Combinatorial optimization - Structures and Algorithms, *GeorgiaTech, Fall 2011*
Problem set 2

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November 16, 2011

1. (a) Prove that a matroid is connected if and only if its dual is connected. (A matroid on ground set S is connected if $r(X) + r(S - X) > r(S)$ for arbitrary $\emptyset \neq X \subsetneq S$). (b) Prove that $(M/Z)^* = M^* - Z$. (M^* is the dual matroid of M, M/Z denotes the contraction of the set Z, and $M - Z$ the deletion of Z.)

2. Let $M = (S, \mathcal{F})$ be a matroid, and assume we have two cost functions $c_1, c_2 : S \to \mathbb{R}_+$. Find a basis that is maximum cost for c_1, and, subject to this, maximum cost for c_2.

3. Consider the following game. In an undirected graph $G = (V, E)$, two players color edges alternately, and color them red and blue, respectively. The red player wins, once all edges in a cut are colored red, and the blue player wins once all edges in a spanning tree are colored blue. Red moves first. Show that the blue player has a winning strategy whenever the graph contains two edge disjoint spanning trees. Otherwise, the red player has a winning strategy.

4. Assume n is odd, and $G = (V, E)$ is a graph with $|V| = n$, $|E| = 2n-2$, such that G is the union of two edge-disjoint spanning trees. Assume furthermore that half of the edges is colored red, the other half blue (without respect to the spanning trees). Show that G contains a spanning tree where exactly half of the edges is red and the half is blue.

5. On the same ground set V, let $\mathcal{A}$ and $\mathcal{B}$ be two different laminar families. Let M be the incidence matrix of $\mathcal{A} \cup \mathcal{B}$. That is, M has $|V|$ columns and $|\mathcal{A}| + |\mathcal{B}|$ rows; assume the i'th row corresponds to the set $X \in \mathcal{A} \cup \mathcal{B}$ and the j'th column to the element $v \in V$; let $M_{ij} = 1$ if $v \in X$ and 0 otherwise. Prove that M is a TU-matrix.

6. Let $G = (V, E)$ be an undirected graph, $S \subseteq V$ an independent set, and let $u : S \to \mathbb{Z}_+$, and $k \geq 1$. Give a polynomial algorithm to decide if the graph contains k edge-disjoint spanning trees, such that the total degree in these trees is at most $u(s)$ for any $s \in S$.

7. Show that every minimally k-edge-connected graph has at least two nodes of degree exactly k.

8. Given an undirected graph $G = (V, E)$ and a set T of terminals, consider a maximum packing of $1/2$ T-paths. That is, we are interested in finding a set of paths $\mathcal{P}$ having both endpoints in T, with weights $w : \mathcal{P} \to \{1/2, 1\}$ such that each arc of the graph is contained in either at most one path with weight 1 or in at most two paths with weight $1/2$. The objective is to maximize $\sum_{P \in \mathcal{P}} w(P)$. Prove that this maximum is equal to $1/2 \sum_{t \in T} \lambda_t$, where λ_t is the maximum number of edge-disjoint paths between t and $T - t$.

   (We **do not** assume that $d(v)$ is even if $v \in V - T$ as in the Lovász-Cherkassky theorem.)