}} d\kappa + \mu_s^2 = S. \tag{A.18}$$

Multiplying both sides of (A.12) by $\lambda z/(1+z)$, and taking inverses, we find

$$\left[1 + \frac{(r+\kappa^*)(1+z)}{\lambda \mu_s^1 z}\right] Q^1 = \left[1 + \frac{(r+\kappa^*)(1+z)}{\lambda \mu_s^2 z}\right] Q^2. \tag{A.19}$$

Moreover, using (1) and (3), we can write Q^1 and Q^2 as

$$Q^1 = 1 + \frac{\int_{\kappa^*}^{\overline{\kappa}} \frac{g(\kappa)\kappa}{1+\frac{\kappa}{\lambda \mu_s^1}} d\kappa}{\lambda (\mu_s^1)^2 z} E_b^1 \left[\frac{1}{1 + \frac{(r+\kappa)(1+z)}{\lambda \mu_s^1 z}}\right] \tag{A.20}$$

$$Q^2 = 1 + \frac{\int_{\underline{\kappa}}^{\kappa^*} \frac{g(\kappa)\kappa}{1+\frac{\kappa}{\lambda \mu_s^2}} d\kappa}{\lambda (\mu_s^2)^2 z} E_b^2 \left[\frac{1}{1 + \frac{(r+\kappa)(1+z)}{\lambda \mu_s^2 z}}\right]. \tag{A.21}$$

We next set $\mu_s^i = \alpha^i/\sqrt{\lambda} + o(1/\sqrt{\lambda})$ and $\kappa^* = \hat{\kappa} + \gamma/\sqrt{\lambda} + o(1/\sqrt{\lambda})$, and consider the asymptotic behavior of (A.17)-(A.19) when λ goes to ∞. Taking limits in (A.17) and (A.18) when λ goes to ∞, we find

$$\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa) d\kappa = \int_{\underline{\kappa}}^{\hat{\kappa}} g(\kappa) d\kappa = S,$$

i.e., (20). Taking limits in (A.19), we find

$$\frac{\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)\kappa d\kappa}{(\alpha^1)^2} = \frac{\int_{\underline{\kappa}}^{\hat{\kappa}} g(\kappa)\kappa d\kappa}{(\alpha^2)^2}. \tag{A.22}$$

Equating terms of order $1/\sqrt{\lambda}$ in (A.17) and (A.18), we find

$$-\frac{1}{\alpha^1}\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)\kappa d\kappa - \gamma g(\hat{\kappa}) + \alpha^1 = 0, \tag{A.23}$$

$$-\frac{1}{\alpha^2}\int_{\underline{\kappa}}^{\hat{\kappa}} g(\kappa)\kappa d\kappa + \gamma g(\hat{\kappa}) + \alpha^2 = 0. \tag{A.24}$$

Combining (A.22)-(A.24), we find (18), (19) and $\gamma = 0$. Given (18) and (19), (A.20) and (A.21) imply that $\lim_{\lambda\to\infty} Q^i = 1 + 1/z$ for $i = 1, 2$. Therefore, (A.8) implies that

$$\lim_{\lambda\to\infty} p^i(\kappa) = \lim_{\lambda\to\infty}\left[\frac{\delta}{r} - \frac{x}{r}\frac{1 - \frac{r}{1+z}\frac{1}{r+\kappa+\lambda\mu_s^i\frac{z}{1+z}}}{Q^i}\right] = \frac{\delta}{r} - \frac{x}{r}\frac{1}{1+\frac{1}{z}} = \frac{\delta}{r} - \frac{x}{r}\frac{z}{1+z}.$$

(A.8) also implies that

$$\begin{aligned}
p^1(\kappa) - p^2(\kappa) &= \frac{x}{r}\left[\frac{1 - \frac{r}{1+z}\frac{1}{r+\kappa+\lambda\mu_s^2\frac{z}{1+z}}}{Q^2} - \frac{1 - \frac{r}{1+z}\frac{1}{r+\kappa+\lambda\mu_s^1\frac{z}{1+z}}}{Q^1}\right] \\
&= \frac{x}{rQ^2}\left[1 - \frac{r}{\lambda\mu_s^2 z}\frac{1}{1+\frac{(r+\kappa)(1+z)}{\lambda\mu_s^2 z}} - \left[1 - \frac{r}{\lambda\mu_s^1 z}\frac{1}{1+\frac{(r+\kappa)(1+z)}{\lambda\mu_s^1 z}}\right]\frac{Q^2}{Q^1}\right] \\
&= \frac{x}{rQ^2}\left[1 - \frac{r}{\lambda\mu_s^2 z}\frac{1}{1+\frac{(r+\kappa)(1+z)}{\lambda\mu_s^2 z}} - \left[1 - \frac{r}{\lambda\mu_s^1 z}\frac{1}{1+\frac{(r+\kappa)(1+z)}{\lambda\mu_s^1 z}}\right]\frac{1+\frac{(r+\kappa^*)(1+z)}{\lambda\mu_s^1 z}}{1+\frac{(r+\kappa^*)(1+z)}{\lambda\mu_s^2 z}}\right],
\end{aligned}$$

where the last step follows from (A.12). Therefore,

$$p^1(\kappa) - p^2(\kappa) = \frac{x}{r\left(1+\frac{1}{z}\right)}\left[1 - \frac{r}{\sqrt{\lambda}\alpha^2 z} - \left[1 - \frac{r}{\sqrt{\lambda}\alpha^1 z}\right]\frac{1+\frac{(r+\hat{\kappa})(1+z)}{\sqrt{\lambda}\alpha^1 z}}{1+\frac{(r+\hat{\kappa})(1+z)}{\sqrt{\lambda}\alpha^2 z}}\right] + o\left(\frac{1}{\sqrt{\lambda}}\right).$$

Simple algebra shows that this is equivalent to (17). ∎

Proof of Corollary 1: Since λ is large, it suffices to show Properties (a)-(c) for the highest-order term (i.e., $1/\sqrt{\lambda}$) in (17). Property (c) follows immediately since z does not enter in $(\alpha^1, \alpha^2, \hat{\kappa})$. For Properties (a) and (b), we note that

$$(\alpha^1)^2 = \int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)\kappa d\kappa = \left[\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)d\kappa\right] \frac{\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)\kappa d\kappa}{\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)d\kappa} = SE_g(\kappa \geq \hat{\kappa})$$

and

$$(\alpha^2)^2 = SE_g(\kappa \leq \hat{\kappa}),$$

where E_g denotes expectation under g. We denote the new distributions by g_ϕ, and the new values of $(\alpha^1, \alpha^2, \hat{\kappa})$ by $(\alpha^1_\phi, \alpha^2_\phi, \hat{\kappa}_\phi)$. The distribution in case (a) does not affect the median ($\hat{\kappa}_\phi = \hat{\kappa}$), but concentrates more weight towards it. Therefore,

$$(\alpha^1_\phi)^2 = SE_{g_\phi}(\kappa \geq \hat{\kappa}) \leq SE_g(\kappa \geq \hat{\kappa}) = (\alpha^1)^2,$$

$$(\alpha^2_\phi)^2 \geq (\alpha^2)^2.$$

(17) then implies that the liquidity premium is lower under g_ϕ than under g. To prove the result in case (b), we consider the distribution $g_\phi(\kappa) \equiv g(\kappa - y)$, which shifts weight up uniformly by y. (This distribution is of the form $g(\kappa) + \phi(\kappa)$, provided that $\phi(\kappa)$ is defined as $g(\kappa - y) - g(\kappa)$ and all distributions are considered in the common support $[\underline{\kappa}, \overline{\kappa} + y]$. A sufficient condition for $\phi(\kappa)$ to change sign only once is that $g(\kappa) = c\kappa^\alpha$ for any two constants (α, c).) We have $\hat{\kappa}_\phi = \hat{\kappa} + y$,

$$(\alpha^1_\phi)^2 = S\left[E_g(\kappa \geq \hat{\kappa}) + y\right],$$

$$(\alpha^2_\phi)^2 = S\left[E_g(\kappa \leq \hat{\kappa}) + y\right].$$

To show that the liquidity premium can be higher or lower under g_ϕ, we differentiate the higher-order term in (17) w.r.t. y at $y=0$. Setting $\hat{\kappa}^1 \equiv E_g(\kappa \geq \hat{\kappa})$ and $\hat{\kappa}^2 \equiv E_g(\kappa \leq \hat{\kappa})$, the derivative has the same sign as

$$\begin{aligned}
&\frac{d}{dy}\left[\left(\frac{r}{1+z}+\hat{\kappa}+y\right)\left(\frac{1}{\sqrt{\hat{\kappa}^2+y}}-\frac{1}{\sqrt{\hat{\kappa}^1+y}}\right)\right]\Bigg|_{y=0} \\
&=\left(\frac{1}{\sqrt{\hat{\kappa}^2}}-\frac{1}{\sqrt{\hat{\kappa}^1}}\right)-\frac{\frac{r}{1+z}+\hat{\kappa}}{2}\left(\frac{1}{(\hat{\kappa}^2)^{\frac{3}{2}}}-\frac{1}{(\hat{\kappa}^1)^{\frac{3}{2}}}\right) \\
&=\left(\frac{1}{\sqrt{\hat{\kappa}^2}}-\frac{1}{\sqrt{\hat{\kappa}^1}}\right)\left[1-\frac{\frac{r}{1+z}+\hat{\kappa}}{2}\left(\frac{1}{\hat{\kappa}^1}+\frac{1}{\hat{\kappa}^2}+\frac{1}{\sqrt{\hat{\kappa}^1\hat{\kappa}^2}}\right)\right].
\end{aligned}$$

The term in brackets is negative for a distribution with $\hat{\kappa} \approx \hat{\kappa}^1$ (i.e., almost all mass concentrated on the upper bound of the support). On the other hand, the term is positive for a distribution with $\hat{\kappa} \approx \hat{\kappa}^2$, provided that r and $\hat{\kappa}^2/\hat{\kappa}^1$ are close enough to zero. Therefore, the derivative can have either sign. ∎

Proof of Proposition 3: In a symmetric equilibrium, (A.12) must hold for all κ^*. This is equivalent to $\mu_s^1 = \mu_s^2 = \mu_s$ (from Lemma 1) and $Q^1 = Q^2$. It is easy to check that there is a continuum of functions $\nu^1(\kappa)$ such that these two scalar equations hold. Plugging these equations into (A.8), we find $p^1(\kappa) = p^2(\kappa)$ for all κ. ∎

Instead of proving Lemma 2, we prove a more general lemma that (i) covers non-steady states (where population measures, expected utilities and prices vary on time), and (ii) does not require that the measures of inactive owners and sellers add up to the asset supply, as must be the case in equilibrium. We extend our welfare criterion to non-steady states as

$$\mathcal{W}_t \equiv \sum_{i=1,2}\left[\int_{\underline{\kappa}}^{\overline{\kappa}}[v_{b,t}^i(\kappa)\mu_{b,t}^i(\kappa)+v_{o,t}^i(\kappa)\mu_{o,t}^i(\kappa)]d\kappa + v_{s,t}^i\mu_{s,t}^i + \int_t^\infty\left[\int_{\underline{\kappa}}^{\overline{\kappa}} v_{b,t'}^i(\kappa)f(\kappa)\nu^i(\kappa)d\kappa\right]e^{-r(t'-t)}dt'\right]$$

where the second subscript denotes time.

Lemma 4 *Welfare is*

$$\mathcal{W}_t = \sum_{i=1}^{2} \int_{t}^{\infty} \left[\delta(\mu^i_{o,t'} + \mu^i_{s,t'}) - x\mu^i_{s,t'}\right] e^{-r(t'-t)} dt'. \tag{A.25}$$

Proof: It suffices to show that

$$\frac{d(\mathcal{W}_t e^{-rt})}{dt} = -\sum_{i=1}^{2} \left[\delta(\mu^i_{o,t} + \mu^i_{s,t}) - x\mu^i_{s,t}\right] e^{-rt}, \tag{A.26}$$

since this integrates to (A.25). Using the definition of $\mathcal{W}_t$, we find

$$\frac{d(\mathcal{W}_t e^{-rt})}{dt} = \sum_{i=1,2} A^i e^{-rt},$$

where

$$\begin{aligned} A^i = &\int_{\underline{\kappa}}^{\overline{\kappa}} \left[\frac{dv^i_{b,t}(\kappa)}{dt}\mu^i_{b,t}(\kappa) + v^i_{b,t}(\kappa)\frac{d\mu^i_{b,t}(\kappa)}{dt} + \frac{dv^i_{o,t}(\kappa)}{dt}\mu^i_{o,t}(\kappa) + v^i_{o,t}(\kappa)\frac{d\mu^i_{o,t}(\kappa)}{dt}\right] d\kappa \\ &+ \frac{dv^i_{s,t}}{dt}\mu^i_{s,t} + v^i_{s,t}\frac{d\mu^i_{s,t}}{dt} - r\left[\int_{\underline{\kappa}}^{\overline{\kappa}} [v^i_{b,t}(\kappa)\mu^i_{b,t}(\kappa) + v^i_{o,t}(\kappa)\mu^i_{o,t}(\kappa)] d\kappa + v^i_{s,t}\mu^i_{s,t}\right] \\ &- \int_{\underline{\kappa}}^{\overline{\kappa}} v^i_{b,t}(\kappa) f(\kappa)\nu^i(\kappa) d\kappa. \end{aligned} \tag{A.27}$$

To simplify (A.27), we compute the derivatives of the population measures and expected utilities. The derivative of a population measure is equal to the difference between the inflow and outflow associated to that population. Proceeding as in Section 3.1, we find

$$\frac{d\mu^i_{b,t}(\kappa)}{dt} = f(\kappa)\nu^i(\kappa) - \kappa\mu^i_{b,t}(\kappa) - \lambda\mu^i_{b,t}(\kappa)\mu^i_{s,t}, \tag{A.28}$$

$$\frac{d\mu^i_{o,t}(\kappa)}{dt} = \lambda\mu^i_{b,t}(\kappa)\mu^i_{s,t} - \kappa\mu^i_{o,t}(\kappa), \tag{A.29}$$

and

$$\frac{d\mu^i_{s,t}}{dt} = \int_{\underline{\kappa}}^{\overline{\kappa}} \left[\kappa\mu^i_{o,t}(\kappa) - \lambda\mu^i_{b,t}(\kappa)\mu^i_{s,t}\right] d\kappa. \tag{A.30}$$

To compute the derivatives of the expected utilities, consider, for example, $v_{b,t}(\kappa)$. For non-steady states, (7) generalizes to

$$v^i_{b,t}(\kappa) = (1 - rdt)\left[\kappa dt 0 + \lambda\mu^i_s dt(v^i_{o,t}(\kappa) - p^i_t(\kappa)) + (1 - \lambda\mu^i_s dt - \kappa dt)v^i_{b,t+dt}(\kappa)\right].$$

Rearranging, we find

$$rv^i_{b,t}(\kappa) - \frac{dv^i_{b,t}(\kappa)}{dt} = -\kappa v^i_{b,t}(\kappa) + \lambda\mu^i_{s,t}(v^i_{o,t}(\kappa) - p^i_t(\kappa) - v^i_{b,t}(\kappa)). \tag{A.31}$$

We similarly find

$$rv^i_{o,t}(\kappa) - \frac{dv^i_{o,t}(\kappa)}{dt} = \delta + \kappa(v^i_{s,t} - v^i_{o,t}(\kappa)), \tag{A.32}$$

and

$$rv^i_{s,t} - \frac{dv^i_{s,t}}{dt} = \delta - x + \int_{\underline{\kappa}}^{\overline{\kappa}} \lambda\mu^i_{b,t}(\kappa)(p^i_t(\kappa) - v^i_{s,t})d\kappa. \tag{A.33}$$

Plugging (A.28)-(A.30) and (A.31)-(A.33) into (A.27), and canceling terms, we find

$$A^i = -\delta\mu^i_{o,t} - (\delta - x)\mu^i_{s,t},$$

which proves (A.26). ∎

Proof of Proposition 4: We only derive (22), as (23) and (24) can be derived using the same procedure. Suppose that at time t the measure of buyers with switching rates in $[\kappa, \kappa + d\kappa]$ in market i is increased by ϵ, while all other measures remain as in the steady state. That is,

$$\mu^i_{b,t}(\kappa) = \mu^i_b(\kappa) + \frac{\epsilon}{d\kappa}, \tag{A.34}$$

$\mu^i_{b,t}(\kappa') = \mu^i_b(\kappa')$ for $\kappa' \notin [\kappa, \kappa + d\kappa]$, $\mu^i_{o,t}(\kappa') = \mu^i_o(\kappa')$ for all κ', and $\mu^i_{s,t} = \mu^i_s$, where measures without the time subscript refer to the steady state.

We determine the change in population measures at time $t + dt$. Consider first the buyers with switching rates in $[\kappa, \kappa + d\kappa]$. (A.28) implies that

$$\mu^i_{b,t+dt}(\kappa) = \mu^i_{b,t}(\kappa) + \left[f(\kappa)\nu^i(\kappa) - \kappa\mu^i_{b,t}(\kappa) - \lambda\mu^i_{b,t}(\kappa)\mu^i_{s,t}\right] dt. \tag{A.35}$$

Plugging (A.34) and $\mu^i_{s,t} = \mu^i_s$ into (A.35), and using the steady-state version of (A.35), i.e.,

$$f(\kappa)\nu^i(\kappa) - \kappa\mu^i_b(\kappa) - \lambda\mu^i_b(\kappa)\mu^i_s = 0,$$

we find

$$\mu^i_{b,t+dt}(\kappa) = \mu^i_b(\kappa) + \frac{\epsilon}{d\kappa}(1 - \kappa dt - \lambda\mu^i_s dt).$$

Thus, the measure of buyers with switching rates in $[\kappa, \kappa + d\kappa]$ increases by $\epsilon(1 - \kappa dt - \lambda\mu^i_s dt) \equiv \epsilon\Delta^i_b(\kappa)$. In a similar manner, (A.29) implies that the measure of inactive owners with switching rates in $[\kappa, \kappa + d\kappa]$ increases by $\epsilon\lambda\mu^i_s dt \equiv \epsilon\Delta^i_o(\kappa)$, and (A.30) implies that the measure of sellers decreases by $\epsilon\lambda\mu^i_s dt \equiv \epsilon\Delta^i_s$. Finally, the measures of buyers and inactive owners with $\kappa' \notin [\kappa, \kappa + d\kappa]$ do not change in order dt.

(A.25) implies that

$$\mathcal{W}_t = \sum_{i=1}^{2} \left[\delta(\mu^i_{o,t} + \mu^i_{s,t}) - x\mu^i_{s,t}\right] dt + (1 - r dt)\mathcal{W}_{t+dt}.$$

The derivative of $\mathcal{W}_t$ w.r.t. ϵ at $\epsilon = 0$ is $V_b^i(\kappa)$. The derivative of the term in brackets is zero since $\mu_{o,t}^i = \mu_o^i$ and $\mu_{s,t}^i = \mu_s^i$. Finally, the derivative of $\mathcal{W}_{t+dt}$ is $\Delta_b^i(\kappa)V_b^i(\kappa) + \Delta_o^i(\kappa)V_o^i(\kappa) - \Delta_s^i V_s^i$. Thus,

$$V_b^i(\kappa) = (1 - rdt)\left[\Delta_b^i(\kappa)V_b^i(\kappa) + \Delta_o^i(\kappa)V_o^i(\kappa) - \Delta_s^i V_s^i\right].$$

Rearranging, we find (22). ∎

Proof of Proposition 5: (22)-(24) are the same as (25)-(27), except that $z/(1+z)$ and $1/(1+z)$ are replaced by 1. Therefore, we can proceed as in Proposition 1 and replace (A.7) by

$$V_b^i(\kappa) = \frac{\lambda \mu_s^i x}{(r+\kappa)\left(r+\kappa+\lambda\mu_s^i\right)\left[1+\lambda\mu_b^i E_b^i\left[\frac{1}{r+\kappa+\lambda\mu_s^i}\right]\right]}.$$

Using this equation, inequality $V_b^1(\kappa^*) < V_b^2(\kappa^*)$ is equivalent to

$$\frac{\mu_s^1}{r+\kappa^*+\lambda\mu_s^1+\lambda\mu_b^1 E_w^1} < \frac{\mu_s^2}{r+\kappa^*+\lambda\mu_s^2+\lambda\mu_b^2 E_w^2}, \tag{A.36}$$

where

$$E_w^i \equiv E_b^i\left[\frac{r+\kappa^*+\lambda\mu_s^i}{r+\kappa+\lambda\mu_s^i}\right].$$

Dividing both sides by (A.13), we obtain the equivalent inequality

$$\frac{r+\kappa^*+\lambda\mu_s^1\frac{z}{1+z}+\lambda\mu_b^1\frac{1}{1+z}E^1}{r+\kappa^*+\lambda\mu_s^1+\lambda\mu_b^1 E_w^1} < \frac{r+\kappa^*+\lambda\mu_s^2\frac{z}{1+z}+\lambda\mu_b^2\frac{1}{1+z}E^2}{r+\kappa^*+\lambda\mu_s^2+\lambda\mu_b^2 E_w^2}. \tag{A.37}$$

Since for $\kappa > \kappa^*$,

$$\frac{r+\kappa^*+\lambda\mu_s^1}{r+\kappa+\lambda\mu_s^1} > \frac{r+\kappa^*+\lambda\mu_s^1\frac{z}{1+z}}{r+\kappa+\lambda\mu_s^1\frac{z}{1+z}},$$

we have $E_w^1 > E^1$. A similar argument implies that $E_w^2 < E^2$. Therefore, to show (A.37), it suffices to show that

$$\frac{r + \kappa^* + \lambda\mu_s^1 \frac{z}{1+z} + \lambda\mu_b^1 \frac{1}{1+z} E^1}{r + \kappa^* + \lambda\mu_s^1 + \lambda\mu_b^1 E^1} < \frac{r + \kappa^* + \lambda\mu_s^2 \frac{z}{1+z} + \lambda\mu_b^2 \frac{1}{1+z} E^2}{r + \kappa^* + \lambda\mu_s^2 + \lambda\mu_b^2 E^2},$$

which is equivalent to

$$\begin{aligned} &(r+\kappa^*)\left[(\mu_s^1 - \mu_s^2)\frac{1}{1+z} + (\mu_b^1 E^1 - \mu_b^2 E^2)\frac{z}{1+z}\right] \\ &+\lambda(\mu_s^2\mu_b^1 E^1 - \mu_s^1\mu_b^2 E^2)\frac{z-1}{1+z} > 0. \end{aligned} \tag{A.38}$$

(A.14) implies that

$$\lambda\frac{1}{1+z}(\mu_s^2\mu_b^1 E^1 - \mu_s^1\mu_b^2 E^2) = (r+\kappa^*)(\mu_s^1 - \mu_s^2). \tag{A.39}$$

Substituting $\mu_s^2\mu_b^1 E^1 - \mu_s^1\mu_b^2 E^2$ from this equation into (A.38), we find the equivalent equation

$$(\mu_s^1 - \mu_s^2)\frac{z^2}{1+z} + (\mu_b^1 E^1 - \mu_b^2 E^2)\frac{z}{1+z} > 0.$$

This equation holds because (i) $\mu_s^1 > \mu_s^2$ and (ii) $\mu_b^1 E^1 > \mu_b^2 E^2$ (from (A.39) and $\mu_s^1 > \mu_s^2$). ∎

Proof of Proposition 6: From Lemma 2, maximizing $\mathcal{W}$ is equivalent to minimizing $\mu_s^1 + \mu_s^2$. We first minimize $\mu_s^1 + \mu_s^2$ through the choice of a "trigger" allocation, i.e., a cutoff κ_w^* such that $\nu^1(\kappa) = 1$ for $\kappa > \kappa_w^*$, and $\nu^1(\kappa) = 0$ for $\kappa < \kappa_w^*$. We show that this constrained problem, $(\mathcal{P}_c)$, has a unique solution. We next show that this solution, together with the symmetric one derived by switching the indices of the two assets, are the only solutions to the unconstrained problem $(\mathcal{P})$.

Lemma 3 implies that the derivative of $\mu_s^1 + \mu_s^2$ w.r.t. κ_w^* is

$$\frac{\lambda f(\kappa_w^*)}{\kappa_w^*}\left[\frac{\mu_s^1}{\kappa_w^* + \lambda\mu_s^1 + \int_{\kappa_w^*}^{\overline{\kappa}} \frac{\lambda f(\kappa)(\kappa_w^*+\lambda\mu_s^1)}{(\kappa+\lambda\mu_s^1)^2} d\kappa} - \frac{\mu_s^2}{\kappa_w^* + \lambda\mu_s^2 + \int_{\underline{\kappa}}^{\kappa_w^*} \frac{\lambda f(\kappa)(\kappa_w^*+\lambda\mu_s^2)}{(\kappa+\lambda\mu_s^2)^2} d\kappa}\right]. \tag{A.40}$$

Multiplying by the denominators, we find that the term in brackets has the same sign as

$$F_w \equiv \mu_s^1 - \mu_s^2 + \mu_s^1 \frac{1}{\kappa_w^*} \int_{\underline{\kappa}}^{\kappa_w^*} \frac{\lambda f(\kappa)(\kappa_w^* + \lambda\mu_s^2)}{(\kappa + \lambda\mu_s^2)^2} d\kappa - \mu_s^2 \frac{1}{\kappa_w^*} \int_{\kappa_w^*}^{\overline{\kappa}} \frac{\lambda f(\kappa)(\kappa_w^* + \lambda\mu_s^1)}{(\kappa + \lambda\mu_s^1)^2} d\kappa.$$

Proceeding as in the existence proof of Theorem 1, we can show that there exists $\kappa_w^* \in (\underline{\kappa}, \overline{\kappa})$ such that $F_w = 0$, and moreover, that for any such κ_w^* we have $\mu_s^1 > \mu_s^2$. Proceeding as in the uniqueness proof of Theorem 1, we can show that for any κ_w^* solving $F_w = 0$, the derivative of F_w w.r.t. κ_w^* is positive. This implies that κ_w^* is unique. It also implies that F_w is negative and then positive, and thus κ_w^* corresponds to a minimum of $\mu_s^1 + \mu_s^2$.

To show that the solution to $(\mathcal{P}_c)$ and its symmetric counterpart are the only solutions to $(\mathcal{P})$, we proceed by contradiction, assuming that $\mu_s^1 + \mu_s^2$ is lower for some non-trigger allocation $\nu^1(\kappa)$. We denote the measures of sellers under $\nu^1(\kappa)$ by $\{\mu_{s\nu}^i\}_{i=1,2}$. Without loss of generality, we assume that $\mu_{s\nu}^1 \geq \mu_{s\nu}^2$, and first consider the case $\mu_{s\nu}^1 > \mu_{s\nu}^2$. Define $\check{\kappa}$ by

$$\int_{\underline{\kappa}}^{\check{\kappa}} \frac{f(\kappa)}{\kappa(\kappa + \lambda\mu_{s\nu}^2)} d\kappa = \int_{\underline{\kappa}}^{\overline{\kappa}} \frac{f(\kappa)\nu^2(\kappa)}{\kappa(\kappa + \lambda\mu_{s\nu}^2)} d\kappa, \tag{A.41}$$

and consider the corresponding trigger allocation. Since $\mu_{s\nu}^2$ solves (6) under the trigger allocation, it coincides with that allocation's μ_s^2. (A.41) implies that

$$\int_{\check{\kappa}}^{\overline{\kappa}} \frac{f(\kappa)}{\kappa(\kappa + \lambda\mu_{s\nu}^2)} d\kappa = \int_{\underline{\kappa}}^{\overline{\kappa}} \frac{f(\kappa)\nu^1(\kappa)}{\kappa(\kappa + \lambda\mu_{s\nu}^2)} d\kappa. \tag{A.42}$$

Since $\nu^1(\kappa)$ gives non-zero weight to values below $\check{\kappa}$ (being a non-trigger allocation), and the function $1/(\kappa + \lambda\mu^1_{s\nu})$ is decreasing in κ, (A.42) implies that

$$\int_{\check{\kappa}}^{\overline{\kappa}} \frac{f(\kappa)}{\kappa(\kappa+\lambda\mu^2_{s\nu})(\kappa+\lambda\mu^1_{s\nu})} d\kappa < \int_{\underline{\kappa}}^{\overline{\kappa}} \frac{f(\kappa)\nu^1(\kappa)}{\kappa(\kappa+\lambda\mu^2_{s\nu})(\kappa+\lambda\mu^1_{s\nu})} d\kappa. \tag{A.43}$$

Multiplying (A.43) by $\lambda(\mu^1_{s\nu} - \mu^2_{s\nu})$, and subtracting it from (A.42), we find

$$\int_{\check{\kappa}}^{\overline{\kappa}} \frac{f(\kappa)}{\kappa(\kappa+\lambda\mu^1_{s\nu})} d\kappa > \int_{\underline{\kappa}}^{\overline{\kappa}} \frac{f(\kappa)\nu^1(\kappa)}{\kappa(\kappa+\lambda\mu^1_{s\nu})} d\kappa. \tag{A.44}$$

(6) then implies that $\mu^1_{s\nu}$ is greater than the trigger allocation's μ^1_s, a contradiction. When $\mu^1_{s\nu} = \mu^2_{s\nu}$, $\mu^1_{s\nu}$ is equal to the trigger allocation's μ^1_s. Therefore, the allocation $\nu^1(\kappa)$ achieves the same welfare as the trigger allocation, but since the latter satisfies $\mu^1_s = \mu^2_s$, it is suboptimal from the first part of the proof. ∎

Proof of Theorem 3: Denote by $\mathcal{W}(\ell)$ the welfare under the trigger allocation with cutoff ℓ. From Proposition 6, $\mathcal{W}(\ell)$ is increasing for $\ell < \kappa^*_w$, and decreasing for $\ell > \kappa^*_w$. The cutoff κ' for which the measure of sellers is equal across markets obviously satisfies $\kappa' < \kappa^*$, where κ^* is the clientele-equilibrium cutoff. Moreover, for small r, we have $\kappa^* < \kappa^*_w$. Indeed, for $r = 0$, (A.36) takes the form

$$\frac{\mu^1_s}{\kappa^* + \lambda\mu^1_s + \lambda\mu^1_b E^1_b\left[\frac{\kappa^*+\lambda\mu^1_s}{\kappa+\lambda\mu^1_s}\right]} < \frac{\mu^2_s}{\kappa^* + \lambda\mu^2_s + \lambda\mu^2_b E^2_b\left[\frac{\kappa^*+\lambda\mu^2_s}{\kappa+\lambda\mu^2_s}\right]}. \tag{A.45}$$

Since

$$E^1_b\left[\frac{\kappa^*+\lambda\mu^1_s}{\kappa+\lambda\mu^1_s}\right] = \frac{1}{\mu^1_b}\int_{\kappa^*}^{\overline{\kappa}} \frac{\kappa^*+\lambda\mu^1_s}{\kappa+\lambda\mu^1_s}\mu^1_b(\kappa) d\kappa = \frac{1}{\mu^1_b}\int_{\kappa^*}^{\overline{\kappa}} \frac{f(\kappa)(\kappa^*+\lambda\mu^1_s)}{(\kappa+\lambda\mu^1_s)^2} d\kappa$$

and

$$E_b^2\left[\frac{\kappa^*+\lambda\mu_s^2}{\kappa+\lambda\mu_s^2}\right]=\frac{1}{\mu_b^2}\int_{\underline{\kappa}}^{\kappa^2}\frac{f(\kappa)(\kappa^*+\lambda\mu_s^2)}{(\kappa+\lambda\mu_s^2)^2}d\kappa,$$

(A.45) implies that the term in brackets in (A.40) is negative when κ_w^* is replaced by κ^*. Therefore, $\mathcal{W}(\ell)$ is increasing at κ^*, implying that $\kappa^*<\kappa_w^*$. Since $\kappa'<\kappa^*<\kappa_w^*$, and $\mathcal{W}(\ell)$ is increasing for $\ell<\kappa_w^*$, we have $\mathcal{W}(\kappa')<\mathcal{W}(\kappa^*)$. Since, in addition, $\mathcal{W}(\kappa')$ is equal to the welfare $\mathcal{W}_s$ in any symmetric equilibrium (see Footnote 12), we have $\mathcal{W}_s<\mathcal{W}(\kappa^*)$. ∎

Proof of Proposition 7: With simultaneous search, the expected utility of a buyer with switching rate κ does not depend on i, and is given by

$$rv_b(\kappa)=-\kappa v_b(\kappa)+\sum_{i=1}^{2}\nu^i(\kappa)\lambda\mu_s^i\left(v_o^i(\kappa)-p^i(\kappa)-v_b(\kappa)\right), \tag{A.46}$$

where $\nu^i(\kappa)$ is the probability that the buyer accepts to trade upon meeting a seller of asset i. This probability is one if the surplus $v_o^i(\kappa)-v_b(\kappa)-v_s^i$ is positive, and zero if it is negative. (Note that $\nu^1(\kappa)$ and $\nu^2(\kappa)$ do not have to sum to one if the buyer accepts to buy both assets.) The utility $v_o^i(\kappa)$ of an inactive owner, the utility v_s^i of a seller, and the price $p^i(\kappa)$ can depend on i, and are given by (9), (10) and (11). In (10), μ_b^i is the measure of buyers in the set $B^i\equiv\{\kappa: v_o^i(\kappa)-v_b(\kappa)-v_s^i>0\}$, and in (11), $v_b^i(\kappa)$ is replaced by $v_b(\kappa)$.

We first show that $v_s^i<\delta/r$. Using (11), we can write (A.46) as

$$\begin{aligned}rv_b(\kappa)&=-\kappa v_b(\kappa)+\sum_{j=1}^{2}\nu^j(\kappa)\lambda\mu_s^j\frac{z}{1+z}\left(v_o^j(\kappa)-v_b(\kappa)-v_s^j\right)\\&\geq-\kappa v_b(\kappa)+\nu^i(\kappa)\lambda\mu_s^i\frac{z}{1+z}\left(v_o^i(\kappa)-v_b(\kappa)-v_s^i\right)\end{aligned}$$

Therefore, for $\kappa \in B^i$,

$$r v_b(\kappa) \geq -\kappa v_b(\kappa) + \lambda \mu_s^i \frac{z}{1+z} \left(v_o^i(\kappa) - v_b(\kappa) - v_s^i \right).$$

Subtracting this equation and (A.2) from (9), and taking expectations over B^i, we find

$$E_b^i(v_o^i(\kappa) - v_b(\kappa)) - v_s^i \leq \frac{x}{r + \kappa + \lambda \mu_s^i \frac{z}{1+z} + \lambda \mu_b^i \frac{1}{1+z}}.$$

Substituting back into (A.2), we find

$$r v_s^i \leq \delta - x + \frac{\lambda \mu_b^i \frac{1}{1+z} x}{r + \kappa + \lambda \mu_s^i \frac{z}{1+z} + \lambda \mu_b^i \frac{1}{1+z}} \Rightarrow v_s^i < \frac{\delta}{r}.$$

We next show that in equilibrium all buyers realize positive surplus from at least one asset. Indeed, suppose that there exists κ such that $v_o^i(\kappa) - v_b(\kappa) - v_s^i \leq 0$ for $i = 1, 2$. Since $\nu^i(\kappa) = 0$ for $i = 1, 2$, (A.46) implies that $v_b(\kappa) = 0$. (9) then implies that

$$v_o^i(\kappa) - v_b(\kappa) - v_s^i = \frac{\delta - r v_s^i}{r + \kappa}.$$

Since $v_s^i < \delta/r$, the surplus is positive, a contradiction.

We finally show that in equilibrium all buyers realize positive surplus from both assets. If not, there exists κ such that, e.g., $v_o^1(\kappa) - v_b(\kappa) - v_s^1 \leq 0$. Since all buyers realize positive surplus from at least one asset, we have $v_o^2(\kappa) - v_b(\kappa) - v_s^2 > 0$. Subtracting one equation from the other, and using (9), we find

$$v_o^2(\kappa) - v_s^2 - (v_o^1(\kappa) - v_s^1) > 0 \Rightarrow \frac{\delta - r v_s^2}{r + \kappa} - \frac{\delta - r v_s^1}{r + \kappa} > 0 \Rightarrow v_s^1 > v_s^2.$$

The same reasoning implies that $B^1 \subset B^2$. Writing (A.2) as

$$
\begin{aligned}
rv_s^i &= \delta - x + \lambda \frac{1}{1+z} \int\limits_{B^i} (v_o^i(\kappa) - v_b(\kappa) - v_s^i)\mu_b(\kappa) d\kappa \\
&= \delta - x + \lambda \frac{1}{1+z} \int\limits_{B^i} \left[\frac{\delta - rv_s^i}{r+\kappa} - v_b(\kappa) \right] \mu_b(\kappa) d\kappa,
\end{aligned}
$$

and subtracting this equation for $i = 1$ from its counterpart for $i = 2$, we find

$$
\begin{aligned}
r(v_s^2 - v_s^1) &= \lambda \frac{1}{1+z} \int\limits_{B^1} \frac{r(v_s^1 - v_s^2)}{r+\kappa} \mu_b(\kappa) d\kappa + \lambda \frac{1}{1+z} \int\limits_{B^2 \setminus B^1} (v_o^i(\kappa) - v_b(\kappa) - v_s^i)\mu_b(\kappa) d\kappa \\
&> \lambda \frac{1}{1+z} \int\limits_{B^1} \frac{r(v_s^1 - v_s^2)}{r+\kappa} \mu_b(\kappa) d\kappa,
\end{aligned}
$$

a contradiction since $v_s^1 > v_s^2$. Therefore, all buyers buy the first asset they find, and sellers' search times are identical across assets. Moreover, since sellers' utilities are identical across assets, so are prices. ∎

Proof of Proposition 8: When z is a function of κ, (A.7) and (A.8) generalize to

$$
v_b^i(\kappa) = \frac{\lambda \mu_s^i \frac{z(\kappa)}{1+z(\kappa)} x}{(r+\kappa)\left[r + \kappa + \lambda \mu_s^i \frac{z(\kappa)}{1+z(\kappa)} \right] Q^i}, \tag{A.47}
$$

$$
p^i(\kappa) = \frac{\delta}{r} - \frac{x}{r} \frac{1 - \frac{r}{1+z(\kappa)} \frac{1}{r+\kappa+\lambda\mu_s^i \frac{z(\kappa)}{1+z(\kappa)}}}{Q^i}, \tag{A.48}
$$

where

$$
Q^i \equiv 1 + \lambda \mu_b^i E_b^i \left[\frac{1}{[1+z(\kappa)](r+\kappa) + \lambda \mu_s^i z(\kappa)} \right].
$$

(A.9) generalizes to

$$
\frac{v_b^1(\kappa)}{v_b^1(\kappa^*)} - \frac{v_b^2(\kappa)}{v_b^2(\kappa^*)} = \frac{r+\kappa^*}{r+\kappa} \frac{\lambda(\mu_s^1 - \mu_s^2) \frac{z(\kappa)}{1+z(\kappa)} \frac{z(\kappa^*)}{1+z(\kappa^*)} \left[(r+\kappa)\frac{1+z(\kappa)}{z(\kappa)} - (r+\kappa^*)\frac{1+z(\kappa^*)}{z(\kappa^*)} \right]}{\left[r+\kappa+\lambda\mu_s^1 \frac{z(\kappa)}{1+z(\kappa)} \right] \left[r+\kappa+\lambda\mu_s^2 \frac{z(\kappa^*)}{1+z(\kappa^*)} \right]}
$$

Therefore, Lemma 1 holds if the function $(r+\kappa)\frac{1+z(\kappa)}{z(\kappa)}$ is increasing in κ. (A.15) generalizes to

$$\mu_s^1 - \mu_s^2 + \mu_s^1 \frac{1}{(r+\kappa^*)} \int_{\underline{\kappa}}^{\kappa^*} \frac{\lambda f(\kappa)\left[r+\kappa^*+\lambda\mu_s^2\frac{z(\kappa^*)}{1+z(\kappa^*)}\right]}{(\kappa+\lambda\mu_s^2)\left[[1+z(\kappa)](r+\kappa)+\lambda\mu_s^2 z(\kappa)\right]} d\kappa$$
$$- \mu_s^2 \frac{1}{(r+\kappa^*)} \int_{\kappa^*}^{\overline{\kappa}} \frac{\lambda f(\kappa)\left[r+\kappa^*+\lambda\mu_s^1\frac{z(\kappa^*)}{1+z(\kappa^*)}\right]}{(\kappa+\lambda\mu_s^1)\left[[1+z(\kappa)](r+\kappa)+\lambda\mu_s^1 z(\kappa)\right]} d\kappa = 0.$$

Proceeding as in Theorem 1, we can show that this equation has a solution. The proof that $\mu_s^1 > \mu_s^2$ goes through if

$$\frac{r+\kappa^*+\lambda\mu_s\frac{z(\kappa^*)}{1+z(\kappa^*)}}{\left[[1+z(\kappa)](r+\kappa)+\lambda\mu_s z(\kappa)\right]} - \frac{\kappa^*}{[1+z(\kappa^*)]\kappa}$$

is positive for $\kappa > \kappa^*$ and negative for $\kappa < \kappa^*$. This is equivalent to the function

$$\frac{\kappa}{[1+z(\kappa)](r+\kappa)+\lambda\mu_s z(\kappa)}$$

being increasing in κ. Finally, the uniqueness proof in Theorem 1 goes through if the function $(r+\kappa)\frac{1+z(\kappa)}{z(\kappa)}$ is increasing in κ. Therefore, if $z(\kappa)$ is decreasing in κ, Lemma 1 and Theorem 1 hold, meaning that a clientele equilibrium exists and is unique.

(A.16) generalizes to

$$G'(\mu) = -\frac{r+\kappa^*}{\mu^2}\left[1 - \frac{r}{[1+z(\kappa)]\left[r+\kappa+\lambda\mu\frac{z(\kappa)}{1+z(\kappa)}\right]} - \frac{r\lambda\mu\frac{z(\kappa)}{1+z(\kappa)}\left[r+\kappa^*+\lambda\mu\frac{z(\kappa^*)}{1+z(\kappa^*)}\right]}{[1+z(\kappa)]\left[r+\kappa+\lambda\mu\frac{z(\kappa)}{1+z(\kappa)}\right]^2(r+\kappa^*)}\right].$$

To show that $G'(\mu) < 0$, it suffices to show that the term in brackets is positive when κ, κ^* are set to zero, holding $z(\kappa), z(\kappa^*)$ fixed. Solving for $z(\kappa^*)$, the term of brackets is positive if

$$\frac{z(\kappa^*)}{1+z(\kappa^*)} < \frac{1}{\lambda\mu}\left[\frac{r+\lambda\mu}{\lambda\mu}\left[r[1+z(\kappa)]+\lambda\mu z(\kappa)\right]-r\right].$$

This condition holds when $z(\kappa)$ is decreasing and $\frac{z(\underline{\kappa})}{1+z(\underline{\kappa})} < z(\overline{\kappa})$. ∎

Proof of Proposition 9: To show that a clientele equilibrium exists, it suffices to find κ^* such that when (a) entry decisions are given by $\nu^1(\kappa) = 1$ for $\kappa > \kappa^*$, and $\nu^1(\kappa) = 0$ for $\kappa < \kappa^*$, (b) population measures are as in Section 3.1, and (c) expected utilities and prices are given by (9) and (28)-(30), $v_b^1(\kappa) - v_b^2(\kappa)$ has the same sign as $\kappa - \kappa^*$. Simple algebra shows that the solution to the system of (9) and (28)-(30) is

$$v_b^i(\kappa) = \frac{\lambda\mu_s^i\left[\frac{r}{r+\kappa}\left(r+\overline{\kappa}^i+\lambda\mu_s^i\frac{z}{1+z}\right)-\frac{r}{1+z}\right]x}{r\left(r+\kappa+\lambda\mu_s^i\right)\left(r+\overline{\kappa}^i+\lambda\mu_b^i\frac{1}{1+z}+\lambda\mu_s^i\frac{z}{1+z}\right)}, \tag{A.49}$$

$$v_o^i(\kappa) = \frac{\delta}{r} - \frac{\kappa\left(r+\overline{\kappa}^i+\lambda\mu_s^i\frac{z}{1+z}\right)x}{r(r+\kappa)\left(r+\overline{\kappa}^i+\lambda\mu_b^i\frac{1}{1+z}+\lambda\mu_s^i\frac{z}{1+z}\right)}, \tag{A.50}$$

$$v_s^i = \frac{\delta}{r} - \frac{\left(r+\overline{\kappa}^i+\lambda\mu_s^i\frac{z}{1+z}\right)x}{r\left(r+\overline{\kappa}^i+\lambda\mu_b^i\frac{1}{1+z}+\lambda\mu_s^i\frac{z}{1+z}\right)}, \tag{A.51}$$

$$p^i = \frac{\delta}{r} - \frac{\left(r\frac{z}{1+z}+\overline{\kappa}^i+\lambda\mu_s^i\frac{z}{1+z}\right)x}{r\left(r+\overline{\kappa}^i+\lambda\mu_b^i\frac{1}{1+z}+\lambda\mu_s^i\frac{z}{1+z}\right)}. \tag{A.52}$$

To show that there exists κ^* such that

$$v_b^1(\kappa^*) - v_b^2(\kappa^*) = 0, \tag{A.53}$$

we treat κ^* as a free variable and $(\overline{\kappa}^i, \mu_s^i, \mu_b^i)$ as functions of κ^*. Condition (C) implies that the LHS of (A.53) is negative for $\kappa^* = \kappa'$. To show that it is positive for $\kappa^* = \overline{\kappa}$, we write it as

$$\left[H(\overline{\kappa}^1, \mu_s^1, \mu_b^1) - H(\overline{\kappa}^2, \mu_s^2, \mu_b^2)\right] \frac{x}{r},$$

where

$$H(\ell, \mu_s, \mu_b) \equiv \frac{\lambda\mu_s \left[\frac{r}{r+\kappa^*}\left(r + \ell + \lambda\mu_s \frac{z}{1+z}\right) - \frac{r}{1+z}\right]}{(r + \kappa^* + \lambda\mu_s)\left(r + \ell + \lambda\mu_b \frac{1}{1+z} + \lambda\mu_s \frac{z}{1+z}\right)}.$$

The function H is increasing in ℓ and μ_s, and decreasing in μ_b. For $\kappa^* = \overline{\kappa}$, we have $\overline{\kappa}^1 = \overline{\kappa}^2 = \overline{\kappa}$, $\mu_s^1 = S$, $\mu_s^2 < S$, $\mu_b^1 = 0$, and $\mu_b^2 > 0$. Therefore, the LHS of (A.53) is positive, and thus this equation has a solution $\kappa^* \in (\kappa', \overline{\kappa})$. Since $\kappa^* > \kappa'$, and since μ_s^1 is increasing in κ^* and μ_s^2 is decreasing, we have $\mu_s^1 > \mu_s^2$.

The sign of $v_b^1(\kappa) - v_b^2(\kappa)$ is the same as of

$$\frac{v_b^1(\kappa)}{v_b^1(\kappa^*)} - \frac{v_b^2(\kappa)}{v_b^2(\kappa^*)}.$$

From (A.49), this has the same sign as

$$\frac{\left(r + \overline{\kappa} + \lambda\mu_s^1 \frac{z}{1+z} - \frac{r+\kappa}{1+z}\right)(r + \kappa^* + \lambda\mu_s^1)}{\left(r + \overline{\kappa} + \lambda\mu_s^1 \frac{z}{1+z} - \frac{r+\kappa^*}{1+z}\right)(r + \kappa + \lambda\mu_s^1)} - \frac{\left(r + \kappa^* + \lambda\mu_s^2 \frac{z}{1+z} - \frac{r+\kappa}{1+z}\right)(r + \kappa^* + \lambda\mu_s^2)}{\left(r + \kappa^* + \lambda\mu_s^2 \frac{z}{1+z} - \frac{r+\kappa^*}{1+z}\right)(r + \kappa + \lambda\mu_s^2)}.$$

After some algebra, we find that this has the same sign as

$$(\kappa - \kappa^*)\left[(\overline{\kappa} - \kappa^*)\frac{1}{1+z}(r + \kappa + \lambda\mu_s^1) + \lambda(\mu_s^1 - \mu_s^2)\frac{z}{1+z}(r + \overline{\kappa} + \lambda\mu_s^1)\right].$$

Since $\mu_s^1 > \mu_s^2$, this has the same sign as $\kappa - \kappa^*$, and thus a clientele equilibrium exists.

We next prove Properties (a)-(c). Property (a) follows from $\mu_s^1 > \mu_s^2$ and Property (b). To prove Property (b), we note that since H is increasing in ℓ,

$$H(\kappa^*, \mu_s^1, \mu_b^1) < H(\overline{\kappa}, \mu_s^1, \mu_b^1) = v_b^1(\kappa^*) = v_b^2(\kappa^*) = H(\kappa^*, \mu_s^2, \mu_b^2).$$

Since

$$H(\kappa^*, \mu_s, \mu_b) = \frac{\lambda\mu_s \frac{r}{r+\kappa^*}\frac{z}{1+z}}{r + \kappa^* + \lambda\mu_b\frac{1}{1+z} + \lambda\mu_s\frac{z}{1+z}},$$

we can write the above inequality as

$$\frac{\mu_s^1}{r + \kappa^* + \lambda\mu_b^1\frac{1}{1+z} + \lambda\mu_s^1\frac{z}{1+z}} < \frac{\mu_s^2}{r + \kappa^* + \lambda\mu_b^2\frac{1}{1+z} + \lambda\mu_s^2\frac{z}{1+z}}.$$

Multiplying by the denominators, and using $\mu_s^1 > \mu_s^2$, we find Property (b).

We finally prove Property (c). Substituting the price from (A.52), we have to prove that

$$\frac{r\frac{z}{1+z} + \overline{\kappa}^1 + \lambda\mu_s^1\frac{z}{1+z}}{r + \overline{\kappa}^1 + \lambda\mu_b^1\frac{1}{1+z} + \lambda\mu_s^1\frac{z}{1+z}} < \frac{r\frac{z}{1+z} + \overline{\kappa}^2 + \lambda\mu_s^2\frac{z}{1+z}}{r + \overline{\kappa}^2 + \lambda\mu_b^2\frac{1}{1+z} + \lambda\mu_s^2\frac{z}{1+z}}.$$

Dividing both sides by $v_b^1(\kappa^*) = v_b^2(\kappa^*)$, we can write this inequality as $K(\overline{\kappa}^1, \mu_s^1) < K(\overline{\kappa}^2, \mu_s^2)$, where

$$K(\ell, \mu) \equiv \frac{\left(r\frac{z}{1+z} + \ell + \lambda\mu\frac{z}{1+z}\right)(r + \kappa^* + \lambda\mu)}{\left(r + \ell + \lambda\mu\frac{z}{1+z} - \frac{r+\kappa^*}{1+z}\right)\lambda\mu}.$$

Since K is decreasing in ℓ and μ, and since $\overline{\kappa}^1 > \overline{\kappa}^2$ and $\mu_s^1 > \mu_s^2$, the inequality holds. ∎

Proof of Proposition 10: Condition (31) can be written as

$$
\begin{aligned}
&v_o^i(\overline{\kappa}^i) - v_b^i(\overline{\kappa}^i) \geq \left[P_b^i(\kappa)(v_o^i(\kappa) - v_b^i(\kappa)) + (1 - P_b^i(\kappa))v_s^i\right] \\
&\Leftrightarrow (1 - P_b^i(\kappa))(v_o^i(\kappa) - v_b^i(\kappa) - v_s^i) \geq v_o^i(\kappa) - v_b^i(\kappa) - (v_o^i(\overline{\kappa}^i) - v_b^i(\overline{\kappa}^i)).
\end{aligned} \tag{A.54}
$$

To show that this holds, we use (A.49)-(A.51). Combining (A.49) and (A.50), we find

$$
v_o^i(\kappa) - v_b^i(\kappa) = \frac{\delta}{r} - \frac{\left[\left(r + \overline{\kappa}^i + \lambda\mu_s^i \frac{z}{1+z}\right)(\kappa + \lambda\mu_s^i) - \lambda\mu_s^i \frac{r}{1+z}\right] x}{r\,(r + \kappa + \lambda\mu_s^i)\left(r + \overline{\kappa}^i + \lambda\mu_b^i \frac{1}{1+z} + \lambda\mu_s^i \frac{z}{1+z}\right)}, \tag{A.55}
$$

which for $\kappa = \overline{\kappa}^i$ simplifies into

$$
v_o^i(\overline{\kappa}^i) - v_b^i(\overline{\kappa}^i) = \frac{\delta}{r} - \frac{\left(\overline{\kappa}^i + \lambda\mu_s^i \frac{z}{1+z}\right) x}{r\left(r + \overline{\kappa}^i + \lambda\mu_b^i \frac{1}{1+z} + \lambda\mu_s^i \frac{z}{1+z}\right)}. \tag{A.56}
$$

(A.51) and (A.55) imply that

$$
v_o^i(\kappa) - v_b^i(\kappa) - v_s^i = \frac{(r + \overline{\kappa}^i + \lambda\mu_s^i)\, x}{(r + \kappa + \lambda\mu_s^i)\left(r + \overline{\kappa}^i + \lambda\mu_b^i \frac{1}{1+z} + \lambda\mu_s^i \frac{z}{1+z}\right)},
$$

and (A.55) and (A.56) imply that

$$
v_o^i(\kappa) - v_b^i(\kappa) - (v_o^i(\overline{\kappa}^i) - v_b^i(\overline{\kappa}^i)) = \frac{(\overline{\kappa}^i - \kappa)x}{(r + \kappa + \lambda\mu_s^i)\left(r + \overline{\kappa}^i + \lambda\mu_b^i \frac{1}{1+z} + \lambda\mu_s^i \frac{z}{1+z}\right)}.
$$

(A.54) is thus equivalent to

$$
\frac{1}{\overline{\kappa}^i - \kappa} \frac{\int_\kappa^{\overline{\kappa}^i} \frac{f(\ell)}{\ell + \lambda\mu_s^i} d\ell}{\int_{\underline{\kappa}^i}^{\overline{\kappa}^i} \frac{f(\ell)}{\ell + \lambda\mu_s^i} d\ell} = \frac{1 - P_b^i(\kappa)}{\overline{\kappa}^i - \kappa} \geq \frac{1}{r + \overline{\kappa}^i + \lambda\mu_s^i}, \tag{A.57}
$$

where $\underline{\kappa}^i$ denotes the minimum switching rate in market i ($\underline{\kappa}^1 = \kappa^*$ and $\underline{\kappa}^2 = \underline{\kappa}$). Proposition 2 shows that when λ goes to ∞, $\lambda\mu_s^i$ is of order $\sqrt{\lambda}$ and thus converges to ∞. Therefore, the RHS of (A.57) converges to zero and the LHS converges to

$$L^i(\kappa) \equiv \frac{1}{\overline{\kappa}^i - \kappa} \frac{\int_{\kappa}^{\overline{\kappa}^i} f(\ell)dy}{\int_{\underline{\kappa}^i}^{\overline{\kappa}^i} f(\ell)dy}.$$

This function is continuous in the compact set $[\underline{\kappa}^i, \overline{\kappa}^i]$, and strictly positive since $f(\kappa) > 0$. Therefore, it is bounded away from zero, implying that (A.57) holds for large λ.

We next show Condition (C). From the definition of κ', we have $\mu_s^1 = \mu_s^2 \equiv \mu_s$. Using (A.49), we can then write inequality $v_b^1(\kappa') < v_b^2(\kappa')$ as

$$\frac{r + \overline{\kappa} + \lambda\mu_s \frac{z}{1+z} - \frac{r+\kappa'}{1+z}}{r + \overline{\kappa} + \lambda\mu_b^1 \frac{1}{1+z} + \lambda\mu_s \frac{z}{1+z}} < \frac{r + \kappa' + \lambda\mu_s \frac{z}{1+z} - \frac{r+\kappa'}{1+z}}{r + \kappa' + \lambda\mu_b^2 \frac{1}{1+z} + \lambda\mu_s \frac{z}{1+z}}.$$

Simple algebra shows that this is equivalent to

$$\lambda(\mu_b^1 - \mu_b^2)\frac{z}{1+z}(r + \kappa' + \lambda\mu_s) > (\overline{\kappa} - \kappa')(r + \kappa' + \lambda\mu_b^2). \tag{A.58}$$

Proceeding as in Proposition 2 we can show that when λ goes to ∞, κ' converges to $\hat{\kappa}$, μ_s is asymptotically equal to $\alpha/\sqrt{\lambda}$, and μ_b^i is asymptotically equal to $\gamma^i/\sqrt{\lambda}$, with

$$\alpha = \sqrt{\frac{\int_{\underline{\kappa}}^{\overline{\kappa}} g(\kappa)\kappa d\kappa}{2}},$$

$$\gamma^1 = \frac{\int_{\hat{\kappa}}^{\overline{\kappa}} g(\kappa)\kappa d\kappa}{\alpha},$$

$$\gamma^2 = \frac{\int_{\underline{\kappa}}^{\hat{\kappa}} g(\kappa)\kappa d\kappa}{\alpha}.$$

Since $\gamma^1 > \gamma^2$, (A.58) holds for large λ. ∎

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