Strongly polynomial algorithms and generalized flows
Problem set 1

Summer School on Combinatorial Optimization
Hausdorff Center for Mathematics
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Exercise 1.1 Consider an instance of the minimum-cost flow problem in the capacitated form

$$\begin{aligned} \min\ & c^\top f \\ \nabla f_i &= b_i \quad \forall i \in V \\ 0 \leq f &\leq u. \end{aligned} \tag{1}$$

with n nodes and m arcs. Show that it can be replaced by an equivalent uncapacitated instance (where all upper capacities are ∞) in a network of $n+m$ nodes and $2m$ arcs.

Exercise 1.2 Show that the existence of a feasible solution to (1) can be decided by a maximum flow computation. Derive Hoffman's circulation theorem for uncapacitated flows *(Theorem 1.4 in the lecture notes)* from the MFMC theorem.

Exercise 1.3 Consider a directed graph $G = (V, E)$, and $p, b : V \to \mathbb{R}$, $p \leq b$. Assume there exists a flow $f \geq 0$ such that $\nabla f \geq p$, and there exists another flow $f' \geq 0$ with $\nabla f' \leq b$. Then, there exists a flow $f'' \geq 0$ that simultaneously satisfies $p \leq \nabla f'' \leq b$.

Hint: One possible approach is to use duality/Farkas's lemma. Alternatively, you can prove this via a combinatorial algorithm, moving from one flow towards the other via path augmentations.

Exercise 1.4 Consider a directed graph $G = (V, E)$ with edge weights $c : E \to \mathbb{R}$. For $k = 0, 1, 2, \ldots, n$, and $i \in V$, we let $\rho_i^{(k)}$ denote the length of the minimum-cost walk of *exactly* k arcs ending in i. These can be computed by a simple algorithm as follows. Start by setting $\rho_i^{(0)} = 0$ for all $i \in V$, and in every iteration, we update

$$\rho_i^{(k+1)} := \min_{j \in \delta^-(i)} \rho_j^{(k)} + c_{ji}.$$

Show that the minimum mean value of a cycle in G equals

$$\min_{i \in V} \max_{0 \leq k \leq n-1} \frac{\rho^n(i) - \rho^k(i)}{n-k}.$$