Strongly polynomial algorithms and generalized flows
Problem set 3

Summer School on Combinatorial Optimization
Hausdorff Center for Mathematics
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Exercise 3.1 Assume we have a strongly polynomial algorithm available for generalized flow maximization under Assumption 3.6, namely, that an initial feasible flow is available. We are given a directed graph $G = (V, E)$ with gain factors $\gamma : E \to \mathbb{R}_{>0}$, and node demands $b : V \to \mathbb{R}$.

(a) Show that using the above subroutine, we can find a feasible solution to the feasibility problem

$$\begin{aligned} \nabla f_i \geq b_i \quad \forall i \in V \\ f \geq 0. \end{aligned} \tag{1}$$

(b) Show that we can also solve the feasibility problem

$$\begin{aligned} \nabla f_i \leq b_i \quad \forall i \in V \\ f \geq 0. \end{aligned} \tag{2}$$

Exercise 3.2 Consider the generalized flow feasibility problem in the equality form.

$$\begin{aligned} \nabla f_i = b_i \quad \forall i \in V \\ f \geq 0. \end{aligned} \tag{3}$$

Show that this problem has a feasible solution if and only if both systems (1) and (2) are feasible. *Hint: see Exercise 1.3. Also, note that for standard flows, the three systems* (1), (2) *and* (3) *are all equivalent, but this does not hold for generalized flows.*

Exercise 3.3 Prove Theorem 3.1: solving LPs with up to two nonzero variables per column in the constraint matrix can be strongly polynomially reduced to the minimum-cost generalized flow problem.

Exercise 3.4 Formulate and prove a generalization of the flow decomposition theorem (Theorem 1.5) to generalized flows. For regular flows, we use paths and cycles in the decomposition; what will be the elementary flows for the generalized flow setting?

Exercise 3.5 Prove Lemma 3.10 in the lecture notes.

Exercise 3.6 To illustrate the importance of maintaining safe labellings, provide a **counterexample** to the following extension of Lemma 3.10.

> *Let (f, μ) be a fitting pair and $\varepsilon > 0$ such that $\nabla f_i^\mu \geq b_i^\mu - \varepsilon$ for all $i \in V \setminus \{t\}$. Then, if $f_{ij}^\mu > Ex(f, \mu) + n\varepsilon$ for some $ij \in E$, then $\gamma_{ij}^{\mu^*} = 1$ must hold for every dual optimal solution μ^*.*

Moreover, show that the statement remains false even with $f_{ij}^\mu > \beta(n)\,(\mathrm{Ex}(f, \mu) + n\varepsilon)$ for any function $\beta : \mathbb{R} \to \mathbb{R}$.