A Preferred-Habitat Model of the Term Structure of Interest Rates

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June 30, 2019*

Abstract

We model the term structure of interest rates that results from the interaction between investors with preferences for specific maturities and risk-averse arbitrageurs. Shocks to the short rate are transmitted to long rates through arbitrageurs’ carry trades. Arbitrageurs earn rents from transmitting the shocks, through bond risk premia that relate positively to the slope of the term structure. When the short rate is the only risk factor, changes in investor demand have the same relative effect on interest rates across maturities regardless of the maturities where they originate. When investor demand is also stochastic, demand effects become more localized. A calibration indicates significant localization, and high volatility and price-elasticity of investor demand.

*We thank Markus Brunnermeier, Andrea Buraschi, Stefania D’Amico, Greg Duffee, Pierre Collin-Dufresne, Peter DeMarzo, Giorgio Fossi, Xavier Gabaix, Ken Garbade, Robin Greenwood, Sam Hanson, Moyeen Islam, Ralph Kuijzen, Arvind Krishnamurthy, Jun Liu, Vasant Naik, Anna Pavlova, Monika Piazzesi, Ricardo Reis, Jeremy Stein, Michael Woodford, seminar participants at the Bank of England, Chicago Fed, ECB, Fed Board, LSE, Manchester, New York Fed, Tilburg, Toulouse, UCLA, and participants at the American Finance Association, Adam Smith Asset Pricing, Brazilian Finance Association, Chicago, CRETE, Gerzensee, Imperial, NBER Asset Pricing, SITE, and York conferences for helpful comments. We are especially grateful to John Cochrane for extensive and valuable comments and encouragement. Financial support from the Paul Woolley Centre at the LSE is gratefully acknowledged. The views expressed in this paper are those of the authors and not of Capula Investment Management. Please address correspondence to Dimitri Vayanos, d.vayanos@lse.ac.uk.
1 Introduction

What determines the term structure of interest rates? In most macro-finance models, the interest rate for a given maturity depends on the willingness of a representative agent to substitute consumption from today towards that maturity. The consumption-based view of the term structure contrasts with a more informal preferred-habitat view, which has been proposed by Culbertson (1957) and Modigliani and Sutch (1966), and is popular within central banks and the financial industry. According to that view, there are investor clienteles for specific maturity segments, and the interest rate for a given maturity is mainly driven by shocks affecting the demand of the corresponding clientele. The term structure thus exhibits a degree of segmentation.

The preferred-habitat view has been used to interpret numerous market episodes. The 2004 U.K. pension reform is one example. The reform required pension funds to evaluate their pension liabilities using the yields of long-maturity bonds. To hedge against drops in long rates, which would raise the value of pension liabilities and trigger regulatory scrutiny, pension funds bought long-maturity bonds in large quantities. This drove long rates to record low levels. A flat term structure in early 2004 became downward-sloping in subsequent years, with the 30-year bond yielding as much as 0.80% below its 10-year counterpart. More recently, the preferred-habitat view informed decisions by major central banks to engage in Quantitative Easing (QE). A stated goal of QE programmes was that large-scale purchases of long-maturity bonds would drive long rates down, stimulating corporate investment.

The preferred-habitat view cannot be correct in its most extreme form, namely, the interest rate for a given maturity cannot be driven only by shocks affecting the demand of the corresponding clientele. Indeed, if that were the case, interest rates for nearby maturities could be very different, generating large profits for term-structure arbitrageurs. At the same time, shocks to clientele demands can affect interest rates. Indeed, because absorbing the shocks exposes arbitrageurs to interest-rate risk, bond prices must change to compensate them for the risk.

How do shocks to clientele demands affect the term structure? What are the effects of large-scale bond purchases by central banks? What are the implications of the preferred-habitat view for the dynamics of interest rates, for bond risk premia, and for the transmission of monetary policy from short to long rates? In this paper we develop a model to answer these questions both

\footnote{For accounts of the 2004 U.K. pension reform and other related episodes, see Tzucker and Islam (2005), Garbade and Rutherford (2007), Islam (2007), and Greenwood and Vayanos (2010).}

\footnote{See, for example, the 2011 speeches on large-scale asset purchases by Janet Yellen, the then Vice-Chair of the U.S. Federal Reserve (Yellen (2011)), and John Williams, the then President of the San Francisco Fed (Williams (2011)).}
qualitatively as well as quantitatively through a calibration exercise. Our model formalizes the preferred-habitat view and embeds it into a modern no-arbitrage term-structure framework.

We describe our model in Section 2. The short rate follows an exogenous mean-reverting process. An exogenous short rate can be interpreted as the return of a linear and instantaneously riskless production technology, or as the instantaneous rate that a (non-modelled) central bank pays on reserves. Bond yields are determined endogenously through trading between preferred-habitat investors and arbitrageurs. Preferred-habitat investors demand zero-coupon bonds with specific maturities, and their demand can be price-elastic. We provide an optimizing foundation for preferred-habitat demand in a setting where investors form overlapping generations consuming at the end of their life, are infinitely risk averse, and can invest in bonds and in a private opportunity with exogenous return (e.g., real estate). Arbitrageurs are competitive and maximize a mean-variance objective over instantaneous changes in wealth. We fix the aggregate risk aversion of arbitrageurs and do not study entry into the arbitrage business.

In Section 3 we solve for equilibrium when the demand of preferred-habitat investors is constant over time and the only risk factor is the short rate. We address three main questions: how shocks to the short rate are transmitted to long rates, how bond risk premia depend on the shape of the term structure, and how changes in preferred-habitat demand affect the term structure. Since demand is constant over time, we take the changes to be unanticipated and permanent.

Shocks to the short rate are transmitted to bond yields through the trades of arbitrageurs. Suppose that the short rate drops. Since investing in bonds becomes more attractive than investing in the short rate, arbitrageurs buy bonds by borrowing short-term. That trade causes bond prices to rise and yields to drop. Because, however, arbitrageurs become exposed to the risk that the short rate will increase, they do not scale up their trade to the point where it earns zero expected profit. Hence, the drop in yields does not fully reflect the drop in the short rate, which means that forward rates under-react to expected future short rates. The under-reaction disappears when arbitrageurs are risk-neutral, or when preferred-habitat demand is price-inelastic since in that case arbitrageurs have infinite price impact.

Bond risk premia (expected returns in excess of the short rate) are positively related to the slope of the term structure, consistent with the empirical findings of Fama and Bliss (FB 1987) and Campbell and Shiller (CS 1991). When the short rate is low, the term structure slopes up, and bonds earn positive risk premia so that arbitrageurs are induced to buy them. The risk premia accrue to arbitrageurs as a rent for transmitting short-rate shocks to long rates. Monetary-policy
actions by central banks affecting the short rate can hence be viewed as a source of arbitrageur rent.\(^3\) That rent is higher when arbitrageurs are more risk-averse and when preferred-habitat demand is more price-elastic.

When the short rate is the only risk factor, changes in preferred-habitat demand have global effects: the effects depend on how the arbitrageurs’ overall exposure to the short rate (“duration risk”) changes, and not on the specific maturities where the demand changes originate. To illustrate this result’s surprising implications, suppose that the demand for short-maturity bonds increases and the demand for long-maturity bonds decreases by the same dollar amount. Since arbitrageurs buy long-maturity bonds, and these are more sensitive to short-rate changes than short-maturity bonds, all yields rise—including those of short-maturity bonds for which demand increases. The same logic implies that all demand changes have the same relative effect across maturities regardless of where they originate. Moreover, the effect is largest at the longest maturity. Indeed, since the longest-maturity bonds are the most sensitive to short-rate changes, their risk premia are also the most sensitive to changes in the arbitrageurs’ exposure to the short rate.

In Section 4 we allow the demand of preferred-habitat investors to vary over time. We maintain a stochastic short rate; with a constant short rate, arbitrageur activity would render all yields equal to the short rate. We mainly focus on the case where demand has a one-factor structure and that factor is independent of the short rate, but we also consider multiple demand factors and correlation. Within the two-factor model, we revisit the same three questions as in Section 3.

Demand risk weakens and can even reverse the transmission of short-rate shocks to long rates. Suppose that the short rate drops, in which case arbitrageurs buy bonds. Arbitrageurs become exposed to the risk that the short rate will increase and that preferred-habitat demand will decrease. Because demand risk becomes dominant for long-maturity bonds, arbitrageurs buy them in small quantities and may even sell them short to hedge the demand risk of their long positions in intermediate maturities. Long-maturity yields may thus rise in response to a short-rate drop.

Demand risk strengthens the positive relationship between bond risk premia and term-structure slope. Indeed, when preferred-habitat demand is low, risk premia are high so that arbitrageurs are induced to buy bonds to make up for the low demand. Because of the high premia, yields are high and the term structure slopes up. As a result of the stronger premia-slope relationship, the model-generated coefficients in the FB and CS regressions have properties closer to their empirical counterparts. For example, the FB coefficient can be larger than one and increasing with maturity.

\(^3\)We thank John Cochrane for suggesting this idea (Cochrane (2008)).
rather than only positive and constant as in the one-factor model.

With multiple risk factors, demand effects become more localized. Changes in the demand for short- (long-) maturity bonds have more pronounced effects on short- (long-) maturity yields. As in the one-factor model, the effects arise through the arbitrageurs’ exposure to the risk factors. They become more localized because demand changes originating at different maturities affect the exposure to each factor differently, and because changes in each factor exposure have a different relative effect across maturities.

In Section 5 we calibrate the two-factor model. We pick the persistence and volatility of short-rate shocks to match their empirical counterparts. The key remaining parameters are the persistence and volatility of demand shocks, the price elasticity of preferred-habitat demand, and the risk aversion of arbitrageurs. To calibrate these four parameters, we match the model-generated volatility of yields and FB and CS regression coefficients, all as function of maturity—a total of 88 moments, to their empirical counterparts. This exercise identifies only three out of the four parameters. Intuitively, yields can be volatile because arbitrageurs are highly risk-averse and demand shocks are small, or because arbitrageurs are less risk-averse and demand shocks are larger. To identify the final parameter, we need observable demand shocks, and use QE purchases.

Our model matches the 88 moments with an average discrepancy of only 7.69% of the moments’ values. Regardless of the value of the QE-identified parameter, we find that demand shocks have a half-life of just below two years and become the dominant source of variation of yields for maturities longer than four years. Moreover, demand effects are localized, with the maturities where the effects are largest being close to the maturities where the demand changes originate. By contrast, if preferred-habitat demand is assumed to be price-inelastic, the average discrepancy in matching the moments more than doubles, and demand effects become almost fully delocalized.

Our model formalizes the preferred-habitat theory of the term structure, proposed by Culbertson (1957) and Modigliani and Sutch (1966). Related to preferred habitat is Tobin’s (1958,1969) portfolio-balance theory, in which financial assets are imperfect substitutes, and investors require a rise in interest rates to absorb an increased supply of government bonds. The portfolio-balance channel is present in our model, with Tobin’s investors being our arbitrageurs. It is the only channel present in the special case of our model where preferred-habitat demand is price-inelastic.

Andres, Lopez-Salido, and Nelson (2004) study demand effects and the portfolio-balance channel in a calibrated macroeconomic model with trading frictions. Greenwood and Vayanos (2014) use our model’s special case with a price-inelastic demand to test for a positive relationship between
the maturity of government debt and future bond returns. Other empirical studies of demand effects in the bond market that build on our model include Hamilton and Wu (2012) and Li and Wei (2013) on QE purchases and the zero lower bound (ZLB);\(^4\) Hanson (2014) and Malkhozov, Mueller, Vedolin, and Venter (2016) on mortgage-backed securities; Gorodnichenko and Ray (2018) on Treasury auctions; Kaminska and Zinna (2019) on purchases by foreign central banks; and King (2019) on non-linearities induced by the ZLB. Hayashi (2018) develops numerical algorithms to solve our model with a general number of risk factors.

The notion that demand shocks can drive asset prices away from fundamental values is emphasized in the literature on the limits of arbitrage, surveyed in Gromb and Vayanos (2010). Closest to our paper is the strand of the literature on price distortions across an asset class. See, for example, Barberis and Shleifer (2003) on style investing; Greenwood (2005) and Hau (2011) on index redefinitions; Gabaix, Krishnamurthy, and Vigneron (2007) on mortgage-backed securities; Garleanu, Pedersen, and Posheshman (2009) on options; and Gabaix and Maggiori (2015) on foreign exchange.

Preferred habitats in our model concern maturities. They could alternatively concern bonds that differ in liquidity or in the type of issuer, e.g., government versus corporate. Preferences for liquidity have been used to explain the on-the-run phenomenon, whereby just-issued government bonds are more expensive than previously-issued bonds maturing on nearby dates.\(^5\) Preferences for government bonds could be arising because of those bonds' safety and wider acceptability as collateral. Krishnamurthy and Vissing-Jorgensen (2012) provide evidence consistent with the existence of an investor clientele pricing those attributes.

Our model belongs to the class of affine no-arbitrage term-structure models (Duffie and Kan (1996)) because yields are affine in the risk factors. Dai and Singleton (2002) and Duffee (2002) develop models within that class that embody the positive relationship between bond risk premia and term-structure slope. We derive such a relationship in an equilibrium model.\(^6\) Our model can address questions that reduced-form models cannot such as how demand shocks affect the term structure and how the effects depend on arbitrageur risk aversion and investor price-elasticity.

\(^4\)For empirical estimates of the effects of QE, see also Gagnon, Raskin, Remache, and Sack (2011), Joyce, Lasaosa, Stevens, and Tong (2011), Krishnamurthy and Vissing-Jorgensen (2011), Swanson (2011), Christensen and Rudebusch (2012), D’Amico and King (2013), Swanson and Williams (2014), and the survey by Williams (2014). Some of these papers emphasize the duration-risk channel. That channel describes demand effects in the one-factor version of our model but not with multiple factors.


\(^6\)Other equilibrium models that generate a positive premia-slope relationship include Wachter (2006), Buraschi and Jiltsov (2007) and Lettau and Wachter (2011) who assume habit formation; Xiong and Yan (2010) who assume heterogeneous beliefs; and Gabaix (2012) who assumes rare disasters with time-varying severity.
2 Model

Time is continuous and goes from zero to infinity. The term structure at time $t$ consists of a continuum of zero-coupon government bonds. The maturities of the bonds lie in the interval $(0, T)$, where $T$ can be finite or infinite. The bond with maturity $\tau$ has face value one, hence paying one dollar at time $t + \tau$. We denote by $P_t^{(\tau)}$ and $y_t^{(\tau)}$, respectively the time-$t$ price and yield of the bond with maturity $\tau$. The yield is the spot rate for maturity $\tau$, and is related to the price through

$$y_t^{(\tau)} = -\frac{\log(P_t^{(\tau)})}{\tau}.$$  

(1)

We denote by $f_t^{(\tau-\Delta\tau, \tau)}$ the time-$t$ forward rate between maturities $\tau - \Delta\tau$ and $\tau$. The forward rate is related to the price through

$$f_t^{(\tau-\Delta\tau, \tau)} = -\frac{\log\left(\frac{P_t^{(\tau)}}{P_t^{(\tau-\Delta\tau)}}\right)}{\Delta\tau}.$$  

(2)

The short rate $r_t$ is the limit of the yield $y_t^{(\tau)}$ when $\tau$ goes to zero. We take $r_t$ as exogenous, and describe its dynamics later in this section (Equation (7)). An exogenous $r_t$ can be interpreted as the return of a linear and instantaneously riskless production technology. Alternatively, $r_t$ can be determined by the central bank in response to exogenous shocks. We sketch the central-bank interpretation in Section 3.3, where we derive some of our model’s implications for monetary policy.

Agents are of two types: arbitrageurs and preferred-habitat investors. Arbitrageurs can invest in the bonds and in the short rate. Denoting their time-$t$ wealth by $W_t$ and their dollar position in the bond with maturity $\tau$ by $X_t^{(\tau)}$, their budget constraint is

$$dW_t = \left(W_t - \int_0^T X_t^{(\tau)}\right) r_t dt + \int_0^T X_t^{(\tau)} \frac{dP_t^{(\tau)}}{P_t^{(\tau)}}.$$  

(3)

Arbitrageurs maximize a mean-variance objective over instantaneous changes in wealth. Their optimization problem is

$$\max_{\{X_t^{(\tau)}\}_{\tau \in (0,T)}} \left[ \mathbb{E}_t(dW_t) - \frac{a}{2} \text{Var}_t(dW_t) \right],$$  

(4)

where $a \geq 0$ is a risk-aversion coefficient that characterizes the trade-off between mean and variance. Arbitrageurs with the objective (4) can be interpreted as overlapping generations living over
infinitesimal periods. The generation born at time \( t \) is endowed with wealth \( W \), invests from \( t \) to \( t + dt \), consumes at \( t + dt \) and then dies. If preferences over consumption are described by the Von Neumann-Morgenstern (VNM) utility function \( U \), and if all uncertainty is Brownian as is the case in equilibrium, utility maximization yields the objective (4) with the risk-aversion coefficient

\[
a = -\frac{U''(W)}{U'(W)}.
\]

Preferred-habitat investors have preferences for specific maturities. For example, pension funds prefer long-maturity bonds because their duration matches the duration of pension liabilities. We model the demand of preferred-habitat investors in reduced form and provide an optimizing foundation in Appendix B.

Investors’ maturity habitats cover the interval \((0, T)\) of bond maturities. The investors with habitat \( \tau \) at time \( t \) hold a dollar position

\[
Z_t(\tau) = -\alpha(\tau) \log(P_t(\tau)) - \beta_t(\tau)
\]

in the bond with maturity \( \tau \) and hold no other bond. Equation (5) is a demand function linear and decreasing in the logarithm of the bond price. The slope coefficient \( \alpha(\tau) \geq 0 \) is constant over time but can depend on maturity \( \tau \). The intercept coefficient \( \beta_t(\tau) \) can depend on both \( t \) and \( \tau \). For simplicity, we refer to \( \alpha(\tau) \) and \( \beta_t(\tau) \) as demand slope and demand intercept, respectively. The actual intercept is \(-\beta_t(\tau)\).

The demand intercept \( \beta_t(\tau) \) takes the form

\[
\beta_t(\tau) = \theta_0(\tau) + \sum_{k=1}^{K} \theta_k(\tau) \beta_{k,t},
\]

where \( \{\theta_k(\tau)\}_{k=0,...,K} \) are constant over time but can depend on maturity \( \tau \), and \( \{\beta_{k,t}\}_{k=1,...,K} \) are time-varying but independent of \( \tau \). We refer to \( \{\beta_{k,t}\}_{k=1,...,K} \) as demand risk factors. The functions \( \{\theta_k(\tau)\}_{k=1,...,K} \) characterize the maturities where demand changes originate. If, for example, \( \theta_k(\tau) \) is independent of \( \tau \), then a change in \( \beta_{k,t} \) impacts demand for all maturities equally, and can be interpreted as a global demand shock. If instead \( \theta_k(\tau) \) peaks at a specific maturity, then a change in \( \beta_{k,t} \) impacts demand for that maturity the most, and can be interpreted as a local demand shock.

To ensure that integrals involving \( \alpha(\tau) \) and \( \{\theta_k(\tau)\}_{k=1,...,K} \) are well-defined, we assume: (i) \( \alpha(\tau) \) and \( \{\theta_k(\tau)\}_{k=1,...,K} \) are continuous, and (ii) if \( T = \infty \) then \( \alpha(\tau) \) and \( \{\theta_k(\tau)\}_{k=1,...,K} \) converge to
zero at exponential rates when \( \tau \) goes to infinity with the rates for \( \{\theta_k(\tau)\}_{k=1,..,K} \) not exceeding that for \( \alpha(\tau) \).

The \((K+1)\times1\) vector \(q_t \equiv (r_t, \beta_{1,t},..,\beta_{K,t})^\top\) follows the process

\[
dq_t = -\Gamma(q_t - \mathbf{\Upsilon}E)dt + \Sigma dB_t, \tag{7}
\]

where \(\mathbf{\Upsilon}\) is a constant, \(E\) is the \((K+1)\times1\) vector \((1,0,..,0)^\top\), \((\Gamma, \Sigma)\) are constant \((K+1)\times(K+1)\) matrices, \(dB_t\) is a \((K+1)\times1\) vector \((dB_{r,t}, dB_{\beta_{1,t}},..,dB_{\beta_{K,t}})^\top\) of independent Brownian motions, and \(^\top\) denotes transpose. Equation (7) nests the case where the short rate \(r_t\) and the \(K\) demand factors \(\{\beta_{k,t}\}_{k=1,..,K}\) are mutually independent, and the case where they are correlated. Independence arises when the matrices \((\Gamma, \Sigma)\) are diagonal. When instead \(\Sigma\) is non-diagonal, shocks to the factors \(r_t\) and \(\{\beta_{k,t}\}_{k=1,..,K}\) are correlated, and when \(\Gamma\) is non-diagonal, the drift (instantaneous expected change) of each factor depends on all other factors. We assume that \(\Sigma\) has full rank so that each factor is not perfectly correlated with the others, and that the eigenvalues of \(\Gamma\) have negative real parts so that \(q_t\) is stationary. Since \(q_t\) is stationary, (7) implies that the long-run means of \(r_t\) and \(\{\beta_{k,t}\}_{k=1,..,K}\) are \(\mathbf{\Upsilon}\) and zero, respectively. Setting the long-run mean of \(\{\beta_{k,t}\}_{k=1,..,K}\) to zero is without loss of generality since we can redefine the function \(\theta_0(\tau)\) to include a non-zero long-run mean.

If government bonds are in positive supply, then we include their supply into the demand function (5). That function becomes a net demand, equal to the demand by preferred-habitat investors for the bond with maturity \(\tau\), net of the government supply of that bond.

Under the assumed demand function (5), the demand by preferred-habitat investors for the bond with maturity \(\tau\) depends only on that bond’s price and not on the prices of other bonds. This begs the question why rational investors buy the bond with maturity \(\tau\) if a bond with maturity close to \(\tau\) is much cheaper. Appendix B shows that the demand function (5), together with the specification (6) and (7) for the demand intercept \(\beta_t(\tau)\), can be given an optimizing foundation when \(T\) is finite. That foundation requires that the term structure satisfies no-arbitrage, which is the case for the equilibrium derived in Sections 3 and 4.

The preferred-habitat investors in Appendix B form overlapping generations living over a period equal to the maximum bond maturity \(T\). The generation born at time \(t\) consumes only at \(t+T\) and then dies. Investors are infinitely risk-averse over consumption, and derive consumption by investing in bonds and in a private opportunity. Infinite risk aversion ensures that investors choose
a bond portfolio that yields a riskless payoff at the time \( t + T \) when they consume. That portfolio consists only of the bond maturing at \( t + T \). No-arbitrage ensures that investors cannot achieve a higher payoff with certainty by investing in bonds with maturities other than \( t + T \): if the payoff is higher with positive probability, then it must also be lower with positive probability.

The demand intercept \( \beta_t^{(\tau)} \) in Appendix B is determined by the return of the private opportunity. When that opportunity offers a higher return, it becomes more attractive relative to the bond. Hence, bond demand decreases and \( \beta_t^{(\tau)} \) increases. The private opportunity could represent, for example, an investment in real estate.\(^7\)

Stepping outside of the optimizing foundation in Appendix B, \( \beta_t^{(\tau)} \) could vary because of shocks to the supply of bonds issued by the government (which is not modelled in Appendix B) and shocks to the composition of the preferred-habitat investor pool (Appendix B assumes that all generations are in equal mass). The demand specification (5)-(7) can capture these shocks if the maturities affected by the shocks remain fixed as time passes. Suppose, for example, that there is a sudden increase at time \( t \) in the demand for the bond with maturity \( \tau \). The specification (5)-(7) requires that this increase translates to an increase at time \( t + 1 \) in the demand for the bond with maturity \( \tau \) rather than \( \tau - 1 \). That is, the shock does not “roll down” over time in the maturity space.

Some shocks roll down in the maturity space. For example, an increase at time \( t \) in the government supply of the bond with maturity \( \tau \) translates to an increase at time \( t + 1 \) in the supply of the bond with maturity \( \tau - 1 \) rather than \( \tau \). For such shocks, the specification (5)-(7) can be viewed as an approximation. Modifying that specification to allow roll down would render the analysis more complicated because bond demand at time \( t \) would depend on the entire history of shocks up to time \( t - T \). (The shocks up to time \( t - T + \tau \) would affect demand for bonds with maturities up to \( \tau \).)

Our model makes a stark distinction between arbitrageurs, who can substitute across maturities, and preferred-habitat investors, who invest only in their maturity habitat. Suppressing this distinction (by making the risk aversion of preferred-habitat investors finite in Appendix B), would complicate the model without changing the basic mechanisms. Preferred-habitat investors would substitute across maturities, acting partly as arbitrageurs, and arbitrage capacity would increase. The analysis would become more complicated because it would involve a continuum of portfolios

\(^7\)An example of preferred-habitat investors substituting from government bonds into real estate comes from the UK’s pension reform of 2004, mentioned in the Introduction. The drop in long rates induced pension funds to substitute towards non-bond investments, including real estate. For example, Marks & Spencer arranged for their pension fund to receive payments based on the leases of their property portfolio (Islam (2007), p.61).
rather than only the portfolio of arbitrageurs.

An additional distinction between arbitrageurs and preferred-habitat investors, which is implicit in the demand specification (5) and explicit in the optimizing foundation in Appendix B, is that the latter can access investment opportunities outside of the bond market while the former cannot. Allowing private opportunities for arbitrageurs would not change our analysis.

3 No Demand Risk

In this section we study the case where there are no demand risk factors \((K = 0)\). Time-variation in yields arises because of the short rate \(r_t\), which is the only risk factor. For \(K = 0\), (7) reduces to

\[
d r_t = \kappa_r (\overline{r} - r_t) dt + \sigma_r dB_{r,t},
\]

where \(\kappa_r \equiv \Gamma_{1,1} > 0\) and \(\sigma_r \equiv \Sigma_{1,1}\).

3.1 Equilibrium without Arbitrageurs

We first derive, as a benchmark, the equilibrium that would prevail in the arbitrageurs’ absence. We refer to it as the segmentation equilibrium because the yield for each maturity is determined solely by the demand of the investors with that maturity habitat. The yield \(y_t(\tau)\) for maturity \(\tau\) is determined by setting the net demand (5) by preferred-habitat investors to zero. Since (1) implies

\[
\log(P_t(\tau)) = -\tau y_t(\tau),
\]

\(y_t(\tau)\) is given by

\[
y_t(\tau) = \frac{\beta_t(\tau)}{\alpha(\tau)\tau} = \frac{\theta_0(\tau)}{\alpha(\tau)\tau},
\]

where the second equality follows by setting \(K = 0\) in (6). The yield \(y_t(\tau)\) for maturity \(\tau\) is constant over time and is disconnected from the time-varying short rate \(r_t\). It depends only on the demand intercept \(\beta_t(\tau) = \theta_0(\tau)\) and demand slope \(\alpha(\tau)\) for maturity \(\tau\). An increase in \(\theta_0(\tau)\) lowers the demand by preferred-habitat investors for the bond with maturity \(\tau\), and hence raises \(y_t(\tau)\). The effect is weaker the larger \(\alpha(\tau)\) is because the demand by preferred-habitat investors is more price-elastic. The segmentation equilibrium corresponds to an extreme form of the preferred-habitat view (Culbertson (1957), Modigliani and Sutch (1966)).
3.2 Equilibrium with Arbitrageurs

We next derive the equilibrium when arbitrageurs are present. We proceed in three steps: (i) conjecture a functional form for equilibrium yields, (ii) derive the arbitrageurs’ first-order condition given the conjectured yields, and (iii) combine the arbitrageurs’ first-order condition with market clearing, and confirm that yields are as conjectured.

We conjecture that equilibrium yields are affine in the single risk factor $r_t$. That is, there exist two functions $(A_r(\tau), C(\tau))$ that depend only on $\tau$ such that the time-$t$ price of the bond with maturity $\tau$ is

$$P_t^{(\tau)} = e^{-[A_r(\tau)r_t + C(\tau)]}. \tag{10}$$

Applying Ito’s Lemma to (10), using the dynamics (8) of $r_t$, and noting that $t + \tau$ stays constant when taking the derivative, we find that the time-$t$ instantaneous return on the bond with maturity $\tau$ is

$$\frac{dP_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)} dt - A_r(\tau) \sigma_r dB_{r,t}, \tag{11}$$

where

$$\mu_t^{(\tau)} \equiv A'_r(\tau) r_t + C'(\tau) - A_r(\tau) \kappa_r(\tau - r_t) + \frac{1}{2} A_r(\tau)^2 \sigma_r^2 \tag{12}$$

is the instantaneous expected return.

To derive the arbitrageurs’ first-order condition, we substitute the bond return (11) into the arbitrageurs’ budget constraint (3) and optimization problem (4). This yields

$$dW_t = \left[ W_t r_t + \int_0^T X_t^{(\tau)} (\mu_t^{(\tau)} - r_t) d\tau \right] dt - \left[ \int_0^T X_t^{(\tau)} A_r(\tau) d\tau \right] \sigma_r dB_{r,t}$$

and

$$\max_{\{X_t^{(\tau)}, \tau \in (0,T)\}} \left\{ \int_0^T X_t^{(\tau)} (\mu_t^{(\tau)} - r_t) d\tau - \frac{a \sigma_r^2}{2} \left[ \int_0^T X_t^{(\tau)} A_r(\tau) d\tau \right]^2 \right\}, \tag{13}$$

respectively. Point-wise maximization of (13) yields the arbitrageurs’ first-order condition.
Lemma 1. The arbitrageurs’ first-order condition is

\[ \mu_t^{(\tau)} - r_t = -A_r(\tau) \lambda_{r,t}, \]  

(14)

where

\[ \lambda_{r,t} \equiv -a \sigma_r^2 \int_0^T X_t^{(\tau)} A_r(\tau) d\tau. \]  

(15)

The arbitrageurs’ first-order condition (14) balances risk and return. The left-hand side is the increase in portfolio expected return if arbitrageurs shift one dollar from the short rate \( r_t \) to the bond with maturity \( \tau \). Portfolio expected return increases by the difference between the bond’s expected return \( \mu_t^{(\tau)} \) and the short rate \( r_t \). The right-hand side is the increase in portfolio risk, times the arbitrageurs’ risk-aversion coefficient \( a \). Portfolio risk increases by the covariance between the return on the additional investment in the bond and the return on the arbitrageurs’ portfolio. With only one risk factor, the covariance is the product of the sensitivities of the two returns to the factor, times the factor’s variance. The risk factor is the short rate, and its variance is \( \sigma_r^2 \). Moreover, (11) implies that the sensitivity of the bond’s return to the short rate is \( -A_r(\tau) \), and the sensitivity of the portfolio’s return is \( -\int_0^T X_t^{(\tau)} A_r(\tau) d\tau \).

The first-order condition (14) can alternatively be interpreted in the context of no-arbitrage models of the term structure.\(^8\) No-arbitrage in continuous time requires that there exist prices specific to each risk factor and common across assets, such that the expected return of any asset in excess of the short rate is equal to the sum across factors of the asset’s sensitivity to each factor times the factor’s price. With only one factor, the no-arbitrage condition boils down to requiring that the factor’s price is equal to the ratio of any asset’s expected excess return to the asset’s factor sensitivity. The no-arbitrage condition in our model is the arbitrageurs’ first-order condition (14), and the price of the short rate factor is \( \lambda_{r,t} \).

Absence of arbitrage is mute on what the prices of the risk factors are. These prices are instead determined by equilibrium arguments. Equation (15) shows that \( \lambda_{r,t} \) is proportional to the factor sensitivity \( -\int_0^T X_t^{(\tau)} A_r(\tau) d\tau \) of the arbitrageurs’ portfolio. To determine that portfolio, we use market clearing.

\(^8\)See, for example, Vasicek (1977) and Cox, Ingersoll, and Ross (1985) for early contributions, and Veronesi (2010) for a textbook treatment.
Market clearing requires that the time-$t$ dollar positions of arbitrageurs and preferred-habitat investors in the bond with maturity $\tau$ sum to zero:

$$X^{(\tau)}_t + Z^{(\tau)}_t = 0. \quad (16)$$

Substituting $X^{(\tau)}_t$ from (16) into (15), we find

$$\lambda_{r,t} = a\sigma^2_r \int_0^T Z^{(\tau)}_t A_r(\tau) d\tau$$

$$= a\sigma^2_r \int_0^T \left[ -\alpha(\tau) \log(P^{(\tau)}_t) - \beta^{(\tau)}_t \right] A_r(\tau) d\tau$$

$$= a\sigma^2_r \int_0^T \left[ \alpha(\tau) [A_r(\tau)t + C(\tau)] - \theta_0(\tau) \right] A_r(\tau) d\tau, \quad (17)$$

where the second equality follows by substituting $Z^{(\tau)}_t$ from (5), and the third equality follows by substituting $P^{(\tau)}_t$ from (10) and using $\beta^{(\tau)}_t = \theta_0(\tau)$ (which follows by setting $K = 0$ in (6)). Equation (17) shows that the price $\lambda_{r,t}$ of the short-rate risk factor depends on the short rate $r_t$ and on the demand intercept $\theta_0(\tau)$ and demand slope $\alpha(\tau)$ of preferred-habitat investors. We return to these effects and their economic implications in Sections 3.3-3.5.

Substituting $\lambda_{r,t}$ and $\mu^{(\tau)}_t$ from (17) and (12), respectively, into (14), we find

$$A'_r(\tau)r_t + C'(\tau) - A_r(\tau)\kappa_r(\tau) - r_t + 1/2 A_r(\tau)\sigma^2_r - r_t$$

$$= a\sigma^2_r A_r(\tau) \int_0^T \left[ \theta_0(\tau) - \alpha(\tau) [A_r(\tau)t + C(\tau)] \right] A_r(\tau) d\tau. \quad (18)$$

Equation (18) must hold for all values of $r_t$. Hence, the linear terms in $r_t$ on both sides must be equal, and the same is true for the terms that are independent of $r_t$. This yields the two first-order linear ordinary differential equations (ODEs)

$$A'_r(\tau) + \kappa_r A_r(\tau) - 1 = -a\sigma^2_r A_r(\tau) \int_0^T \alpha(\tau) A_r(\tau)^2 d\tau, \quad (19)$$

$$C'(\tau) - \kappa_r \tau A_r(\tau) + 1/2 \sigma^2_r A_r(\tau)^2 = a\sigma^2_r A_r(\tau) \int_0^T \left[ \theta_0(\tau) - \alpha(\tau) C(\tau) \right] A_r(\tau) d\tau, \quad (20)$$

in the functions $(A_r(\tau), C(\tau))$. Equations (19) and (20) must be solved with the initial conditions $A_r(0) = C(0) = 0$, which follow from (10) because a bond with zero maturity trades at its face value.
of one. A complicating feature of (19) and (20) is that the coefficient of \( A_r(\tau) \) in each equation depends on an integral involving the functions \((A_r(\tau), C(\tau))\). To solve (19) and (20), we proceed in two steps. First, we take the integrals as given and solve (19) and (20) as linear ODEs with constant coefficients. Second, we require that the solution is consistent with the value of the integrals.

The first step yields

\[
A_r(\tau) = \frac{1 - e^{-\kappa_r^* \tau}}{\kappa_r^*},
\]

(21)

\[
C(\tau) = \kappa_r^* \tau^\tau \int_0^T A_r(u) du - \frac{\sigma_r^2}{2} \int_0^T A_r(u)^2 du,
\]

(22)

where the scalars \((\kappa_r^*, \tau^\tau)\) are defined by

\[
\kappa_r^* \equiv \kappa_r + a\sigma_r^2 \int_0^T \alpha(\tau) A_r(\tau)^2 d\tau,
\]

(23)

\[
\kappa_r^* \tau^\tau \equiv \kappa_r \tau^\tau + a\sigma_r^2 \int_0^T [\theta_0(\tau) - \alpha(\tau)C(\tau)] A_r(\tau) d\tau.
\]

(24)

We use the star subscript because \((\kappa_r^*, \tau^\tau)\) are the counterparts of \((\kappa_r, \tau)\) under the risk-neutral measure. The second step requires that \((\kappa_r^*, \tau^\tau)\) solve (23) and (24) when \((A_r(\tau), C(\tau))\) are substituted in from (21) and (22). Proposition 1 shows that this requirement determines \((\kappa_r^*, \tau^\tau)\) uniquely.

**Proposition 1.** The functions \((A_r(\tau), C(\tau))\) are given by (21) and (22), respectively, where \(\kappa_r^*\) is the unique solution to

\[
\kappa_r^* = \kappa_r + a\sigma_r^2 \int_0^T \alpha(\tau) \left( \frac{1 - e^{-\kappa_r^* \tau}}{\kappa_r^*} \right)^2 d\tau,
\]

(25)

and \(\tau^\tau\) is given by

\[
\tau^\tau = \tau^\tau + a\sigma_r^2 \int_0^T \alpha(\tau) \left[ \int_0^T \frac{1 - e^{-\kappa_r^* u}}{\kappa_r^*} du \right] \frac{1 - e^{-\kappa_r^* \tau}}{\kappa_r^*} d\tau
\]

\[
+ \int_0^T \alpha(\tau) \left[ \int_0^T \frac{1 - e^{-\kappa_r^* u}}{\kappa_r^*} du \right] \frac{1 - e^{-\kappa_r^* \tau}}{\kappa_r^*} d\tau.
\]

(26)

We next explore the economic implications of the equilibrium derived in Proposition 1. Section 3.3 examines how shocks to the short rate are transmitted to longer maturities. Section 3.4 examines how bond expected excess returns depend on the short rate and on the shape of the term structure. Section 3.5 examines how changes in bond demand affect the term structure.
3.3 Monetary Policy Transmission and Carry Trades

In the segmentation equilibrium, in which there are no arbitrageurs, bond yields $y_t^{(\tau)}$ are disconnected from the short rate $r_t$. By contrast, when arbitrageurs are present, they transmit short-rate shocks to bond yields, ensuring that yields are informative about the current and expected future short rates.

Arbitrageurs transmit short-rate shocks to bond yields through their *carry trades*. Suppose that a shock causes the short rate to drop below the value that bond yields would take in the segmentation equilibrium. To benefit from the discrepancy between bond yields and the short rate, arbitrageurs buy bonds and finance their position by borrowing short-term. Their activity causes bond prices to rise and yields to drop, thus reflecting the drop in the short rate. Conversely, following a shock that causes the short rate to exceed the value that bond yields would take under segmentation, arbitrageurs short-sell bonds and invest short-term. Their activity causes bond prices to drop and yields to rise, thus reflecting the rise in the short rate. In both cases, arbitrageurs engage in carry trades—trades that are profitable when prices do not move. For example, buying a bond and financing that position by short-term borrowing is profitable when the short rate remains below the bond’s yield until the bond’s maturity.

The extent to which arbitrageurs transmit short-rate shocks to bond yields depends on three main parameters of our model: the arbitrageurs’ risk-aversion coefficient $a$, the volatility $\sigma_r$ of the short rate, and the slope $\alpha(\tau)$ of the demand by preferred-habitat investors. The parameters $a$ and $\sigma_r^2$ concern the arbitrageurs’ ability to bear the risk that carry trades entail, namely, that the short rate can rise when arbitrageurs borrow short-term to buy bonds, and that the short rate can drop when arbitrageurs short-sell bonds and invest short-term. When $a$ and $\sigma_r^2$ are small, arbitrageurs are better able to bear that risk and hence engage in larger carry trades, transmitting the shocks more fully. The parameter $\alpha(\tau)$ concerns the arbitrageurs’ ability to impact prices. When $\alpha(\tau)$ is small for all $\tau \in (0, T)$, the demand by preferred-habitat investors is less price-elastic. Hence trades by arbitrageurs have larger price impact, transmitting the shocks more fully.

We measure the extent to which arbitrageurs transmit short-rate shocks to bond yields by comparing the reaction of forward rates to that of expected future short rates. We evaluate how a time-$t$ shock to the short rate $r_t$ affects the expected short rate $E_t(r_{t+\tau})$ at time $t + \tau$ and the instantaneous forward rate $f_t^{(\tau)}$ for maturity $\tau$. The latter rate is defined as the limit of the forward
rate \( f_t^{(\tau - \Delta \tau, \tau)} \) between maturities \( \tau - \Delta \tau \) and \( \tau \) when \( \Delta \tau \) goes to zero:

\[
f_t^{(\tau)} \equiv \lim_{\Delta \tau \to 0} f_t^{(\tau - \Delta \tau, \tau)} = -\frac{\partial \log(P_t^{(\tau)})}{\partial \tau} = A'_r(\tau)r_t + C'(\tau),
\]

(27)

where the second step follows from (2), and the third from (10). When the expectations hypothesis (EH) of the term structure holds, forward rates move one-to-one with expected future short rates. Proposition 2 shows that forward rates instead under-react and hence arbitrageurs transmit short-rate shocks to bond yields only partially.

Formally, a unit shock to \( r_t \) raises \( E_t(r_{t+\tau}) \) by \( e^{-\kappa_r \tau} \) because the short rate mean-reverts at rate \( \kappa_r \). Equation (27) implies that \( f_t^{(\tau)} \) rises by \( A'_r(\tau) = e^{-\kappa^*_r \tau} \), where the equality follows from (21). Under-reaction occurs because the short rate’s mean-reversion parameter \( \kappa^*_r \) under the risk-neutral measure exceeds its counterpart \( \kappa_r \) under the physical measure. Equation (25) implies that the difference \( \kappa^*_r - \kappa_r \), and hence the extent of under-reaction, increases in \( a, \sigma^2_r \) and \( \alpha(\tau) \).

**Proposition 2 (Under-Reaction of Forward Rates).** A unit shock to the short rate \( r_t \): 

- Raises the expected short rate \( E_t(r_{t+\tau}) \) at time \( t + \tau \) by \( \frac{\partial E_t(r_{t+\tau})}{\partial r_t} = e^{-\kappa_r \tau} \).
- Raises the instantaneous forward rate \( f_t^{(\tau)} \) for maturity \( \tau \) by \( \frac{\partial f_t^{(\tau)}}{\partial r_t} = e^{-\kappa^*_r \tau} \).

The forward rate under-reacts (\( \kappa^*_r > \kappa_r \)) provided that arbitrageurs are risk-averse (\( a > 0 \)) and the demand by preferred-habitat investors is price-elastic (\( \alpha(\tau) > 0 \) in a positive-measure subset of \( (0, T) \)). The extent of under-reaction \( \kappa^*_r - \kappa_r \) increases in \( a, \sigma^2_r \) and \( \alpha(\tau) \).

Our results have implications for the transmission of monetary policy. Suppose that the central bank conducts monetary policy by changing the rate that it pays on bank reserves. Suppose also that arbitrageurs are banks, in which case the short rate \( r_t \) that they earn on their wealth is the rate paid on reserves. Our model implies that the transmission of monetary-policy shocks to the yields of long-maturity bonds is done by arbitrageurs. Moreover, the transmission mechanism is weaker when arbitrageurs are more risk-averse, central bank actions are more uncertain (the short rate is more volatile), or the demand by preferred-habitat investors is more price-elastic. An additional implication is that in transmitting monetary-policy shocks, arbitrageurs earn a rent. That rent arises from the returns on the carry trades, and reflects bond risk premia, as we explain in Section 3.4. In that section we also show that bond risk premia are larger, resulting in a larger rent for arbitrageurs, under the exact same conditions that generate a weaker transmission mechanism.
3.4 Bond Risk Premia

The expected excess returns that bonds earn in equilibrium mirror the carry trades of arbitrageurs. This is because arbitrageurs enter into the carry trades only if they expect to earn sufficiently high returns as compensation for the risk they take. When the short rate is low, bonds earn positive expected excess returns so that arbitrageurs are induced to buy them. When instead the short rate is high, bonds earn negative expected excess returns so that arbitrageurs are induced to sell them short. We refer to expected excess returns as risk premia because they compensate arbitrageurs for risk.

Since in the absence of demand risk factors, the short rate is the only source of time-variation, bond risk premia are positively related to the slope of the term structure: a low (high) short rate implies both an upward-sloping (downward-sloping) term structure and positive (negative) bond risk premia. The positive premia-slope relationship is a widely documented empirical fact in the term-structure literature, starting with Fama and Bliss (FB, 1987). FB perform the regression

\[
\frac{1}{\Delta \tau} \log \left( \frac{P_t^{(\tau - \Delta \tau)}}{P_t^{(\tau)}} \right) - y_t^{(\Delta \tau)} = a_{FB} + b_{FB} \left( f_t^{(\tau - \Delta \tau, \tau)} - y_t^{(\Delta \tau)} \right) + \epsilon_{t+\Delta \tau}.
\]

The dependent variable is the return on a zero-coupon bond with maturity \( \tau \) held over a period \( \Delta \tau \), in excess of the spot rate for maturity \( \Delta \tau \). The independent variable is the slope of the term structure as measured by the difference between the forward rate between maturities \( \tau - \Delta \tau \) and \( \tau \), and the spot rate for maturity \( \Delta \tau \). FB find that \( b_{FB} \) is positive and that it exceeds one for some \( \tau \). The implied time-variation of risk premia is economically significant: predicted premia have a standard deviation of about 1-1.5% per year, while average premia are about 0.5% per year.

The behavior of bond risk premia is related to the predictability of changes to long rates. Campbell and Shiller (CS 1991) find that the slope of the term structure predicts changes in long rates, but to a weaker and typically opposite extent than implied by the EH. CS perform the regression

\[
y_t^{(\tau - \Delta \tau)} - y_t^{(\tau)} = a_{CS} + b_{CS} \frac{\Delta \tau}{\tau - \Delta \tau} \left( y_t^{(\tau)} - y_t^{(\Delta \tau)} \right) + \epsilon_{t+\Delta \tau}.
\]

The dependent variable is the change, between times \( t \) and \( t + \Delta \tau \), in the yield of a zero-coupon bond that has maturity \( \tau \) at time \( t \). The independent variable is the difference between the spot rates for maturities \( \tau \) and \( \Delta \tau \), normalized so that the regression coefficient \( b_{CS} \) is equal to one under the EH. CS find that \( b_{CS} \) is smaller than one, negative for most \( \tau \), and decreasing in \( \tau \). This
finding is related to the positive premia-slope relationship. Indeed, suppose that the term structure is upward sloping. Because bonds earn positive expected excess returns, their yields increase by less than under the EH, implying a regression coefficient $b_{CS}$ smaller than one.\footnote{For more material and references on bond return predictability, see the survey by Cochrane (1999). See also Cochrane and Piazzesi (2005) who find that a tent-shaped factor of yields explains bond risk premia even better than the slope of the term structure does.}

Proposition 3 computes the FB and CS regression coefficients $b_{FB}$ and $b_{CS}$ in the analytically convenient case where $\Delta \tau$ is small. The proposition confirms that $b_{FB}$ is positive and $b_{CS}$ is smaller than one. It also shows that $b_{FB}$ increases in the arbitrageurs’ risk-aversion coefficient $a$, the volatility $\sigma_{r}$ of the short rate, and the slope $\alpha(\tau)$ of the demand by preferred-habitat investors. Intuitively, when $a$ and $\sigma_{r}^{2}$ are small, arbitrageurs are better able to bear risk and require smaller premia to enter into carry trades. When instead $\alpha(\tau)$ is small, arbitrageurs have larger price impact. Hence, they enter into smaller carry trades and require smaller premia.

**Proposition 3 (Positive Premia-Slope Relationship).** For $\Delta \tau \to 0$ and for all $\tau$:

- The FB regression coefficient in (28) is $b_{FB} = \frac{\kappa^{*} - \kappa_{r}}{\sigma_{r}^{2}}$. It is positive provided that arbitrageurs are risk-averse ($a > 0$) and the demand by preferred-habitat investors is price-elastic ($\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$). It increases in $a$, $\sigma_{r}^{2}$ and $\alpha(\tau)$.

- The CS regression coefficient in (29) is $b_{CS} = 1 - \frac{(\kappa^{*} - \kappa_{r})A_{t}(\tau)^{\tau}}{\tau - A_{t}(\tau)}$. It is smaller than one under the same condition that ensures $b_{FB} > 0$, and it increases in $\tau$.

Proposition 3 has the additional implications that $b_{FB}$ is independent of $\tau$ and is smaller than one, and that $b_{CS}$ increases in $\tau$. In the data, by contrast, $b_{FB}$ increases in $\tau$ and exceeds one for most maturities, and $b_{CS}$ decreases in $\tau$. Our model can match these empirical properties in the presence of demand risk, as we show in Sections 4 and 5.

### 3.5 Demand Effects

In the segmentation equilibrium, in which there are no arbitrageurs, the yield $y_{t}(\tau)$ for maturity $\tau$ depends only on the demand intercept $\beta_{t}(\tau) = \theta_{0}(\tau)$ and demand slope $\alpha(\tau)$ for that maturity. The presence of arbitrageurs changes that aspect of the equilibrium dramatically. The yield $y_{t}(\tau)$ depends on the demand intercept and slope for all maturities. Moreover, a change in the demand intercept and slope for maturity $\tau$ can have its largest effects for maturities other than $\tau$.\footnote{For more material and references on bond return predictability, see the survey by Cochrane (1999). See also Cochrane and Piazzesi (2005) who find that a tent-shaped factor of yields explains bond risk premia even better than the slope of the term structure does.}
Suppose that the demand intercept $\theta_0(\tau)$ changes to $\theta_0(\tau) + \Delta \theta_0(\tau)$, where $\Delta \theta_0(\tau)$ is a general function of $\tau$ and represents an unanticipated and permanent change. Maturities for which $\Delta \theta_0(\tau) > 0$ experience a drop in demand because (5) defines the demand intercept with a negative sign. Proposition 1 implies that $\kappa_\tau^r$ and $A_r(\tau)$ do not change, that the change $\Delta r^\tau$ in $r^\tau$ has the same sign as $a \kappa_\tau^2 \int_0^\tau \Delta \theta_0(\tau) A_r(\tau) d\tau$, and that $C(\tau)$ changes by $\kappa_\tau^r \Delta r^\tau \int_0^\tau A_r(\tau) d\tau$. Hence, the yield $y_t(\tau)$ for maturity $\tau$ changes by $\Delta y_t(\tau) \equiv \kappa_\tau^r \Delta r^\tau \int_0^\tau A_r(\tau) d\tau$. This implies Proposition 4.

**Proposition 4 (Global Demand Effects).** A change in the demand intercept from $\theta_0(\tau)$ to $\theta_0(\tau) + \Delta \theta_0(\tau)$ affects yields provided that arbitrageurs are risk-averse ($a > 0$). Spot rates for all maturities rise if $\int_0^T \Delta \theta_0(\tau) A_r(\tau) d\tau > 0$ and drop otherwise. The relative effect across maturities is independent of the maturities where the demand change originates ($\Delta y_t(\tau_2)/\Delta y_t(\tau_1)$ is independent of $\Delta \theta_0(\tau)$). Yields for longer maturities are more affected ($\Delta y_t(\tau_2)/\Delta y_t(\tau_1) > 1$ for $\tau_1 < \tau_2$).

Proposition 4 shows that the effects of the change $\Delta \theta_0(\tau)$ are characterized fully by the integral $\int_0^T \Delta \theta_0(\tau) A_r(\tau) d\tau$. If that integral is positive, then yields for all maturities rise—even for maturities for which demand increases because $\Delta \theta_0(\tau) < 0$. Thus, demand effects are global: demand intercepts across all maturities are aggregated into the one-dimensional index $\int_0^T \theta_0(\tau) A_r(\tau) d\tau$, and changes to that index move all yields in the same direction. These global effects are the polar opposite of the local effects derived in the segmentation equilibrium.

Demand effects are represented by a one-dimensional index because there is only one risk factor, the short rate. The index relates to the sensitivity of arbitrageurs’ portfolio to that factor. Suppose that following a change in preferred-habitat demand, arbitrageurs are induced to hold a portfolio that realizes more losses when the short rate increases. Arbitrageurs then view bonds as riskier and require higher expected excess returns to hold them, causing yields to increase for all maturities.

The index is derived by multiplying the demand intercept $\theta_0(\tau)$ for maturity $\tau$ by the function $A_r(\tau) = \frac{1-e^{-\kappa_\tau^r \tau}}{\kappa_\tau^r}$ that characterizes the sensitivity of the $\tau$-maturity bond to the short rate, and integrating across maturities. If a change in the demand intercept raises that integral, then the sensitivity-weighted demand for bonds by preferred-habitat investors declines and the sensitivity of arbitrageurs’ portfolio increases. Since $A_r(\tau)$ increases in $\tau$, demand intercepts for longer-maturity bonds receive a larger weight in the index. Hence, changes to the demand for these bonds have a larger effect on the term structure.
While changes to the demand for longer-maturity bonds have a larger effect on yields, the relative effect across maturities is the same as when the demand for shorter-maturity bonds changes. Moreover, any demand change has its largest effect on the yield for the longest maturity. Intuitively, a decrease in demand raises the instantaneous expected returns of long-maturity bonds more than of short-maturity bonds. This is because expected excess returns compensate arbitrageurs for risk, and long-maturity bonds are riskier ($A_r(\tau)$ increases in $\tau$). The increase in expected returns causes yields to increase: the yield for maturity $\tau$ involves an average of instantaneous expected returns that the bond with maturity $\tau$ earns during its life $[t, t + \tau]$. Since demand changes are permanent, the average of instantaneous expected returns increases more for longer-maturity bonds. Hence, yields for longer maturities are more affected by demand changes.

4 Demand Risk

In this section we generalize our analysis to the case where demand is time-varying. Since demand affects yields only when arbitrageurs are risk-averse, we assume $a > 0$. Time-variation in yields arises because of the short rate $r_t$ and the $K$ demand factors $\{\beta_{k,t}\}_{k=1,..,K}$.

4.1 Equilibrium

We derive the equilibrium following the same three steps as in Section 3.2. We conjecture that there exist $K + 2$ functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..,K}, C(\tau))$ that depend only on $\tau$ such that the time-$t$ price of the bond with maturity $\tau$ is

$$P^\tau_t = e^{-[A(\tau)^T q_t + C(\tau)]}, \quad (30)$$

where $A(\tau)$ is the $(K + 1) \times 1$ vector $(A_r(\tau), A_{\beta,1}(\tau), .., A_{\beta,K}(\tau))^\top$. Applying Ito’s Lemma to (10), using the dynamics (7) of $q_t$, and noting that $t + \tau$ stays constant when taking the derivative, we find that the time-$t$ instantaneous return on the bond with maturity $\tau$ is

$$\frac{dP^\tau_t}{P^\tau_t} = \mu^\tau_t dt - A(\tau)^T \Sigma dB_t, \quad (31)$$

where

$$\mu^\tau_t = A'(\tau)^T q_t + C'(\tau) + A(\tau)^T \Gamma(q_t - \tau E) + \frac{1}{2} A(\tau)^T \Sigma \Sigma^T A(\tau) \quad (32)$$
is the instantaneous expected return. Substituting the bond return (31) into the arbitrageurs’
optimization problem (4) yields

$$\max_{\{X_t^{(\tau)}\}_{\tau \in (0,T)}} \left\{ \int_0^T X_t^{(\tau)} (\mu_t^{(\tau)} - r_t) d\tau - \frac{a}{2} \left[ \int_0^T X_t^{(\tau)} A(\tau) d\tau \right]^\top \Sigma \Sigma^\top \left[ \int_0^T X_t^{(\tau)} A(\tau) d\tau \right] \right\}.$$  (33)

Point-wise maximization of (33) yields the arbitrageurs’ first-order condition.

**Lemma 2.** The arbitrageurs’ first-order condition is

$$\mu_t^{(\tau)} - r_t = a A(\tau)^\top \Sigma \Sigma^\top \left[ \int_0^T X_t^{(\tau)} A(\tau) d\tau \right].$$  (34)

Equation (34) is the multi-factor counterpart of (14). The left-hand side is the increase in
portfolio expected return if arbitrageurs shift one dollar from the short rate $r_t$ to the bond with
maturity $\tau$. The right-hand side is the increase in portfolio risk, times the arbitrageurs’ risk aversion
coefficient $a$. The increase in portfolio risk is equal to the covariance between the return on the
additional investment in the bond and the return on the arbitrageurs’ portfolio. With multiple risk
factors, the covariance is the product of the sensitivity vectors $A(\tau)$ and $\int_0^T X_t^{(\tau)} A(\tau) d\tau$ of the two
returns to the factors, times the factors’ covariance matrix $\Sigma \Sigma^\top$. To show the full analogy between
(34) and (14), we can write (34) in terms of factor prices. Denoting the $(K+1) \times 1$ vector of factor
prices by $\lambda_t \equiv (\lambda_{r,t}, \lambda_{\beta,1,t}, ..., \lambda_{\beta,K,t})^\top$, we can write (34) as $\mu_t^{(\tau)} - r_t = a A(\tau)^\top \lambda_t$ and deduce that
factor prices are $\lambda_t = \Sigma \Sigma^\top \left[ \int_0^T X_t^{(\tau)} A(\tau) d\tau \right]$.

Substituting $X_t^{(\tau)}$ from the market-clearing equation (16) into (34), using (5), (6), (30) and (32),
and denoting by $\Theta(\tau)$ the $1 \times (K+1)$ vector $(0, \theta_1(\tau), ..., \theta_K(\tau))$, we find the following counterpart
of (18):

$$V'(\tau)^\top q_t + C'(\tau) + A(\tau)^\top \Gamma (q_t - \bar{\tau} E) + \frac{1}{2} A(\tau)^\top \Sigma \Sigma^\top A(\tau) - r_t$$

$$= a A(\tau)^\top \Sigma \Sigma^\top \left[ \theta_0(\tau) + \Theta(\tau) q_t - \alpha(\tau) \left(A(\tau)^\top q_t + C(\tau)\right) \right] A(\tau) d\tau.$$  (35)

Setting the linear terms in $q_t$ on both sides of (35) to be equal yields the system of $K+1$ first-order
linear ODEs

$$A'(\tau) + MA(\tau) - E = 0,$$  (36)
where $M$ is the $(K + 1) \times (K + 1)$ matrix

$$M \equiv \Gamma^T - a \int_0^T \left[ \Theta(\tau)^T A(\tau)^T - \alpha(\tau) A(\tau) A(\tau)^T \right] \Sigma \Sigma^T. \tag{37}$$

Setting the terms that are independent of $q_t$ on both sides of (35) to be equal yields the first-order linear ODE

$$C'(\tau) - \tau A(\tau)^T \Gamma E + \frac{1}{2} A(\tau)^T \Sigma \Sigma^T A(\tau) = a A(\tau)^T \Sigma \Sigma^T \int_0^T [\theta_0(\tau) - \alpha(\tau) C(\tau)] A(\tau) d\tau. \tag{38}$$

Equations (36) and (38) must be solved with the initial conditions $A(0) = C(0) = 0$. To solve (36) and (38), we follow the same two steps as in Section 3. The first step is to take the integrals in (36) and (38) as given and solve these equations as linear ODEs with constant coefficients. The solution is in Lemma 3.

**Lemma 3.** Suppose that the matrix $M$ defined in (37) has $K + 1$ distinct eigenvalues $(\nu_1, \ldots, \nu_{K+1})$. The function $A(\tau) = (A_r(\tau), A_{\beta,1}(\tau), \ldots, A_{\beta,K}(\tau))^T$ is given by

$$A_r(\tau) = \frac{1 - e^{-\nu_1 \tau}}{\nu_1} + \sum_{k'=1}^{K} \phi_{r,k'} \left( \frac{1 - e^{-\nu_{k'+1} \tau}}{\nu_{k'+1}} - \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \right), \tag{39}$$

$$A_{\beta,k}(\tau) = \sum_{k'=1}^{K} \phi_{\beta,k,k'} \left( \frac{1 - e^{-\nu_{k'+1} \tau}}{\nu_{k'+1}} - \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \right), \tag{40}$$

where $(\{\phi_{r,k'}\}_{k'=1}^{K}, \{\phi_{\beta,k,k'}\}_{k,k'=1}^{K})$ are scalars derived from the eigenvectors of $M$. The function $C(\tau)$ is given by

$$C(\tau) = \left[ \int_0^\tau A(u)^T du \right] \chi - \frac{1}{2} \int_0^\tau A(u)^T \Sigma \Sigma^T A(u) du, \tag{41}$$

where $\chi \equiv (\chi_r, \chi_{\beta,1}, \ldots, \chi_{\beta,K})^T$ is the $(K + 1) \times 1$ vector

$$\chi \equiv \tau \Gamma E + a \Sigma \Sigma^T \int_0^T [\theta_0(\tau) - \alpha(\tau) C(\tau)] A(\tau) d\tau. \tag{42}$$

The second step is to ensure that the solution derived in Lemma 3 is consistent with the value of the integrals. There are $(K + 1)^2$ integrals in (36). These integrals involve the $K + 1$ functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1}^{K})$, and determine the elements of the $(K + 1) \times (K + 1)$ matrix
$M$ defined in (37). In turn, the eigenvalues and eigenvectors of $M$ determine the solution for $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1...K})$ in Lemma 3, and that solution determines the value of the integrals. This yields a nonlinear system of $(K+1)^2$ equations in the $(K+1)^2$ integrals. Given a solution to that system, the elements $(\chi_r, \chi_{\beta,1}, ..., \chi_{\beta,K})$ of the vector $\chi$ in the solution for $C(\tau)$ in Lemma 3 can be derived from a linear system of $K+1$ equations.

In the remainder of this section, we show analytically general properties of the model. We focus on the case where there is one demand factor ($K=1$, four nonlinear equations) and omit the subscript $k$ from that factor. We additionally assume that the short rate and the demand factor are independent. This corresponds to the matrices $(\Gamma, \Sigma)$ being diagonal. We denote their diagonal elements by $(\kappa_r, \kappa_\beta, \sigma_r, \sigma_\beta) \equiv (\Gamma_{1,1}, \Gamma_{2,2}, \Sigma_{1,1}, \Sigma_{2,2})$. The case with one independent demand factor is a natural first case to analyze, and it yields a rich set of results. We analyze the same case numerically in Section 5, where we perform a calibration exercise.\textsuperscript{10} We discuss the general case briefly at the end of Section 4.4.

Two useful assumptions for deriving some of our analytical results are that the maximum maturity $T$ is infinite and the functions $(\alpha(\tau), \{\theta_k(\tau)\}_{k=1...K})$ are exponentials or linear combinations of exponentials. Under these assumptions, the integrals in (36) involve Laplace transforms of the functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1...K})$ and of those functions’ pairwise products. Moreover, by multiplying the ODE system (36) by the exponentials in $(\alpha(\tau), \{\theta_k(\tau)\}_{k=1...K})$ and by the products of these exponentials with the functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1...K})$, we find equations that involve the same Laplace transforms. This yields a system of equations in the Laplace transforms, derived in Appendix A for the general case (Lemma A.1). While that system remains nonlinear, a key advantage of the Laplace-transform approach is that we do not need to compute the eigenvalues and eigenvectors of $M$, which can be real or complex.

We begin our analytical investigation by showing existence of equilibrium. We set $T=\infty$ and take the demand elasticity $\alpha(\tau)$ to be the declining exponential $\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}$, where $(\alpha, \delta_\alpha)$ are positive constants. We take the impact $\theta(\tau)$ of the single demand factor on the demand intercept to be a difference between two exponentials $\theta(\tau) = \theta (e^{-\delta_\theta \tau} - e^{\delta_\theta \tau})$, where $(\theta, \delta_\theta)$ are positive constants and $\delta_\alpha < \delta_\theta$. A unit increase in the demand factor $\beta_i$ raises the spot rate for maturity $\tau$

\textsuperscript{10}Hayashi (2018) derives two alternative numerical algorithms for solving our model in the case $\alpha(\tau) = 0$. Both algorithms discretize the functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1...K})$, without imposing the structure derived in Lemma 3. They have the advantage of handling large values of $K$ as easily as small values.
in the segmentation equilibrium by

\[ \frac{\theta(\tau)}{\alpha(\tau)\tau} = \frac{\theta (1 - e^{(\delta_\theta-\delta_\alpha)\tau})}{\alpha\tau}. \]

This function has a positive limit at \( \tau = 0 \) and decreases in \( \tau \).

**Theorem 1 (Equilibrium Existence).** Suppose that there is one demand factor, the matrices \((\Gamma, \Sigma)\) are diagonal, \( T = \infty \), \( \alpha(\tau) = \alpha e^{-\delta_\alpha\tau} \) and \( \theta(\tau) = \theta (e^{-\delta_\theta\tau} - e^{\delta_\theta\tau}) \), where \((\alpha, \theta, \delta_\alpha, \delta_\theta)\) are positive constants and \( \delta_\theta \) is large. An equilibrium exists under either of the following sufficient conditions:

- \( \kappa_\beta \) is close to zero.
- \( \delta_\alpha(\delta_\alpha + \kappa_\gamma)(\delta_\alpha + \kappa_\beta) > 2a\theta\sigma_r\sigma_\beta \).

In equilibrium, \( M_{1,1} > \kappa_\gamma, M_{1,2} > 0, M_{2,1} < 0 \) and \( M_{2,2} > \frac{\kappa_\beta-\delta_\theta}{2} \).

We complement the existence result in Theorem 1 by computing in Appendix A (Lemma A.2) the equilibrium in closed form when the arbitrageurs’ risk-aversion coefficient \( a \) is close to zero or to infinity and other parameters, including \( T \), can take any values. For our analysis of \( a \approx 0 \) and \( a \approx \infty \), we require that \( \alpha(\tau) \) and \( \frac{\theta(\tau)}{\tau} \) have a positive and a finite limit, respectively, at \( \tau = 0 \). That restriction is satisfied by the specification in Theorem 1. We next examine how the results of Sections 3.3-3.5 are modified in the presence of demand risk.

### 4.2 Carry Trades and Hedging

Demand risk weakens the transmission of short-rate shocks to bond yields. This is because the carry trades through which arbitrageurs transmit the shocks become riskier. To hedge against demand risk, arbitrageurs scale down their carry trades or even convert them into *butterfly trades*, reversing the sign of their positions for long maturities. Because of hedging, short-rate shocks can move yields for long maturities in the direction opposite to the shocks.

To explain hedging in our model, suppose as in Section 3.3 that a shock causes the short rate to drop below the value that bond yields would take in the segmentation equilibrium. Arbitrageurs can benefit from the discrepancy between bond yields and the short rate by buying bonds and borrowing short-term. This carry trade leaves them exposed to a rise in the short rate, as in Section 3.3, and to a drop in bond demand by preferred-habitat investors. The importance of
demand risk relative to short-rate risk rises with maturity. This is shown in Proposition 5, and can be partly anticipated from the one-factor model, in which short-rate shocks have an effect on yields that declines with maturity, while permanent demand changes have an increasing effect. Because long-maturity bonds are highly exposed to demand risk, arbitrageurs can short-sell them to hedge the demand risk of their aggregate position. Such short-selling occurs when arbitrageurs are sufficiently risk-averse, and causes yields for long maturities to rise despite the drop in current and expected future short rates. Buying intermediate-maturity bonds and short-selling long-maturity ones and very short-maturity ones (i.e., borrowing short-term) is a butterfly trade, common in term-structure arbitrage.\footnote{An example of a butterfly trade comes from the 2007-2008 financial crisis. Short-rate cuts triggered by the crisis rendered the US term structure steeply upward sloping. Term structure arbitrageurs took the view that forward rates did not drop enough to reflect the low expected future spot rates—the under-reaction result of Proposition 2. For example, a Barclays Capital report by Pradhan (2009), p.2., points out that while the two-year spot rate was 258 basis points (bps) lower than the ten-year spot rate, the difference between their two-year forward counterparts was only 93bps. The report goes on to advise lending at the two-year rate two years forward and borrowing at the ten-year rate two years forward. Lending at the two-year rate two years forward is a carry trade: it amounts to shorting two-year bonds and buying four-year bonds. Borrowing at the ten-year rate two years forward amounts to buying two-year bonds and shorting twelve-year bonds. That position is layered to the carry trade to hedge term-structure movements at intermediate maturities, and is for a smaller notional amount since the twelve-year bond is more sensitive to such movements than the four-year bond. The overall trade is a butterfly: a short position in two-year bonds, a long position in four-year bonds, and a short position in twelve-year bonds. It exerts upward pressure on the twelve-year spot rate, even though it is triggered by a drop in the short rate.}

Proposition 5 characterizes the response of yields to short-rate and demand shocks. The proposition assumes \( M_{2,1} < 0 \), a property shown to hold for the equilibrium derived in Theorem 1. The assumptions of Theorem 1 are not needed as long as that property holds.

The characterization is simple when the two eigenvalues of \( M \) are real. The function \( A_\beta(\tau) \) is positive, which implies that a drop in demand causes yields for all maturities to rise, and increases in \( \tau \). The function \( A_r(\tau) \) is either positive, or switches sign from positive to negative when \( \tau \) crosses a threshold \( \bar{\tau} \). In the latter case, a drop in the short rate causes yields for maturities \( \tau > \bar{\tau} \) to rise. The ratio \( \frac{A_r(\tau)}{A_\beta(\tau)} \) decreases in \( \tau \), which implies that the effect of demand shocks relative to short-rate shocks rises with maturity.

When the two eigenvalues of \( M \) are complex, the functions \((A_r(\tau), A_\beta(\tau))\) exhibit an oscillating pattern driven by the arbitrageurs’ hedging activity. Following a rise in the short rate, prices of short-maturity bonds drop. Prices of long-maturity bonds can instead rise because arbitrageurs can buy them to hedge demand risk. Long-maturity bonds can thus hedge the short-rate risk of a portfolio with long positions in bonds, and earn negative expected excess returns when arbitrageurs hold such a portfolio in equilibrium. Since arbitrageurs hold long positions when demand by preferred-habitat investors is low, low demand can cause, through the cumulation of negative
expected returns, the prices of bonds of even longer ("very long") maturities to rise. In that case, arbitrageurs do not use the very-long-maturity bonds to hedge demand risk, and those bonds’ prices rise following a drop in the short rate. This yields an oscillating pattern of price sensitivity to the short rate as a function of maturity. The properties shown for real eigenvalues carry through to complex ones for the first half-cycle of the oscillation (which can be longer than the maximum maturity $T$). The functions $(A_r(\tau), A_\beta(\tau))$ begin by being increasing in $\tau$. The function $A_r(\tau)$ eventually reaches a maximum, and the function $A_\beta(\tau)$ does so at a larger value $\bar{\tau}$ which marks the end of the first half-cycle. We set $\bar{\tau} = \infty$ when the two eigenvalues of $M$ are real.

**Proposition 5 (Effect of Short-Rate and Demand Shocks).** Suppose that there is one demand factor, the matrices $(\Gamma, \Sigma)$ are diagonal, and $M_{2,1} < 0$.

- If the two eigenvalues of $M$ are real, then $A_\beta(\tau) > 0$, $A'_\beta(\tau) > 0$ and \[ \left[ \frac{A_r(\tau)}{A_\beta(\tau)} \right]' < 0. \] Moreover, $A_r(\tau) > 0$ for $\tau \in (0, \bar{\tau})$ and $A_r(\tau) < 0$ for $\tau \in (\bar{\tau}, T)$, where $\bar{\tau} = T$ when $a \approx 0$ or $\alpha(\tau) = 0$, and $\bar{\tau} < T$ when $a \approx \infty$.

- If the two eigenvalues of $M$ are complex, then $A_\beta(\tau) > 0$ for $\tau \in (0, \bar{\tau})$, and $A'_\beta(\tau) > 0$ and \[ \left[ \frac{A_r(\tau)}{A_\beta(\tau)} \right]' < 0 \] for $\tau \in (0, \bar{\tau})$, where $\bar{\tau} > \hat{\tau}$. If $\bar{\tau} < T$, then $A_r(\tau) > 0$ for $\tau \in (0, \bar{\tau})$, where $\bar{\tau} \in (0, \bar{\tau})$. When the thresholds $(\bar{\tau}, \bar{\bar{\tau}}, \hat{\tau})$ are crossed, the respective inequalities switch.

### 4.3 Bond Risk Premia

Demand risk strengthens the positive premia-slope relationship derived in Section 3.4. Indeed, low demand by preferred-habitat investors implies positive bond risk premia because arbitrageurs must be induced to buy the bonds to make up for the low investor demand. Because of the positive premia, yields are high and the term structure is upward-sloping.

Proposition 6 computes the FB and CS coefficients $b_{FB}$ and $b_{CS}$. It shows that $b_{FB}$ is positive and $b_{CS}$ is smaller than one for at least all maturities such that the functions $(A_r(\tau), A_\beta(\tau))$ are positive and $A_\beta(\tau)$ increases in $\tau$, and for all maturities when $a$ is close to zero or to infinity. Moreover, when $a \approx \infty$ and the average maturity where demand shocks originate is sufficiently long, $b_{FB}$ exceeds one and increases in $\tau$, while $b_{CS}$ is negative and decreases in $\tau$.

**Proposition 6 (Demand Risk Strengthens Positive Premia-Slope Relationship).** Suppose that there is one demand factor, the matrices $(\Gamma, \Sigma)$ are diagonal, $M_{1,2} \geq 0$, $M_{2,1} < 0$ and $\Delta \tau \to 0$. 

• The FB regression coefficient in (28) is positive for \( \tau < \min\{\bar{\tau}, \hat{\tau}\} \), and for all \( \tau \) when \( a \approx 0 \) or \( a \approx \infty \). When \( a \approx \infty \) and

\[
\int_{T_0}^{T} \theta(\tau)\tau d\tau > \frac{\int_{T_0}^{T} \alpha(\tau)\tau^2 d\tau}{\int_{T_0}^{T} \alpha(\tau)\tau d\tau},
\]

(43)

then \( b_{FB} \) exceeds one and increases in \( \tau \).

• The CS regression coefficient in (29) is smaller than one for \( \tau < \min\{\bar{\tau}, \hat{\tau}\} \), and for all \( \tau \) when \( a \approx 0 \) or \( a \approx \infty \). When \( a \approx 0 \), \( b_{CS} \) is close to one and increases in \( \tau \). When \( a \approx \infty \) and (43) holds, \( b_{CS} \) is negative and decreases in \( \tau \).

### 4.4 Demand Effects

Suppose, as in Section 3.5, that the demand intercept \( \theta_0(\tau) \) changes to \( \theta_0(\tau) + \Delta \theta_0(\tau) \), where \( \Delta \theta_0(\tau) \) is a general function of \( \tau \). The functions \((A_r(\tau), A_\beta(\tau))\) do not change, and the effects on yields are entirely through \( C(\tau) \). Because there are two risk factors, the effects are represented by two one-dimensional indices. The indices are \( \int_{T_0}^{T} \theta_0(\tau)A_r(\tau) d\tau \) and \( \int_{T_0}^{T} \theta_0(\tau)A_\beta(\tau) d\tau \), and relate to the sensitivity of arbitrageurs’ portfolio to the short-rate and the demand factor, respectively.

While demand effects retain a global flavor because they are represented by only two indices across a continuum of maturities, they become more localized relative to the one-factor case. Recall from Section 3.5 that with only one factor, demand changes have the same relative effect across maturities regardless of the maturities where they originate. This independence result does not extend to two factors. The maturities where demand shocks originate matter because they influence how the shocks affect one index relative to the other, and because changes to each index have a different relative effect across maturities. Changes to the demand for long-maturity bonds have a large effect on \( \int_{T_0}^{T} \theta_0(\tau)A_\beta(\tau) d\tau \) relative to \( \int_{T_0}^{T} \theta_0(\tau)A_r(\tau) d\tau \), and changes to \( \int_{T_0}^{T} \theta_0(\tau)A_\beta(\tau) d\tau \) have a large effect on long rates relative to short rates. Hence, the effects of long-maturity bond demand are more pronounced at the long end of the term structure. In comparison, changes to the demand for short-maturity bonds have a larger relative effect on \( \int_{T_0}^{T} \theta_0(\tau)A_r(\tau) d\tau \), and changes to that index have a larger relative effect on short rates. Hence, the effects of short-maturity bond demand are more pronounced at the short end.

The economic intuition is as follows. Suppose that the demand by preferred-habitat investors for long-maturity bonds declines, in which case arbitrageurs take up the slack by purchasing those.
bonds. Since bonds’ sensitivity to demand shocks relative to short-rate shocks rises with maturity, arbitrageurs’ exposure to demand risk increases significantly, while their exposure to short-rate risk increases more mildly. The expected excess returns that arbitrageurs require to bear demand risk increase significantly as well. Since bonds’ sensitivity to demand shocks rises faster with maturity than their sensitivity to short-rate shocks, long-maturity bonds experience a sharp increase in their expected excess returns relative to short-maturity bonds. Hence, long rates increase sharply. By contrast, when the demand by preferred-habitat investors for short-maturity bonds declines, long rates increase less than short rates.

To show a formal result on localization, we consider the simple case where the change \( \Delta \theta_0(\tau) \) represents a decrease in demand for a specific short maturity \( \tau_1 \) or a specific long maturity \( \tau_2 > \tau_1 \).

We denote the resulting changes in the yield \( y_{t,\tau} \) by \( \Delta y_{t,\tau_1} \) and \( \Delta y_{t,\tau_2} \), respectively.

**Proposition 7 (Localization of Demand Effects).** When there is one demand factor, a change in the demand intercept from \( \theta_0(\tau) \) to \( \theta_0(\tau) + \Delta \theta_0(\tau) \) affects yields only through \( \int_0^T \Delta \theta_0(\tau) A_r(\tau) d\tau \) and \( \int_0^T \Delta \theta_0(\tau) A_\beta(\tau) d\tau \). When additionally the matrices \((\Gamma, \Sigma)\) are diagonal, \( M_{2,1} < 0 \), \( \alpha(\tau) \) is non-increasing, and the change \( \Delta \theta_0(\tau) \) is a Dirac function with point mass at \( \tau_1 < \hat{\tau} \) or at \( \tau_2 \in (\tau_1, \hat{\tau}) \),

\[
\Delta y_{t,\tau_1} \Delta y_{t,\tau_2} > \Delta y_{t,\tau_1} \Delta y_{t,\tau_2}.
\]

Equation (44) states that the product of the “local” effects that the changes have on the maturity where they originate exceeds the product of the “cross” effects on the other maturity. Local effects are thus stronger than cross effects.

We expect full localization when there is a large number of demand factors and arbitrageurs are highly risk-averse. Indeed, suppose that a demand shock originating at maturity \( \tau_1 \) has its largest effect at maturity \( \tau_2 \neq \tau_1 \). For this to happen, arbitrageurs must hold non-zero positions in at least the bonds of one of the two maturities. Highly risk-averse arbitrageurs, however, hold non-zero positions only if their exposure to all risk factors is zero, which is infeasible with a large number of factors. Proposition 1 implies a full localization result for the effects of short-rate shocks: since the function \( A_r(\tau) \) converges to zero when the arbitrageurs’ risk-aversion coefficient \( a \) goes to infinity, the effects of short-rate shocks become localized at the zero maturity. We can derive the same localization result with one and two demand factors, using closed-form solutions for the large \( a \) limit. Extending the full localization result to demand shocks requires extending our solutions to a large number of demand factors and is left for future research.
5 Calibration

In this section we perform a calibration exercise. We assume that there is one demand factor which is independent of the short rate. We omit the factor subscript and set \((\kappa_r, \kappa_\beta, \sigma_r, \sigma_\beta) \equiv (\Gamma_{1,1}, \Gamma_{2,2}, \Sigma_{1,1}, \Sigma_{2,2})\). The equilibrium term structure is determined by the parameters \((\kappa_r, \kappa_\beta, \sigma_r, \sigma_\beta)\) of the short-rate process, the parameters \((\kappa_\beta, \sigma_\beta)\) of the demand-factor process, the risk-aversion coefficient \(a\) of arbitrageurs, the functions \((\alpha(\tau), \theta_0(\tau), \theta(\tau))\) that describe the demand slope and intercept of preferred-habitat investors, and the maximum bond maturity \(T\).

The values of \((\kappa_r, \sigma_r)\) affect only the long-run average of yields and not how yields respond to shocks. They do not matter for the analysis in this section which concerns the response to shocks.

We set \(T = 30\) years and \((\kappa_r, \sigma_r) = (0.18, 0.02)\). The values of \((\kappa_r, \sigma_r)\) are derived in Chan, Karolyi, Longstaff, and Sanders (1992), who estimate the Ornstein-Uhlenbeck process (8) over the period 1964-1989, proxying the short rate by the one-month Treasury Bill rate. Greenwood and Vayanos (2014) find similar estimates over the period 1952-2007.

Since \((\theta(\tau), \beta_t)\) affect the demand of preferred-habitat investors only through their product, \((\theta(\tau), \sigma_\beta)\) matter only through their product as well. We can hence normalize \(\sigma_\beta\) to an arbitrary value, and we set it equal to \(\sigma_r\). Setting the standard deviation of shocks to \(\beta_t\) equal to that of shocks to \(r_t\), makes it easier to compare the response of the term structure across the two types of shocks.

We adopt the same exponential specification as in Theorem 1 for \((\alpha(\tau), \theta(\tau))\), setting \(\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}\) and \(\theta(\tau) = \theta(e^{-\delta_\alpha \tau} - e^{-\delta_\theta \tau})\). This leaves six parameters to calibrate: the parameters \((\alpha, \theta, \kappa_\beta, \delta_\alpha, \delta_\theta)\) describing the demand shocks, and the risk-aversion coefficient \(a\) of arbitrageurs. Since it is difficult to measure directly bond demand and arbitrageur risk aversion, we calibrate the six parameters indirectly, based on (i) moments of bond yields and returns, and (ii) the effect of demand shocks on the average level of yields. We show that even with this indirect approach, our model provides strong restrictions on the data.

We use three sets of moments of bond yields and returns: the coefficient of the Fama-Bliss (FB) regression (28), the coefficient of the Campbell-Shiller (CS) regression (29), and the unconditional volatility (standard deviation) of bond yields. For the FB regression, we consider the return of the \(\tau\)-year zero-coupon bond held over one year in excess of the one-year yield \((\Delta \tau = 1)\). For the CS regression, we consider the change in the yield of the \(\tau\)-year zero-coupon bond over one
year ($\Delta \tau = 1$). In both cases, we use the regression coefficients for all $\tau = 2, 3, \ldots, 30$. For the unconditional volatility, we consider the $\tau$-year yield for all $\tau = 1, 2, \ldots, 30$. We estimate the three sets of moments for the U.S. using the Gurkaynak, Sack, and Wright (2007) dataset. That dataset reports daily spot rates extracted from government bond prices from 1961 to the present. We start our sample in November 1985 because this is the earliest date when the reported term structure goes all the way to 30 years, and end it in December 2018.

In our model, the three sets of moments depend on the standard deviations ($\sigma_r, \sigma_\beta$) of the shocks to $r_t$ and $\beta_t$, and on the functions ($A_r(\tau), A_\beta(\tau)$) that describe how the shocks affect the term structure. Equation (37) shows that the matrix $M$ that determines ($A_r(\tau), A_\beta(\tau)$) through the ODE (36) depends on $(a, \alpha, \theta)$ only through the products $(a\alpha, a\theta)$. Hence, the three sets of moments are informative about the products $(a\alpha, a\theta)$ and not about the separate values $(a, \alpha, \theta)$. Intuitively, yields can be volatile because arbitrageurs are highly risk-averse and demand shocks are small, or because arbitrageurs are less risk-averse and demand shocks are larger. To separate $a$ from $(\alpha, \theta)$, we use information on observable demand shocks at the end of this section.

We determine the five parameters ($a\alpha, a\theta, \kappa_\beta, \delta_\alpha, \delta_\theta$) by minimizing the sum $S$ of absolute-value differences

$$S \equiv \sum_{\tau=2}^{30} \left| b_{FB, model}^{(\tau)} - b_{FB, data}^{(\tau)} \right| + \sum_{\tau=2}^{30} \left| b_{CS, model}^{(\tau)} - b_{CS, data}^{(\tau)} \right| + \sum_{\tau=1}^{30} \left| \sigma_{model} \left( y_t^{(\tau)} \right) - \sigma_{data} \left( y_t^{(\tau)} \right) \right| , \quad (45)$$

where the superscript $\tau$ denotes a regression coefficient corresponding to maturity $\tau$, and the subscript $\{model, data\}$ denotes whether a moment is computed within our model or is estimated in the data. When computing the volatility, we express the yields in percentages (e.g., a yield of 2.50% is 2.50) so that the three terms in the sum are of comparable magnitude. The formulas for the model-generated moments are in Appendix C.

The minimum value of $S$ is 12.51. Since there are 88 ($=29+29+30$) moments in total, the average discrepancy per moment is 0.14 ($=12.51/88$). The average absolute value per moment is 1.82, so the average discrepancy per moment is 7.69% ($=0.14/1.82$) in relative terms. Our model can hence match simultaneously a diverse set of moments with a small average error. Figure 1 confirms the close match by plotting the model-generated moments, in blue solid lines, and their empirical counterparts, in black crosses. The green dashed lines are the model-generated moments when the minimization is performed over a constrained set of parameter values, considered later in this section.

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Figure 1: Fama-Bliss regression coefficient (top left panel), Campbell-Shiller regression coefficient (top right panel), and unconditional volatility of yields (bottom panel), as a function of maturity. The black crosses represent the moments estimated on U.S. government bond data from November 1985 to December 2018. The blue solid lines represent the moments computed within our model for the values of \((a_\alpha, a_\theta, \kappa_\beta, \delta_\alpha, \delta_\theta)\) that minimize the sum \(S\) in (45). The green dashed lines represent the moments computed within our model when \((a_\alpha, \delta_\alpha, \delta_\theta) = (0.2, 0.05, 0.25)\) and \((a_\theta, \kappa_\beta)\) minimize \(S\).

The parameter values minimizing \(S\) are \((a_\alpha, a_\theta, \kappa_\beta, \delta_\alpha, \delta_\theta) = (6, 107, 0.37, 0.05, 0.25)\). The implied eigenvalues of the matrix \(M\) are real and equal to \((\nu_1, \nu_2) = (0.16, 0.07)\). The minimization is sensitive to the values of \((a_\alpha, a_\theta, \kappa_\beta)\) and relatively insensitive to those of \((\delta_\alpha, \delta_\theta)\). The mean-reversion rate \(\kappa_\beta = 0.37\) implies a half-life of demand shocks of 1.87 \((\log(2)/0.37)\) years. Hence, demand shocks mean-revert twice as fast as short-rate shocks, whose half-life is 3.85 \((\log(2)/0.18)\) years.

We next determine how shocks to the short rate \(r_t\) and the demand factor \(\beta_t\) affect yields. The left panel of Figure 2 plots the functions \(\left(\frac{A_r(\tau)}{\tau}, \frac{A_\beta(\tau)}{\tau}\right)\) that describe how unit shocks to \(r_t\) and \(\beta_t\)
Figure 2: The response of yields to shocks to the short rate $r_t$ (function $A_r(\tau)$) and to the demand factor $\beta_t$ (function $A_\beta(\tau)$). The left panel plots the response in the equilibrium with arbitrageurs, and the right panel does the same for $\beta_t$-shocks in the segmentation equilibrium. The red and blue solid lines represent the functions $A_r(\tau)$ and $A_\beta(\tau)$, respectively, for the values of $(a\alpha, a\theta, \kappa_\beta, \delta_\alpha, \delta_\theta)$ that minimize the sum $S$ in (45). The purple (magenta) and green dashed lines represent the functions $A_r(\tau)$ and $A_\beta(\tau)$, respectively, when $(a\alpha, \delta_\alpha, \delta_\theta) = (0.2, 0.05, 0.25)$ and $(a\theta, \kappa_\beta)$ minimize $S$.

affect the yield for maturity $\tau$. (The response to one-standard-deviation shocks can be derived by multiplying those functions by $\sigma_r = \sigma_\beta = 0.02$.) We focus on the solid lines, which are drawn for the parameter values minimizing $S$, and consider the dashed lines later in this section. A unit shock to $r_t$ is reflected almost one-to-one in short-maturity yields. The effect, illustrated by the red solid line, declines with maturity, and turns slightly negative for maturities longer than nineteen years due to hedging by arbitrageurs, as explained in Section 4.2. A unit shock to $\beta_t$ raises all yields. The effect, illustrated by the blue solid line, is almost zero for short-maturity yields, increases with maturity until sixteen years, and then declines. The effect of the $\beta_t$-shock dominates that of the $r_t$-shock for maturities longer than four years.

The right panel of Figure 2 plots the function $A_\beta(\tau)$ in the segmentation equilibrium. (The function $A_r(\tau)$ is zero in that equilibrium.) A shock to $\beta_t$ has a larger effect than in the arbitrageurs’ presence, as shown by comparing the blue solid lines across the left and the right panel. The precise comparison depends on maturity. Averaging across maturities, the effect of a $\beta_t$-shock is twice as large under segmentation.

We next determine how unanticipated demand changes $\Delta\theta_0(\tau)$ affect yields. In each of the four panels of Figure 3, the purple (magenta), red, green, blue and black lines illustrate how a decrease
Figure 3: The response of yields to unanticipated demand changes. In each panel, the purple (magenta), red, green, blue and black lines represent the effect of an unanticipated decrease in demand for the two-, five-, ten-, fifteen- and twenty-year bond, respectively, on the yield for maturity $\tau$. In all panels, the decrease in demand is $\Delta \theta_0(\tau) = 0.2$. In the top panels, $a = 75$ and the values of $(a, a\alpha, a\theta, \kappa_{\beta}, \delta_{\alpha}, \delta_{\theta})$ minimize the sum $S$ in (45). In the bottom panels, $(a, a\alpha, \delta_{\alpha}, \delta_{\theta}) = (2.5, 0.2, 0.05, 0.25)$ and $(a\theta, \kappa_{\beta})$ minimize $S$. In the left panels, the unanticipated demand change is permanent, while in the right panels it reverts to zero deterministically at the rate $\kappa_{\theta} = 0.25$.

In demand for the two-, five-, ten-, fifteen- and twenty-year bond, respectively, affects the yield for maturity $\tau$. We focus on the top panels, which are drawn for the parameter values minimizing $S$, and turn to the bottom panels later in this section.

In the top-left panel, demand changes are assumed to be permanent. Consistent with Proposition 7, the changes have localized effects. Changes to the demand for two- and for five-year bonds have their maximum effect at the eight-year yield; changes for ten-year bonds have maximum effect at the ten-year yield; for fifteen-year bonds at the twenty-year yield; and for twenty-year bonds at the thirty-year yield. In their study of U.S. yields during Quantitative Easing (QE) purchases,
D’Amico and King (2013) find near-full localization, in the sense that the purchases have their maximum effect on yields at or close to the maturities where they occur. Our model does not fully match that evidence but comes close.

In the top-right panel, demand changes are assumed to revert deterministically to zero at the rate \( \kappa_\theta = 0.25 \), corresponding to a half-life of \( 2.77 = \log(2)/0.25 \) years. For example, if the demand changes correspond to QE purchases, mean reversion corresponds to the QE’s unwinding. The analysis in Sections 3.5 and 4.4 can be generalized to mean-reverting demand changes, with the corresponding formulas derived in Appendix C. Mean-reversion renders the effects more localized on average. Changes to the demand for two- and for five-year bonds have their maximum effect at the six- rather than the eight-year yield; changes for ten-year bonds have maximum effect at the eight- rather than the ten-year yield; for fifteen-year bonds at the sixteen- rather than the twenty-year yield; and for twenty-year bonds at the twenty-five rather than the thirty-year yield.

We next turn to the dashed lines in Figures 1-3, which illustrate the effects of low risk aversion of arbitrageurs and low demand elasticity of preferred-habitat investors. We set the product of the respective parameters \( a \) and \( \alpha \) to 0.2, rather than to the value \( a\alpha = 6 \) that minimizes \( S \) and corresponds to the solid lines. We determine \( (a\theta, \kappa_\beta) \) by minimizing \( S \), and we keep \( (\delta_a, \delta_\theta) = (0.05, 0.25) \) because the minimization is relatively insensitive to these parameters. The minimizing values are \( (a\theta, \kappa_\beta) = (11, 0.35) \). The minimum value of \( S \) is 27.54, implying an average discrepancy per moment of 17.20% in relative terms, more than twice as large as under the original values.

To explain why small values of \( a\alpha \) do not fit the data well, take the case where \( a\alpha \) is low because arbitrageur risk aversion \( a \) is low. With low \( a \), bond risk premia are small, and demand shocks (measured by \( \theta \)) must be offsettingly large to generate the average size of the FB and CS coefficients in the data. Since the effects of demand shocks are more pronounced at longer maturities, the FB and CS coefficients become convex and concave, respectively, as function of maturity, while the opposite is true in the data. Also at odds with the data is that the volatility of yields becomes increasing with maturity for long maturities (above fourteen years).

For small values of \( a\alpha \), unanticipated demand changes have very different effects on yields than for larger values. One difference is that the effects are almost fully delocalized. For example, when demand changes are permanent, changes to the demand for ten-, for fifteen- and for twenty-year bonds have their maximum effect at the thirty-year yield, while changes for two-year bonds have maximum effect at the fifteen-year yield. A second difference is that a change in the demand for a long-maturity bond impacts the yields for all maturities more than the same change for a shorter-
A maturity bond. A third difference is that the effects are much larger when the demand changes are permanent than when they mean-revert.

To explain why larger values of $a\alpha$ generate more localization, consider a drop in demand for the twenty-year bond. Arbitrageurs respond to the change by buying the bond, and by shorting bonds of nearby maturities to hedge demand risk which is the dominant type of risk for long maturities. Their positions in shorter-maturity bonds expose them to the risk that the short rate may drop, and they may hedge that risk by buying bonds of even shorter maturities. Hence, yields for short maturities rise only mildly, and may even drop as they do in the top-right panel of Figure 3. Yields for short maturities rise more, by contrast, in response to a drop in demand for the two-year bond. The hedging mechanism ceases to operate for small values of $a\alpha$. Indeed, when $a$ is low, arbitrageurs are not concerned with short-rate risk (they are concerned with demand risk because demand shocks are large, as required to fit the data). When instead $\alpha$ is low, arbitrageurs do not trade with preferred-habitat investors and hence have no positions to hedge.

In summary, demand effects exhibit substantial localization, in line with the empirical evidence, when arbitrageurs are sufficiently risk-averse and preferred-habitat investors are sufficiently price-elastic. Our model shows that the same parameter values fit best the empirical evidence on bond return predictability (FB and CS regression coefficients) and on the volatility of yields. Seemingly unrelated properties of the term structure are thus related in our model.

We end this section by calibrating $a$ and separating it from $(\alpha, \theta)$. Equations (41) and (42) show that an increase in $a$, holding $(a\alpha, a\theta)$ (and hence $A(\tau)$) constant, results in a proportional increase in the effect of $\Delta\theta_0(\tau)$ on the yield for each maturity. Hence, different values of $a$ shift the lines in Figure 3 up or down, but do so proportionately, i.e., each line is multiplied by the same scalar. In particular, the localization properties are independent of $a$, holding $(a\alpha, a\theta)$ constant, as implicitly assumed in the previous paragraphs. The $y$-values of the lines are fully comparable across the top panels and across the bottom panels, while the comparison across the two sets of panels is a comparative static with respect to $a$ holding $\alpha$ constant. (In the top panels $a = 75$ and $a\alpha = 6$, and in the bottom panels $a = 2.5$ and $a\alpha = 0.2$). To calibrate $a$, we use the overall level of the $y$-values in the lines in the top panels. We need an observable demand shock, whose effect on the level of yields is also observable. We take that shock to be QE purchases.

Williams (2014) summarizes the large number of QE studies in the U.S. as suggesting that bond purchases of $600$ billion by the Fed reduced the ten-year yield by $0.15\%-0.25\%$ (15-25 basis points). Taking U.S. GDP at that time to be $15$ trillion, the $600$ billion purchases are $4\%\\ (=600/15000)$
of GDP. Figure 3 assumes a demand change $\Delta \theta_0(\tau) = 0.2$, which is 20% of GDP when using GDP as the unit of account. Using the parameter values in the top panels of Figure 3, and the maturity distribution of QE purchases in Figure 1 of D’Amico and King (2013), we find that bond purchases of $600 billion reduce the ten-year yield by 0.25% if they are permanent and by 0.18% if they mean-revert at the rate $\kappa_\theta = 0.25$. These numbers are in the ballpark of the empirical estimates, so we calibrate $a$ with its value in the top panels of Figure 3, which is 75.

To map $a$ into a coefficient of relative risk aversion (RRA), we recall that if arbitrageurs have wealth $W$ and a VNM utility function $U$, then $a = -\frac{U''(W)}{U'(W)}$. Hence, the coefficient of RRA is $\gamma = -\frac{U''(W)W}{U'(W)} = aW$. Since we use GDP as the unit of account, $W$ is arbitrageur wealth as fraction of GDP. Suppose that we identify arbitrageurs with hedge funds, which are sophisticated investors with relatively unconstrained mandates. Hedge Fund Research reports that hedge funds controlled about $2 trillion during the QE period, which was about 13% (=2/15) of GDP at that time. This yields $\gamma = 9.75$ (=13% $\times$ 75). Estimates for CRRA in the macro-finance literature range from 0.2 to 10, so the CRRA implied by our calibration is on the high side but not outside the range. Accounting for the fact that only a subset of hedge funds engage in term-structure arbitrage would lower the CRRA, bringing it more in line with the average estimate in the literature.

The risk-aversion coefficient $a = 75$ implies a demand elasticity parameter $\alpha = 6/75 = 0.08$. To map $\alpha$ into an actual elasticity, we compute in Appendix C the increase in preferred-habitat demand across all maturities if all yields rise by 1%. When $\alpha = 0.08$, demand increases by 14.15% of GDP. By comparison, Krishnamurthy and Vissing-Jorgensen (KV 2012) find that a 1% decline in the yield spread between corporate and government bonds raises the government-bond demand of insurance companies and pension funds by 37.2% of GDP. Accounting for the fact that insurance companies and pension funds are not the only preferred-habitat investors would raise the elasticity. At the same time, a 1% rise in government bond yields would cause the corporate-government yield spread to decline by less than 1% because corporate yields would rise. Hence, the elasticity in our calibration may be in line with KV’s estimate.

6 Conclusion

We model the term structure of interest rates that results from the interaction between investors with preferences for specific maturities and risk-averse arbitrageurs. Our model formalizes the preferred-habitat view of the term structure and embeds it into a modern no-arbitrage framework.
We use our model to study three main questions: how shocks to the short rate, including monetary-policy actions by central banks, are transmitted to long rates; how bond risk premia depend on the shape of the term structure; and how changes in preferred-habitat demand, including large-scale bond purchases by central banks, affect the term structure. We provide qualitative answers as well as quantitative ones through a calibration exercise.

Our approach can be extended in a number of directions. One direction is to derive optimal debt issuance by governments or corporations when investors have preferences for specific maturities. Work along these lines includes Greenwood, Hanson, and Stein (2010), Guibaud, Nosbusch, and Vayanos (2013) and Bigio, Nuno, and Passadore (2019). Another direction is to determine how central banks should select the bonds that they purchase in their QE operations to engineer a given shift in the term structure. A final and related direction is to embed this analysis in a broader macro-economic model, in which term-structure shifts affect investment and output, and central banks target output rather than the shifts. Work along these lines includes Ray (2019), who nests our model within a New Keynesian framework.
Appendix

A Proofs

Proof of Lemma 1: The proof is in the text.

Proof of Proposition 1: Equations (21) and (22) follow from integrating the linear ODEs (19) and (20) with the initial conditions $A_r(0) = C(0) = 0$. Substituting $A_r(\tau)$ from (21) into (23), we find (25). The left-hand side of (25) is increasing in $\kappa^*_r$, is zero for $\kappa^*_r = 0$, and converges to infinity when $\kappa^*_r$ goes to infinity. The right-hand side of (25) is decreasing in $\kappa^*_r$, exceeds $\kappa_r > 0$ for $\kappa^*_r = 0$, and converges to $\kappa_r$ when $\kappa^*_r$ goes to zero. Therefore, (25) has a unique solution for $\kappa^*_r$, which is positive.

Substituting $C(\tau)$ from (22) into (24), we find

\[
\kappa^*_r \tau^* \left[ 1 + a\sigma^2_r \int_0^\tau \alpha(\tau) \left( \int_0^\tau A_r(u)du \right) A_r(\tau)d\tau \right]
\]

\[
= \kappa_r \tau + a\sigma^2_r \int_0^\tau \theta_0(\tau) A_r(\tau)d\tau + a\sigma^4_r \int_0^\tau \alpha(\tau) \left( \int_0^\tau A_r(u)^2du \right) A_r(\tau)d\tau. \tag{A.1}
\]

Since

\[
\kappa^*_r \tau^* = \kappa^*_r \tau \left[ 1 + a\sigma^2_r \int_0^\tau \alpha(\tau) \left( \int_0^\tau A_r(u)du \right) A_r(\tau)d\tau \right]
\]

\[
+ (\kappa_r - \kappa^*_r) \tau^* - \kappa^*_r a \sigma^2_r \int_0^\tau \alpha(\tau) \left( \int_0^\tau A_r(u)du \right) A_r(\tau)d\tau,
\]

and

\[
(\kappa_r - \kappa^*_r) \tau^* - \kappa^*_r a \sigma^2_r \int_0^\tau \alpha(\tau) \left( \int_0^\tau A_r(u)du \right) A_r(\tau)d\tau
\]

\[
= -\tau a \sigma^2_r \int_0^\tau \alpha(\tau) A_r(\tau)^2d\tau - \kappa^*_r a \sigma^2_r \int_0^\tau \alpha(\tau) \left( \int_0^\tau A_r(u)du \right) A_r(\tau)d\tau
\]

\[
= -\tau a \sigma^2_r \int_0^\tau \alpha(\tau) \left[ A_r(\tau) + \kappa^*_r \int_0^\tau A_r(u)du \right] A_r(\tau)d\tau
\]

\[
= -\tau a \sigma^2_r \int_0^\tau \alpha(\tau) \tau A_r(\tau)d\tau,
\]

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where the first step follows from (21) and (25), and the third step follows from integrating (19) from zero to $\tau$ and using (21) and (25), we can write (A.1) as
\[
\kappa_r \left[ 1 + a\sigma_r^2 \int_0^T \alpha(\tau) \left[ \int_0^\tau A_r(u)du \right] A_r(\tau) d\tau \right]
\]
\[
= \kappa_r \left[ 1 + a\sigma_r^2 \int_0^T \alpha(\tau) \left[ \int_0^\tau A_r(u)du \right] A_r(\tau) d\tau \right] - \tau a\sigma_r^2 \int_0^T \alpha(\tau) A_r(\tau) d\tau
\]
\[
+ a\sigma_r^2 \int_0^T \theta_0(\tau) A_r(\tau) d\tau + \frac{a\sigma_r^4}{2} \int_0^T \alpha(\tau) \left[ \int_0^\tau A_r(u)^2 du \right] A_r(\tau) d\tau.
\] (A.2)

Equations (21) and (A.2) imply (26).

**Proof of Proposition 2:** Taking expectations conditional on time $t$ in (8), we find
\[
dE_t(r_{t+\tau}) = \kappa_r (\overline{r} - E_t(r_{t+\tau})) d\tau
\]
\[
\Rightarrow E_t(r_{t+\tau}) = (1 - e^{-\kappa_r \tau}) \overline{r} + e^{-\kappa_r \tau} r_t.
\] (A.3)

Equation (A.3) implies
\[
\frac{\partial E_t(r_{t+\tau})}{\partial r_t} = e^{-\kappa_r \tau}.
\] (A.4)

Equation (27) likewise implies
\[
\frac{\partial f^{(\tau)}}{\partial r_t} = A_r'(\tau) = e^{-\kappa_r' \tau},
\] (A.5)

where the second step follows from (21).

Equation (25) implies that if $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$, then $\kappa_r' > \kappa_r$. Since the right-hand side of (25) increases in $a$, $\sigma_r^2$ and $\alpha(\tau)$, and the difference between the left-hand side and the right-hand side increases in $\kappa_r'$, $\kappa_r$ increases in $a$, $\sigma_r^2$ and $\alpha(\tau)$.

**Proof of Proposition 3:** Equations (1), (2) and (10) imply that the dependent variable in (28) is
\[
\frac{1}{\Delta \tau} \{ A_r(\tau) r_t + C(\tau) - [A_r(\tau - \Delta \tau) r_{t+\Delta \tau} + C(\tau - \Delta \tau)] - [A_r(\Delta \tau) r_t + C(\Delta \tau)] \}
\]
and the independent variable is
\[
\frac{1}{\Delta \tau} \{ A_r(\tau) r_t + C(\tau) - [A_r(\tau - \Delta \tau) r_t + C(\tau - \Delta \tau)] - [A_r(\Delta \tau) r_t + C(\Delta \tau)] \}.
\]
Therefore, the FB regression coefficient is

$$b_{FB} = \frac{\text{Cov} \{ [A_r(\tau) - A_r(\Delta \tau)]r_t - A_r(\tau - \Delta \tau)r_{t+\Delta \tau}, [A_r(\tau) - A_r(\tau - \Delta \tau) - A_r(\Delta \tau)]r_t \}}{\text{Var} \{ [A_r(\tau) - A_r(\tau - \Delta \tau) - A_r(\Delta \tau)]r_t \}}$$

$$= \frac{[A_r(\tau) - A_r(\Delta \tau)]\text{Var}(r_t) - A_r(\tau - \Delta \tau)\text{Cov}(r_{t+\Delta \tau}, r_t)}{[A_r(\tau) - A_r(\tau - \Delta \tau) - A_r(\Delta \tau)]\text{Var}(r_t)}.$$  \hspace{1cm} (A.6)

Since (A.3) implies

$$\text{Cov}(r_{t+\Delta \tau}, r_t) = \text{Var}(r_t)e^{-\kappa_r \Delta \tau},$$  \hspace{1cm} (A.7)

we can write (A.6) as

$$b_{FB} = \frac{A_r(\tau) - A_r(\tau - \Delta \tau)e^{-\kappa_r \Delta \tau} - A_r(\Delta \tau)}{A_r(\tau) - A_r(\tau - \Delta \tau) - A_r(\Delta \tau)}.$$  

Taking the limit $\Delta \tau \to 0$ and noting from (21) that $\frac{A_r(\Delta \tau)}{\Delta \tau} \to 1$, we find

$$b_{FB} \to \frac{A_r(\tau) + \kappa_r A_r(\tau) - 1}{A_r(\tau) - 1} = \frac{(\kappa_r^* - \kappa_r)A_r(\tau)}{\kappa_r^* A_r(\tau)} = \frac{\kappa_r^* - \kappa_r}{\kappa_r^*},$$  \hspace{1cm} (A.8)

where the second step follows from (19) and (25). Since $\kappa_r^* > \kappa_r$ when $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$, (A.8) implies $b_{FB} > 0$. Since $\kappa_r^*$ increases in $a$, $\sigma_r^2$ and $\alpha(\tau)$, (A.8) implies that $b_{FB}$ increases in the same variables.

Equations (1) and (10) imply that the dependent variable in (29) is

$$\frac{A_r(\tau - \Delta \tau)r_{t+\Delta \tau} + C(\tau - \Delta \tau) - A_r(\tau)\tau + C(\tau)}{\Delta \tau}$$

and the independent variable is

$$\frac{\Delta \tau}{\tau - \Delta \tau} \left[ \frac{A_r(\tau)r_t + C(\tau)}{\tau} - \frac{A_r(\Delta \tau)r_t + C(\Delta \tau)}{\Delta \tau} \right].$$

Therefore, the CS regression coefficient is

$$b_{CS} = \frac{\text{Cov} \{ \frac{A_r(\tau - \Delta \tau)}{\tau - \Delta \tau} r_{t+\Delta \tau} - \frac{A_r(\tau)}{\tau} r_t, \frac{\Delta \tau}{\tau - \Delta \tau} \left[ \frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta \tau)}{\Delta \tau} \right] r_t \}}{\text{Var} \left\{ \frac{\Delta \tau}{\tau - \Delta \tau} \left[ \frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta \tau)}{\Delta \tau} \right] r_t \right\}}$$

$$= \frac{\frac{A_r(\tau - \Delta \tau)}{\tau - \Delta \tau} \text{Cov}(r_{t+\Delta \tau}, r_t) - \frac{A_r(\tau)}{\tau} \text{Var}(r_t)}{\frac{\Delta \tau}{\tau - \Delta \tau} \left[ \frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta \tau)}{\Delta \tau} \right] \text{Var}(r_t)}.$$  \hspace{1cm} (A.9)
Using (A.7), we can write (A.9) as

\[
b_{CS} = \frac{A_r(\tau - \Delta \tau) e^{-\kappa_r \Delta \tau} - A_r(\tau)}{\frac{\Delta \tau}{\tau - \Delta \tau} \left[ \frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta \tau)}{\Delta \tau} \right]}.
\]

Taking the limit \( \Delta \tau \to 0 \), we find

\[
b_{CS} \to \frac{A_r(\tau)}{\tau} - \frac{[A_r'(\tau) + \kappa_r A_r(\tau)]}{\frac{A_r(\tau)}{\tau} - 1} = 1 - \frac{A_r'(\tau) + \kappa_r A_r(\tau) - 1}{\frac{A_r(\tau)}{\tau} - 1} = 1 - \frac{(\kappa^*_r - \kappa_r) A_r(\tau) \tau}{\tau - A_r(\tau)},
\]

(A.10) where the third step follows from (19) and (25). Since \( \kappa^*_r > \kappa_r \) when \( a > 0 \) and \( \alpha(\tau) > 0 \) in a positive-measure subset of \((0, T)\), (A.10) implies \( b_{CS} < 1 \). Since

\[
\frac{A_r(\tau) \tau}{\tau - A_r(\tau)} = \frac{1 - e^{-\kappa^*_r \tau}}{\kappa^*_r \left( 1 - \frac{1 - e^{-\kappa^*_r \tau}}{\kappa^*_r \tau} \right)},
\]

(A.10) implies that \( b_{CS} \) increases in \( \tau \) if the function

\[
K(x) \equiv \frac{1 - \frac{1 - e^{-x}}{x}}{1 - e^{-x}} = \frac{1}{1 - e^{-x}} - \frac{1}{x}
\]

is increasing for \( x > 0 \). The derivative \( K'(x) \) has the same sign as the function

\[
\hat{K}(x) \equiv 1 - e^{-x} - xe^{-\frac{x}{2}}.
\]

The function \( \hat{K}(x) \) is equal to zero for \( x = 0 \), and its derivative \( \hat{K}'(x) \) has the same sign as \( e^{-\frac{x}{2}} - 1 + \frac{x}{2} \) which is positive for all \( x \). Therefore, \( \hat{K}(x) > 0 \) for \( x > 0 \), and \( K(x) \) is increasing.  

**Proof of Proposition 4:** The argument in the text shows that \( \Delta y_t^{(\tau)} = \kappa^*_r \Delta \mathbf{r}^t \int_0^T A_r(u) du \) and \( \Delta \mathbf{r}^t \) has the same sign as \( a \sigma^2 \int_0^T \Delta \theta_0(\tau) A_r(\tau) d\tau \). Hence, when \( a > 0 \), the change \( \Delta \theta_0(\tau) \) raises all yields if \( \int_0^T \Delta \theta_0(\tau) A_r(\tau) d\tau > 0 \) and lowers them otherwise. The relative effect across maturities is

\[
\frac{\Delta y_t^{(\tau_2)}}{\Delta y_t^{(\tau_1)}} = \frac{\int_{\tau_1}^{\tau_2} A_r(u) du}{\int_{\tau_1}^{\tau_2} A_r(u) du},
\]

and is independent of \( \Delta \theta_0(\tau) \). Since the function \( A_r(\tau) \) increases in \( \tau \), the function \( \int_{\tau_1}^{\tau_2} A_r(u) du \) also increases, and hence the relative effect across maturities is larger than one for \( \tau_1 < \tau_2 \).  

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**Proof of Lemma 2:** The proof is in the text.

**Proof of Lemma 3:** Using the diagonalization

\[ M = P^{-1} \text{Diag}(\nu_1, \nu_2, ..., \nu_{K+1}) P, \]

where \( \text{Diag}(z_1, z_2, ..., z_N) \) is the \( N \times N \) diagonal matrix with elements \((z_1, z_2, ..., z_N)\), and multiplying the ODE system (36) from the left by \( P \), we can write it as

\[ PA'(\tau) + \text{Diag}(\nu_1, \nu_2, ..., \nu_{K+1}) PA(\tau) - PE = 0. \]  

(A.11)

Integrating (A.11) with the initial condition \( A(0) = 0 \) yields

\[ PA(\tau) = \text{Diag} \left( \frac{1 - e^{-\nu_1 \tau}}{\nu_1}, \frac{1 - e^{-\nu_2 \tau}}{\nu_2}, ..., \frac{1 - e^{-\nu_{K+1} \tau}}{\nu_{K+1}} \right) PE. \]

(A.12)

Using

\[ \text{Diag} \left( \frac{1 - e^{-\nu_1 \tau}}{\nu_1}, \frac{1 - e^{-\nu_2 \tau}}{\nu_2}, ..., \frac{1 - e^{-\nu_{K+1} \tau}}{\nu_{K+1}} \right) \]

\[ = \frac{1 - e^{-\nu_1 \tau}}{\nu_1} I_{K+1} + \text{Diag} \left( 0, \frac{1 - e^{-\nu_2 \tau}}{\nu_2}, ..., \frac{1 - e^{-\nu_{K+1} \tau}}{\nu_{K+1}}, \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \right), \]

where \( I_N \) is the \( N \times N \) identity matrix, we can write (A.12) as

\[ A(\tau) = \frac{1 - e^{-\nu_1 \tau}}{\nu_1} E + P^{-1} \text{Diag} \left( 0, \frac{1 - e^{-\nu_2 \tau}}{\nu_2}, ..., \frac{1 - e^{-\nu_{K+1} \tau}}{\nu_{K+1}}, \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \right) PE \]

\[ \Rightarrow \begin{bmatrix} A_{r}(\tau) \\ A_{\beta,1}(\tau) \\ \vdots \\ A_{\beta,K}(\tau) \end{bmatrix} = \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

\[ + P^{-1} \text{Diag} \left( 0, \frac{1 - e^{-\nu_2 \tau}}{\nu_2}, ..., \frac{1 - e^{-\nu_{K+1} \tau}}{\nu_{K+1}}, \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \right) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \]

(A.13)

Equation (A.13) implies (39) and (40). Integrating (38) with the initial condition \( C(0) = 0 \) yields (41).

We next derive the system of equations in the Laplace transforms. We consider the general case where there are \( K \) demand factors. We assume \( T = \infty \), \( \alpha(\tau) = \alpha e^{-\delta \tau} \) and \( \theta_k(\tau) = \)
\[
\sum_{n=1}^{N} \theta_{k,n} e^{-\delta_{\theta n} \tau}, \text{ where } N \geq 1, (\alpha, \delta_{\alpha}, \{\theta_{k,n}\}_{k=1,..,K, n=1,..,N}, \{\delta_{\theta n}\}_{n=1,..,N}) \text{ are scalars and } (\alpha, \delta_{\alpha}, \{\delta_{\theta n}\}_{n=1,..,N}) \text{ are positive. We set }
\]

\[
I \equiv \int_{0}^{\infty} \alpha(\tau)A(\tau) d\tau,
\]

\[
J \equiv \int_{0}^{\infty} \alpha(\tau)A(\tau)A(\tau)^{\top} d\tau,
\]

For \( n = 1, \ldots, N \), we set

\[
I_{n} \equiv \int_{0}^{\infty} e^{-\delta_{\theta n} \tau} A_{r}(\tau) d\tau,
\]

and denote by \( \Theta_{n} \) the \( 1 \times (K + 1) \) vector \((0, \theta_{1,n}, \ldots, \theta_{K,n})\). Since the vectors \((I, I_{1}, \ldots, I_{N})\) are \((K + 1) \times 1\), and since the matrix \( J \) is \((K + 1) \times (K + 1)\) and symmetric, there are a total of

\[
K + 1 + \frac{(K + 1)(K + 2)}{2} + (K + 1)N = (K + 1) \left( \frac{K}{2} + N + 2 \right)
\]

distinct elements. These elements are Laplace transforms of the functions \((A_{r}(\tau), \{A_{\beta,k}(\tau)\}_{k=1,..,K})\)

and of those functions’ pairwise products. Using \((J, \{I_{n}\}_{n=1,..,N}, \{\Theta_{n}\}_{n=1,..,N})\), we can write the matrix \( M \) defined in (37) as

\[
M \equiv \Gamma^{\top} - a \int_{0}^{T} \left( \sum_{n=1}^{N} \Theta_{n}^{\top} I_{n}^{\top} - J \right) \Sigma \Sigma^{\top}.
\] (A.14)

**Lemma A.1.** Suppose that \( T = \infty \), \( \alpha(\tau) = \alpha e^{-\delta_{\alpha} \tau} \) and \( \theta_{k}(\tau) = \sum_{j=1}^{N} \theta_{k,j} e^{-\delta_{\theta j} \tau} \), where \( N \geq 1 \), \((\alpha, \delta_{\alpha}, \{\theta_{k,n}\}_{k=1,..,K, n=1,..,N}, \{\delta_{\theta n}\}_{n=1,..,N}) \text{ are scalars and } (\alpha, \delta_{\alpha}, \{\delta_{\theta n}\}_{n=1,..,N}) \text{ are positive. The } (K + 1) \left( \frac{K}{2} + N + 2 \right) \text{ elements of } (I, J, \{I_{n}\}_{n=1,..,N}) \text{ solve the system of }
\]

\[
(\delta_{\alpha} I_{K+1} + M) I = \frac{\alpha}{\delta_{\alpha}} E,
\] (A.15)

\[
(\delta_{\theta n} I_{K+1} + M) I_{n} = \frac{1}{\delta_{\theta n}} E,
\] (A.16)

for \( n = 1, \ldots, N \), and

\[
(\delta_{\alpha} I_{K+1} + M) J + JM^{\top} = EI^{\top} + IE^{\top}.
\] (A.17)
Proof of Lemma A.1: To derive (A.15), we multiply the ODE system (36) by \( \alpha(\tau) \) and integrate from zero to infinity. This yields

\[
\int_0^\infty \alpha(\tau)A'(\tau)d\tau + MI - \left[ \int_0^\infty \alpha(\tau)d\tau \right] E = 0. \tag{A.18}
\]

Integration by parts implies

\[
\int_0^\infty \alpha(\tau)A'(\tau)d\tau = \left[ \alpha(\tau)A(\tau) \right]_0^\infty - \int_0^\infty \alpha'(\tau)A(\tau)d\tau
\]

\[
= \lim_{\tau \to \infty} \alpha(\tau)A(\tau) - \alpha(0)A(0) + \delta_a \int_0^\infty \alpha(\tau)A(\tau)d\tau
\]

\[
= \lim_{\tau \to \infty} \alpha(\tau)A(\tau) + \delta_a \int_0^\infty \alpha(\tau)A(\tau)d\tau,
\]

where the second step follows from \( \alpha'(\tau) = -\delta_a \alpha(\tau) \) and the third step from \( A(0) = 0 \). Assuming \( \lim_{\tau \to \infty} \alpha(\tau)A(\tau) = 0 \), a property that is required for the matrix \( M \) to be finite (and that holds for the solution in Theorem 1, as we show at the end of that theorem’s proof), we find

\[
\int_0^\infty \alpha(\tau)A'(\tau)d\tau = \delta_a \int_0^\infty \alpha(\tau)A(\tau)d\tau = \delta_a I. \tag{A.19}
\]

Using (A.18), (A.19) and \( a(\tau) = \alpha e^{-\delta_a \tau} \), we find (A.15).

To derive (A.16), we likewise multiply the ODE system (36) by \( e^{-\delta_{\theta_n} \tau} \) and integrate from zero to infinity. This yields

\[
\int_0^\infty e^{-\delta_{\theta_n} \tau}A'(\tau)d\tau + MI_n - \left[ \int_0^\infty e^{-\delta_{\theta_n} \tau}d\tau \right] E = 0. \tag{A.20}
\]

Integration by parts and a zero limit at infinity imply

\[
\int_0^\infty e^{-\delta_{\theta_n} \tau}A'(\tau)d\tau = \delta_{\theta_n} \int_0^\infty e^{-\delta_{\theta_n} \tau}A(\tau)d\tau = \delta_{\theta_n} I_n. \tag{A.21}
\]

Using (A.20) and (A.21), we find (A.16).

To derive (A.17), we multiply the ODE system (36) from the left by \( \alpha(\tau)A(\tau)^\top \), add to the resulting \((K + 1) \times (K + 1)\) matrix its transpose, and integrate from zero to infinity. This yields

\[
\int_0^\infty \alpha(\tau) \left[ A'(\tau)A(\tau)^\top + A(\tau)A'(\tau)^\top \right] d\tau + MJ + JM^\top - EI^\top - IE^\top = 0. \tag{A.22}
\]
Integration by parts and a zero limit at infinity imply
\[
\int_0^\infty \alpha(\tau) \left[ A'(\tau)A(\tau)^\top + A(\tau)A'(\tau)^\top \right] d\tau = \delta_\alpha \int_0^\infty \alpha(\tau)A(\tau)A(\tau)^\top d\tau = \delta_\alpha J. \quad (A.23)
\]

Using (A.22) and (A.23), we find (A.17).

The total number of equations is \((K+1)(\frac{3K}{2} + N + 2)\), same as the number of unknown Laplace transforms: the vector equation (A.15) yields \(K + 1\) scalar equations, the vector equations (A.16) for \(n = 1, \ldots, N\) yield \((K + 1)N\) scalar equations, and the matrix equation (A.17) yields \(\frac{(K+1)(K+2)}{2}\) scalar equations because the matrices in it are symmetric.

**Proof of Theorem 1:** The theorem specializes Lemma A.1 to the case \(K = 1\), \(N = 2\), \(\theta_{11} = -\theta_{12} = \theta\), \(\delta_{\theta_1} = \delta_\alpha\), \(\delta_{\theta_2} = \delta_\beta\), \(\Gamma = \text{Diag}(\kappa_r, \kappa_\beta)\) and \(\Sigma = \text{Diag}(\sigma_r^2, \sigma_\beta^2)\). Since \(K = 1\) and \(N = 2\), there are nine unknown Laplace transforms, which reduce to seven because \(\delta_{\theta_1} = \delta_\alpha\) implies \(I_1 = \frac{I}{\alpha}\). Setting \(I \equiv (I_r, I_\beta)^\top\), \(I_2 \equiv (I_{r,2}, I_{\beta,2})^\top\) and

\[
J \equiv \begin{bmatrix} I_r & I_{r,\beta} \\ I_{r,\beta} & I_{\beta,\beta} \end{bmatrix},
\]

the seven unknown Laplace transforms are \((I_r, I_\beta, I_{r,2}, I_{\beta,2}, I_{r,\beta}, I_{r,\beta,2}, I_{\beta,\beta})\). Setting

\[
\Delta I_{r,\theta} \equiv \theta \left( \frac{I_r}{\alpha} - I_{r,2} \right) - I_{r,\beta}, \quad (A.24)
\]
\[
\Delta I_{\beta,\theta} \equiv \theta \left( \frac{I_\beta}{\alpha} - I_{\beta,2} \right) - I_{\beta,\beta}, \quad (A.25)
\]

we can write the matrix \(M\) given by (A.14) as

\[
\begin{bmatrix}
\kappa_r + a\sigma_r^2I_{r,r} & a\sigma_r^2I_{r,\beta} \\
-a\sigma_r^2\Delta I_{r,\theta} & \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}
\end{bmatrix}.
\]

(A.26)

The vector equation (A.15) yields the two scalar equations

\[
(\delta_\alpha + \kappa_r + a\sigma_r^2I_{r,r}) I_r + a\sigma_r^2I_{r,\beta}I_\beta = \frac{\alpha}{\delta_\alpha}, \quad (A.27)
\]
\[
-a\sigma_r^2\Delta I_{r,\theta}I_r + (\delta_\alpha + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}) I_\beta = 0.
\]

(A.28)
The vector equation (A.16) yields the two scalar equations

\[
\begin{align*}
\left( \delta_\theta + \kappa_r + a\sigma_r^2 I_{r,r} \right) I_{r,2} + a\sigma_r^2 I_{r,\beta} I_{\beta,2} &= \frac{1}{\delta_\theta}, \\
-a\sigma_r^2 \Delta I_{r,\theta} I_{r,2} + \left( \delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right) I_{\beta,2} &= 0.
\end{align*}
\] (A.29)

The matrix equation (A.17) yields the three scalar equations

\[
\begin{align*}
\frac{\delta_\alpha}{2} + \kappa_r + a\sigma_r^2 I_{r,r} I_{r,r} + a\sigma_r^2 I_{r,\beta} I_{\beta,\beta} &= I_r, \\
\left( \delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r} - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right) I_{r,\beta} + a\sigma_r^2 I_{r,\beta} I_{\beta,\beta} - a\sigma_r^2 \Delta I_{r,\theta} I_{r,r} &= I_\beta, \\
-a\sigma_r^2 \Delta I_{r,\theta} I_{r,\beta} + \left( \frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right) I_{\beta,\beta} &= 0.
\end{align*}
\] (A.30)

Equations (A.27)-(A.32) constitute a system of seven equations in the seven unknowns \((I_r, I_\beta, I_{r,r}, I_{r,\beta}, I_{\beta,\beta}, I_{r,2}, I_{\beta,2}).\) We next reduce this system into one of four equations in the four unknowns \((I_{r,r}, I_{r,\beta}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}).\)

The system of (A.27) and (A.28) is linear in \((I_r, I_\beta)\) and its solution is

\[
\begin{align*}
I_r = \frac{\frac{\delta_\alpha}{a\sigma_r^2 I_{r,r}} \left( \delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right)}{\left( \delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r} \right) \left( \delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right) + a^2\sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}, \\
I_\beta = \frac{\frac{\delta_\alpha}{a\sigma_r^2 I_{r,r}} \left( \delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r} \right) \left( \delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right) + a^2\sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\left( \delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r} \right) \left( \delta_\alpha + \kappa_r + a\sigma_r^2 I_{r,r} - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right) + a^2\sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}.
\end{align*}
\] (A.34)

Likewise, the system of (A.29) and (A.30) is linear in \((I_{r,2}, I_{\beta,2})\) and its solution is

\[
\begin{align*}
I_{r,2} = \frac{\frac{1}{\delta_\theta} \left( \delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right)}{\left( \delta_\theta + \kappa_r + a\sigma_r^2 I_{r,r} \right) \left( \delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right) + a^2\sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}, \\
I_{\beta,2} = \frac{\frac{1}{\delta_\theta} a\sigma_r^2 \Delta I_{r,\theta}}{\left( \delta_\theta + \kappa_r + a\sigma_r^2 I_{r,r} \right) \left( \delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta} \right) + a^2\sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}.
\end{align*}
\] (A.36)

Equation (A.33) is linear in \(I_{\beta,\beta}\) and its solution is

\[
I_{\beta,\beta} = \frac{a\sigma_r^2 I_{r,\beta} \Delta I_{r,\theta}}{\frac{\delta_\alpha}{2} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}}.
\] (A.38)
Substituting $I_r$ from (A.34), we can write (A.31) as
\[
\left( \frac{\delta}{2} + \kappa_r + a\sigma^2_r I_{r,r} \right) I_{r,r} + a\sigma^2_r I_{r,\beta}
= - \frac{\alpha}{\delta} \left( \delta + \kappa - a\sigma^2_\beta \Delta I_{\beta,\theta} \right)
\]
\[
\frac{\delta + \kappa + a\sigma^2_\beta I_{r,r}}{(\delta + \kappa + a\sigma^2_\beta I_{r,r}) \left( \delta + \kappa - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) + a^2\sigma^2_r \sigma^2_\beta I_{r,\beta} \Delta I_{r,\theta}} = 0. \tag{A.39}
\]

Substituting $(I_r, I_{r,2})$ from (A.34) and (A.36), respectively, into the definition (A.24) of $\Delta I_{r,\theta}$, we find
\[
\Delta I_{r,\theta} = \frac{\theta}{\delta_\alpha} \left( \delta + \kappa + a\sigma^2_\beta I_{r,r} \right) \left( \delta + \kappa - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) + a^2\sigma^2_r \sigma^2_\beta I_{r,\beta} \Delta I_{r,\theta}
+ \frac{\theta}{\delta_\theta} \left( \delta + \kappa + a\sigma^2_\beta I_{r,r} \right) \left( \delta + \kappa - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) + a^2\sigma^2_r \sigma^2_\beta I_{r,\beta} \Delta I_{r,\theta} = 0. \tag{A.40}
\]

Substituting $(I_\beta, I_{\beta,2}, I_{\beta,\beta})$ from (A.34), (A.36) and (A.38), respectively, into the definition (A.25) of $\Delta I_{\beta,\theta}$, we find
\[
\Delta I_{\beta,\theta} = \frac{\theta}{\delta_\alpha} a\sigma^2_\beta \Delta I_{r,\theta}
\frac{\theta}{\delta_\beta} a\sigma^2_\beta \Delta I_{r,\theta}
\]
\[
\frac{\theta}{\delta_\alpha} a\sigma^2_\beta \Delta I_{r,\theta}
\frac{\theta}{\delta_\beta} a\sigma^2_\beta \Delta I_{r,\theta}
\]
\[
\left( \delta + \kappa + a\sigma^2_\beta I_{r,r,\theta} - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) a\sigma^2_\beta \theta + a\sigma^2_\beta \right) \Delta I_{r,\theta} = 0. \tag{A.41}
\]

Substituting $(I_\beta, I_{\beta,\beta})$ from (A.34) and (A.38), respectively, we can write (A.31) as
\[
\left( \delta + \kappa + a\sigma^2_\beta I_{r,r,\theta} - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) a\sigma^2_\beta \theta + a\sigma^2_\beta \right) \Delta I_{r,\theta} = 0. \tag{A.42}
\]

Equations (A.39)-(A.42) form the system of four equations in the four unknowns $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta})$.

Given a solution to that system, we can determine $(I_r, I_\beta, I_{r,2}, I_{\beta,2}, I_{\beta,\beta})$ from (A.34)-(A.38).

To show that the system (A.39)-(A.42) has a solution, we proceed in two steps. In Step 1 we take $I_{r,\beta} > 0$ as given, and construct $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$ and $\Delta I_{\beta,\theta} < \frac{\delta + \kappa + a\sigma^2_\beta \Delta I_{r,\theta}}{2a\sigma^2_\beta}$ uniquely from
Step 1: We first take \( \Delta I_{r,\theta} = (A.42) \) has a solution \( I_{r,\beta} > 0 \). We denote the left-hand sides of (A.39), (A.40), (A.41) and (A.42) by \( L_{r,r}, L_{r,\theta}, L_{\beta,\theta} \) and \( L_{r,\beta} \), respectively, and set

\[
D_j \equiv (\delta_j + \kappa_r + a\sigma_r^2 I_{r,r}) (\delta_j + \kappa_r - a\sigma_r^2 \Delta I_{\beta,\theta}) + a^2 \sigma_r^2 \sigma_{\beta}^2 I_{r,\beta} \Delta I_{r,\theta}
\]

for \( j = \alpha, \theta \). For \( I_{r,r} \geq 0, \Delta I_{r,\theta} \geq 0, \Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta} \) and \( I_{r,\beta} > 0 \), \( D_\theta > D_\alpha > 0 \), and hence \( (L_{r,r}, L_{r,\theta}, L_{\beta,\theta}, L_{r,\beta}) \) are continuous functions of \( (I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta}) \).

We next take \( \Delta I_{r,\theta} \geq 0, \Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta} \) and \( I_{r,\beta} > 0 \) as given, and construct \( I_{r,r} > 0 \) from (A.39). Equation (A.39) implies

\[
\frac{\partial L_{r,r}}{\partial I_{r,r}} = \frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma_r^2 I_{r,r} + \frac{\alpha_\alpha}{\delta_\alpha} \left( \frac{\delta_\alpha + \kappa_r - a\sigma_r^2 \Delta I_{\beta,\theta}}{D_\alpha^2} \right)^2 a\sigma_r^2, \tag{A.43}
\]

which in turn implies \( \frac{\partial L_{r,r}}{\partial I_{r,r}} > 0 \) for \( I_{r,r} \geq 0 \). Hence, if \( L_{r,r} < 0 \) for \( I_{r,r} = 0 \), and \( L_{r,r} > 0 \) for \( I_{r,r} \) large enough, then (A.39) has a unique positive solution for \( I_{r,r} \). Equation (A.39) implies that \( L_{r,r} \) converges to infinity when \( I_{r,r} \) goes to infinity. We assume that \( (\Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta}) \) are such that \( L_{r,r} < 0 \) for \( I_{r,r} = 0 \), and return to this issue in Step 2.

We next take \( \Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma_\beta} \) and \( I_{r,\beta} > 0 \) as given, treat \( I_{r,r} > 0 \) as an implicit function of \( (\Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta}) \), and construct \( \Delta I_{r,\theta} > 0 \) from (A.40). Equation (A.40) implies that the partial derivative of \( L_{r,\theta} \) with respect to \( \Delta I_{r,\theta} \) when the variation of \( I_{r,r} \) is taken into account is

\[
\hat{L}_{r,\theta} \equiv \frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}}.
\]

We show that if \( L_{r,\theta} = 0 \) for a value \( \Delta I_{r,\theta} > 0 \), then \( \hat{L}_{r,\theta} > 0 \) for the same value. Equation (A.40) implies

\[
\frac{\partial L_{r,\theta}}{\partial I_{r,r}} = \left( \frac{\alpha_\alpha}{\delta_\alpha} \left( \frac{\delta_\alpha + \kappa_r - a\sigma_r^2 \Delta I_{\beta,\theta}}{D_\alpha^2} \right)^2 - \frac{\alpha_\theta}{\delta_\theta} \left( \frac{\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}}{D_\theta^2} \right)^2 \right) a\sigma_r^2, \tag{A.44}
\]

\[
\frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} = 1 + \left( \frac{\alpha_\alpha}{\delta_\alpha} \left( \frac{\delta_\alpha + \kappa_r - a\sigma_r^2 \Delta I_{\beta,\theta}}{D_\alpha^2} \right) - \frac{\alpha_\theta}{\delta_\theta} \left( \frac{\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}}{D_\theta^2} \right) \right) a^2 \sigma_r^2 \sigma_\beta^2 I_{r,\beta}. \tag{A.45}
\]
Equation (A.39) implies
\[
\frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} = \frac{\alpha}{\sigma} \left( \delta_\alpha + \kappa_\beta - a_\sigma^2 \delta I_{r,\theta} \right) \frac{\partial^2 \sigma_\tau^2 \sigma_\beta^2}{D_\alpha^2}.
\]
(A.46)

Since \( \Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a_\sigma^2} \) and \( I_{r,\theta} > 0 \), (A.46) implies \( \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} > 0 \) and hence
\[
\frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} = -\frac{\partial I_{r,\theta}}{\partial I_{r,\theta}} < 0.
\]
(A.47)

Combining (A.44) and (A.45) with
\[
\frac{\partial}{\partial I_{r,\theta}} \left( \frac{\phi}{\delta_\alpha} \left( \delta_j + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta} \right)^2 \right) \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} + \frac{\partial}{\partial I_{r,\theta}} \left( \frac{\phi}{\delta_\theta} \left( \delta_j + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta} \right)^2 \right) \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} + \frac{\partial}{\partial I_{r,\theta}} \left( \frac{\phi}{\delta_\beta} \left( \delta_j + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta} \right)^2 \right) \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} = 0,
\]
for \( j = \alpha, \theta, \beta, \)
\[
(\delta_\alpha + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta}) a_\sigma^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} + a_\sigma^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} > (\delta_\theta + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta}) a_\sigma^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} + (\delta_\beta + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta}) a_\sigma^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}},
\]
which follows from \( \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} < 0 \), \( \Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a_\sigma^2} \) and \( \delta_\theta > \delta_\alpha \),
\[
(\delta_\alpha + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta}) a_\sigma^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} + a_\sigma^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} = 0,
\]
\[
\frac{\alpha}{\sigma} \left( \delta_\alpha + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta} \right)^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} + \frac{\alpha}{\sigma} \left( \delta_\theta + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta} \right)^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} + \frac{\alpha}{\sigma} \left( \delta_\beta + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta} \right)^2 \frac{\partial I_{r,\theta}}{\partial \Delta I_{r,\theta}} = a_\sigma^2 \sigma_\beta^2 \sigma_\tau^2 \frac{\delta_\alpha + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta}}{D_\alpha^2} > 0,
\]
which follows from (A.43), (A.46) and (A.47), \( D_\theta > D_\alpha > 0 \), and
\[
\frac{\alpha}{\sigma} \left( \delta_\alpha + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta} \right) - \frac{\alpha}{\sigma} \left( \delta_\theta + \kappa_\beta - a_\sigma^2 \Delta I_{\beta,\theta} \right) = \Delta I_{r,\theta} + I_{r,\theta} > 0,
\]

which follows from $L_{r,\theta} = 0$ (i.e., (A.40)), we find

$$
\hat{L}_{r,\theta} = \frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{r,\theta}} + \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} > 1 > 0.
$$

(A.48)

Since $\hat{L}_{r,\theta} > 0$ at any point where $L_{r,\theta} = 0$, $L_{r,\theta}$ can be equal to zero only once. Hence, if $L_{r,\theta} < 0$ for $\Delta I_{r,\theta} = 0$, and $L_{r,\theta} > 0$ for $\Delta I_{r,\theta} = \overline{\Delta I}_{r,\theta}$ sufficiently large, and if all values of $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$ yield $I_{r,r} > 0$, then (A.40) yields a unique solution for $\Delta I_{r,\theta} \in (0, \overline{\Delta I}_{r,\theta})$. We assume that $(\Delta I_{\beta,\theta}, I_{r,\beta})$ are such that these conditions hold, and return to this issue in Step 2.

We finally take $I_{r,\beta} > 0$ as given, treat $I_{r,r} > 0$ and $\Delta I_{r,\theta} > 0$ as implicit functions of $(\Delta I_{\beta,\theta}, I_{r,\beta})$, and construct $\Delta I_{\beta,\theta} < \frac{\delta_{r,\beta} + \kappa_{\beta}}{2\sigma_{\beta}^2}$ from (A.41). Equation (A.41) implies that the partial derivative of $L_{\beta,\theta}$ with respect to $\Delta I_{\beta,\theta}$ when the variation of $(I_{r,r}, \Delta I_{r,\theta})$ is taken into account is

$$
\hat{L}_{\beta,\theta} = \frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} \frac{\partial \Delta I_{r,\theta}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{\beta,\theta}}.
$$

(A.49)

We show that if $L_{\beta,\theta} = 0$ for a value $\Delta I_{\beta,\theta} < \frac{\delta_{r,\beta} + \kappa_{\beta}}{2\sigma_{\beta}^2}$, then $\hat{L}_{\beta,\theta} > 0$ for the same value. Differentiating (A.39) and (A.40) at the values of $(I_{r,r}, \Delta I_{r,\theta})$ that render $(L_{r,r}, L_{r,\theta})$ equal to zero, we find

$$
\frac{\partial L_{r,r}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} \frac{\partial \Delta I_{r,\theta}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{r,r}}{\partial \Delta I_{\beta,\theta}} = 0,
$$

(A.50)

$$
\frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial I_{r,r}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} \frac{\partial \Delta I_{r,\theta}}{\partial \Delta I_{\beta,\theta}} + \frac{\partial L_{r,\theta}}{\partial \Delta I_{\beta,\theta}} = 0,
$$

(A.51)

respectively. Equations (A.50) and (A.51) form a linear system in the unknowns $(\frac{\partial I_{r,r}}{\partial \Delta I_{\beta,\theta}}, \frac{\partial \Delta I_{r,\theta}}{\partial \Delta I_{\beta,\theta}})$. The determinant of that system is

$$
\frac{\partial L_{r,r}}{\partial I_{r,r}} \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} - \frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} = \frac{\partial L_{r,r}}{\partial I_{r,r}} \left( \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} - \frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} \right)
$$

$$
= \frac{\partial L_{r,r}}{\partial I_{r,r}} \left( \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} + \frac{\partial L_{r,\theta}}{\partial I_{r,r}} \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} \right)
$$

$$
= \frac{\partial L_{r,r}}{\partial I_{r,r}} L_{r,\theta},
$$

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and is positive because $\frac{\partial L_{r,r}}{\partial I_{r,r}} > 0$ and $\hat{L}_{r,\theta} > 0$. Substituting the solution of the system (A.50)-(A.51) into (A.49), we find that (A.49) has the same sign as the Jacobian determinant

$$\begin{vmatrix}
\frac{\partial L_{r,r}}{\partial I_{r,r}} & \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} & \frac{\partial L_{r,\theta}}{\partial I_{r,r}} & \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} \\
\frac{\partial L_{r,\theta}}{\partial I_{r,r}} & \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} & \frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} & \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} \\
\frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} & \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} & \frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} & \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} \\
\frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} & \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} & \frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} & \frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} \\
\end{vmatrix}. \quad (A.52)$$

The partial derivatives $(\frac{\partial L_{r,r}}{\partial I_{r,r}}, \frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}}, \frac{\partial L_{r,\theta}}{\partial I_{r,r}}, \frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}})$ are given by (A.43), (A.46), (A.44) and (A.45), respectively. Equations (A.39), (A.40) and (A.41) imply that the remaining partial derivatives are

$$\frac{\partial L_{r,r}}{\partial \Delta I_{r,\theta}} = \frac{\sigma_r^2 \alpha^2 \sigma_{r,\beta} I_{r,\beta} \Delta I_{r,\theta}}{\alpha D_\alpha \alpha^2 \sigma_r^2 \alpha^2 \sigma_{r,\beta} I_{r,\beta} \Delta I_{r,\theta}}, \quad (A.53)$$

$$\frac{\partial L_{r,\theta}}{\partial \Delta I_{r,\theta}} = \left(\frac{\sigma_r^2 \alpha^2 \sigma_{r,\beta} I_{r,\beta} \Delta I_{r,\theta}}{\alpha D_\alpha \alpha^2 \sigma_r^2 \alpha^2 \sigma_{r,\beta} I_{r,\beta} \Delta I_{r,\theta}} \right) \cdot (A.54)$$

$$\frac{\partial L_{\beta,\theta}}{\partial I_{r,r}} = \begin{pmatrix} \frac{\sigma_r^2 \alpha^2 \sigma_{r,\beta} I_{r,\beta} \Delta I_{r,\theta}}{\alpha D_\alpha \alpha^2 \sigma_r^2 \alpha^2 \sigma_{r,\beta} I_{r,\beta} \Delta I_{r,\theta}} \end{pmatrix} \cdot (A.55)$$

$$\frac{\partial L_{\beta,\theta}}{\partial \Delta I_{r,\theta}} = 1 \quad (A.56)$$

The sign of the Jacobian determinant (A.52) does not change if we multiply the last row by $(\delta_\alpha + \kappa_\beta - a \sigma_r^2 \Delta I_{r,\theta})$. The resulting determinant does not change if we subtract the middle row times $a \sigma_r^2 \Delta I_{r,\theta}$ from the last row, and then the first row times $\frac{\sigma_r^2 \alpha^2 \sigma_{r,\beta} I_{r,\beta} \Delta I_{r,\theta}}{\alpha D_\alpha \alpha^2 \sigma_r^2 \alpha^2 \sigma_{r,\beta} I_{r,\beta} \Delta I_{r,\theta}}$ from the middle row. In the resulting determinant, the elements (1,1), (1,2) and (1,3) are given by (A.43), (A.46) and
(A.53), respectively, the element (2,2) is given by
\[
\left[ \frac{\theta}{\sigma^{2}} \left( \frac{\delta_{T} + \kappa_{\beta} - a^{2} \Delta I_{\beta,\theta}}{\theta^{2}} \right) \left( \frac{\delta_{T} + \kappa_{\beta} - a^{2} \Delta I_{\beta,\theta}}{\theta^{2}} \right) \right] a^{2} \sigma^{2}_{r}
\]
\[
- \frac{\theta}{\alpha} \left( 1 - \frac{\delta_{\alpha} D_{\alpha}^{2}}{\delta_{\theta} D_{\theta}^{2}} \right) \frac{\delta_{\alpha} + \kappa_{r} + 2 a^{2} \sigma^{2}_{r} I_{r,r} + \frac{\alpha}{\delta_{\alpha}} \left( \frac{\delta_{\alpha} + \kappa_{\beta} - a^{2} \Delta I_{\beta,\theta}}{\theta^{2}} \right) a^{2} \sigma^{2}_{r}}{\theta^{2}} = 0,
\]
the element (2,2) by
\[
1 + \left( \frac{\theta}{\sigma^{2}} \left( \frac{\delta_{T} + \kappa_{\beta} - a^{2} \Delta I_{\beta,\theta}}{\theta^{2}} \right) \left( \frac{\delta_{T} + \kappa_{\beta} - a^{2} \Delta I_{\beta,\theta}}{\theta^{2}} \right) \right) a^{2} \sigma^{2}_{r} = 0,
\]
the element (2,3) by
\[
\left[ \frac{\delta_{T}}{D_{\alpha}^{2} - \delta_{\theta} D_{\theta}^{2}} \right] a^{3} \sigma^{2}_{r} \sigma^{2}_{\beta} I_{r,\beta} \Delta I_{r,\theta} - \frac{\theta}{\alpha} \left( 1 - \frac{\delta_{\alpha} D_{\alpha}^{2}}{\delta_{\theta} D_{\theta}^{2}} \right) a^{3} \sigma^{2}_{r} \sigma^{2}_{\beta} I_{r,\beta} \Delta I_{r,\theta} = 0,
\]
the element (3,1) by
\[
\left[ \frac{\delta_{T}}{D_{\alpha}^{2}} \right] a^{2} \sigma^{2}_{r} \Delta I_{r,\theta} - \frac{\delta_{T}}{\theta} \left( \frac{\delta_{T} + \kappa_{\beta} - a^{2} \Delta I_{\beta,\theta}}{\theta^{2}} \right) a^{2} \sigma^{2}_{r} \Delta I_{r,\theta}
\]
\[
- \left( \frac{\delta_{T}}{\delta_{\alpha} D_{\alpha}^{2}} \right) \left( \frac{\delta_{T} + \kappa_{\beta} - a^{2} \Delta I_{\beta,\theta}}{\theta^{2}} \right) a^{2} \sigma^{2}_{r} \Delta I_{r,\theta} - \frac{\delta_{T}}{D_{\theta}^{2}} a^{2} \sigma^{2}_{r} \Delta I_{r,\theta}
\]
\[
= \frac{\delta_{T}}{\theta} \left( \frac{\delta_{T} + \kappa_{\beta} - a^{2} \Delta I_{\beta,\theta}}{\theta^{2}} \right) a^{2} \sigma^{2}_{r} \Delta I_{r,\theta},
\]
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the element $(3,2)$ by

\[
\begin{align*}
&\left[ -\frac{\theta}{D_\alpha} + \frac{\delta_\theta}{D_\theta} + \frac{I_{r,\beta}}{\frac{2}{\alpha} + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta}} \right] a\sigma_r^2 + \left( \frac{\theta}{D_\alpha} - \frac{\delta_\theta}{D_\theta} \right) a^2 \sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta} \\
&- \left[ 1 + \left( \frac{\theta}{D_\alpha} \left( \delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta} \right) - \frac{\delta_\theta + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta}}{D_\theta} \right) \right] a^2 \sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta} \\
&= -a\sigma_r^2 \Delta I_{r,\theta} + \frac{\theta}{D_\alpha} \left( \delta_\theta - \delta_\alpha \right) a^3 \sigma_r^4 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta} - \frac{\Delta I_{r,\theta}}{\Delta I_{r,\theta}} \left( \delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta} \right),
\end{align*}
\]

where we use $L_{\beta,\theta} = 0$ (i.e., (A.41)), and the element $(3,3)$ by

\[
\begin{align*}
&\left[ 1 - \left( \frac{\theta}{D_\alpha} \left( \delta_\alpha + \kappa_\beta + a\sigma_\beta^2 I_{r,r} \right) - \frac{\delta_\theta + \kappa_\beta + a\sigma_\beta^2 I_{r,r}}{D_\theta} \right) \right] a^2 \sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta} \\
&+ \frac{\left( \delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta} \right) a^2 \sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\left( \frac{\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta}}{D_\alpha} \right)^2} - \left( \frac{\theta}{D_\alpha} - \frac{\delta_\theta}{D_\theta} \right) a^2 \sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta} \\
&= \delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta} - \left( \frac{\theta}{D_\alpha} \left( \delta_\theta - \delta_\alpha \right) \left( \delta_\theta + \kappa_\beta + a\sigma_\beta^2 I_{r,r} \right) \right) a^2 \sigma_r^2 \sigma_\beta^2 \Delta I_{r,\theta} \\
&+ \left( \frac{\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta}}{D_\alpha} \right)^2 \frac{a^2 \sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\left( \frac{\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta}}{D_\alpha} \right)^2} \\
&= \delta_\alpha + \kappa_\beta - 2a\sigma_\beta^2 \Delta I_{r,\theta} - \frac{\theta}{D_\alpha} \left( \delta_\theta - \delta_\alpha \right) \left( \delta_\theta + \kappa_\beta + a\sigma_\beta^2 I_{r,r} \right) a^2 \sigma_r^2 \sigma_\beta^2 \Delta I_{r,\theta} + \frac{\delta_\theta}{D_\alpha} \left( \frac{a^2 \sigma_r^2 \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\left( \frac{\delta_\alpha + \kappa_\beta - a\sigma_\beta^2 \Delta I_{r,\theta}}{D_\alpha} \right)^2} \right),
\end{align*}
\]

where the last step follows from $L_{\beta,\theta} = 0$.

For large $\delta_\theta$, all the terms with $D_\theta$ in the denominator are close to zero, and the determinant
obtained by multiplying (A.52) by \( \left( \delta_\alpha + \kappa_\beta - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) \) becomes

\[
\begin{vmatrix}
\frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma^2_\alpha I_{r,r} & \frac{\alpha}{D_\alpha} a^2\sigma^2_\alpha \sigma^2_\beta \Delta I_{\beta,\theta} & \frac{\alpha}{D_\alpha} a^3\sigma^2_\alpha \sigma^2_\beta \sigma^4_\beta \Delta I_{r,\theta} \\
\frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma^2_\alpha I_{r,r} & 1 & 0 \\
0 & -a\sigma^2_\alpha \Delta I_{r,\theta} - \frac{\Delta I_{\beta,\theta}}{\Delta I_{r,\theta}} \left( \delta_\alpha + \kappa_\beta - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) & \frac{\delta_\alpha + \kappa_\beta - 2a\sigma^2_\beta \Delta I_{\beta,\theta}}{\left( \frac{\alpha}{D_\alpha} a^2\sigma^2_\alpha \sigma^2_\beta I_{r,\beta} \Delta I_{r,\theta} \right)^2} \\
\end{vmatrix}
\]

\[
= \frac{\alpha}{D_\alpha} \left( \frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma^2_\alpha I_{r,r} \right) \left( a^2\sigma^2_\alpha \Delta I_{r,\theta} + \frac{\Delta I_{\beta,\theta}}{\Delta I_{r,\theta}} \left( \delta_\alpha + \kappa_\beta - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) \right) a^3\sigma^2_\alpha \sigma^4_\beta \sigma^4_\beta \Delta I_{r,\theta} \\
+ \left( \delta_\alpha + \kappa_\beta - 2a\sigma^2_\beta \Delta I_{\beta,\theta} + \frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma^2_\alpha I_{r,r} \right) \left( 1 + \frac{\alpha}{\left( \frac{\alpha}{D_\alpha} a^2\sigma^2_\alpha \sigma^2_\beta I_{r,\beta} \Delta I_{r,\theta} \right)^2} \left( a^2\sigma^2_\alpha \sigma^2_\beta I_{r,\beta} \right) \right) \\
\]  \tag{A.58}

To show that (A.58) is positive, and hence \( \hat{L}_{\beta,\theta} > 0 \), we distinguish cases. When \( \Delta I_{\beta,\theta} < 0 \), the only negative term in (A.58) is the one generated by \( \frac{\Delta I_{\beta,\theta}}{\Delta I_{r,\theta}} \left( \delta_\alpha + \kappa_\beta - a\sigma^2_\beta \Delta I_{\beta,\theta} \right) \). We group it together with the term generated by one of the two \(-a\sigma^2_\beta \Delta I_{\beta,\theta} \) in \( \left( \delta_\alpha + \kappa_\beta - 2a\sigma^2_\beta \Delta I_{\beta,\theta} \right) \) and note that (A.58) exceeds

\[
\frac{\alpha}{D_\alpha} \left( \frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma^2_\alpha I_{r,r} \right) a^3\sigma^2_\alpha \sigma^4_\beta I_{r,\beta} \Delta I_{r,\theta} \\
+ \left( \delta_\alpha + \kappa_\beta - a\sigma^2_\beta \Delta I_{\beta,\theta} + \frac{\delta_\alpha}{2} + \kappa_r + 2a\sigma^2_\alpha I_{r,r} \right) \left( 1 + \frac{\alpha}{\left( \frac{\alpha}{D_\alpha} a^2\sigma^2_\alpha \sigma^2_\beta I_{r,\beta} \Delta I_{r,\theta} \right)^2} \left( a^2\sigma^2_\alpha \sigma^2_\beta I_{r,\beta} \right) \right),
\]

which is positive. When instead \( \Delta I_{\beta,\theta} \in \left( 0, \frac{\delta_\alpha + \kappa_\beta}{2a\sigma^2_\beta} \right) \), all the terms in (A.58), with \( \left( \delta_\alpha + \kappa_\beta - 2a\sigma^2_\beta \Delta I_{\beta,\theta} \right) \) counted as a single term, are positive. Hence, \( \hat{L}_{\beta,\theta} > 0 \) at any point \( \Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a\sigma^2_\beta} \) where \( L_{\beta,\theta} = 0, \)
which implies that $L_{\beta, \theta}$ can be equal to zero only once. Moreover, if $L_{\beta, \theta} < 0$ for $\Delta I_{\beta, \theta} = \Delta I_{\beta, \theta}$ sufficiently negative, and $L_{\beta, \theta} > 0$ for $\Delta I_{\beta, \theta} = \frac{\delta_{\alpha} + \kappa_{\beta}}{2a\sigma_{\beta}}$, and if all values of $\Delta I_{\beta, \theta} \in \left( \frac{\Delta I_{\beta, \theta}}{\Delta I_{\beta, \theta}}, \frac{\delta_{\alpha} + \kappa_{\beta}}{2a\sigma_{\beta}} \right)$ yield $I_{r,r} > 0$ and $\Delta I_{r,\theta} > 0$, then (A.40) yields a unique solution for $\Delta I_{\beta, \theta} \in \left( \frac{\Delta I_{\beta, \theta}}{\Delta I_{\beta, \theta}}, \frac{\delta_{\alpha} + \kappa_{\beta}}{2a\sigma_{\beta}} \right)$. We assume that $I_{r,\beta}$ is such that these conditions hold, and return to this issue in Step 2.

**Step 2:** Suppose that $I_{r,\beta} > 0$ satisfies

$$a\sigma_{\beta}^2 I_{r,\beta}^2 < \frac{\theta}{\delta_{\alpha} + \kappa_{r}}$$

and define $\bar{I}_{r,r} > 0$ by

$$\left( \frac{\delta_{\alpha}}{2} + \kappa_{r} + a\sigma_{r} \bar{I}_{r,r} \right) \bar{I}_{r,r} + a\sigma_{\beta}^2 I_{r,\beta}^2 - \frac{\theta}{\delta_{\alpha} + \kappa_{r} + a\sigma_{r}^2 \bar{I}_{r,r}} = 0.$$  

Equation (A.60) defines $\bar{I}_{r,r} > 0$ uniquely because the left-hand side increases for $\bar{I}_{r,r} \geq 0$, converges to infinity when $\bar{I}_{r,r}$ goes to infinity, and is negative for $\bar{I}_{r,r} = 0$ because of (A.59). Suppose that $I_{r,\beta}$ satisfies additionally

$$a\sigma_{\beta}^2 I_{r,\beta} < \frac{\theta}{\delta_{\alpha} + \kappa_{r}},$$

and is positive because of (A.59). Equation (A.39) implies that for $I_{r,\beta} = 0$,

$$a\sigma_{\beta}^2 I_{r,r}^2 - \frac{\theta}{\delta_{\alpha} + \kappa_{r}} \left( \frac{\delta_{\alpha} + \kappa_{\beta} - a\sigma_{\beta}^2 \Delta I_{\beta, \theta}}{\delta_{\alpha} + \kappa_{r}} \right) a\sigma_{r}^2 \sigma_{\beta}^2 \bar{I}_{r,r} \bar{I}_{r,\theta} = 0$$

and is positive because of (A.59). Equation (A.39) implies that for $I_{r,\beta} = 0$,

$$L_{r,r} = a\sigma_{\beta}^2 I_{r,\beta}^2 - \frac{\theta}{\delta_{\alpha} + \kappa_{r}} \left( \frac{\delta_{\alpha} + \kappa_{\beta} - a\sigma_{\beta}^2 \Delta I_{\beta, \theta}}{\delta_{\alpha} + \kappa_{r}} \right) a\sigma_{r}^2 \sigma_{\beta}^2 \bar{I}_{r,r} \bar{I}_{r,\theta} < 0.$$
where the inequality follows from $\Delta I_{r,\theta} \in (0, \Delta I_{r,\theta})$ and (A.63). Equation (A.39) and (A.60) imply that for $I_{r,r} = \bar{I}_{r,r}$,

$$L_{r,r} = \left( \frac{\delta_\alpha}{2} + \kappa_r + a\sigma_r I_{r,r} \right) I_{r,r} + a\sigma_\beta I_{r,\beta}^2$$

$$= \frac{\alpha}{\delta_\alpha} \left( \delta_\alpha + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right)$$

$$\left( \delta_\alpha + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right) + a^2\sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}$$

$$= \frac{\alpha}{\delta_\alpha} \left( \delta_\alpha + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right) > 0.$$

Hence (A.39) has a unique positive solution for $I_{r,r} \in (0, \bar{I}_{r,r})$.

Take next $\Delta I_{r,\theta} < \frac{\delta_\alpha + \kappa_r}{2a\sigma_\beta}$ and $I_{r,\beta} > 0$ as given, and treat $I_{r,r} \in (0, \bar{I}_{r,r})$ as an implicit function of $(\Delta I_{r,\theta}, \Delta I_{r,\theta}, I_{r,\beta})$. For $\Delta I_{r,\theta} = 0$, (A.39) and (A.60) imply $I_{r,r} = \bar{I}_{r,r}$, and (A.40) implies

$$L_{r,\theta} = -\frac{\theta}{\delta_\theta} \left( \delta_\theta + \kappa_r + a\sigma_\beta I_{r,r} \right) + \frac{\theta}{\delta_\theta} \left( \delta_\theta + \kappa_r + a\sigma_\beta I_{r,r} \right) + I_{r,\beta} < 0,$$

where the inequality follows from (A.61). For $\Delta I_{r,\theta} = \Delta I_{r,\theta}$, (A.39) and (A.63) imply $I_{r,r} = 0$, and (A.40) implies

$$L_{r,\theta} = \Delta I_{r,\theta} - \frac{\theta}{\delta_\theta} \left( \delta_\theta + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right)$$

$$\left( \delta_\theta + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right) + a^2\sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}$$

$$+ \frac{\theta}{\delta_\theta} \left( \delta_\theta + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right)$$

$$\left( \delta_\theta + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right) + a^2\sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}$$

$$= \Delta I_{r,\theta} - \frac{\theta}{\alpha} a\sigma_\beta I_{r,\beta}^2 + \frac{\theta}{\delta_\theta} \left( \delta_\theta + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right)$$

$$\left( \delta_\theta + \kappa_r + a\sigma_\beta \Delta I_{r,\theta} \right) + a^2\sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}$$

$$> I_{r,\beta} \left( 1 - \frac{\theta}{\alpha} a\sigma_\beta I_{r,\beta} \right) > 0,$$

(A.64)

where the second step follows from (A.63) and the fourth from (A.62). Hence, (A.40) has a unique solution for $\Delta I_{r,\theta} \in (0, \Delta I_{r,\theta})$.  

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Take finally $I_{r,\beta} > 0$ as given, and treat $I_{r,r} \in (0, \bar{I}_{r,r})$ and $\Delta I_{r,\theta} \in (0, \bar{\Delta} I_{r,\theta})$ as implicit functions of $(\Delta I_{\beta,\theta}, I_{r,\beta})$. When $\Delta I_{\beta,\theta}$ goes to minus infinity, (A.63) implies that $\frac{\overline{\Delta} I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}}$ converges to a positive limit. Since, in addition, $\bar{I}_{r,r}$ is independent of $\Delta I_{\beta,\theta}, I_{r,\beta} \in (0, \bar{I}_{r,r})$ and $\Delta I_{r,\theta} \in (0, \bar{\Delta} I_{r,\theta})$, (A.41) implies that $L_{\beta,\theta}$ converges to minus infinity. We next determine conditions so that $L_{\beta,\theta} > 0$ for $\Delta I_{\beta,\theta} = \frac{\delta_\alpha + \kappa_\beta}{2a \sigma_\beta^2}$. Equations (A.40) and (A.41) imply

$$L_{\beta,\theta} = \Delta I_{\beta,\theta} - \frac{a \sigma_\beta^2 (\Delta I_{r,\theta} + I_{r,\beta}) \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}}$$

$$- \frac{\theta}{\delta_\theta} \left( \delta_\theta - \delta_\alpha \right) \left( \delta_\theta + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta} \right) \frac{a \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}}$$

$$= \Delta I_{\beta,\theta} - \frac{a \sigma_\beta^2 \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}}$$

$$- \frac{\theta}{\delta_\theta} \left( \delta_\theta - \delta_\alpha \right) \left( \delta_\theta + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta} \right) \frac{a \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}} + \frac{\delta_\theta a \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}}.$$ 

Hence, $L_{\beta,\theta} > 0$ for large $\delta_\theta$ if

$$\Delta I_{\beta,\theta} - \frac{a \sigma_\beta^2 \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}} + \frac{\delta_\theta a \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}} > 0. \quad \text{(A.65)}$$

Setting $\Delta I_{\beta,\theta} = \frac{\delta_\alpha + \kappa_\beta}{2a \sigma_\beta^2}$ in (A.65), we can write it as

$$\frac{\delta_\alpha + \kappa_\beta}{2a \sigma_\beta^2} - \frac{a \sigma_\beta^2 \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}} + \frac{\delta_\theta a \sigma_\beta^2 I_{r,\beta} \Delta I_{r,\theta}}{\delta_\alpha + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}} > 0. \quad \text{(A.66)}$$

Equation (A.66) is satisfied for $\kappa_\beta \approx 0$. It is also satisfied for a general value of $\kappa_\beta$ if

$$\frac{\delta_\alpha + \kappa_\beta}{2a \sigma_\beta^2} - \frac{a \sigma_\beta^2 \left( \frac{\theta}{\delta_\theta} \right)^2}{\delta_\alpha + \kappa_\beta} > 0 \Leftrightarrow \delta_\alpha (\delta_\alpha + \kappa_\beta) (\delta_\alpha + \kappa_\beta) > 2a \theta \sigma_\tau \sigma_\beta, \quad \text{(A.67)}$$

which follows from (A.66) by noting that (A.40) implies $\Delta I_{r,\theta} < \frac{\theta}{\delta_\theta + \kappa_\tau}$. Under either $\kappa_\beta \approx 0$ or $\delta_\alpha (\delta_\alpha + \kappa_\beta) (\delta_\alpha + \kappa_\beta) > 2a \theta \sigma_\tau \sigma_\beta$, $L_{\beta,\theta} > 0$ for $\Delta I_{\beta,\theta} = \frac{\delta_\alpha + \kappa_\beta}{2a \sigma_\beta^2}$, and hence (A.41) has a unique solution for $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\beta}{2a \sigma_\beta^2}$.
Inequalities (A.59), (A.61) and (A.62) hold for \( I_{r,\beta} \) close to zero. Consider the largest value \( \bar{I}_{r,\beta} \) such that (A.59), (A.61) and (A.62) hold for all \( I_{r,\beta} < \bar{I}_{r,\beta} \). The implicit function theorem ensures that the functions \( (I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}) \) are continuous in \( I_{r,\beta} \leq \bar{I}_{r,\beta} \). For \( I_{r,\beta} \) close to zero, (A.39) and (A.40) imply that \( I_{r,r} \) and \( \Delta I_{r,\theta} \) are bounded away from zero. Since, in addition, \( \Delta I_{\beta,\theta} \) is bounded above by \( \frac{\delta_{\alpha} + \kappa_{\beta}}{2a\sigma_{\beta}^2} \), (A.42) implies \( L_{r,\beta} < 0 \). We next determine a value \( I^{\ast}_{r,\beta} \geq \bar{I}_{r,\beta} \) such that \( L_{r,\beta} > 0 \) (and such that \( (I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta}) \) are well-defined and continuous in \( I_{r,\beta} \in (\bar{I}_{r,\beta}, I^{\ast}_{r,\beta}) \)). Continuity then ensures that a solution \( I_{r,\beta} < I^{\ast}_{r,\beta} \) to (A.42) exists, and hence a solution \( (I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}, I_{r,\beta}) \) to the system (A.39)-(A.42) also exists.

The inequality among (A.59), (A.61) and (A.62) that switches to an equality at \( \bar{I}_{r,\beta} \) cannot be (A.59). Indeed, if (A.59) switches to an equality at \( \bar{I}_{r,\beta} \), then (A.60) implies \( \bar{I}_{r,r} = 0 \), and (A.61) becomes

\[
\bar{I}_{r,\beta} < \frac{\theta}{\delta_{\alpha} + \kappa_{r}} - \frac{\theta}{\delta_{\theta} + \kappa_{r}}.
\]  

(A.68)

Multiplying (A.62) by (A.68), we find

\[
a\sigma_{\beta}^2 I_{r,\beta}^2 < \frac{\theta}{\delta_{\alpha} + \kappa_{r}} - \frac{\theta}{\delta_{\theta} + \kappa_{r}} < \frac{\theta}{\delta_{\alpha} + \kappa_{r}},
\]

which implies that (A.59) holds, a contradiction.

If (A.61) switches to an equality at \( \bar{I}_{r,\beta} \), then \( L_{r,\theta} = 0 \) for \( \Delta I_{r,\theta} = 0 \), and hence the solution to (A.40) is \( \Delta I_{r,\theta} = 0 \). Equation (A.42) then implies \( L_{r,\beta} > 0 \) for \( I_{r,\beta} = \bar{I}_{r,\beta} = \bar{I}^{\ast}_{r,\beta} \).

Suppose instead that (A.62) switches to an equality at \( \bar{I}_{r,\beta} \). Consider a value of \( I_{r,\beta} > \bar{I}_{r,\beta} \) such that (A.59) and (A.61) hold. Define \( \bar{\Delta} I_{r,\theta} > 0 \) by (A.63) and consider the set of \( \Delta I_{\beta,\theta} < \frac{\delta_{\alpha} + \kappa_{\beta}}{2a\sigma_{\beta}^2} \) such that \( L_{r,\theta} > 0 \) for \( \Delta I_{r,\theta} = \bar{\Delta} I_{r,\theta} \). Proceeding as in (A.64) and substituting

\[
\bar{\Delta} I_{r,\theta} \quad \text{from (A.63),}
\]

we can write the condition defining that set as

\[
\frac{\theta}{\delta_{\alpha} + \kappa_{r}} (\delta_{\theta} + \kappa_{\beta} - a\sigma_{\beta}^2 \Delta I_{\beta,\theta}) + I_{r,\beta} \left( 1 - \frac{\theta}{\alpha} a\sigma_{\beta}^2 I_{r,\beta} \right)
\]

\[
+ \frac{\theta}{\delta_{\beta}} \left( \delta_{\theta} + \kappa_{\beta} - a\sigma_{\beta}^2 \Delta I_{\beta,\theta} \right) \left( \delta_{\alpha} + \kappa_{r} - a\sigma_{\beta}^2 \Delta I_{\beta,\theta} \right) > 0.
\]

(A.69)
If (A.69) holds for all $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\alpha}{2\alpha\beta}$, then we can proceed as in the case where (A.59), (A.61) and (A.62) hold, and construct $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$ and $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\alpha}{2\alpha\beta}$ uniquely. Denote by $\bar{I}_{r,\beta} > \bar{I}_{r,\beta}$ the maximum value of $I_{r,\beta}$ such that (A.69) holds for all $\Delta I_{\beta,\theta} < \frac{\delta_\alpha + \kappa_\alpha}{2\alpha\beta}$ and for all $I_{r,\beta} \in [\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$.

If (A.61) switches to an equality at $\bar{I}_{r,\beta} \in (\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$ and (A.59) holds for all $I_{r,\beta} \in [\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$, then $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$ are well-defined and continuous in $I_{r,\beta} \in [\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$ and $L_{r,\theta} > 0$ for $I_{r,\beta} = \bar{I}_{r,\beta} = \bar{I}_{r,\beta}$.

Suppose instead that (A.61) holds for all $I_{r,\beta} \in [\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$. Then (A.59) also holds for all $I_{r,\beta} \in [\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$. Indeed, if (A.59) switches to an equality at $\bar{I}_{r,\beta} \in (\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$, then (A.60) implies $\bar{I}_{r,\beta} = 0$, and (A.64) implies

$$
\bar{I}_{r,\beta} \left(1 - \frac{\theta}{\alpha} a\sigma_\beta^2 \bar{I}_{r,\beta}\right) + \frac{\theta}{\delta_\theta + \kappa_\beta} > 0
$$

$$
\Rightarrow \bar{I}_{r,\beta} - \frac{\theta}{\delta_\theta + \kappa_\beta} > 0
$$

$$
\Rightarrow \bar{I}_{r,\beta} = \frac{\theta}{\delta_\theta + \kappa_\beta} > 0,
$$

(A.70)

where the first and third steps follow from (A.59) switching to an equality at $\bar{I}_{r,\beta}$. Hence, (A.61) holds in the opposite direction, a contradiction. Since (A.59) and (A.61) hold for all $I_{r,\beta} \in [\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$, $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$ are well-defined and continuous in $I_{r,\beta} \in [\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$. For $I_{r,\beta} = \bar{I}_{r,\beta}$, (A.64) switches to an equality for a single value $\Delta I_{\beta,\theta}$. (Since the left-hand side is convex in $\Delta I_{\beta,\theta}$, if (A.64) switches to an equality for two values of $\Delta I_{\beta,\theta}$, then it switches to an inequality in the opposite direction for values of $\Delta I_{\beta,\theta}$ in-between, which contradicts the definition of $\bar{I}_{r,\beta}$.) Suppose without loss of generality that the solution $\Delta I_{\beta,\theta}$ is to the right of $\Delta I_{\beta,\theta}$, in which case $L_{\beta,\theta} < 0$ for $\Delta I_{\beta,\theta} = \Delta L_{\beta,\theta}$. Consider a value of $I_{r,\beta} > \bar{I}_{r,\beta}$ such that (A.59) and (A.61) hold, and denote by $\Delta L_{\beta,\theta}$ the minimum value of $\Delta I_{\beta,\theta}$ such that (A.69) holds for all $\Delta I_{\beta,\theta} \in \left(\Delta L_{\beta,\theta}, \frac{\delta_\alpha + \kappa_\alpha}{2\alpha\beta}\right)$.

Proceeding as in the case where (A.59), (A.61) and (A.62) hold, we can construct $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$ and $\Delta I_{\beta,\theta} \in \left(\Delta L_{\beta,\theta}, \frac{\delta_\alpha + \kappa_\alpha}{2\alpha\beta}\right)$ uniquely. Consider the largest value $\bar{I}_{r,\beta} > \bar{I}_{r,\beta}$ such that for all $I_{r,\beta} \in [\bar{I}_{r,\beta}, \bar{I}_{r,\beta}]$, (A.59) and (A.61) hold and $L_{\beta,\theta} < 0$ for $\Delta I_{\beta,\theta} = \Delta L_{\beta,\theta}$. The functions $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$
are well-defined and continuous in $I_{r,\beta} \in (\bar{I}_{r,\beta}', \bar{I}_{r,\beta}')$. The same argument as in (A.70) implies that
the inequality among (A.59), (A.61) and $L_{\beta,\theta} < 0$ for $\Delta I_{\beta,\theta} = \Delta L_{\beta,\theta}$ that switches to an equality at
$\bar{I}_{r,\beta}'$ cannot be (A.59). If (A.61) switches to an equality at $\bar{I}_{r,\beta}'$, then $L_{r,\beta} > 0$ for $I_{r,\beta} = \bar{I}_{r,\beta}'$. If instead, $L_{\beta,\theta} = 0$ for $\Delta I_{\beta,\theta} = \Delta L_{\beta,\theta}$, then $(I_{r,r}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta}) = (0, \overline{\Delta I_{r,\theta}}, \Delta L_{\beta,\theta})$. Hence,

$$L_{r,\beta} = \left( \delta_{\alpha} + \kappa_{r} + \kappa_{\beta} - a \sigma_{r}^{2} \Delta I_{r,\beta,\theta} + a \sigma_{\beta}^{2} \frac{a \sigma_{\beta}^{2} \Delta I_{r,\beta,\theta}}{2} + \kappa_{\beta} - a \sigma_{\beta}^{2} \Delta I_{r,\beta,\theta} \right) \bar{I}_{r,\beta}'$$

$$- \frac{\frac{\gamma_{r}}{\theta_{r}} a \sigma_{r}^{2} \Delta I_{r,\theta}}{\left( \delta_{\alpha} + \kappa_{r} - a \sigma_{r}^{2} \Delta I_{r,\beta,\theta} \right) + a^{2} \sigma_{r}^{2} \sigma_{\beta}^{2} \Delta I_{r,\beta,\theta}}$$

$$> \left( \delta_{\alpha} + \kappa_{r} + \kappa_{\beta} - a \sigma_{r}^{2} \Delta I_{r,\beta,\theta} + a \sigma_{\beta}^{2} \frac{a \sigma_{\beta}^{2} \Delta I_{r,\beta,\theta}}{2} + \kappa_{\beta} - a \sigma_{\beta}^{2} \Delta I_{r,\beta,\theta} \right) \bar{I}_{r,\beta}'$$

$$- \frac{\frac{\gamma_{r}}{\theta_{r}} a \sigma_{r}^{2} \sigma_{\beta}^{2} \Delta I_{r,\theta}}{\left( \delta_{\alpha} + \kappa_{r} - a \sigma_{r}^{2} \Delta I_{r,\beta,\theta} \right) + a^{2} \sigma_{r}^{2} \sigma_{\beta}^{2} \Delta I_{r,\beta,\theta}}$$

$$= \left( \delta_{\alpha} + \kappa_{r} + \kappa_{\beta} - 2 a \sigma_{r}^{2} \Delta I_{r,\beta,\theta} \right) \bar{I}_{r,\beta}' - \frac{\frac{\gamma_{r}}{\theta_{r}} a \sigma_{r}^{2} \sigma_{\beta}^{2} \Delta I_{r,\theta}}{\left( \delta_{\theta} + \kappa_{r} - a \sigma_{r}^{2} \Delta I_{r,\beta,\theta} \right) + a^{2} \sigma_{r}^{2} \sigma_{\beta}^{2} \Delta I_{r,\beta,\theta}}$$

where the first step follows from $\bar{I}_{r,\beta}' > \bar{I}_{r,\beta} = \frac{\alpha_{r}}{\theta_{r} a \sigma_{r}^{2}}$ and the second step from (A.41). For large $\delta_{\theta}$,
$L_{r,\beta} > 0$ if

$$\delta_{\alpha} + \kappa_{r} + \kappa_{\beta} - 2 a \sigma_{r}^{2} \Delta I_{r,\beta,\theta} > 0,$$

(A.71)

which holds because $\Delta I_{r,\beta,\theta} < \frac{\delta_{r} + \kappa_{r}}{2 a \sigma_{r}^{2}}$. Hence, $I_{r,\beta} = \bar{I}_{r,\beta}' = \bar{I}_{r,\beta}$. The solution satisfies $I_{r,r} > 0$,
$\Delta I_{r,\theta} > 0$, $\Delta I_{\beta,\theta} < \frac{\delta_{r} + \kappa_{r}}{2 a \sigma_{r}^{2}}$ and $I_{r,\beta} > 0$. Combining these inequalities with (A.26), we find $M_{1,1} > \kappa_{r}$,
$M_{1,2} > 0$, $M_{2,1} < 0$ and $M_{2,2} > \frac{\kappa_{r} - \delta_{\alpha}}{2}$.

To complete the existence proof, we show that the integrals in the Laplace transforms $(I_{r}, I_{r,\beta}, I_{r,r}, I_{r,\beta,\beta}, I_{r,2}, I_{r,\beta,2})$ converge. That property is assumed when performing the integration by parts
in Lemma A.1. Since $\delta_{\theta} > \delta_{\alpha}$, the Laplace-transform integrals converge if the real parts of the
eigenvalues of $M$ exceed $-\frac{\delta_{\alpha}}{2}$. Using (A.26), we find that the characteristic polynomial of $M$ is

$$P(\lambda) = (\kappa_{r} + a \sigma_{r}^{2} I_{r,r} - \lambda) (\kappa_{\beta} - a \sigma_{\beta}^{2} \Delta I_{r,\beta,\theta} - \lambda) + a^{2} \sigma_{r}^{2} \sigma_{\beta}^{2} I_{r,\beta,\theta,\Delta I_{r,\beta,\theta}}.$$

(A.72)
Since $I_{r,r} > 0$, $\Delta I_{r,\theta} > 0$, $\Delta I_{\beta,\theta} < \frac{\delta_{r}+\kappa_{r}}{2\alpha_{r}^{2}}$ and $I_{r,\beta} > 0$, $P(\lambda) > 0$ for all $\lambda < -\frac{\delta_{r}}{2\alpha_{r}^{2}}$. Hence, if the eigenvalues are real, they must exceed $-\frac{\delta_{r}}{2}$. If the eigenvalues are complex, their real part is
\[
\frac{\kappa_{r} + a\sigma_{r}^{2} I_{r,r} + \kappa_{\beta} - a\sigma_{\beta}^{2} \Delta I_{\beta,\theta}}{2}
\]
and exceeds $-\frac{\delta_{r}}{2}$ because $I_{r,r} > 0$ and $\Delta I_{\beta,\theta} < \frac{\delta_{r}+\kappa_{r}}{2\alpha_{r}^{2}}$.

**Proof of Proposition 5:** Using $K = 1$ and (A.26), we can write the system (36) as
\[
\begin{align*}
A_{r}'(\tau) + (\kappa_{r} + a\sigma_{r}^{2} I_{r,r}) A_{r}(\tau) + a\sigma_{\beta}^{2} I_{r,\beta} A_{\beta}(\tau) - 1 &= 0, \quad (A.73) \\
A_{\beta}'(\tau) - a\sigma_{r}^{2} \Delta I_{r,\theta} A_{r}(\tau) + (\kappa_{\beta} - a\sigma_{\beta}^{2} \Delta I_{\beta,\theta}) A_{\beta}(\tau) &= 0, \quad (A.74)
\end{align*}
\]
and the solution to that system, given in Lemma 3, as
\[
\begin{align*}
A_{r}(\tau) &= \frac{1 - e^{-\nu_{1}\tau}}{\nu_{1}} + \phi_{r} \left( \frac{1 - e^{-\nu_{2}\tau}}{\nu_{2}} - \frac{1 - e^{-\nu_{1}\tau}}{\nu_{1}} \right), \quad (A.75) \\
A_{\beta}(\tau) &= \phi_{\beta} \left( \frac{1 - e^{-\nu_{2}\tau}}{\nu_{2}} - \frac{1 - e^{-\nu_{1}\tau}}{\nu_{1}} \right). \quad (A.76)
\end{align*}
\]
Equations (A.73) and (A.74), together with the initial conditions $A_{r}(0) = A_{\beta}(0) = 0$, imply $A_{r}'(0) = 1$ and $A_{\beta}'(0) = 0$. Differentiating (A.74) at zero and using $\Delta I_{r,\theta} > 0$, which follows from $M_{2,1} < 0$ and (A.26), we find $A_{\beta}''(0) > 0$. Hence, $A_{r}(\tau) > 0$, $A_{\beta}'(\tau) > 0$ and $A_{\beta}(\tau) > 0$ for small $\tau$.

Suppose that the two eigenvalues of $M$ are real, and without loss of generality set $\nu_{1} > \nu_{2}$. Since the function $(\nu, \tau) \longrightarrow \frac{1 - e^{-\nu\tau}}{\nu}$ decreases in $\nu$, the term in parenthesis in (A.76) is positive. Since, in addition, $A_{\beta}'(\tau) > 0$ for small $\tau$, $\phi_{\beta} > 0$ and hence $A_{\beta}(\tau) > 0$ for all $\tau$. Since
\[
A_{\beta}'(\tau) = \phi_{\beta} \left( e^{-\nu_{2}\tau} - e^{-\nu_{1}\tau} \right)
\]
and $\phi_{\beta} > 0$, $A_{\beta}'(\tau) > 0$. Since
\[
\frac{A_{r}(\tau)}{A_{\beta}(\tau)} = \frac{\frac{1 - e^{-\nu_{1}\tau}}{\nu_{1}}}{\phi_{\beta} \left( \frac{1 - e^{-\nu_{2}\tau}}{\nu_{2}} - \frac{1 - e^{-\nu_{1}\tau}}{\nu_{1}} \right)} + \frac{\phi_{r}}{\phi_{\beta}} = \frac{1}{\nu_{2}} \frac{1 - e^{-\nu_{2}\tau}}{1 - e^{-\nu_{1}\tau}} = \frac{\phi_{r}}{\phi_{\beta}},
\]
and the function $(\nu_{1}, \nu_{2}, \tau) \longrightarrow \frac{1 - e^{-\nu_{2}\tau}}{1 - e^{-\nu_{1}\tau}}$ increases in $\tau$ because its derivative has the same sign as $\frac{e^{\nu_{1}\tau}-1}{\nu_{1}} - \frac{e^{\nu_{2}\tau}-1}{\nu_{2}}$, $\left[ \frac{A_{r}(\tau)}{A_{\beta}(\tau)} \right]' < 0$. Since
\[
A_{r}'(\tau) = e^{-\nu_{1}\tau} + \phi_{r} \left( e^{-\nu_{2}\tau} - e^{-\nu_{1}\tau} \right),
\]

the sign of \( A'_r(\tau) \) can change at most once. Hence, \( A'_r(\tau) \) is either positive, or is positive for \( \tau \in (0, \bar{\tau}) \) and negative for \( \tau \in (\bar{\tau}, T) \), where \( \bar{\tau} \) is a threshold in \( (0, T) \). The function \( A_r(\tau) \) has the same behavior for a different threshold \( \bar{\tau} \).

When \( a \approx 0, A_r(\tau) > 0 \) because Lemma A.2 implies \( \phi_r \approx 0, \nu_1 \approx \kappa_r > 0 \) and \( \nu_2 \approx \kappa_\beta > 0 \). When \( \alpha(\tau) = 0, I_{r,\tau} = I_{r,\beta} = 0 \), and hence (A.73) implies \( A_r(\tau) = \frac{1-e^{-\bar{\tau}r}}{r} > 0 \). In both cases, \( \bar{\tau} = T \). When \( a \approx \infty \), Lemma A.2 implies that for \( \tau \) bounded away from zero

\[
A_r(\tau) \approx \frac{1}{a} \left( \frac{1}{n_1} + \bar{\tau} \frac{1 - e^{-\bar{\tau}r}}{v_2} \right)
\]

\[
= \frac{1}{a \frac{1}{n_1} \bar{\tau} \left( 1 - \int_0^\bar{\tau} \frac{\alpha(\tau') \frac{1 - e^{-\bar{\tau}r'}}{v_2} d\tau'}{\int_0^\bar{\tau} \alpha(\tau') \left( \frac{1 - e^{-\bar{\tau}r'}}{v_2} \right)^2 d\tau'} \right)
\]

\[
= \frac{1}{a \frac{1}{n_1} \bar{\tau} \left( \int_0^\bar{\tau} \alpha(\tau')(1 - e^{-\bar{\tau}r'})^2 d\tau' \right)}
\]

Since this is negative for \( \tau \) close to \( T, \bar{\tau} < T \).

Suppose that the two eigenvalues of \( M \) are complex. Since they are conjugates, we set \( \nu_1 = \mu + i\xi \) and \( \nu_2 = \mu - i\xi \) for real numbers \((\mu, \xi)\). Equations (A.75) and (A.76) imply that \((A_r(\tau), A_\beta(\tau))\) take the form

\[
A_r(\tau) = \phi_{r,0} + \phi_{r,1} e^{-\mu \tau} \cos(\xi \tau) + \phi_{r,2} e^{-\mu \tau} \sin(\xi \tau), \tag{A.77}
\]

\[
A_\beta(\tau) = \phi_{\beta,0} + \phi_{\beta,1} e^{-\mu \tau} \cos(\xi \tau) + \phi_{\beta,2} e^{-\mu \tau} \sin(\xi \tau), \tag{A.78}
\]

for real numbers \( \{\phi_{j,n}\}_{j=r, \beta, n=0,1,2} \). Since the initial conditions \( A_r(0) = A_\beta(0) = 0 \) imply \( \phi_{j,0} + \phi_{j,1} = 0 \) for \( j = r, \beta \), condition \( A'_r(0) = 1 \) implies \( -\phi_{r,1} \mu + \phi_{r,2} \xi = 1 \), and condition \( A'_\beta(0) = 0 \) implies \( -\phi_{\beta,1} \mu + \phi_{\beta,2} \xi = 0 \), we can write (A.77) and (A.78) as

\[
A_r(\tau) = \phi_{r,0} \left[ 1 - \frac{\mu}{\xi} e^{-\mu \tau} \sin(\xi \tau) \right] + \frac{1}{\xi} e^{-\mu \tau} \sin(\xi \tau), \tag{A.79}
\]

\[
A_\beta(\tau) = \phi_{\beta,0} \left[ 1 \right] + \frac{1}{\xi} e^{-\mu \tau} \sin(\xi \tau) \]. \tag{A.80}
\]

Differentiating (A.79) and (A.80), we find

\[
A'_r(\tau) = \phi_{r,0} \frac{\mu^2 + \xi^2}{\xi} e^{-\mu \tau} \sin(\xi \tau) + e^{-\mu \tau} \left[ \cos(\xi \tau) - \frac{\mu}{\xi} \sin(\xi \tau) \right], \tag{A.81}
\]

\[
A'_\beta(\tau) = \phi_{\beta,0} \frac{\mu^2 + \xi^2}{\xi} e^{-\mu \tau} \sin(\xi \tau). \tag{A.82}
\]
Since $A_\beta'(\tau) > 0$ for small $\tau$, $\phi_{\beta,0} > 0$, and hence $A_\beta'(\tau) > 0$ for $\tau \in (0, \frac{\pi}{|\xi|})$. The derivative $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]'$ has the same sign as

$$A_r'(\tau)A_\beta(\tau) - A_r(\tau)A_\beta'(\tau)$$

$$= e^{-\mu \tau} \left[ \cos(\xi \tau) - \frac{\mu}{\xi} \sin(\xi \tau) \right] \phi_{\beta,0} \left[ 1 - \frac{\mu}{\xi} e^{-\mu \tau} \sin(\xi \tau) - e^{-\mu \tau} \cos(\xi \tau) \right]$$

$$- \frac{1}{\xi} e^{-\mu \tau} \sin(\xi \tau) \phi_{\beta,0} \left[ \frac{\mu^2 + \xi^2}{\xi} e^{-\mu \tau} \sin(\xi \tau) \right]$$

$$= \phi_{\beta,0} e^{-\mu \tau} \left[ \cos(\xi \tau) - \frac{\mu}{\xi} \sin(\xi \tau) - e^{-\mu \tau} \right],$$

(A.83)

where the second step follows from (A.79)-(A.82) and the third by rearranging. Since $\phi_{\beta,0} > 0$, $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]'$ is negative if the term in brackets in (A.83) is negative. That term is concave in $\mu$ and is maximized for $\mu$ given by

$$- \frac{1}{\xi} \sin(\xi \tau) + \tau e^{-\mu \tau} = 0 \Leftrightarrow e^{-\mu \tau} = \frac{\sin(\xi \tau)}{\xi \tau}.$$

The maximum is

$$\cos(\xi \tau) - \frac{\sin(\xi \tau)}{\xi \tau} \left[ 1 - \log \left( \frac{\sin(\xi \tau)}{\xi \tau} \right) \right] = H(\xi \tau) \frac{\sin(\xi \tau)}{\xi \tau},$$

(A.84)

where

$$H(x) \equiv \frac{x \cos(x)}{\sin(x)} - 1 + \log \left( \frac{\sin(x)}{x} \right).$$

The function $H(x)$ is equal to zero for $x = 0$, and its derivative is

$$H'(x) = -\frac{x}{\sin^2(x)} + \frac{\cos(x)}{\sin(x)} + \frac{x \cos(x) - \sin(x)}{x^2 \sin(x)} = -\frac{x^2 - 2x \cos(x) \sin(x) + \sin^2(x)}{x \sin^2(x)}.$$

Since

$$x^2 - 2x \cos(x) \sin(x) + \sin^2(x) > x^2 - 2|x \sin(x)| + \sin^2(x) = (|x| - |\sin(x)|)^2 > 0$$

for $x \neq 0$, $H'(x) > 0$ for $x < 0$, and $H'(x) < 0$ for $x > 0$. Since, in addition, $H(0) = 0$, $H(x) < 0$. Hence, the maximum (A.84) is negative for $\tau \in (0, \frac{\pi}{|\xi|})$, and so is $\left[\frac{A_r(\tau)}{A_\beta(\tau)}\right]'$. This establishes the
results in the proposition for $A'_\beta(\tau)$ and $\frac{A_\beta(\tau)}{A'_\beta(\tau)}$ and for the threshold $\bar{\tau} = \frac{\nu}{|\sigma|}$. The result for $A_\beta(\tau)$ and for a threshold $\bar{\tau} > \bar{\tau}$ follows because $A_\beta(0) = 0$ and $A'_\beta(\tau) > 0$ for $\tau \in (0, \bar{\tau})$ imply $A_\beta(\tau) > 0$ for $\tau \in (0, \bar{\tau}]$

If $\bar{\tau} < T$, then $A_\beta(\bar{\tau}) = 0$ and $A'_\beta(\bar{\tau}) \leq 0$. If $A'_\beta(\bar{\tau}) < 0$, then $\Delta I_{r,\theta} > 0$ and (A.74) imply $A_\beta(\bar{\tau}) = 0$, and (A.74) implies $A'_\beta(\bar{\tau}) = 1$. Hence, in both cases, $A_\beta(\tau) < 0$ for $\tau$ smaller than and close to $\bar{\tau}$. This yields the result in the proposition for $A_\beta(\tau)$ and for a threshold $\bar{\tau} < \bar{\tau}$.

Lemma A.2 derives the asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ when $a \approx 0$ and $a \approx \infty$. To state and prove the lemma, we define the functions

\[
F(\nu, \nu') \equiv \int_0^T \alpha(\tau) \frac{1 - e^{-\nu \tau}}{\nu} \frac{1 - e^{-\nu' \tau}}{\nu'} d\tau,
\]

\[
\hat{F}(\nu, \nu') \equiv F(\nu, \nu') - F(\nu, \nu),
\]

\[
\hat{F}(\nu, \nu') \equiv F(\nu, \nu) + F(\nu', \nu') - 2F(\nu, \nu'),
\]

\[
G(\nu) \equiv \int_0^T \theta(\tau) \frac{1 - e^{-\nu \tau}}{\nu} d\tau,
\]

\[
\hat{G}(\nu, \nu') \equiv G(\nu') - G(\nu).
\]

We also note that the definitions of $(J, I_{r,r}, I_{r,\beta})$ imply

\[
I_{r,r} = \int_0^\infty \alpha(\tau)A_\tau(\tau)^2 d\tau, 
\]

\[
I_{r,\beta} = \int_0^\infty \alpha(\tau)A_\tau(\tau)A_\beta(\tau) d\tau.
\]

**Lemma A.2.** Suppose that there is one demand factor, the matrices $(\Gamma, \Sigma)$ are diagonal, and $\alpha(\tau)$ and $\frac{\theta(\tau)}{\tau}$ have a positive and a finite limit, respectively, at $\tau = 0$. The asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ when $a \approx 0$ and $a \approx \infty$ is as follows:

- When $a \approx 0$, $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (\kappa_r, \kappa_\beta, a^3 \xi_r, a\xi_\beta)$, where

\[
\xi_r = -\frac{c_r^2 \sigma^2_\Gamma \hat{F}(\kappa_r, \kappa_\beta)}{\kappa_r - \kappa_\beta},
\]

\[
\xi_\beta = \frac{\sigma^2_r G(\kappa_r)}{\kappa_r - \kappa_\beta}.
\]
When $a \approx \infty$, $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx \left(\frac{1}{a} \overline{w}_1, \nu_2, \phi_r, \phi_\beta\right)$, where

$$\overline{w}_1 = \sigma_r^2 \left[ \int_0^T \alpha(\tau) d\tau - \frac{\int_0^T \alpha(\tau) \frac{1-e^{-\nu_2 \tau}}{\nu_2} d\tau}{\int_0^T \alpha(\tau) \left(1 - \frac{1-e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau} \right] > 0. \quad (A.89)$$

$$\overline{w}_r = -\frac{1}{\overline{w}_1} \frac{\int_0^T \alpha(\tau) \frac{1-e^{-\nu_2 \tau}}{\nu_2} d\tau}{\int_0^T \alpha(\tau) \left(1 - \frac{1-e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau} < 0, \quad (A.90)$$

$$\phi_\beta = \frac{\int_0^T \theta(\tau) \frac{1-e^{-\nu_2 \tau}}{\nu_2} d\tau}{\int_0^T \alpha(\tau) \left(1 - \frac{1-e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau}, \quad (A.91)$$

and $\overline{v}_2$ solves

$$\frac{\int_0^T \theta(\tau) \frac{1-e^{-\nu_2 \tau}}{\nu_2} d\tau}{\int_0^T \alpha(\tau) \left(1 - \frac{1-e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau} = \frac{\int_0^T \alpha(\tau) \left(1 - \frac{1-e^{-\nu_2 \tau}}{\nu_2}\right)^2 d\tau}{\int_0^T \alpha(\tau) \frac{1-e^{-\nu_2 \tau}}{\nu_2} d\tau}. \quad (A.92)$$

**Proof:** Substituting (A.75) and (A.76) into (A.73) and identifying terms in $\frac{1-e^{-\nu_1 \tau}}{\nu_1}$ and $\left(\frac{1-e^{-\nu_2 \tau}}{\nu_2} - \frac{1-e^{-\nu_1 \tau}}{\nu_1}\right)$, we find

$$\phi_r (\nu_1 - \nu_2) - \nu_1 + \kappa_r + a \sigma_r^2 I_{r,r} = 0, \quad (A.93)$$

$$- \phi_r \nu_2 + \phi_r (\kappa_r + a \sigma_r^2 I_{r,r}) + \phi_\beta a \sigma_\beta^2 I_{r,\beta} = 0, \quad (A.94)$$

respectively. Using (A.93), we can write (A.94) as

$$\phi_r (1 - \phi_r) (\nu_1 - \nu_2) + \phi_\beta a \sigma_\beta^2 I_{r,\beta} = 0. \quad (A.95)$$

Substituting (A.75) and (A.76) into (A.74) and identifying terms, we find

$$\phi_\beta (\nu_1 - \nu_2) - a \sigma_\beta^2 \Delta I_{r,\theta} = 0, \quad (A.96)$$

$$- \phi_\beta \nu_2 - \phi_\beta \Delta I_{r,\theta} + \phi_\beta (\kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta}) = 0, \quad (A.97)$$

respectively. Using (A.96), we can write (A.97) as

$$- \nu_2 - \phi_r (\nu_1 - \nu_2) + \kappa_\beta - a \sigma_\beta^2 \Delta I_{\beta,\theta} = 0. \quad (A.98)$$

Equations (A.93), (A.95), (A.96) and (A.98) constitute a system of four equations in the four unknowns $(\nu_1, \nu_2, \phi_r, \phi_\beta)$. Substituting (A.75) and (A.76) into the definitions (A.85), (A.86), (A.24)
and (A.25) of \((I_{r,r}, I_{r,\beta}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})\), we can write that system as

\[
\phi_r(\nu_1 - \nu_2) - \nu_1 + \kappa_r + a\sigma^2_r \left[ F(\nu_1, \nu_1) + 2\phi_r \hat{F}(\nu_1, \nu_2) + \phi_r^2 \hat{F}(\nu_1, \nu_2) \right] = 0, \tag{A.99}
\]

\[
\phi_r(1 - \phi_r)(\nu_1 - \nu_2) + \phi^2_r a\sigma^2_\beta \left[ \hat{F}(\nu_1, \nu_2) + \phi_r \hat{F}(\nu_1, \nu_2) \right] = 0, \tag{A.100}
\]

\[
\phi_\beta(\nu_1 - \nu_2) - a\sigma^2_r \left[ G(\nu_1) + \phi_r \hat{G}(\nu_1, \nu_2) - \phi_\beta \left[ \hat{F}(\nu_1, \nu_2) + \gamma_r \hat{F}(\nu_1, \nu_2) \right] \right] = 0, \tag{A.101}
\]

\[
-\nu_2 - \phi_r(\nu_1 - \nu_2) + \kappa_\beta - \phi_\beta a\sigma^2_\beta \left[ \hat{G}(\nu_1, \nu_2) - \phi_\beta \hat{F}(\nu_1, \nu_2) \right] = 0. \tag{A.102}
\]

Suppose that \(a \approx 0\). Setting \((\phi_r, \phi_\beta) = (a^3 c_r, a c_\beta)\), we can write (A.99)-(A.102) as

\[
a^3 c_r(\nu_1 - \nu_2) - \nu_1 + \kappa_r + a\sigma^2_r \left[ F(\nu_1, \nu_1) + 2a^3 c_r \hat{F}(\nu_1, \nu_2) + a^6 c_r^2 \hat{F}(\nu_1, \nu_2) \right] = 0, \tag{A.103}
\]

\[
c_r(1 - a^3 c_r)(\nu_1 - \nu_2) + c^2_r a\sigma^2_\beta \left[ \hat{F}(\nu_1, \nu_2) + c^3_r a \hat{F}(\nu_1, \nu_2) \right] = 0, \tag{A.104}
\]

\[
c_\beta(\nu_1 - \nu_2) - a\sigma^2_r \left[ G(\nu_1) + a^3 c_r \hat{G}(\nu_1, \nu_2) - a c_\beta \left[ \hat{F}(\nu_1, \nu_2) + a^3 c_r \hat{F}(\nu_1, \nu_2) \right] \right] = 0, \tag{A.105}
\]

\[
-\nu_2 - a^3 c_r(\nu_1 - \nu_2) + \kappa_\beta - a^2 c_\beta a\sigma^2_\beta \left[ \hat{G}(\nu_1, \nu_2) - a c_\beta \hat{F}(\nu_1, \nu_2) \right] = 0. \tag{A.106}
\]

The asymptotic behavior of \((\nu_1, \nu_2, \phi_r, \phi_\beta)\) is as in the lemma if (A.103)-(A.106) has a non-zero solution \((\nu_1, \nu_2, c_r, c_\beta)\) for \(a = 0\). For \(a = 0\), (A.103) implies \(\nu_1 = \kappa_r\), (A.106) implies \(\nu_2 = \kappa_\beta\), (A.105) implies \(c_\beta = c_\beta^*\) and (A.104) implies \(c_r = c_r^*\).

Suppose that \(a \approx \infty\). Setting \((\nu_1, \phi_r) = (a^\pm n_1, a^{-\frac{1}{3}} c_r)\), we can write (A.99)-(A.102) as

\[
a^{-\frac{2}{3}} c_r \left( a^{\frac{1}{3}} n_1 - \nu_2 \right) - n_1 + a^{-\frac{1}{3}} \kappa_r + a^{\frac{2}{3}} \sigma^2_r \left[ F \left( a^{\frac{1}{3}} n_1, a^{\frac{1}{3}} n_1 \right) \right]

+ 2a^{-\frac{1}{3}} c_r \hat{F} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) + a^{-\frac{2}{3}} c_r^2 \hat{F} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) \right] = 0, \tag{A.107}
\]

\[
a^{-1} c_r(1 - a^{-\frac{1}{3}} c_r) \left( a^{\frac{1}{3}} n_1 - \nu_2 \right) + a^{-\frac{1}{3}} \phi^2_\beta a\sigma^2_\beta \left[ \hat{F} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) + a^{-\frac{1}{3}} c_r \hat{F} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) \right] = 0, \tag{A.108}
\]

\[
a^{-\frac{2}{3}} \phi_\beta \left( a^{\frac{1}{3}} n_1 - \nu_2 \right) - a^{\frac{1}{3}} \sigma^2_r \left[ G \left( a^{\frac{1}{3}} n_1 \right) + a^{-\frac{1}{3}} c_r \hat{G} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) \right]

- \phi_\beta \left[ \hat{F} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) + a^{-\frac{1}{3}} c_r \hat{F} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) \right] = 0, \tag{A.109}
\]

\[
a^{-1} \left[ -\nu_2 - a^{-\frac{1}{3}} c_r \left( a^{\frac{1}{3}} n_1 - \nu_2 \right) + \kappa_\beta \right] - \phi_\beta a\sigma^2_\beta \left[ \hat{G} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) - \phi_\beta \hat{F} \left( a^{\frac{1}{3}} n_1, \nu_2 \right) \right] = 0. \tag{A.110}
\]
The asymptotic behavior of \((\nu_1, \nu_2, \phi_r, \phi_\beta)\) is as in the lemma if (A.107)-(A.110) has a non-zero solution \((n_1, \nu_2, c_r, \phi_\beta)\) for \(a = \infty\). Noting that
\[
\lim_{a \to \infty} a^2 F\left(a^{\frac{1}{3}}n_1, a^{\frac{1}{3}}n_1\right) = \frac{1}{n_1} \int_0^T \alpha(\tau) d\tau,
\]
\[
\lim_{a \to \infty} a^\frac{1}{3} F\left(a^{\frac{1}{3}}n_1, \nu_2\right) = \frac{1}{n_1} \int_0^T \alpha(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau,
\]
\[
\lim_{a \to \infty} a^\frac{1}{3} G\left(a^{\frac{1}{3}}n_1\right) = \frac{1}{n_1} \int_0^T \theta(\tau) d\tau,
\]
we can write (A.107)-(A.110) for \(a = \infty\) as
\[
n_1 - \sigma_r^2 \left[ \frac{1}{n_1} \int_0^T \alpha(\tau) d\tau + 2c_r \frac{1}{n_1} \int_0^T \alpha(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau + c_r^2 \int_0^T \alpha(\tau) \left( \frac{1 - e^{-\nu_2 \tau}}{\nu_2} \right)^2 d\tau \right] = 0,
\]
\[
\left(\text{A.111}\right)
\]
\[
\frac{1}{n_1} \int_0^T \alpha(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau + c_r \int_0^T \alpha(\tau) \left( \frac{1 - e^{-\nu_2 \tau}}{\nu_2} \right)^2 d\tau = 0,
\]
\[
\left(\text{A.112}\right)
\]
\[
\frac{1}{n_1} \int_0^T \theta(\tau) d\tau + c_r \int_0^T \theta(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau
\]
\[
- \phi_\beta \left[ \frac{1}{n_1} \int_0^T \alpha(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau + c_r \int_0^T \alpha(\tau) \left( \frac{1 - e^{-\nu_2 \tau}}{\nu_2} \right)^2 d\tau \right] = 0,
\]
\[
\left(\text{A.113}\right)
\]
\[
\int_0^T \theta(\tau) \frac{1 - e^{-\nu_2 \tau}}{\nu_2} d\tau - \phi_\beta \int_0^T \alpha(\tau) \left( \frac{1 - e^{-\nu_2 \tau}}{\nu_2} \right)^2 d\tau = 0.
\]
\[
\left(\text{A.114}\right)
\]
Equations (A.112) and (A.113) imply (A.92). Equation (A.92) has a solution \(\bar{\nu}_2\). Indeed, when \(\nu_2\) goes to infinity, the left-hand side is
\[
\frac{1}{\nu_2} \left[ 1 - \frac{\int_0^T \theta(\tau) e^{-\nu_2 \tau} d\tau}{\int_0^T \theta(\tau) d\tau} \right] = \frac{1}{\nu_2} \left[ 1 + o \left( \frac{1}{\nu_2} \right) \right]
\]
because \(\frac{\theta(\tau)}{\tau}\) has a finite limit at zero, and the right-hand side is
\[
\frac{1}{\nu_2} \left[ 1 - \frac{\int_0^T \alpha(\tau)(1 - e^{-\nu_2 \tau}) e^{-\nu_2 \tau} d\tau}{\int_0^T \alpha(\tau)(1 - e^{-\nu_2 \tau}) d\tau} \right] = \frac{1}{\nu_2} \left[ 1 - \frac{\alpha(0)}{\nu_2 \int_0^T \alpha(\tau) d\tau} + o \left( \frac{1}{\nu_2} \right) \right]
\]
because $\alpha(\tau)$ has a positive limit at zero. Hence, the left-hand side exceeds the right-hand side. When $T$ is finite and $\nu_2$ goes to minus infinity, the left-hand side is

$$\frac{e^{-\nu_2 T} \int_0^T \theta(\tau) \left[ e^{\nu_2 T} - e^{\nu_2(T-r)} \right] d\tau}{\nu_2} = \frac{e^{-\nu_2 T} \int_0^T \theta(T) d\tau}{\nu_2} + o \left( \frac{1}{\nu_2^2} \right),$$

and is smaller than the right-hand side, which is

$$\frac{e^{-\nu_2 T} \int_0^T \alpha(\tau) \left[ e^{\nu_2 T} - e^{\nu_2(T-r)} \right]^2 d\tau}{\nu_2} = \frac{e^{-\nu_2 T} \int_0^T \alpha(\tau) d\tau}{-2\nu_2} + o \left( \frac{1}{\nu_2} \right).$$

Hence, a solution $\nu_2 \in (-\infty, \infty)$ to (A.92) exists. When $T = \infty$, $(\alpha(\tau), \theta(\tau)) \approx (\alpha e^{-\delta_\alpha \tau}, \theta e^{-\delta_\alpha \tau})$ for $\tau$ large and for $0 < \delta_\alpha \leq \delta'_\alpha$. When $\nu_2$ goes to $-\frac{\delta_\alpha}{2}$, the right-hand side goes to infinity, while the left-hand side remains finite. Hence, a solution $\nu_2 \in (-\frac{\delta_\alpha}{2}, \infty)$ to (A.92) exists.

Using (A.112) to eliminate $c_r$ in (A.111), we find $n_1 = \pi_1$. Equations (A.112) and (A.114) imply $c_r = \bar{c}_r$ and $\phi_\beta = \bar{\phi}_\beta$, respectively. The Cauchy-Schwarz inequality implies $\pi_1 > 0$, and hence $\bar{c}_r < 0$.

**Proof of Proposition 6:** Proceeding as in the proof of Proposition 3, we find that the FB regression coefficient is

$$b_{FB} = \frac{N_{FB,r} \text{Var}(r_1) + N_{FB,\beta} \text{Var}(\beta_1)}{[A_r(\tau) - A_r(\tau - \Delta \tau) - A_r(\Delta \tau)]^2 \text{Var}(r_1) + [A_\beta(\tau) - A_\beta(\tau - \Delta \tau)]^2 \text{Var}(\beta_1)}$$

$$= \frac{N_{FB,r} \frac{\sigma_r^2}{\kappa_r} + N_{FB,\beta} \frac{\sigma_\beta^2}{\kappa_\beta}}{[A_r(\tau) - A_r(\tau - \Delta \tau) - A_r(\Delta \tau)]^2 \frac{\sigma_r^2}{\kappa_r} + [A_\beta(\tau) - A_\beta(\tau - \Delta \tau)]^2 \frac{\sigma_\beta^2}{\kappa_\beta}},$$

(A.115)

where

$$N_{FB,j} = \left[ A_j(\tau) - A_j(\tau - \Delta \tau) e^{-\kappa_j \Delta \tau} - A_j(\Delta \tau) \right] \left[ A_j(\tau) - A_j(\tau - \Delta \tau) - A_j(\Delta \tau) \right]$$

for $j = r, \beta$. Taking the limit in (A.115) when $\Delta \tau \to 0$, and noting from (A.75) and (A.76) that

$$\frac{A_j(\Delta \tau)}{\Delta \tau} \to 1 \quad \text{and} \quad \frac{A_j'(\Delta \tau)}{\Delta \tau} \to 0,$$

we find

$$b_{FB} = \frac{\left[ A'_r(\tau) + \kappa_r A_r(\tau) - 1 \right] [A'_r(\tau) - 1] \frac{\sigma_r^2}{\kappa_r} + \left[ A'_\beta(\tau) + \kappa_\beta A_\beta(\tau) \right] A'_\beta(\tau) \frac{\sigma_\beta^2}{\kappa_\beta}}{[A'_r(\tau) - 1]^2 \frac{\sigma_r^2}{\kappa_r} + A'_\beta(\tau))^2 \frac{\sigma_\beta^2}{\kappa_\beta}}.$$
For $\tau < \min\{\tilde{\tau}, \bar{\tau}\}$, $A_\tau(\tau) > 0$, $A_\beta(\tau) > 0$ and $A'\beta(\tau) > 0$. Moreover, (A.73) implies

$$A'_i(\tau) + \kappa_i A_\tau(\tau) - 1 = -a \sigma_i^2 I_{\tau,r} A_\tau(\tau) - a \sigma_i^2 I_{\tau,\beta} A_\beta(\tau) \leq 0,$$

$$A'_\beta(\tau) - 1 = - (\kappa_\tau + a \sigma_\tau^2 I_{\tau,r}) A_\tau(\tau) - a \sigma_\beta^2 I_{\tau,\beta} A_\beta(\tau) < 0,$$

where the inequalities follow from $A_\tau(\tau) > 0$, $A_\beta(\tau) > 0$, $I_{\tau,r} \geq 0$ and $I_{\tau,\beta} \geq 0$, which in turn follows from $M_{1,2} \geq 0$ and (A.26). Equations (A.116), $A_\beta(\tau) > 0$, $A'_\beta(\tau) > 0$, (A.117) and (A.118) imply $b_{FB} > 0$.

When $a \approx 0$, (A.75), (A.76) and $(\nu_1, \nu_2, \phi_\tau, \phi_\beta) \approx (\kappa_\tau, \kappa_\beta, a^3 \kappa_\tau, a^3 \kappa_\beta)$ (Lemma A.2) imply

$$b_{FB} = \frac{\nu_1 - \kappa_\tau}{\nu_1} (1 - e^{-\kappa_\tau})^{2} \frac{2}{\kappa_\tau} + a^{2} \frac{2}{\nu_1} \left[ L'_\beta(\tau) + \kappa_\beta L_\beta(\tau) \right] \left[ L'_\beta(\tau) \right]^{2} \frac{\sigma^2}{\kappa_\beta} + o(a^2),$$

where

$$L_\beta(\tau) \equiv \frac{1 - e^{-\kappa_\beta \tau}}{\kappa_\beta} - \frac{1 - e^{-\kappa_\tau \tau}}{\kappa_\tau}.$$

Since $L_\beta(\tau)L'_\beta(\tau) > 0$, and (A.85) and (A.93) imply

$$\nu_1 - \kappa_\tau = a \sigma_\tau^2 \int_0^{T} (1 - e^{-\kappa_\tau \tau}) d\tau + o(a^2),$$

$$b_{FB} > 0.$$

When $a \approx \infty$, (A.75), (A.76) and $(\nu_1, \nu_2, \phi_\tau, \phi_\beta) \approx (a^{1/2} \nu_1, a^{-1/2} \nu_2, \nu_2^{1/2} \tau, \nu_2^{1/2} \tau)$ (Lemma A.2) imply that for $\tau$ bounded away from zero

$$b_{FB} = \frac{\sigma^2}{\kappa_\tau} + \frac{\sigma^2}{\kappa_\beta} \left( e^{-\nu \tau} + \kappa_\beta \frac{1 - e^{-\nu \tau \nu}}{\nu^2} \right) e^{-\nu \tau} \frac{\sigma^2}{\kappa_\beta} + o(1) = 1 + \frac{\sigma^2}{\kappa_\beta} (1 - e^{-\nu \tau}) e^{-\nu \tau} \frac{\sigma^2}{\kappa_\beta} + o(1).$$

Hence, $b_{FB} > 1$. We next show that $b_{FB}$ increases in $\tau$ if (43) holds. Equation (43) implies that the left-hand side of (A.92) exceeds the right-hand side for $\nu_2 = 0$, and hence (A.92) has a solution $\nu_2 < 0$. We write (A.120) as

$$b_{FB} = 1 + \frac{\sigma^2}{\kappa_\beta} N_{FB}(\tau) \frac{\sigma^2}{\kappa_\beta} + o(1),$$

$$b_{FB} = 1 + \frac{\sigma^2}{\kappa_\beta} N_{FB}(\tau) \frac{\sigma^2}{\kappa_\beta} + o(1),$$

(43)
\[ N_{FB}(\tau) \equiv e^{2\tau} \frac{t}{z}, \]

\[ D_{FB}(\tau) \equiv e^{2\tau}, \]

and \( z \equiv -\eta_2 > 0 \), and consider the derivative

\[ \left[ \frac{\phi_2 N_{FB}(\tau) \sigma_{FB}^2}{\sigma_{FB}^2 + \phi_2 D_{FB}(\tau) \sigma_{FB}^2} \right]' = \frac{\sigma_{FB}^2 \phi_2^2 + \phi_2 D_{FB}(\tau) \sigma_{FB}^2}{\sigma_{FB}^2 + \phi_2 D_{FB}(\tau) \sigma_{FB}^2} \]

Since

\[ \frac{N_{FB}(\tau)}{D_{FB}(\tau)}' = \frac{1 - e^{-\tau}}{z} \]

\[ N_{FB}'(\tau) D_{FB}(\tau) - N_{FB}(\tau) D_{FB}'(\tau) > 0. \]

Since, in addition,

\[ N_{FB}'(\tau) = 2e^{2\tau} - e^{\tau} > 0, \]

\( b_{FB} \) increases in \( \tau \).

Proceeding as in the proof of Proposition 3, we find that the CS regression coefficient is

\[ b_{CS} = \frac{N_{CS,j} \text{Var}(r_t) + N_{CS,\beta} \text{Var}(\beta_t)}{\Delta \tau - \Delta \tau} \left\{ \left[ A_j(\tau) - \frac{A_j(\Delta \tau)}{\Delta \tau} \right]^2 \text{Var}(r_t) + \left[ A_\beta(\tau) - \frac{A_\beta(\Delta \tau)}{\Delta \tau} \right]^2 \text{Var}(\beta_t) \right\} \]

\[ = \frac{N_{CS,j} \sigma_{r}^2 + N_{CS,\beta} \sigma_{\beta}^2}{\Delta \tau - \Delta \tau} \left\{ \left[ A_j(\tau) - \frac{A_j(\Delta \tau)}{\Delta \tau} \right]^2 \sigma_{r}^2 + \left[ A_\beta(\tau) - \frac{A_\beta(\Delta \tau)}{\Delta \tau} \right]^2 \sigma_{\beta}^2 \right\}, \quad \text{(A.122)} \]

where

\[ N_{CS,j} = \left[ \frac{A_j(\tau - \Delta \tau)}{\tau - \Delta \tau} e^{-\kappa_j \Delta \tau} - A_j(\tau) \right] \left[ \frac{A_j(\tau)}{\tau} - \frac{A_j(\Delta \tau)}{\Delta \tau} \right] \]

for \( j = r, \beta \). Taking the limit in (A.122) when \( \Delta \tau \to 0 \), we find

\[ b_{CS} \rightarrow \left[ \frac{A_j(\tau)}{\tau} - [A_j'(\tau) + \kappa_j A_j(\tau)] \right] \left[ \frac{A_j(\tau)}{\tau} - 1 \right] \sigma_{r}^2 + \left[ \frac{A_\beta(\tau)}{\tau} - [A_\beta'(\tau) + \kappa_\beta A_\beta(\tau)] \right] \frac{A_\beta(\tau)}{\tau} \sigma_{\beta}^2 \]

\[ = 1 - [A_j'(\tau) + \kappa_j A_j(\tau) - 1] \left[ \frac{A_j(\tau)}{\tau} - 1 \right] \sigma_{r}^2 + [A_\beta'(\tau) + \kappa_\beta A_\beta(\tau)] \frac{A_\beta(\tau)}{\tau} \sigma_{\beta}^2. \quad \text{(A.123)} \]
For \( \tau < \min\{\bar{\tau}, \hat{\tau}\} \), \( A_\beta(\tau) > 0 \), \( A'_\beta(\tau) > 0 \), and (A.117) and (A.118) hold. Equation (A.118) and the initial condition \( A_r(0) = 0 \) imply \( A_r(\tau) - \tau < 0 \). Equations (A.9), \( A_\beta(\tau) > 0 \), \( A'_\beta(\tau) > 0 \), (A.117) and \( A_r(\tau) - \tau < 0 \) imply \( b_{cs} < 1 \).

When \( a \approx 0 \), (A.75), (A.76), \((\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (\kappa_r, \kappa_\beta, a^{d_c}, a^{c_2}) \) (Lemma A.2) and (A.119) imply

\[
b_{cs} = 1 - a \frac{\sigma_r^2 (1 - e^{-\kappa_r \tau})}{\kappa_r (1 - \frac{1 - e^{-\kappa_r \tau}}{\kappa_r \tau})} \int_0^{\tau} a(\tau) \left( \frac{1 - e^{-\kappa_r \tau}}{\kappa_r \tau} \right)^2 d\tau + o(a).
\]

Hence, \( b_{cs} \) is smaller than and close to one. Moreover, \( b_{cs} \) increases in \( \tau \) because the function \( K(x) \) defined in Proposition 3 is increasing for \( x > 0 \).

When \( a \approx \infty \), (A.75), (A.76) and \((\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (a^{d_1}, a^{1/2}, a^{-1/2}, \phi_\beta) \) (Lemma A.2) imply that for \( \tau \) bounded away from zero

\[
b_{cs} = 1 - \frac{\sigma_r^2 + \phi_\beta^2 \left( e^{-\nu_2 \tau} + \kappa_\beta \frac{1 - e^{-\nu_2 \tau}}{\nu_2 \tau} \right) \left( \frac{1 - e^{-\nu_2 \tau}}{\nu_2 \tau} \right)^2 \sigma_\beta^2}{\sigma_r^2 + \phi_\beta^2 \left( \frac{1 - e^{-\nu_2 \tau}}{\nu_2 \tau} \right)^2 \sigma_\beta^2} + o(1).
\]  

(A.124)

Hence, \( b_{cs} < 1 \). We next show that \( b_{cs} \) is negative and decreasing in \( \tau \) if (43) holds. We write (A.124) as

\[
b_{cs} = 1 - \frac{\sigma_r^2 + \phi_\beta N_{cs}(\tau) \sigma_\beta^2}{\sigma_r^2 + \phi_\beta D_{cs}(\tau) \sigma_\beta^2} + o(1),
\]

(A.125)

where

\[
N_{cs}(\tau) \equiv \left( e^{z\tau} + \kappa_\beta \frac{e^{z\tau} - 1}{z} \right) \left( \frac{e^{z\tau} - 1}{z} \right),
\]

\[
D_{cs}(\tau) \equiv \left( \frac{e^{z\tau} - 1}{z} \right)^2,
\]

and \( z \equiv -\nu_2 > 0 \). Equation (A.125) implies

\[
b_{cs} = - \frac{\phi_\beta^2 \left[ N_{cs}(\tau) - D_{cs}(\tau) \right] \sigma_\beta^2}{\sigma_r^2 + \phi_\beta D_{cs}(\tau) \sigma_\beta^2} + o(1).
\]

(A.126)
Since
\[ N_{CS}(\tau) - D_{CS}(\tau) = \left[ e^{z\tau} + \left( \frac{\kappa_\beta - 1}{\tau} \right) \frac{e^{z\tau} - 1}{z} \right] \frac{e^{z\tau} - 1}{z\tau} \]

and \( xe^x - e^x + 1 > 0 \) for all \( x \), (A.126) implies \( b_{CS} < 0 \). Consider next the derivative
\[
\left[ \frac{\sigma^2}{\kappa_r} + \frac{\phi_\beta}{\kappa_r} N_{CS}(\tau) \frac{\sigma^2}{\kappa_r} \right]' = \frac{\sigma^2}{\kappa_r} \phi_\beta \left[ N'_{CS}(\tau) - D'_{CS}(\tau) \right] + \phi_\beta \left[ N'_{CS}(\tau) D_{CS}(\tau) - N_{CS}(\tau) D'_{CS}(\tau) \right].
\]

Since
\[ N'_{CS}(\tau) - D'_{CS}(\tau) = \left[ ze^{z\tau} + \left( \frac{\kappa_\beta - 1}{\tau} \right) \frac{e^{z\tau} - 1}{z} \right] \frac{e^{z\tau} - 1}{z\tau} + \left[ e^{z\tau} + \left( \frac{\kappa_\beta - 1}{\tau} \right) \frac{e^{z\tau} - 1}{z} \right] \frac{2\tau e^{z\tau} - z (e^{z\tau} - 1)}{z^2} \]

and \( x^2 e^x - xe^x + e^x - 1 > 0 \) for all \( x \), \( N'_{CS}(\tau) - D'_{CS}(\tau) > 0 \). Since
\[ \left[ \frac{N_{CS}(\tau)}{D_{CS}(\tau)} \right]' = \left[ \frac{z\tau e^{z\tau}}{e^{z\tau} - 1} \right]' = ze^{z\tau} \frac{(1 + z\tau)(e^{z\tau} - 1) - z\tau e^{z\tau}}{(e^{z\tau} - 1)^2} = ze^{z\tau} \frac{e^{z\tau} - 1 - z\tau}{(e^{z\tau} - 1)^2} \]

and \( e^x - 1 - x > 0 \) for all \( x \), \( N'_{CS}(\tau) D_{CS}(\tau) - N_{CS}(\tau) D'_{CS}(\tau) > 0 \). Hence, \( b_{CS} \) decreases in \( \tau \).

**Proof of Proposition 7:** Substituting \( C(\tau) \) from (41) into (42), using \( \Gamma = \text{Diag}(\kappa_r, \kappa_\beta) \) and \( \Sigma = \text{Diag}(\sigma^2_{\tau r}, \sigma^2_{\beta}) \), and dropping the subscript 1 from functions of the single demand factor, we
The system of (A.127) and (A.128) is linear in \((\chi_r, \chi_\beta)\) and its solution is
\[
\chi_r = \frac{1}{D} \left\{ \kappa_r T + a\sigma_r^2 \int_0^T \theta_0(\tau) A_r(\tau) d\tau + C_r \right\} + \frac{1}{a\sigma_r^2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(\tau) d\tau \right) A_r(\tau) d\tau
- \frac{a\sigma_r^2}{2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(\tau) d\tau \right) A_r(\tau) d\tau
- \frac{a\sigma_r^2}{2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(\tau) d\tau \right) A_r(\tau) d\tau
- \frac{a\sigma_r^2}{2} \int_0^T \alpha(\tau) A_r(\tau) d\tau
+ \frac{a\sigma_r^2}{2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(\tau) d\tau \right) A_r(\tau) d\tau
\]
(A.127)
\[
\chi_\beta = \frac{1}{D} \left\{ \kappa_\beta T + a\sigma_\beta^2 \int_0^T \theta_0(\tau) A_\beta(\tau) d\tau + C_\beta \right\} + \frac{1}{a\sigma_\beta^2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(\tau) d\tau \right) A_\beta(\tau) d\tau
- \frac{a\sigma_\beta^2}{2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(\tau) d\tau \right) A_\beta(\tau) d\tau
- \frac{a\sigma_\beta^2}{2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(\tau) d\tau \right) A_\beta(\tau) d\tau
- \frac{a\sigma_\beta^2}{2} \int_0^T \alpha(\tau) A_\beta(\tau) d\tau
+ \frac{a\sigma_\beta^2}{2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(\tau) d\tau \right) A_\beta(\tau) d\tau
\]
(A.128)
where
\[
D \equiv \left[ 1 + a\sigma_r^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(\tau) d\tau \right) A_r(\tau) d\tau \right] + a\sigma_r^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(\tau) d\tau \right) A_\beta(\tau) d\tau
\]
and
\[
C_j = \frac{a\sigma_r^2 a\sigma_\beta^2}{2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(\tau) d\tau \right) A_j(\tau) d\tau
+ \frac{a\sigma_r^2 a\sigma_\beta^2}{2} \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(\tau) d\tau \right) A_j(\tau) d\tau
\]
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for \( j = r, \beta \). The effect of a change in the demand intercept from \( \theta_0(\tau) \) to \( \theta_0(\tau) + \Delta \theta_0(\tau) \) on the yield \( y_i(\tau) \) for maturity \( \tau \) is \( \frac{\Delta C(\tau)}{\tau} \), which from (41), (A.129) and (A.130) is

\[
\Delta y_i^{(\tau)} = \frac{1}{D} \left\{ a\sigma_r^2 \int_0^T \Delta \theta_0(\tau) A_r(\tau) d\tau \left[ 1 + a\sigma_\beta^2 \int_0^T A_\beta(u) du \right] \right\} \int_0^T A_r(u) du \frac{d\tau}{\tau} - \left[ a\sigma_r^2 \int_0^T \Delta \theta_0(\tau) A_\beta(\tau) d\tau \left[ a\sigma_r^2 \int_0^T A_r(u) du \right] \right] \int_0^T A_r(u) du \frac{d\tau}{\tau} + \frac{1}{D} \left\{ \left[ a\sigma_\beta^2 \int_0^T \Delta \theta_0(\tau) A_r(\tau) d\tau \left[ 1 + a\sigma_r^2 \int_0^T A_r(u) du \right] \right] \right\} \int_0^T A_r(u) du \frac{d\tau}{\tau}
\]

(A.131)

Hence, the change \( \Delta \theta_0(\tau) \) affects yields only through \( \int_0^T \Delta \theta_0(\tau) A_r(\tau) d\tau \) and \( \int_0^T \Delta \theta_0(\tau) A_\beta(\tau) d\tau \).

When the change \( \Delta \theta_0(\tau) \) is a Dirac function with point mass at \( \tau^* \),

\[
\int_0^T \Delta \theta_0(\tau) A_j(\tau) d\tau = A_j(\tau^*)
\]

for \( j = r, \beta \), and (A.131) becomes

\[
\Delta y_i^{(\tau^*)} = \frac{1}{D} \left\{ \Lambda_r(\tau^*) \int_0^T A_r(u) du \frac{d\tau}{\tau} + \Lambda_\beta(\tau^*) \int_0^T A_\beta(u) du \frac{d\tau}{\tau} \right\}, \tag{A.132}
\]

where

\[
\Lambda_r(\tau^*) \equiv a\sigma_r^2 A_r(\tau^*) \left[ 1 + a\sigma_\beta^2 \int_0^T A_\beta(u) du \right] \left[ a\sigma_r^2 \int_0^T A_r(u) du \right] \left[ a\sigma_r^2 \int_0^T A_r(u) du \right] \frac{d\tau}{\tau} - \left[ a\sigma_\beta^2 A_\beta(\tau^*) \left[ a\sigma_r^2 \int_0^T A_r(u) du \right] \left[ a\sigma_r^2 \int_0^T A_r(u) du \right] \frac{d\tau}{\tau} \right],
\]

\[
\Lambda_\beta(\tau^*) \equiv a\sigma_\beta^2 A_\beta(\tau^*) \left[ 1 + a\sigma_r^2 \int_0^T A_r(u) du \right] \left[ a\sigma_r^2 \int_0^T A_r(u) du \right] \left[ a\sigma_r^2 \int_0^T A_r(u) du \right] \frac{d\tau}{\tau} - \left[ a\sigma_\beta^2 A_r(\tau^*) \left[ a\sigma_\beta^2 \int_0^T A_\beta(u) du \right] \left[ a\sigma_r^2 \int_0^T A_r(u) du \right] \frac{d\tau}{\tau} \right].
\]
Using (A.132), we can write (44) in the equivalent form

\[
\left[ \Lambda_r(\tau_1) \int_0^{\tau_1} A_r(u) du \right. \\
\left. \int \Lambda_\beta(\tau_1) \int_0^{\tau_1} A_\beta(u) du \right]
\left[ \Lambda_r(\tau_2) \int_0^{\tau_2} A_r(u) du \right. \\
\left. \int \Lambda_\beta(\tau_2) \int_0^{\tau_2} A_\beta(u) du \right]
= \Lambda_\beta(\tau_1) \int_0^{\tau_1} A_\beta(u) du \int_0^{\tau_1} A_\beta(u) du
\]

To show that (A.133) holds, we show that each of the two terms in brackets is positive. The second term is positive because it has the same sign as

\[
\int_0^{\tau_1} A_r(\tau_1) \int_0^{\tau_1} A_\beta(\tau_1) du \\
\int_0^{\tau_1} A_\beta(\tau_1) \int_0^{\tau_1} A_\beta(\tau_1) du
\]

where the second step follows because \( A_\beta(\tau) > 0 \) and \( \left[ \frac{A_\beta(\tau)}{A_\beta(\tau)} \right]' < 0 \) for \( \tau \in (0, \hat{\tau}) \). The first term is equal to

\[
[A_r(\tau_1)A_\beta(\tau_2) - A_r(\tau_2)A_\beta(\tau_1)] D,
\]

and is positive if \( D > 0 \) since \( A_\beta(\tau) > 0 \) and \( \left[ \frac{A_\beta(\tau)}{A_\beta(\tau)} \right]' < 0 \) for \( \tau \in (0, \hat{\tau}) \). Integration by parts implies that for \( j = r, \beta \)

\[
\int_0^T \alpha(\tau) \left( \int_0^\tau A_j(u) du \right) A_j(\tau) d\tau
= \left[ \alpha(\tau) \left( \int_0^\tau A_j(u) du \right)^2 \right]_0^T - \int_0^T \left[ \alpha'(\tau) \left( \int_0^\tau A_j(u) du \right) + \alpha(\tau) A_j(\tau) \right] \left( \int_0^\tau A_j(u) du \right) d\tau,
\]

(A.134)

When \( T \) is finite,

\[
\left[ \alpha(\tau) \left( \int_0^\tau A_j(u) du \right)^2 \right]_0^T = \alpha(T) \left( \int_0^T A_j(u) du \right)^2.
\]

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When $T = \infty$, 

\[
\left[ \alpha(\tau) \left( \int_0^\tau A_j(u) du \right)^2 \right]^T_0 = \lim_{\tau \to \infty} \left[ \alpha(\tau) \left( \int_0^\tau A_j(u) du \right)^2 \right] = 0,
\]

where the second step follows because $M$ is finite. Denoting by $d\hat{\alpha}(\tau)$ the measure generated by the non-decreasing function $\hat{\alpha}(\tau)$ defined by $\hat{\alpha}(\tau) = -\alpha(\tau)$ for $\tau \in (0, T)$ and $\hat{\alpha}(T) = 0$ for $\tau = T$, we can nest the cases $T$ finite and $T = \infty$ by writing (A.134) as

\[
\int_0^T \alpha(\tau) \left( \int_0^\tau A_j(u) du \right) A_j(\tau) d\tau \\
= \int_0^T \left( \int_0^\tau A_j(u) du \right)^2 d\hat{\alpha}(\tau) - \int_0^T \alpha(\tau) \left( \int_0^\tau A_j(u) du \right) A_j(\tau) d\tau \\
\Rightarrow \int_0^T \alpha(\tau) \left( \int_0^\tau A_j(u) du \right) A_j(\tau) d\tau = \frac{\int_0^T \left( \int_0^\tau A_j(u) du \right)^2 d\hat{\alpha}(\tau)}{2} \geq 0, \quad (A.135)
\]

Likewise,

\[
\int_0^T \alpha(\tau) \left( \int_0^\tau A_r(u) du \right) A_\beta(\tau) d\tau + \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(u) du \right) A_r(\tau) d\tau \\
= 2 \left[ \alpha(\tau) \left( \int_0^\tau A_r(u) du \right) \left( \int_0^\tau A_\beta(u) du \right) \right]^T_0 \\
- \int_0^T \left[ \alpha'(\tau) \left( \int_0^\tau A_r(u) du \right) + \alpha(\tau) A_r(\tau) \right] \left( \int_0^\tau A_\beta(u) du \right) d\tau \\
- \int_0^T \left[ \alpha'(\tau) \left( \int_0^\tau A_\beta(u) du \right) + \alpha(\tau) A_\beta(\tau) \right] \left( \int_0^\tau A_r(u) du \right) d\tau \\
\Rightarrow \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(u) du \right) A_\beta(\tau) d\tau + \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(u) du \right) A_r(\tau) d\tau \\
= \int_0^T \left( \int_0^\tau A_r(u) du \right) \left( \int_0^\tau A_\beta(u) du \right) d\hat{\alpha}(\tau),
\]

and hence

\[
\left[ \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(u) du \right) A_\beta(\tau) d\tau \right] \left[ \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(u) du \right) A_r(\tau) d\tau \right] \\
\leq \frac{\int_0^T \left( \int_0^\tau A_r(u) du \right) \left( \int_0^\tau A_\beta(u) du \right) d\hat{\alpha}(\tau)}{4}. \quad (A.136)
\]
Equations (A.135) and (A.136) imply that $D > 0$ if
\[
\left[ \int_0^T \left( \int_0^\tau A_r(u) du \right)^2 d\alpha(\tau) \right] \left[ \int_0^T \left( \int_0^\tau A_\beta(u) du \right)^2 d\alpha(\tau) \right] \\
\geq \left[ \int_0^T \left( \int_0^\tau A_r(u) du \right) \left( \int_0^\tau A_\beta(u) du \right) d\alpha(\tau) \right]^2,
\]
which holds because of the Cauchy-Schwarz inequality.
B Demand of Preferred-Habitat Investors

There are overlapping generations of preferred-habitat investors living for a period of length $T$, and of arbitrageurs living for a period of length $dt$. Thus, at each point in time there is a continuum of investor generations and one arbitrageur generation. Arbitrageurs and investors receive endowment $W$ at the beginning of their life and consume at the end of their life. Arbitrageurs use their endowment to buy bonds. Investors use their endowment to buy bonds and to invest in a private opportunity (“real estate”) that pays at the end of their life. To ensure that the slope of the investors’ demand for bonds is finite, we require that substitution between bonds and the private opportunity is imperfect. We model imperfect substitution by assuming that bonds pay in a good 1 (“money”) and the private opportunity pays in a different good 2 (“real estate services”). The endowment $W$ is in good 1. Arbitrageurs and investors can use good 1 to invest in bonds and in the private opportunity.

Consider the optimization problem of an investor $n$ born at time 0. We denote by $\hat{Z}_{n,t}^{(\tau)}$ the number of units of the bond with maturity $\tau$ that the investor holds at time $t \in [0, T]$, where one unit of the bond is an investment in the bond with face value one. We denote by $W_{n,t}$ the value of the investor’s bond portfolio at time $t$ and by $dc_{n,t}$ the investment in the private opportunity between $t$ and $t + dt$, both expressed in units of good 1. We denote by $(\hat{W}_{n,t}, \hat{d}c_{n,t})$ the counterparts of $(W_{n,t}, dc_{n,t})$ when expressed in units of the bond maturing at time $T$:

$$\hat{W}_{n,t} \equiv \frac{W_{n,t}}{P_t(T-t)},$$

$$\hat{d}c_{n,t} \equiv \frac{dc_{n,t}}{P_t(T-t)}.$$

We finally denote by $\hat{\beta}_{n,t}^{(T-t)} > 0$ the number of units of good 2 that an investment of one unit of good 1 at time $t$ yields at time $T$. The investor’s budget constraint is

$$d\hat{W}_{n,t} = \int_0^T \hat{Z}_{n,t}^{(\tau)} d \left( \frac{P_t^{(\tau)}}{P_t(T-t)} \right) d\tau - d\hat{c}_{n,t}. \quad (B.1)$$

The investor’s utility at time $T$ is

$$u(C_T) + \int_0^T \hat{\beta}_{n,t}^{(T-t)} P_t^{(T-t)} d\hat{c}_{n,t}, \quad (B.2)$$
and consists of two parts: a utility $u(C_T)$ that is an increasing and concave function of the consumption $C_T$ of good 1 at time $T$, and a utility $\int_0^T \beta_n^{(T-t)} P_t^{(T-t)} \, d\hat{c}_{n,t}$ that is equal to the consumption of good 2 at time $T$ and is derived from the accumulated investment in the private opportunity between times 0 and $T$. The marginal utility $u'(C_T)$ converges to infinity when $C_T$ goes to a lower bound $C$ and to zero when $C_T$ goes to infinity. The investor has max-min preferences. At each time $t \in [0, T]$, the investor chooses $(\hat{Z}^{(T-t)}_{n,t}, \hat{c}_{n,t})$ to maximize the minimum of (B.2) over sample paths of $q_t = (r_t, \beta_{1,t}, ..., \beta_{K,t})^\top$ and $\hat{\beta}_{n,t}^{(T-t)}$, subject to the budget constraint (B.1) and the terminal condition $C_T = \hat{W}_T$.

**Proposition B.1.** Assume that the term structure involves no arbitrage, i.e., (34) holds, and that $\beta_{n,t}^{(T-t)}$ can depend on $q_t$ only through $(\beta_{1,t}, ..., \beta_{K,t})^\top$. At time $t$, the investor holds only the bond maturing at time $T$ and no other bonds. The number $\hat{Z}^{(T-t)}_{n,t}$ of units of the bond held by the investor solves

$$u'(\hat{Z}^{(T-t)}_{n,t}) = P_t^{(T-t)} \beta_{n,t}^{(T-t)}.$$  \hfill (B.3)

**Proof:** Defining $(\mu_{Z,n,t}, \sigma_{Z,n,t})$ by

$$\int_0^T \hat{Z}^{(T-t)}_{n,t} \, d \left( \frac{P_t^{(T-t)}}{P_{t}^{(T-t)}} \right) \, d\tau \equiv \mu_{Z,n,t} \, dt + \sigma_{Z,n,t} \, dB_t,$$

where $dB_t = (dB_{r,t}, dB_{\beta_{1,t}}, ..., dB_{\beta_{K,t}})^\top$, we write the budget constraint (B.1) as

$$d\hat{W}_{n,t} = \mu_{Z,n,t} \, dt + \sigma_{Z,n,t} \, dB_t - d\hat{c}_{n,t}.$$  \hfill (B.4)

Integrating (B.4) from 0 to $T$ and using the terminal condition $C_T = \hat{W}_T$, we write the investor’s optimization problem at $t = 0$ as

$$\max_{\hat{Z}^{(T-t)}_{n,t}, \hat{c}_{n,t} \in \hat{\beta}_{n,t}^{(T-t)}} \left[ u \left( \hat{W}_0 + \int_0^T \mu_{Z,n,t} \, dt + \int_0^T \sigma_{Z,n,t} \, dB_t - \Delta \hat{c}_{0,n} - \int_0^T d\hat{c}_{n,t} \right) + \beta_{0,n}^{(T)} P_0^{(T)} \Delta \hat{c}_{n,t} + \int_0^T P_t^{(T-t)} \, d\hat{c}_{n,t} \right],$$  \hfill (B.5)

where we allow for the possibility that $\hat{c}_t$ has a discrete change $\Delta \hat{c}_{0,n}$ at $t = 0$. Since $\beta_{n,t}^{(T-t)}$ can depend on $q_t$ only through $(\beta_{1,t}, ..., \beta_{K,t})$, and $r_t$ is not perfectly correlated with $(\beta_{1,t}, ..., \beta_{K,t})$, sample
paths of $q_t$ and $\hat{\beta}(T-t)$ exist such that $\hat{\beta}(T-t)P_t(T-t) = u'(\hat{W}_0 - \Delta \hat{c}_0)$ for $t > \epsilon$ and for any $\epsilon > 0$. Hence, the minimum in (B.5) is smaller than

$$\min_{q_t, \hat{\beta}(T-t)} \left[ u \left( \hat{W}_0 + \int_0^T \mu_{\hat{Z},n,t} dt + \int_0^T \sigma_{\hat{Z},n,t} dB_t - \Delta \hat{c}_{0,n} - \int_0^T d\hat{c}_{n,t} \right) + \hat{\beta}_{0,n} P_0^{(T)} \hat{c}_{0,n} + u' \left( \hat{W}_0 - \Delta \hat{c}_0 \right) \int_0^T d\hat{c}_{n,t} \right],$$

which in turn is smaller than

$$\min_{q_t, \hat{\beta}(T-t)} \left[ u \left( \hat{W}_0 - \Delta \hat{c}_0 \right) + u' \left( \hat{W}_0 - \Delta \hat{c}_0 \right) \left( \int_0^T \mu_{\hat{Z},n,t} dt + \int_0^T \sigma_{\hat{Z},n,t} dB_t \right) + \hat{\beta}_{0,n} P_0^{(T)} \Delta \hat{c}_{0,n} \right] \quad \text{(B.6)}$$

because $u$ is concave. If $\sigma_{\hat{Z},n,t} \neq 0$ for any interval in $(0, T)$, then the minimum in (B.6) is minus infinity because the Brownian motion has infinite variation. Therefore, $\sigma_{\hat{Z},n,t} = 0$, i.e., the investor holds the bond maturing at time $T$ and zero units of all other bonds. Since absence of arbitrage requires $\mu_{\hat{Z},n,t} = 0$, (B.6) is smaller than

$$u \left( \hat{W}_0 - \Delta \hat{c}_0 \right) + \hat{\beta}_{0,n} P_0^{(T)} \Delta \hat{c}_{0,n},$$

and hence

$$\max_{\hat{Z}_{n,t}, \hat{c}_{n,t}} \min_{q_t, \hat{\beta}(T-t)} \left[ u \left( \hat{W}_0 + \int_0^T \mu_{\hat{Z},n,t} dt + \int_0^T \sigma_{\hat{Z},n,t} dB_t - \Delta \hat{c}_{0,n} - \int_0^T d\hat{c}_{n,t} \right) + \hat{\beta}_{0,n} P_0^{(T)} \Delta \hat{c}_{0,n} + \int_0^T \hat{\beta}_{0,n} P_0^{(T-t)} d\hat{c}_{n,t} \right] \leq \max_{\Delta \hat{c}_{0,n}} \left[ u \left( \hat{W}_0 - \Delta \hat{c}_0 \right) + \hat{\beta}_{0,n} P_0^{(T)} \Delta \hat{c}_{0,n} \right]. \quad \text{(B.7)}$$

Setting $\hat{Z}_{n,t}(\tau) = 0$ for $t \geq 0$ and $\tau \neq T - t$, and $d\hat{c}_{n,t} = 0$ for $t > 0$, in (B.5), we find that (B.7) holds also in the reverse sense, and is therefore an equality. The optimal $\Delta \hat{c}_{0,n}$ thus satisfies

$$u' \left( \hat{W}_0 - \Delta \hat{c}_0 \right) = \hat{\beta}_{0,n} P_0^{(T)} \Delta \hat{c}_{0,n}. \quad \text{(B.8)}$$

Since $\hat{W}_0 - \Delta \hat{c}_{0,n}$ represents units of the bond maturing at time $T$ that the investor holds at time $0$, (B.8) yields (B.3) for $t = 0$. The same argument yields (B.3) for $t > 0$.  

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Proposition B.1 implies that preferred-habitat investors demand only the bond whose maturity coincides with the time when they consume. To ensure that the demand by preferred-habitat investors takes the specific functional form (5)-(7), we assume specific functions for the utility \( u \) and the return \( \hat{\beta}_{n,t}^{(T-t)} \) on the private opportunity.

Suppose \( C = -\infty \) \( u(C_T) = -e^{-C_T} \) and \( \hat{\beta}_{n,t}^{(T)} = e^{\beta_{n,t}^{(T)}} \), where \( \beta_{n,t}^{(T)} \) is given by (6) and (7). Proposition B.1 implies that the number \( \hat{Z}_{n,t}^{(T-t)} \) of units of the bond maturing at time \( T \) and held at time \( t \) by an investor born at time 0 is given by

\[
e^{-\hat{Z}_{n,t}^{(T-t)}} = P^T_t(T-t) \hat{\beta}_{n,t}^{(T-t)} \iff \hat{Z}_{n,t}^{(T-t)} = -\log \left( P^T_t(T-t) \right) \hat{\beta}_{n,t}^{(T-t)}.
\]

This coincides with the demand (5)-(7) with \( \alpha(T) = 1 \), except that (5)-(7) concern the dollar value of the bond rather than the number of units of the bond. To derive the demand (5)-(7) for the dollar value, we modify the assumed functions for \( u \) and \( \hat{\beta}_{n,t}^{(T-t)} \). We can obtain the demand (5)-(7) for a set of values of \( q_t \) whose probability can be made arbitrarily close to one.

Suppose that there are two types of preferred-habitat investors born at each time \( t \), in equal measure. For type 1 investors, \( C = 0 \), \( u(C_t + T) = \log(C_t + T) \) and \( \hat{\beta}_{n,t}^{(T-t)} = -\frac{1}{\min(\beta_{n,t}^{(T-t)}, -\epsilon)} \), where \( \beta_{n,t}^{(T-t)} \) is given by (6) and (7), and \( \epsilon \) is positive and small. For type 2 investors, \( C = -\infty \) and \( \hat{\beta}_{n,t}^{(T-t)} = 1 \). To define \( u(C_t + T) \) for type 2 investors, we start with the function

\[
N(x) \equiv -\frac{\log(x)}{x},
\]

defined for \( x > 0 \). The function \( N(x) \) converges to infinity when \( x \) goes to zero, and to zero when \( x \) goes to infinity. It decreases for \( x \in (0, e) \), and increases for \( x \in (e, \infty) \). Its minimum value, obtained for \( x = e \), is \(-\frac{1}{e}\). We take \( x \) to represent marginal utility \( u'(C_t + T) \), and \( N(x) \) to represent \( C_t + T \). This defines \( u(C_t + T) \) for \( C_t + T > -\frac{1}{e} \) and \( u'(C_t + T) \in (0, e) \). To define \( u(C_t + T) \) for \( C_t + T < -\frac{1}{e} \) and \( u'(C_t + T) > e \), we extend \( u'(C_t + T) \) as a linear function of \( C_t \). (Other extensions are possible as well.) We set the derivative of the linear function so that \( u'(C_t + T) \) is continuously differentiable at the extension point, and take the extension point to be \( u'(C_t + T) = e(1 - \epsilon) \) (rather
than \( u'(C_{t+T}) = e \) so that the derivative is finite. We thus set
\[
u'(C_{t+T}) = \begin{cases} N^{-1}(C_{t+T}) & \text{for } C_{t+T} \geq N[e(1 - \epsilon)], \\ e(1 - \epsilon) - \frac{e^2(1 - \epsilon)^2}{\log(1 - \epsilon)}[C_{t+T} - N[e(1 - \epsilon)]] & \text{for } C_{t+T} < N[e(1 - \epsilon)]. \end{cases}
\]

Since \( u'(C_{t+T}) \) is positive and decreasing, \( u(C_{t+T}) \) is increasing and concave.

Proposition B.1 implies that the number \( \hat{Z}_{n,t}^{(T-t)} \) of units of the bond maturing at time \( T \) and held at time \( t \) by a type 1 investor born at time 0 is given by
\[
\frac{1}{Z_{n,t}^{(T-t)}} = P_{t}^{(T-t)} \hat{\beta}_{n,t}^{(T-t)}.
\]
This yields the dollar demand
\[
P_{t}^{(T-t)} \hat{Z}_{n,t}^{(T-t)} = \frac{1}{\hat{\beta}_{n,t}^{(T-t)}} = -\beta_{t}^{(T-t)}
\]
when \( \beta_{t}^{(T-t)} < -\epsilon \). Proposition B.1 implies that the number \( \hat{Z}_{n,t}^{(T-t)} \) of units of the bond maturing at time \( T \) and held at time \( t \) by a type 2 investor born at time 0 is given by
\[
N^{-1} \left( \hat{Z}_{n,t}^{(T-t)} \right) = P_{t}^{(T-t)}
\]
when \( P_{t}^{(T-t)} < e(1 - \epsilon) \). This yields the dollar demand
\[
P_{t}^{(T-t)} \hat{Z}_{n,t}^{(T-t)} = P_{t}^{(T-t)} N \left( P_{t}^{(T-t)} \right) = -\log \left( P_{t}^{(T-t)} \right).
\]
The aggregate dollar demand across type 1 and type 2 investors when \( \beta_{t}^{(T-t)} < -\epsilon \) and \( P_{t}^{(T-t)} < e(1 - \epsilon) \) is
\[
-\log \left( P_{t}^{(T-t)} \right) - \beta_{t}^{(T-t)}
\]
and coincides with the demand (5)-(7) with \( \alpha(\tau) = 1 \). Condition \( \beta_{t}^{(T-t)} < -\epsilon \) requires that the demand intercept in (5) is negative (smaller than \( -\epsilon \)). Condition \( P_{t}^{(T-t)} < e(1 - \epsilon) \) requires that zero-coupon bonds trade below \( e(1 - \epsilon) \) and hence below par value. The probability of the set of values of \( q_{t} \) such that the two conditions hold simultaneously can be made arbitrarily close to one if \( \bar{T} \) is sufficiently large and \( \theta_{0}(\tau) \) sufficiently small.
C Calibration

The FB and CS regression coefficients in Figure 1 are derived from (A.115) and (A.122), respectively, by setting $\Delta \tau = 1$. Equations (1) and (30) imply that when there is one demand factor which is independent of the short rate, the yield for maturity $\tau$ is

$$y_t(\tau) = \frac{A_r(\tau)r_t + A_\beta(\tau)\beta_t + C(\tau)}{\tau}$$

and its unconditional volatility is

$$\sigma\left(y_t(\tau)\right) = \sqrt{\frac{A_r(\tau)^2\text{Var}(r_t) + A_\beta(\tau)^2\text{Var}(\beta_t)}{\tau} + \frac{\sqrt{A_r(\tau)^2\frac{\sigma_\tau^2}{\kappa\tau} + A_\beta(\tau)^2\frac{\sigma_\tau^2}{\kappa\tau}}}{\tau}.}

Consider next an unanticipated change $\Delta \theta_0(\tau)$ in the demand intercept at time zero that reverts deterministically to zero at the rate $\kappa_\theta$. Writing bond prices at time $t$ as

$$P_t(\tau) = e^{-\left[A_r(\tau)r_t + A_\beta(\tau)\beta_t + A_\theta(\tau)\Delta \theta_0(\tau)\right] + \frac{\sigma_\theta^2}{2}C(\tau)}$$

and proceeding as in Sections 3 and 4, we find that $A_\theta(\tau)$ solves the ODE

$$A_\theta'(\tau) + \kappa_\theta A_\theta(\tau) = \alpha \sigma_\tau^2 A_r(\tau) \int_0^T \left[ \Delta \theta_0(\tau) - \alpha(\tau) A_\theta(\tau) \right] A_r(\tau)(\tau) + \alpha \sigma_\tau^2 A_\beta(\tau) \int_0^T \left[ \Delta \theta_0(\tau) - \alpha(\tau) A_\theta(\tau) \right] A_\beta(\tau).$$

Proceeding as in the proofs of Lemma 3 and Proposition 7, we find that the solution to the ODE is

$$A_\theta(\tau) = \chi_r \int_0^T A_r(u)e^{-\kappa_\theta(\tau-u)}du + \chi_\beta \int_0^T A_\beta(u)e^{-\kappa_\theta(\tau-u)}du,$$

where

$$\chi_r = \frac{1}{D} \left\{ \alpha \sigma_\tau^2 \left[ \int_0^T \Delta \theta_0(\tau) A_r(\tau)d\tau \right] \left[ 1 + \alpha \sigma_\tau^2 \int_0^T \alpha(\tau) \left( \int_0^T A_\beta(u)e^{-\kappa_\theta(\tau-u)}du \right) A_\beta(\tau)d\tau \right] - \alpha \sigma_\tau^2 \left[ \int_0^T \Delta \theta_0(\tau) A_\beta(\tau)d\tau \right] \left[ \alpha \sigma_\tau^2 \int_0^T \alpha(\tau) \left( \int_0^T A_\beta(u)e^{-\kappa_\theta(\tau-u)}du \right) A_\beta(\tau)d\tau \right] \right\},$$

$$\chi_\beta = \frac{1}{D} \left\{ \alpha \sigma_\beta^2 \left[ \int_0^T \Delta \theta_0(\tau) A_\beta(\tau)d\tau \right] \left[ 1 + \alpha \sigma_\tau^2 \int_0^T \alpha(\tau) \left( \int_0^T A_r(u)e^{-\kappa_\theta(\tau-u)}du \right) A_r(\tau)d\tau \right] - \alpha \sigma_\tau^2 \left[ \int_0^T \Delta \theta_0(\tau) A_r(\tau)d\tau \right] \left[ \alpha \sigma_\tau^2 \int_0^T \alpha(\tau) \left( \int_0^T A_r(u)e^{-\kappa_\theta(\tau-u)}du \right) A_\beta(\tau)d\tau \right] \right\},$$

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and

$$D = \left[ 1 + a\sigma_r^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du \right) A_r(\tau)d\tau \right]$$

$$\times \left[ 1 + a\sigma_\beta^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du \right) A_\beta(\tau)d\tau \right]$$

$$- \left[ a\sigma_r^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du \right) A_r(\tau)d\tau \right]$$

$$\times \left[ a\sigma_\beta^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du \right) A_\beta(\tau)d\tau \right].$$

When the change $\Delta \theta_0(\tau)$ is a Dirac function with point mass at $\tau^*$,

$$\int_0^T \Delta \theta_0(\tau) A_j(\tau)d\tau = A_j(\tau^*)$$

for $j = r, \beta$. Hence, the time-zero change in the yield for maturity $\tau$ is

$$\Delta y_{i,\tau^*} = \frac{1}{D} \left[ A_r(\tau^*) \int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du \right] + \left. \frac{\int_0^T A_\beta(u)e^{-\kappa_\theta(\tau-u)}du}{\tau} \right|_\tau^\tau,$$

where

$$A_r(\tau^*) \equiv a\sigma_r^2 A_r(\tau^*) \left[ 1 + a\sigma_\beta^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_\beta(u)e^{-\kappa_\theta(\tau-u)}du \right) A_\beta(\tau)d\tau \right]$$

$$- a\sigma_\beta^2 A_\beta(\tau^*) \left[ a\sigma_r^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du \right) A_r(\tau)d\tau \right],$$

$$A_\beta(\tau^*) \equiv a\sigma_\beta^2 A_\beta(\tau^*) \left[ 1 + a\sigma_r^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du \right) A_r(\tau)d\tau \right]$$

$$- a\sigma_r^2 A_r(\tau^*) \left[ a\sigma_\beta^2 \int_0^T \alpha(\tau) \left( \int_0^\tau A_r(u)e^{-\kappa_\theta(\tau-u)}du \right) A_\beta(\tau)d\tau \right].$$

Consider finally the change in the demand of preferred-habitat investors when yields across all maturities change by $\Delta y$. Using (1) and (5), and integrating across maturities, we find that the demand change is

$$\Delta y \int_0^T \alpha(\tau) \tau d\tau = \alpha \Delta y \int_0^T e^{-\delta_\alpha \tau} \tau d\tau = \alpha \Delta y \frac{1-e^{-\delta_\alpha T} - \delta_\alpha Tem^{-\delta_\alpha T}}{\delta_\alpha^2}.$$
Setting \((T, \alpha, \delta, \Delta y) = (30, 0.08, 0.05, 0.01)\), we find that the demand change is 0.1415, or 14.15% of GDP.
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