SIAM J. DISCRETE MATH.
Vol. 25, No. 2, pp. 695–718

AUGMENTING UNDIRECTED NODE-CONNECTIVITY BY ONE*

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Dedicated to András Frank on the occasion of his 60th birthday.

Abstract. We present a min-max formula for the problem of augmenting the node-connectivity of a graph by one and give a polynomial time algorithm for finding an optimal solution. We also solve the minimum-cost version for node-induced cost functions.

Key words. node-connectivity, connectivity augmentation

AMS subject classifications. 05C40, 05C85, 68R10

DOI. 10.1137/100787507

1. Introduction.

An undirected graph $G = (V, E)$ is *k-node-connected*, or for short, *k-connected* if $|V| \geq k+1$, and after the deletion of any set of at most $k-1$ nodes, the remaining graph is still connected. By Menger's well-known theorem, a graph is k-connected if and only if it contains k internally disjoint paths between any two nodes. The node-connectivity augmentation problem consists of finding a minimum number of new edges whose addition to a given graph G results in a k-connected graph. The complexity of this problem is a longstanding open question. In this paper we give a min-max formula and a polynomial time algorithm for augmenting connectivity by one, the special case when the input graph G is already $(k-1)$-connected. This special case has itself attracted considerable attention; see, for example, [15], [16], [12], [18], [17].

Besides node-connectivity, one may study edge-connectivity as well, and both augmentation problems can also be addressed for directed graphs. The other three among these four basic connectivity augmentation problems were solved before now: undirected edge-connectivity by Watanabe and Nakamura [22], directed edge-connectivity by Frank [6], and directed node-connectivity by Frank and Jordán [9].

For the undirected node-connectivity version, the best previously known result is due to Jackson and Jordán [14]. They gave a polynomial time algorithm for finding an optimal augmentation for any fixed k. The running time is bounded by $O(|V|^5 + f(k)|V|^3)$, where $f(k)$ is an exponential function of k. For some special classes of graphs they prove even stronger results: for example, the running time of the algorithm is polynomial in $|V|$ if the minimum degree is at least $2k-2$. Liberman and Nutov [18] gave a polynomial time algorithm for augmenting connectivity by one for the graphs satisfying the following property: there exists a set $B \subseteq V$ with $|B| = k-1$ so that $G - B$ has at least k connected components. (It can be decided in polynomial time whether a graph contains such a set; see Cheriyan and Thurimella [3].)

Prior to these results, the cases $k = 2, 3, 4$ were solved by Eswaran and Tarjan [4], Watanabe and Nakamura [23], and Hsu [12], respectively. For $k = |V| - 2$ it is easy to verify that connectivity augmentation is equivalent to finding a maximum matching in the complement graph of G. Similarly, the case $k = |V| - 3$ is equivalent to finding a

*Received by the editors September 20, 2010; accepted for publication (in revised form) April 4, 2011; published electronically June 29, 2011. This research was supported by the Hungarian National Foundation for Scientific Research (OTKA), grants K60802 and NK67867.
http://www.siam.org/journals/sidma/25-2/78750.html

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maximum square-free 2-matching in the complement. This is still open; however, augmentation by one (or, equivalently, finding a maximum square-free 2-matching in a subcubic graph) was recently solved by Bérczi and Kobayashi [1]; see also [2].

It is straightforward to give a 2-approximation for connectivity augmentation by replacing each edge by two oppositely directed edges, and using that directed node-connectivity can be augmented optimally (see [9]). For augmenting connectivity by one, Jordán [15], [16] gave an algorithm finding an augmenting edge set larger than the optimum by at most $\lceil\frac{k-2}{2}\rceil$. Jackson and Jordán [13] extended this result for general connectivity augmentation with an additive term of $\lceil\frac{k(k-1)+4}{2}\rceil$. (The running time of these algorithms can be bounded by polynomials of $|V|$.)

Let us now formulate our theorem, conjectured by Frank and Jordán [8] in 1994. In the $(k-1)$-connected graph $G=(V,E)$, a subpartition $X=(X_1,\dots,X_t)$ of V with $t\geq 2$ is called a *clump* if $|V-\bigcup X_i|=k-1$ and E contains no edge between different sets X_i. The sets X_i are called the *pieces* of X while $|X|$ is used to denote t, the number of pieces. If $t=2$, then X is a *small clump*, while for $t\geq 3$ it is a *large clump*.

An edge $uv\in\binom{V}{2}$ *connects* X if u and v lie in different pieces of X.[1] Two clumps are said to be *independent* if there is no edge in $\binom{V}{2}$ connecting both. The following straightforward claim gives a reformulation of these definitions.

Claim 1.1. *Let $X=(X_1,\dots,X_t)$ and $Y=(Y_1,\dots,Y_h)$ be two clumps.*

(i) X and Y are dependent if and only if there exists indices $1\leq a,b\leq t$, $1\leq c,d\leq h$, $a\neq b$, $c\neq d$ with $X_a\cap Y_c\neq\varnothing$, $X_b\cap Y_d\neq\varnothing$.

(ii) X and Y are independent if and only if either at most one piece of X intersects at least one piece of Y, or at most one piece of Y intersects at least one piece of X. $\square$

As an example, consider $G=K_{k-1,k-1}$, the complete bipartite graph on two color classes of size $k-1$. The subpartition consisting of singleton nodes in one color class forms a clump of size $k-1$; the two clumps corresponding to the two color classes are independent.

Clumps are defined in order to tackle node-cuts in undirected graphs. Assume $B\subseteq V$ is a node-cut in G with $|B|=k-1$; that is, $V-B$ has $t\geq 2$ connected components. The components of $V-B$ form a clump, and any partition of these components to at least two sets forms a clump as well, since in the definition, pieces are not required to be connected. In order to make G k-connected, we have to add at least $t-1$ edges between different components of $V-B$. For $t=2$, an arbitrary edge suffices between the two components; however, the situation is more complicated if $t\geq 3$. Such a set B is often called *separator* in the literature, and *shredder* if $t\geq 3$.

A set $\mathcal{B}$ of clumps is *independent* if any two clumps in $\mathcal{B}$ are independent; that is, every edge in $\binom{V}{2}$ connects at most one clump in $\mathcal{B}$. $\mathcal{B}$ is *semi-independent* if every edge in $\binom{V}{2}$ connects at most two clumps in $\mathcal{B}$.

Definition 1.2. *By a* grove *we mean a subpartition $\Pi=\{\mathcal{B}_0,\mathcal{B}_1,\dots,\mathcal{B}_\ell\}$ of clumps satisfying the following properties.*

(1) *$\mathcal{B}_0$ is a set of arbitrary clumps, while $\mathcal{B}_1,\dots,\mathcal{B}_\ell$ are sets of small clumps. $\mathcal{B}_0$ might be empty, and $\ell=0$ is allowed.*

(2) *$\mathcal{B}_i$ is semi-independent for $i=1,\dots,\ell$.*

(3) *If $X\in\mathcal{B}_i$, $Y\in\mathcal{B}_j$, and $i\neq j$ or $i=j=0$, then X and Y are independent.*

Let us define the deficiency of Π by

[1] By $\binom{V}{2}$ we denote the set of all undirected edges on V, while V^2 stands for the set of all arcs (directed edges) on V.

$$\operatorname{def}(\Pi) = \sum_{X \in \mathcal{B}_0} (|X| - 1) + \sum_{i=1}^{\ell} \left\lceil \frac{|\mathcal{B}_i|}{2} \right\rceil.$$

For a $(k-1)$-connected graph $G = (V, E)$, let $\tau(G)$ denote the minimum number of edges whose addition makes G k-connected, and let $\nu(G)$ denote the maximum value of $\operatorname{def}(\Pi)$ over all groves Π.

THEOREM 1.3. *Let $G = (V, E)$ be a $(k-1)$-connected graph with $|V| \geq k+1$. Then $\nu(G) = \tau(G)$.*

The theorem is illustrated in Figure 1.1. Let us now prove the easy direction $\nu(G) \leq \tau(G)$. Let Π be an arbitrary grove. We need at least $|X| - 1$ edges connecting each clump X in Π. By property (3), an edge cannot connect clumps in different $\mathcal{B}_i$'s. The clumps in $\mathcal{B}_0$ being pairwise independent, the first term in $\operatorname{def}(\Pi)$ gives a lower bound on the number of edges needed to connect the clumps in $\mathcal{B}_0$. For $i \geq 1$, each $\mathcal{B}_i$ is a semi-independent set of small clumps; hence an edge may connect at most two of them. Therefore, we need at least $\lceil \frac{|\mathcal{B}_i|}{2} \rceil$ edges to connect all clumps in $\mathcal{B}_i$. (Notice that $|\mathcal{B}_i| = \sum_{X \in \mathcal{B}_i} (|X| - 1)$ since all clumps in $\mathcal{B}_i$ are small.) The theorem states that $\nu(G) \leq \tau(G)$ holds indeed with equality.

The algorithm uses a simple dual scheme based on this theorem. We construct a subroutine determining the dual optimum value $\nu(G)$. Given that, the algorithm proceeds as follows. First compute $\nu(G)$, and let $J = \binom{V}{2} - E$ be the complement of E. In each step choose an edge $e \in J$, compute $\nu(G+e)$, and remove e from J. If $\nu(G+e) = \nu(G) - 1$, then add e to E; otherwise keep the same G. Note that Theorem 1.3 ensures the existence of an edge e with $\nu(G+e) = \nu(G) - 1$.

Both the proof and the algorithm are motivated by the algorithm given by Frank and the author [11] for augmenting directed node-connectivity by one. Let us now state the min-max formula for this problem. In a digraph $D = (V, A)$, an ordered pair (X^-, X^+) of disjoint nonempty subsets of V is called a *one-way pair* if $|V - (X^- \cup X^+)| = k - 1$ and there is no arc in A from X^- to X^+. An arc $uv \in V^2$

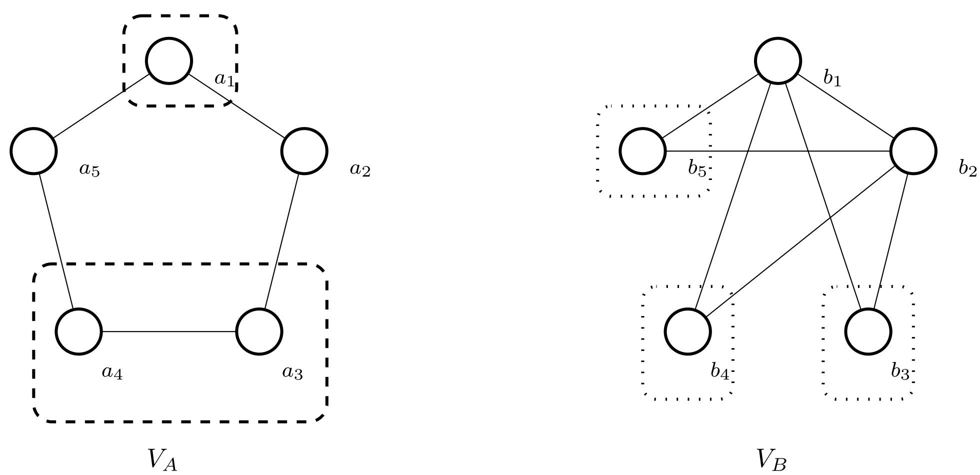


FIG. 1.1. *Let G be the graph on the figure with the addition of a complete bipartite graph between V_A and V_B, and let $k = 8$. G is 7-connected, and it can be made 8-connected by adding the five edges a_1a_3, a_2a_4, a_3a_5, b_3b_4, and b_4b_5. Two clumps $(\{a_1\}, \{a_3, a_4\})$ and $(\{b_3\}, \{b_4\}, \{b_5\})$ are shown in the figure. $\Pi = \{\mathcal{B}_0, \mathcal{B}_1\}$ is a grove with $\operatorname{def}(\Pi) = 5$ for $\mathcal{B}_0 = \{(\{b_3\}, \{b_4\}, \{b_5\})\}$ and $\mathcal{B}_1 = \{(\{a_1\}, \{a_3, a_4\}), (\{a_2\}, \{a_4, a_5\}), (\{a_3\}, \{a_5, a_1\}), (\{a_4\}, \{a_1, a_2\}), (\{a_5\}, \{a_2, a_3\})\}$.*

covers (X^-, X^+) if $u \in X^-$, $v \in X^+$, and two one-way pairs are *independent* if they cannot be covered by the same arc.

THEOREM 1.4 (see [9]). *For a $(k-1)$-connected digraph $D = (V, A)$ with $|V| \geq k+1$, the minimum number of new arcs whose addition results in a k-connected digraph equals the maximum number of pairwise independent one-way pairs.*

Let us briefly outline the argument of [11]. A natural partial order $\preccurlyeq$ can be defined on the set of one-way pairs: let $X = (X^-, X^+) \preccurlyeq Y = (Y^-, Y^+)$ if $X^- \subseteq Y^-$, $X^+ \supseteq Y^+$. A subset $\mathcal{K}$ of one-way pairs is called *cross-free* if any two nonindependent pairs in $\mathcal{K}$ are comparable w.r.t. $\preccurlyeq$; such a $\mathcal{K}$ maximal for inclusion is called a *skeleton*. The two main ingredients of the proof are as follows: (i) for a cross-free $\mathcal{K}$, the maximum number of pairwise independent one-way pairs in $\mathcal{K}$ along with an arc set F of the same cardinality covering all one-way pairs in $\mathcal{K}$ can be determined using Dilworth's theorem on finding a maximum antichain and a minimum chain cover of a poset; (ii) an arc set F covering all one-way pairs in a skeleton $\mathcal{K}$ can be transformed to an arc set F' of the same cardinality covering every one-way pair in D.

Our proof for Theorem 1.3 will roughly follow the same lines. Although no natural partial order can be defined on the set of clumps, we define nestedness in section 2 as a natural analogue to comparability. A cross-free system will be a set of clumps so that any two nonindependent clumps are nested, and by skeleton we mean a maximal cross-free system. For a cross-free $\mathcal{K}$, we will be able to determine an edge set F covering all clumps in $\mathcal{K}$ along with a grove with deficiency $|F|$, consisting of clumps in $\mathcal{K}$. Instead of Dilworth's theorem, we apply a reduction to Fleiner's theorem [5] on covering a symmetric poset by symmetric chains. For part (ii), the argument of [11] is adapted with minor modifications.

While Dilworth's theorem can be derived from the König–Hall theorem on finding a maximum matching in bipartite graphs, Fleiner's theorem may be deduced from the Berge–Tutte theorem on the size of a maximum matching in general graphs. The relation between directed and undirected connectivity augmentation is somewhat analogous: for example, the formula in Theorem 1.3 involves parity. This is a reason why the strikingly simple proof of Theorem 1.4 by Frank and Jordán [9] cannot be adapted for the undirected case.

Another difficulty is that in contrast to one-way pairs, clumps may have more than two pieces. Fortunately, it turns out that large clumps are nested with every other clump they are dependent with. Therefore, although large clumps will cause certain difficulties in the first part of the proof, they play only a minor role in the second part.

The paper is organized as follows. We introduce the necessary concepts and prove some basic claims in section 2. Section 3 contains the proof of Theorem 1.3, while the algorithm is given in section 4. The minimum-cost version for node-induced cost functions is also described in this section. Finally, section 5 discusses possible further directions.

2. Preliminaries. For the undirected graph $G = (V, E)$ and a subset $B \subseteq V$, $d(B) = d_G(B) = d_E(B)$ denotes the degree of B, and $N(B) = N_G(B)$ the set of neighbors of B, that is, $\{v \in V - B, \exists u \in B, uv \in E\}$. For subsets B, $C \subseteq V$, $d(B, C)$ is the number of edges between $B - C$ and $C - B$. For $u \in V$, u sometimes refers to the set $\{u\}$; for example, $B + v$ and $B - v$ denote the sets $B \cup \{v\}$ and $B - \{v\}$, respectively. Similar notation is used concerning edges. Let $n = |V|$, the number of nodes.

Let us list some definitions concerning clumps. For a clump $X = (X_1, X_2, \ldots, X_t)$, let $N_X = V - \bigcup_i X_i$. Observe that for $1 \leq i \leq t$, $N(X_i) = N_X$, since G is $(k-1)$-connected. X is called *basic* if all pieces X_i are connected. The clump Y is *derived* from the basic clump X if each piece of Y is the union of some pieces of X. By $D(X)$ we mean the set of all clumps derived from X, while $D_2(X)$ is used for the set of small clumps derived from X. Let $\mathcal{C}$ denote the set of all basic clumps. For a set $\mathcal{F} \subseteq \mathcal{C}$, $D(\mathcal{F})$ denotes the union of the sets $D(X)$ with $X \in \mathcal{F}$. The clumps being in the same $D(X)$ can easily be characterized (see, e.g., [15], [16], [18]).

CLAIM 2.1. (i) *Two clumps X and Y are derived from the same basic clump if and only if $N_X = N_Y$.* (ii) *If two basic clumps X and Y have a piece in common, then $X = Y$.* $\square$

We say that the edge set F *covers* the clump X if we obtain a connected graph from (V, F) by deleting N_X and shrinking each piece X_i to a single node. Note that at least $|X| - 1$ edges in F connecting X are needed to cover X. However, if X is a small clump, then F covers X if and only if F connects X. We say that F covers (resp., connects) $\mathcal{H} \subseteq D(\mathcal{C})$ if it covers (resp., connects) all clumps in $\mathcal{H}$. Clearly, F is an augmenting edge set if and only if it covers $D(\mathcal{C})$. The following simple claim shows that in order to cover a set $\mathcal{F}$ of clumps, it suffices to connect every small clump derived from the members of $\mathcal{F}$.

CLAIM 2.2. *For an edge set $F \subseteq \binom{V}{2}$ and $\mathcal{F} \subseteq \mathcal{C}$, the following three statements are equivalent:* (i) *F covers $\mathcal{F}$;* (ii) *F covers $D(\mathcal{F})$; and* (iii) *F connects $D_2(\mathcal{F})$.* $\square$

We have already defined when two clumps are independent: if no edge in $\binom{V}{2}$ connects both. Two clumps are *dependent* if they are not independent. In the rest of the section we introduce the concept of nestedness of clumps and uncrossing for dependent clumps, and furthermore we define crossing and cross-free subsets of clumps. The reader may find it useful to compare these to the concepts related to one-way pairs in the case of directed connectivity augmentation as in [9], which will also be defined later in this section, as we will also use them directly. A major difference between the undirected and directed setting is that, in the directed case, a natural partial order can be defined for the one-way pairs, which cannot be done for clumps. Nestedness will be the natural analogue of comparability for clumps.

We say that two clumps $X = (X_1, \ldots, X_t)$ and $Y = (Y_1, \ldots, Y_h)$ are *nested* if $X = Y$ or for some $1 \leq a \leq t$ and $1 \leq b \leq h$, $Y_i \nsubseteq X_a$ for all $i \neq b$ and $X_j \nsubseteq Y_b$ for all $j \neq a$ (see Figure 2.1). We call X_a the *dominant piece* of X w.r.t. Y, and Y_b the dominant piece of Y w.r.t. X. The following important lemma shows that a large basic clump is automatically nested with any other basic clump (see also [18]).

LEMMA 2.3. *Assume X is a large basic clump and Y is an arbitrary basic clump. If X and Y are dependent, then X and Y are nested.*

To prove this, first we need two simple claims.

CLAIM 2.4. *For the basic clumps $X = (X_1, \ldots, X_t)$ and $Y = (Y_1, \ldots, Y_h)$, $X_i \cap N_Y = \varnothing$ implies $X_i \subseteq Y_j$ for some $1 \leq j \leq h$.* $\square$

CLAIM 2.5. *Let $X = (X_1, \ldots, X_t)$ and $Y = (Y_1, \ldots, Y_h)$ be two different clumps both basic or both small. If $X_s \nsubseteq Y_b$ for some $1 \leq s \leq t$, $1 \leq b \leq h$, then X and Y are nested with Y_b being the dominant piece of Y w.r.t. X.*

Proof. Consider an $\ell \neq b$. $X_s \subseteq Y_b$ implies $d(X_s, Y_\ell) = 0$; thus $Y_\ell \cap N_X = \varnothing$. Hence $Y_\ell \subseteq X_a$ for some $a \neq s$ follows either by Claim 2.4 or by $t = 2$. We claim that a is always the same independently from the choice of ℓ. Indeed, assume that for some $\ell' \notin \{b, \ell\}$, $Y_{\ell'} \subseteq X_{a'}$ with $a' \neq a$.

The same argument applied with changing the role of X and Y (by making use of $Y_\ell \subseteq X_a$) shows that $X_{a'} \subseteq Y_j$ for some j, giving $Y_{\ell'} \subseteq Y_j$, a contradiction. $X_i \subseteq Y_b$ for $i \neq a$ can be proved by changing the role of X and Y again. Thus X and Y are nested with dominant pieces X_a and Y_b. $\square$

Proof of Lemma 2.3. The dependence implies $X_1 \cap Y_1 \neq \varnothing$, $X_2 \cap Y_2 \neq \varnothing$ by possibly changing the indices. Let $x_i = |N_Y \cap X_i|$, $y_i = |N_X \cap Y_i|$, $n_0 = |N_X \cap N_Y|$. Then $k-1 \leq |N(X_1 \cap Y_1)| \leq n_0 + x_1 + y_1$. Since $k-1 = |N_X| = n_0 + \sum_i y_i$, this implies $\sum_{i \neq 1} y_i \leq x_1$ and, similarly, $\sum_{i \neq 1} x_i \leq y_1$. The same argument for $X_2 \cap Y_2$ gives $\sum_{i \neq 2} y_i \leq x_2$ and $\sum_{i \neq 2} x_i \leq y_2$.

Thus we have $x_i = y_i = 0$ for $i \geq 3$. This gives $X_3 \cap N_Y = \varnothing$, and hence $X_3 \subseteq Y_i$ for some i by Claim 2.4. The nestedness of X and Y follows by the previous claim. ∎

The notion of one-way pairs from the directed connectivity augmentation setting will also be used. A *one-way pair* $K = (K^-, K^+)$ is an ordered pair of disjoint sets with $|V - (K^- \cup K^+)| = k-1$ and $d(K^-, K^+) = 0$, or equivalently, the subpartition consisting of K^- and K^+ forms a (small) clump. K^- is called the *tail*, while K^+ the *head* of K. For each small clump X, there are two corresponding one-way pairs, called the orientations of X. For a large clump X, we mean by the orientations of X the *orientations* of the small clumps in $D_2(X)$.

For a one-way pair K, $\underline{K}$ denotes the corresponding small clump. An arc (directed edge) $uv \in V^2$ *covers* the one-way pair $K = (K^-, K^+)$ if $u \in K^-$, $v \in K^+$. Note that if the arc uv covers K, then vu does not cover it. If an edge $uv \in \binom{V}{2}$ connects a small clump X, then the arc $uv \in V^2$ covers exactly one of its two orientations (in the directed sense). For the one-way pair $K = (K^-, K^+)$, its reverse is $\overleftarrow{K} = (K^+, K^-)$.

Two one-way pairs are *independent* if no arc covers both, or, equivalently, if either their tails or their heads are disjoint. Two nonindependent set pairs are called *dependent*. Let us define a partial order $\preccurlyeq$ on the one-way pairs as follows. For one-way pairs $K = (K^-, K^+)$ and $L = (L^-, L^+)$, $K \preccurlyeq L$ if $K^- \subseteq L^-$, $K^+ \supseteq L^+$. For dependent one-way pairs K and L, let $K \wedge L = (K^- \cap L^-, K^+ \cup L^+)$ and $K \vee L = (K^- \cup L^-, K^+ \cap L^+)$. A simple argument (e.g., see [9]) shows that these are also one-way pairs.

Take two dependent small clumps $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$. We say that their orientations L_X and L_Y are *compatible* if they are dependent one-way pairs. Clearly, any two dependent one-way pairs admit compatible orientations, and if L_X and L_Y are compatible, then so are $\overleftarrow{L_X}$ and $\overleftarrow{L_Y}$. X and Y are *strongly dependent* if $X_i \cap Y_j \neq \varnothing$ for every $i, j \in \{1, 2\}$. X and Y are *simply dependent* if dependent but not strongly dependent (see Figure 2.2). The following claim is easy to see.

Claim 2.6. *Let X and Y be two small clumps, and let L_X be an orientation of X.*

- (i) X and Y are simply dependent if and only if there is exactly one orientation L_Y of Y compatible with L_X.
- (ii) X and Y are strongly dependent if and only if both orientations of Y are compatible with L_X.
- (iii) X and Y are nested if and only if Y has an orientation L_Y with $L_Y \preccurlyeq L_X$ or $L_Y \succcurlyeq L_X$. ∎

We are ready to define uncrossing of basic clumps. By uncrossing the dependent one-way pairs K and L we mean replacing them by $K \wedge L$ and $K \vee L$ (which coincide with K and L if K and L are comparable). For dependent basic clumps X and Y, we define a set $\Upsilon(X, Y)$ consisting of two or four pairwise nested clumps in the analogous sense. If X and Y are nested, then let $\Upsilon(X, Y) = \{X, Y\}$. By Lemma 2.3, this is always the case if one of X and Y is large. For the small basic clumps X and Y, consider some compatible orientations L_X and L_Y. If X and Y are simply dependent, then let $\Upsilon(X, Y) = \{\underline{L_X \wedge L_Y}, \underline{L_X \vee L_Y}\}$. (Altough there are two possible choices for L_X and L_Y, the set $\Upsilon(X, Y)$ will be the same.) If they are strongly dependent, then L_X is also compatible with $\overleftarrow{L_Y}$. In this case let $\Upsilon(X, Y) = \{\underline{L_X \wedge L_Y}, \underline{L_X \vee L_Y}, \underline{L_X \overleftarrow{L_Y}}, \underline{L_X \vee \overleftarrow{L_Y}}\}$.

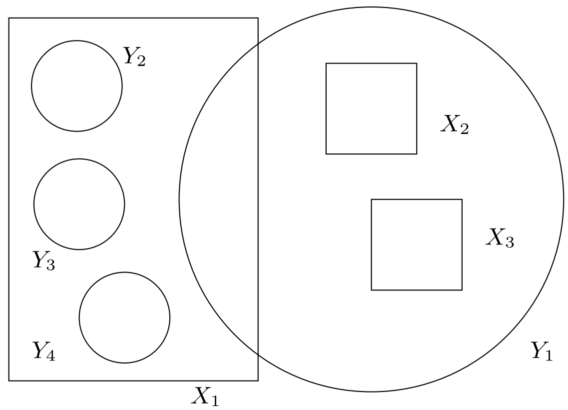


FIG. 2.1. *The nested clumps $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3, Y_4)$ with dominant pieces X_1 and Y_1.*

It is easy to see that the clumps in $\Upsilon(X, Y)$ are nested with X and Y and with each other in both cases. The following property is straightforward.

CLAIM 2.7. *For dependent basic clumps X, Y, if an edge uv connects a clump in $\Upsilon(X, Y)$, then it connects at least one of X and Y.* $\square$

We say that two clumps are *crossing* if they are dependent but not nested. Again by Lemma 2.3, two basic clumps may be crossing only if both are small. A subset $\mathcal{F} \subseteq \mathcal{C}$ is called *crossing* if for any two dependent clumps $X, Y \in \mathcal{F}$, $\Upsilon(X, Y) \subseteq D(\mathcal{F})$. (The reason for assuming containment in $D(\mathcal{F})$ instead of $\mathcal{F}$ is that $\Upsilon(X, Y)$ might contain nonbasic clumps.) Note that $\mathcal{C}$ itself is crossing. For a crossing system $\mathcal{F}$ and a clump $K \in \mathcal{F}$, let $\mathcal{F} \div K$ denote the set of clumps in $\mathcal{F}$ independent from or nested with K. Similarly, for a subset $\mathcal{K} \subseteq \mathcal{F}$, $\mathcal{F} \div \mathcal{K}$ denotes the set of clumps in $\mathcal{F}$ not crossing any clump in $\mathcal{K}$. An $\mathcal{F} \subseteq \mathcal{C}$ is *cross-free* if it contains no crossing clumps; that is, any two dependent clumps in $\mathcal{F}$ are nested. (Note that a cross-free system is crossing as well.) A cross-free $\mathcal{K}$ is called a *skeleton* of $\mathcal{F}$ if it is maximal cross-free in $\mathcal{F}$; that is, $\mathcal{F} \div \mathcal{K} = \mathcal{K}$. By Lemma 2.3, a skeleton of $\mathcal{C}$ should contain every large clump.

LEMMA 2.8. *For a crossing system $\mathcal{F} \subseteq \mathcal{C}$ and $K \in \mathcal{F}$, $\mathcal{F} \div K$ is also a crossing system.*

Proof. Let $\mathcal{F}' = \mathcal{F} \div K$. If K is large, then $\mathcal{F}' = \mathcal{F}$ by Lemma 2.3; therefore K is assumed to be small in what follows. Let us fix an orientation L_K of K. Take crossing basic clumps $X, Y \in \mathcal{F}'$. Again by Lemma 2.3, if a clump in $\Upsilon(X, Y)$ is not basic, then it is automatically in $D(\mathcal{F}')$. We consider all possible cases as follows.

(I) Both are nested with K. Choose orientations L_X and L_Y compatible with L_K (but not necessarily with each other).

(a)

$L_X \preccurlyeq L_K \preccurlyeq L_Y$ or $L_Y \preccurlyeq L_K \preccurlyeq L_X$, then X and Y are nested by Claim 2.6(iii).

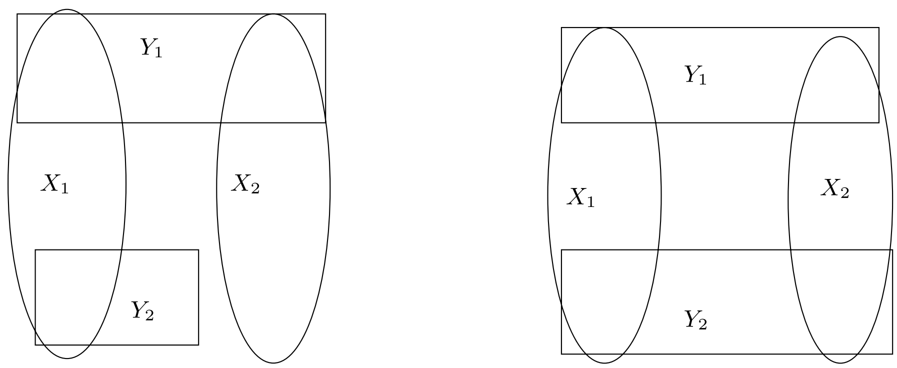
(b)

Let $L_X, L_Y \preccurlyeq L_K$. If L_X and L_Y are dependent, then $L_X \wedge L_Y, L_X \vee L_Y \preccurlyeq L_K$. If L_X and $\overleftarrow{L_Y}$ are dependent, then $L_X \wedge \overleftarrow{L_Y} \preccurlyeq L_K$ and $\overleftarrow{L_K} \preccurlyeq L_X \vee \overleftarrow{L_Y}$. These arguments show $\Upsilon(X, Y) \subseteq D(\mathcal{F}')$.

(c)

In the case of $L_X, L_Y \succcurlyeq L_K$, the claim follows analogously.

(II) Both X and Y are independent from K. By Claim 2.7, all clumps in $\Upsilon(X, Y)$ are independent from K.



(a) (b)

FIG. 2.2. *Simply dependent one-way pairs* (a)*, and strongly dependent ones* (b)*.*

(III) One of them, say X, is nested with K, and the other, Y, is independent from K. Let L_X be an orientation of X compatible with L_K, and letL_Y be an orientation of Y compatible with L_X. By symmetry, we may assume $L_X \preccurlyeq L_K$. Now $L_X \wedge L_Y \preccurlyeq L_K$, and we show that $\underline{L_X \vee L_Y}$ is independent from K. L_Y being an arbitrary orientation compatible with L_X, these again imply $\Upsilon(X, Y) \subseteq D(\mathcal{F}')$. L_Y and L_K are independent, but $L_K^- \cap L_Y^- \neq \varnothing$. Thus $L_K^+ \cap L_Y^+ = \varnothing$, and hence the one-way pairs $L_X \vee L_Y$ and L_K are independent. We also need to show that $\overleftarrow{L_X \vee L_Y}$ and L_K are independent. Indeed, their dependence would imply $L_Y^+ \cap L_K^- \neq \varnothing$, $L_Y^- \cap L_K^+ \neq \varnothing$, contradicting the independence of K and Y. ◻

Finally, the sequence $K_1, K_2, \ldots, K_\ell$ of clumps is called a *chain* if they admit orientations $L_1, L_2, \ldots, L_\ell$ with $L_1 \preccurlyeq L_2 \preccurlyeq \ldots \preccurlyeq L_\ell$. If $u \in L_1^-$, $v \in L_\ell^+$, then the edge uv connects all members of the chain.

3. Proof of Theorem 1.3. For a crossing system $\mathcal{F} \subseteq \mathcal{C}$, let $\tau(\mathcal{F})$ denote the minimum cardinality of an edge set covering $\mathcal{F}$. Let $\nu(\mathcal{F})$ denote the maximum of $\mathrm{def}(\Pi)$ over groves consisting of clumps in $D(\mathcal{F})$. First, we give the proof of the following slight generalization of Theorem 1.3 based on two lemmas proved in the following subsections.

THEOREM 3.1. *For a crossing system $\mathcal{F} \subseteq \mathcal{C}$, $\nu(\mathcal{F}) = \tau(\mathcal{F})$.*

The two lemmas are as follows.

LEMMA 3.2. *For a cross-free system $\mathcal{F}$, $\nu(\mathcal{F}) = \tau(\mathcal{F})$.*

LEMMA 3.3. *For a crossing system $\mathcal{F}$ and a $K \in \mathcal{F}$, if an edge set F covers $\mathcal{F} \div K$, then there exists an F' covering $\mathcal{F}$ with $|F'| = |F|$, and furthermore $d_{F'}(v) = d_F(v)$ for every $v \in V$.*

Proof of Theorem 3.1. $\nu \leq \tau$ is straightforward. The proof of $\nu \geq \tau$ is by induction on $|\mathcal{F}|$. If $\mathcal{F}$ is cross-free, we are done by Lemma 3.2. Otherwise, consider two crossing clumps $K, K' \in \mathcal{F}$ and let $\mathcal{F}' = \mathcal{F} \div K$, a crossing system by Lemma 2.8. As $K' \notin \mathcal{F}'$, we may apply the inductive statement for $\mathcal{F}'$ giving a grove Π and an edge set F covering $\mathcal{F}'$ with $\mathrm{def}(\Pi) = |F|$. The proof is finished using Lemma 3.3. ◻

The following theorem may be seen as a reformulation of this proof; however, it will be more convenient for the aim of the algorithm and to handle the minimum-cost version for node-induced cost functions.

THEOREM 3.4. *For a crossing system $\mathcal{F} \subseteq \mathcal{C}$ and a skeleton $\mathcal{K}$ of $\mathcal{F}$, $\nu(\mathcal{K}) = \nu(\mathcal{F})$. Furthermore, if an edge set F covers $\mathcal{K}$, then there exists an F' covering $\mathcal{F}$ with $|F'| = |F|$ and $d_{F'}(v) = d_F(v)$ for every $v \in V$.*

Proof. Let $\mathcal{K} = \{K_1, \dots, K_\ell\}$. Let $\mathcal{F}_0 = \mathcal{F}$ and for $i = 1, \dots, \ell$, let $\mathcal{F}_i = \mathcal{F} \div \{K_1, \dots, K_i\}$. Lemma 2.8 implies that $\mathcal{F}_i$ is a crossing system as well. $\mathcal{F}_\ell = \mathcal{K}$ since $\mathcal{K}$ is a skeleton. By Lemma 3.2, $\mathcal{K}$ admits a cover F_ℓ with $|F_\ell| = \tau(\mathcal{K}) = \nu(\mathcal{K})$. Applying Lemma 3.3 inductively for $\mathcal{F}_{i-1}$, K_i and F_i for $i = \ell, \ell-1, \dots, 1$, we get a cover F_{i-1} of $\mathcal{F}_{i-1}$ with $|F_{i-1}| = |F_\ell|$. Finally, F_0 is a cover of $\mathcal{F} = \mathcal{F}_0$, and hence $\nu(\mathcal{F}) \le |F_0| = |F_\ell| = \nu(\mathcal{K})$, implying the first part of the theorem. The identity of the degree sequences follows by the second part of Lemma 3.3.

3.1. Covering cross-free systems. This section is devoted to the proof of Lemma 3.2. The analogous statement in the case of directed connectivity augmentation simply follows by Dilworth's theorem, which is a well-known consequence of the König–Hall theorem on the size of a maximum matching in a bipartite graph. In contrast, Lemma 3.2 is deduced from Fleiner's theorem, which is proved via a reduction to the Berge–Tutte theorem on maximum matchings in general graphs.

We need the following notion to formulate Fleiner's theorem. A triple $P = (U, \preccurlyeq, M)$ is called a *symmetric poset* if $(U, \preccurlyeq)$ is a finite poset and M is a perfect matching on U with the property that $u \preccurlyeq v$ and $uu', vv' \in M$ implies $u' \succcurlyeq v'$. The edges of M are called *matches*. A subset $\{u_1v_1, \dots, u_kv_k\} \subseteq M$ is called a *symmetric chain* if $u_1 \preccurlyeq u_2 \preccurlyeq \dots \preccurlyeq u_k$ (and thus $v_1 \succcurlyeq v_2 \succcurlyeq \dots \succcurlyeq v_k$). The symmetric chains $S_1, S_2, \dots, S_t$ *cover* P if $M = \bigcup S_i$.

A set $\mathcal{L} = \{L_1, L_2 \dots, L_\ell\}$ of disjoint subsets of M forms a *legal subpartition* if $uv \in L_i$, $u'v' \in L_j$, $u \preccurlyeq u'$ yields $i = j$, and no symmetric chain of length three is contained in any L_i. The value of $\mathcal{L}$ is $\mathrm{val}(\mathcal{L}) = \sum_i \lceil \frac{|L_i|}{2} \rceil$.

THEOREM 3.5 (see Fleiner [5]). *Let $P = (U, \preccurlyeq, M)$ be a symmetric poset. The minimum number of symmetric chains covering P is equal to the maximum value of a legal subpartition of P.*

Note that the max $\le$ min direction follows easily since a symmetric chain may contain at most two matches belonging to one class of a legal subpartition. This theorem gives a common generalization of Dilworth's theorem and of the well-known min-max formula on the minimum edge cover of a graph (a theorem equivalent to the Berge–Tutte theorem).

First we show that Lemma 3.2 is a straightforward consequence if $\mathcal{F}$ contains only small clumps. Consider the cross-free family $\mathcal{F}$ of clumps, and let U be the set of all orientations of one-way pairs in $\mathcal{F}$. The matches in M consist of the two orientations of the same clump, while $\preccurlyeq$ is the usual partial order on one-way pairs. A symmetric chain corresponds to a chain of clumps. Since all clumps in a chain can be connected by a single edge, a symmetric chain cover gives a cover of $\mathcal{F}$ of the same size. On the other hand, consider a legal subpartition $\mathcal{L} = \{L_1, L_2 \dots, L_\ell\}$. Define the grove $\Pi = \{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_\ell\}$ with $\mathcal{B}_0 = \varnothing$ and $\mathcal{B}_i$ consisting of the clumps corresponding to the matches in L_i for $i = 1, \dots, \ell$. The independence of clumps contained in different $\mathcal{B}_i$'s follows by Claim 2.6, since poset elements in different L_i's are independent. The semi-independece of each $\mathcal{B}_i$ follows by the following claim.

CLAIM 3.6. *If an edge $uv \in \binom{V}{2}$ connects three small clumps in a cross-free system $\mathcal{F}$, then these clumps form a chain.*

Proof. Let X, Y, Z be the three clumps. Let us denote the part of X containing u by X_1 and the one containing v by X_2; let $Y = (Y_1, Y_2)$ and $Z = (Z_1, Z_2)$ in the analogous sense. Clearly, any two clumps among X, Y, and Z are dependent, hence nested. We show that the sets X_1, Y_1, Z_1 are pairwise comparable and thus form a chain; then we are done by Claim 2.5. Indeed, $X_1 \cap Y_1 \neq \varnothing$ since it contains u. If they are not comparable, then $X_1 - Y_1$, $Y_1 - X_1 \neq \varnothing$; thus X_1 and Y_1 should be the dominant pieces of X and Y w.r.t. the other. Consequently, $v \in X_2 \subseteq Y_1$, a contradiction. ◻

Let us now turn to the general case when $\mathcal{F}$ may contain large clumps as well. For an arbitrary set $B \subseteq V$, let $B^* = V - (B \cup N(B))$. An edge set F *pseudocovers* the clump $X = (X_1, \dots, X_t)$ if F contains at least $|X| - 1$ edges connecting X, and furthermore each clump (X_i, X_i^*) is connected for $i = 1, \dots, t$. (Note that $X_i^* = \bigcup_{j \neq i} X_j$.) F pseudocovers $\mathcal{F}$ if it pseudocovers every $X \in \mathcal{F}$. Although a pseudocover is not necessarily a cover, the following lemma shows that it can be transformed into a cover of the same size.

Lemma 3.7. *If F is a pseudocover of $\mathcal{F}$, then there exists an edge set H covering $\mathcal{F}$ with $|F| = |H|$ and $d_H(v) = d_F(v)$ for every $v \in V$.*

Proof. We are done if F covers all clumps in $\mathcal{F}$. Otherwise, consider a clump $X \in \mathcal{F}$ pseudocovered but not covered. X is large, since a pseudocovered small clump is automatically covered. Let F/X be the graph obtained from (V, F) by deleting N_X and shrinking each X_i to a single node. Let $c_F(X)$ denote the number of connected components of F/X. Note that F covers X if and only if $c_F(X) = 1$.

Since X is connected by at least $|X| - 1$ edges of F, there is an edge $e = x_1y_1 \in F$ connecting X with $c_F(X) = c_{F-e}(X)$. Each (X_i, X_i^*) is connected; hence there exists an edge $x_2y_2 \in F$ connecting X with x_2y_2 being in a component of F/X different from the one containing x_1y_1. Let $F' = F - \{x_1y_1, x_2y_2\} + \{x_1y_2, x_2y_1\}$. Clearly, $c_{F'}(X) = c_F(X) - 1$. We show that $c_{F'}(Y) \leq c_F(Y)$ for every $Y \in \mathcal{F} - X$; hence by a sequence of such steps we finally arrive at an H covering $\mathcal{F}$.

Indeed, assume $c_{F'}(Y) > c_F(Y)$ for some $Y \in \mathcal{F}$. X and Y are dependent since at least one of x_1y_1 and x_2y_2 connects both. By Lemma 2.3, X and Y are nested; let X_a and Y_b denote their dominant pieces. The nodes x_1, y_1, x_2, y_2 lie in four different pieces of X, and thus at least three of them are contained in Y_b. Consequently, $c_{F'}(Y) = c_F(Y)$ yields a contradiction. ◻

In what follows, we show how a pseudocover F of $\mathcal{F}$ can be found based on a reduction to Fleiner's theorem. For a basic clump $X = (X_1, \dots, X_t)$, let $u_i^X = (X_i, X_i^*)$, $v_i^X = (X_i^*, X_i)$, and $U^X = \{u_i^X, v_i^X : i = 1, \dots, t\}$. Let $U = \bigcup_{X \in \mathcal{F}} U^X$. We say that the members of U^X are of type X. Let the matching M consist of the matches $u_i^X v_i^X$; such a match is called an *X-match*.

If X is small ($t = 2$), then $u_1^X = v_2^X$ and $v_1^X = u_2^X$; thus $|U^X| = 2$. If X is large, then $|U^X| = 2t$. In this case, let u_1^X and v_1^X be called the *special one-way pairs* w.r.t. X. $u_1^X v_1^X$ is called a *special match*. Note that it matters here which piece of X is denoted by X_1 (arbitrarily chosen though). Let the partial order $\preccurlyeq'$ on U be defined as follows. If x and y are one-way pairs of different type, then let $x \preccurlyeq' y$ if and only if $x \preccurlyeq y$ for the standard partial order $\preccurlyeq$ on one-way pairs. If x and y are both of type X for a large clump X, then let $x \preccurlyeq' y$ if either $x = u_1^X$, $y = v_i^X$, or $x = u_i^X$, $y = v_1^X$ for some $i > 1$. In other words, $\preccurlyeq'$ is the same as $\preccurlyeq$ except that x and y are incomparable whenever x and y are of the same type X, $x \neq y$, and neither of them is special.

Claim 3.8. *$P = (U, \preccurlyeq', M)$ is a symmetric poset.*

Proof. The only nontrivial property to verify is transitivity: $x \preccurlyeq' y$ and $y \preccurlyeq' z$ imply $x \preccurlyeq' z$. By $x \preccurlyeq y$ and $y \preccurlyeq z$ the transitivity of $\preccurlyeq$ yields $x \preccurlyeq z$. Hence $x \preccurlyeq' z$ holds unless x

and z are different one-way pairs of the same type X, and neither of them is special. Thus X is a large clump. Both x and z have the form $u_i^X = (X_i, X_i^*)$ or $v_i^X = (X_i^*, X_i)$ for some $i \geq 2$. $x \prec z$ implies that x is of the first and z is of the second form. By possibly changing the indices, we may assume $x = u_2^X$, $z = v_3^X$. Since $x \preccurlyeq' y$, $y \preccurlyeq' z$, and nonspecial one-way pairs of the same type are incomparable, y could be of type X only if it were special. $y = u_1^X$ is excluded by $x = u_2^X \not\preccurlyeq u_1^X$, and $y = v_1^X$ is excluded by $z = v_3^X \not\succcurlyeq v_1^X$. Hence y is of a different type Y.

Assume first $y = u_i^Y$ for some i. Now $X_2 \subseteq Y_i \subseteq X_3^*$, and thus $N_X \cap Y_i = \varnothing$, giving $Y_i \subseteq X_j$ for some $j \neq 3$ by Claim 2.4. Consequently, $X_2 = Y_i$, a contradiction as it would lead to $X = Y$ by Claim 2.1. Next, assume $y = v_i^Y$. $X_3 \subseteq Y_i \subseteq X_2^*$ gives a contradiction the same way. $\square$

The following simple claim establishes the connection between dependency of clumps and comparability in P.

Claim 3.9. *In a cross-free system $\mathcal{F}$, the clumps X, $Y \in \mathcal{F}$ are dependent if and only if for arbitrary i, j, u_i^X is comparable with either u_j^Y or v_j^Y.* $\square$

Proof. If u_i^X is comparable with either u_j^Y or v_j^Y, then X and Y are clearly dependent. For the other direction, since $\mathcal{F}$ is cross-free, X and Y are nested if dependent. Let X_a and Y_b be their dominant pieces. If $i = a$ and $j = b$, then $v_j^Y \prec u_i^X$; if $i = a$ and $j \neq b$, then $u_j^Y \prec u_i^X$. In case of $i \neq a$ and $j = b$ we have $u_i^X \prec u_j^Y$, while if $i \neq a$ and $j \neq b$, then $u_i^X \prec v_j^Y$. $\square$

Take a symmetric chain cover $S_1, \ldots, S_t$ and a legal subpartition $\mathcal{L} = \{L_1, \ldots, L_\ell\}$ with $\mathrm{val}(\mathcal{L}) = t$. Let us choose $\mathcal{L}$ so that ℓ is maximal, and subject to this, $\bigcup_{i=1}^{\ell} L_i$ contains the maximum number of special matches. A symmetric chain S_i naturally corresponds to a chain of the clumps (X_j, X_j^*) for $u_j^X v_j^X \in S_i$. These can be covered by a single edge; hence a symmetric chain cover corresponds to an edge set F of the same size. A symmetric chain may contain both $u_j^X v_j^X$ and $u_{j'}^X v_{j'}^X$ for $j \neq j'$ only if $j = 1$ or $j' = 1$. Consequently, F is a pseudocover, as there are at least $|X| - 1$ different edges in F connecting X, and all (X_j, X_j^*)'s are connected.

It is left to show that $\mathcal{L}$ can be transformed to a grove Π with $\mathrm{def}(\Pi) = \mathrm{val}(\mathcal{L})$. For a clump X, let $B(X)$ denote the set of indices j with $u_j^X v_j^X \in \bigcup_{i=1}^{\ell} L_i$.

Claim 3.10. *For any clump X, the X-matches corresponding to $B(X)$ are either all contained in the same L_i or are all singleton L_i's. $1 \in B(X)$ always gives the first alternative.*

Proof. There is nothing to prove for $|X| = 2$, so let us assume $|X| \geq 3$. As $\mathcal{L}$ is chosen with ℓ maximal, if $u_j^X v_j^X \in L_i$ with $|L_i| > 1$, then there is an $u_h^Y v_h^Y \in L_i$ with u_h^Y comparable with either u_j^X or v_j^X. If $Y \neq X$, then Claim 3.9 gives that u_h^Y is also comparable with $u_{j'}^X$ or $v_{j'}^X$ for any $j' \in B(X)$. If $Y = X$, then either $j = 1$ or $h = 1$ follows, implying $u_{j'} v_{j'} \in L_i$ for every $j' \in B(X)$. This argument also shows that $1 \in B(X)$ leads to the first alternative. $\square$

Let $\beta(X) = i$ in the first alternative if L_i is not a singleton, and let $\beta(X) = 0$ otherwise. Let $\mathcal{I}$ denote the set of indices for which L_i is a singleton. Let us construct the grove Π as follows. For any X with $\beta(X) = 0$, $B(X) \neq \varnothing$, let $\tilde{X} \in D(X)$ denote the clump consisting of pieces X_i with $i \in B(X)$ and the piece $\bigcup_{j \notin B(X)} X_j$. The latter set is nonempty since either $1 \notin B(X)$ or $B(X) = \{1\}$ by Claim 3.10; thus $|\tilde{X}| - 1 = |B(X)|$. Define $\mathcal{B}_0 = \{\tilde{X} : \beta(X) = 0\}$. For $i \notin \mathcal{I}$, let $\mathcal{B}_i = \{(X_j, X_j^*) : u_j^X v_j^X \in L_i\}$. The next lemma completes the proof.

Lemma 3.11. *Π consists of $\mathcal{B}_0$ and the $\mathcal{B}_i$'s for $i \notin \mathcal{I}$ is a grove with* $\mathrm{def}(\Pi) = \mathrm{val}(\mathcal{L})$.

Proof. Once proved that Π is a grove, $\mathrm{def}(\Pi) = \mathrm{val}(\mathcal{L})$ follows immediately: $\sum_{\tilde{X}\in\mathcal{B}_0}(|\tilde{X}|-1) = \sum_{X:\beta(X)=0}|B(X)| = |\mathcal{I}|$, and if $i \notin \mathcal{I}$, then $|\mathcal{B}_i| = |L_i|$.

Let us verify that Π is a grove. Property (1) in Definition 1.2 holds since $\mathcal{B}_i$ for $i \notin \mathcal{I}$ contains only small clumps. Property (3) is implied by Claim 3.9 since the elements of different L_i's are pairwise incomparable. However, verifying property (2) needs some further technical efforts.

Consider a clump X with $\beta(X) = i > 0$ (and thus $i \notin \mathcal{I}$). Let us say that a piece X_j is a *dominant piece* of X if, for some $Y \neq X$ with $\beta(Y) = i$, X_j is the dominant piece of X w.r.t. Y. Let $U(X)$ denote the set of the indices of the dominant pieces of X; note that the set $U(X) - B(X)$ is possibly nonempty. We shall prove the following claim.

Claim 3.12. *If $\beta(X) > 0$ and $|B(X)| \geq 2$, then $B(X) \cap U(X) = \varnothing$.*

Before proving the claim, we show how property (2) follows. Assume for a contradiction that an edge $xy \in \binom{V}{2}$ covers three clumps in some $\mathcal{B}_i$. Using Claim 3.6, these three clumps form a chain for the partial order $\preccurlyeq$. If these three clumps were derived from three different basic clumps, then L_i would contain a symmetric chain of length three, since $\preccurlyeq$ and $\preccurlyeq'$ coincide for such clumps.

Thus we need to have two clumps derived from the same basic clump X: $xy \in \binom{V}{2}$ connects (X_j, X_j^*), $(X_{j'}, X_{j'}^*)$, and (Y_h, Y_h^*) for $\beta(X) = \beta(Y) = i$. Note that $|B(X)| \geq 2$ holds. Hence one of x and y is in X_j and the other in $X_{j'}$; this also yields $X \neq Y$. Let X_a be the dominant piece of X w.r.t. Y. Either $Y_h \subseteq X_a$ or $Y_h^* \subseteq X_a$. Consequently, $x \in X_a$ or $y \in X_a$, implying $a = j$ or $a = j'$. This contradicts Claim 3.12.

Proof of Claim 3.12. Assume, contrary to the claim, that $B(X) \cap U(X) \neq \varnothing$. We first show that $|U(X)| \geq 2$ leads to a contradiction. Consider arbitrary $j \in B(X) \cap U(X)$ and $j' \in U(X) - \{j\}$, say, X_j is the dominant piece of X w.r.t. Y and $X_{j'}$ the one w.r.t. Z with $\beta(Y) = \beta(Z) = i$. Let $a \in B(Y)$ and $b \in B(Z)$. Then $Y_a \subseteq X_j$ or $Y_a^* \subseteq X_j$, and $X_j \subseteq Z_b$ or $X_j \subseteq Z_b^*$; hence L_i contains a symmetric chain of length three.

Hence $|U(X)| = 1$. Let $U(X) = \{j\}$. We next show $1 \notin B(X)$. Assume again that X_j is the dominant piece of X w.r.t. Y with $\beta(Y) = i$. If $1 \in B(X)$ and $j \neq 1$, then a Y-match, $u_j^X v_j^X$ and $v_1^X u_1^X$ form a symmetric chain in L_i. If $j = 1$, then a Y-match, $u_1^X v_1^X$ and $v_h^X u_h^X$ form a symmetric chain for arbitrary $h \in B(X) - \{1\}$.

Let us replace L_i by $L_i' = L_i - \{u_j^X v_j^X\} + \{u_1^X v_1^X\}$. By Claim 3.9, any element of L_i' is incomparable to any element of L_h for $h \neq i$. We shall prove that L_i' does not contain any symmetric chain of length three given that L_i did not contain any. This gives a contradiction as $\mathcal{L}$ was chosen to contain the maximal possible number of special matches. Indeed, assume there existed such a symmetric chain containing $u_1^X v_1^X$. There is an edge $xy \in \binom{V}{2}$ with $x \in X_1$ connecting the small clumps corresponding to the three matches in the chain. Since $U(X) = \{j\}$, X_j is the dominant piece of X w.r.t. the other clumps in the chain, which easily gives $y \in X_j$. Now xy connects (X_j, X_j^*) and the two clumps in the original chain different from (X_1, X_1^*), and thus Claim 3.6 yields a chain of length three in L_i. □ □

3.2. Proof of Lemma 3.3. First we need the following lemmas.

Lemma 3.13. *Assume that for three small clumps $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, $Z = (Z_1, Z_2)$, all four sets $X_1 \cap Y_1 \cap Z_1$, $X_1 \cap Y_2 \cap Z_2$, $X_2 \cap Y_1 \cap Z_2$, $X_2 \cap Y_2 \cap Z_1$ are nonempty. Then all of X, Y, and Z are derived from the same basic clump (and thus none of them is basic itself).*

Proof. Let $X_c = N_X$, $Y_c = N_Y$, $Z_c = N_Z$. By A_s for a sequence s of three literals each 1,2 or c, we mean the intersection of the corresponding sets. For example, $A_{12c} = X_1 \cap Y_2 \cap Z_c$.

The conditions mean that the sets A_{111}, A_{122}, A_{212}, A_{221} are nonempty. $V - (A_{111} \cup N(A_{111})) \neq \varnothing$, as there is no edge between A_{111} and X_2; thus $|N(A_{111})| \geq k - 1$ as G is $(k-1)$-connected. This implies

$$k - 1 \leq |A_{c11} \cup A_{1c1} \cup A_{11c} \cup A_{1cc} \cup A_{c1c} \cup A_{cc1} \cup A_{ccc}|,$$

as $N(A_{111})$ is a subset of the set on the right-hand side. Let us take the sum of these types of inequalities for all A_{111}, A_{122}, A_{212}, A_{221}. This gives $4(k-1) \leq S_1 + 2S_2 + 4|A_{ccc}|$, where S_1 is the sum of the cardinalities of the sets having exactly one c in their indices, while S_2 is the same for two c's.

On the other hand, $|X_c| = |Y_c| = |Z_c| = k - 1$. This gives $3(k-1) = S_1 + 2S_2 + 3|A_{ccc}|$. These together imply $S_1 = S_2 = 0$, $|A_{ccc}| = k - 1$. We are done by Claim 2.1, since $N_X = N_Y = N_Z = A_{ccc}$. $\square$

Lemma 3.14. (see [11]) (i) *Let L_1, L_2, L_3 be one-way pairs with L_1 and L_2 dependent, $L_1 \wedge L_2$ and L_3 also dependent, but L_2 and L_3 independent. Then $L_1^- - L_2^- \subseteq L_3^-$.* (ii) *Let L_1, L_2, L_3 be one-way pairs with L_1 and L_2 dependent, $L_1 \vee L_2$ and L_3 also dependent, but L_2 and L_3 independent. Then $L_1^+ - L_2^+ \subseteq L_3^+$.*

Proof. (i) The dependence of $L_1 \wedge L_2$ and L_3 implies $L_2^- \cap L_3^- \neq \varnothing$, so L_2 and L_3 can only be independent if $L_2^+ \cap L_3^+ = \varnothing$. Consider now the pair $N = (L_1 \wedge L_2) \vee L_3$. $N^+ = (L_1^+ \cup L_2^+) \cap L_3^+ = L_1^+ \cap L_3^+$; hence $N^+ \subseteq L_1^+$. Applying Claim 2.5 for the small clumps $\underline{L_1}$ and $\underline{N}$, we get $N^- \supseteq L_1^-$, implying the claim. (ii) follows from (i) by reverting the orientations of all pairs. $\square$

Proof of Lemma 3.3. Let $\mathcal{F}' = \mathcal{F} \div K$. If K is large, then $\mathcal{F}' = \mathcal{F}$ by Lemma 2.3; therefore K will be assumed to be small with an orientation L_K.

If F covers $\mathcal{F}'$ but not $\mathcal{F}$, then by Claim 2.2 there exists a small clump $X \in D_2(\mathcal{F}) - D_2(\mathcal{F}')$ not connected by F; thus X and K are crossing. Choose X with the orientation L_X compatible with L_K so that L_X is minimal to these properties w.r.t. $\preccurlyeq$ (that is, there exists no other uncovered X' with orientation $L_{X'}$ compatible with L_K so that $L_{X'} \prec L_X$). Choose Y not connected by F with $L_X \preccurlyeq L_Y$ and L_Y maximal in the analogous sense ($X = Y$ is allowed).

$\underline{L_X \wedge L_K}$ and $\underline{L_Y \vee L_K}$ are nested with L_K and thus connected by edges $x_1y_1, x_2y_2 \in F$ with $x_1 \in L_X^- \cap L_K^-$, $y_2 \in L_Y^+ \cap L_K^+$. As X and Y are not connected, $y_1 \in L_K^+ - L_X^+$, $x_2 \in L_K^- - L_Y^-$ follows. Let $F' = F - \{x_1y_1, x_2y_2\} + \{x_1y_2, x_2y_1\}$ denote the flipping of x_1y_1 and x_2y_2. F' connects X and Y, and we shall prove that F' connects all small clumps in $D_2(\mathcal{F})$ connected by F. Hence after a finite number of such operations all small clumps in $D_2(\mathcal{F})$ will be connected, so by Claim 2.2, $\mathcal{F}$ will be covered.

For a contradiction, assume there is a small clump S connected by F but not by F'. No edge in $F \cap F'$ may connect S; hence either exactly one of x_1y_1 and x_2y_2 connects it, or if both, then x_1 and y_2 are in the same piece, and y_1 and x_2 are in the other piece of S. In this latter case, K and S are strongly dependent.

We claim that S is basic. Indeed, assume S is not basic but derived from the basic clump S_0. S_0 and K are dependent since x_1y_1 or x_2y_2 connects both; hence Lemma 2.3 implies that S_0 and K are nested. It it easy to see that in this case S and K are comparable, and consequently, F' also covers S.

(I) First, assume that x_1y_1 connects S, and choose the orientation L_S with $x_1 \in L_S^-$, $y_1 \in L_S^+$. We claim that L_S and L_Y are also dependent. Indeed, if they are independent, then Lemma 3.14(i) is applicable for $L_1 = L_K$, $L_2 = L_Y$, $L_3 = L_S$, since $L_K \wedge L_Y$ and L_S are dependent because x_1y_1 covers both. This gives $x_2 \in L_K^- - L_Y^- \subseteq L_S^-$; that is, x_2y_1 connects S, a contradiction.

Hence we may consider the one-way pair $L_S \vee L_Y$. $L_S \vee L_Y$ is strictly larger than L_Y. Indeed, if $L_S \preccurlyeq L_Y$ held, then $y_2 \in L_Y^+ \subseteq L_S^+$; thus $x_1 y_2$ would connect S. By the maximal choice of L_Y, $\underline{L_S \vee L_Y}$ is connected by some edge $f \in F$. By Claim 2.7, f also connects S or Y. (It can be shown that Y is also basic the same way as for S above.)

As Y was not connected by F, f must connect S. $f = x_1 y_1$ gives a contradiction, as $x_1 \in L_S^- \cup L_Y^-$ and $y_1 \notin L_X^+$ implies $y_1 \notin L_S^+ \cap L_Y^+$. Hence $f = x_2 y_2$, and since F' does not connect S, we get $x_2 \in L_S^+$, $y_2 \in L_S^-$. Also, since f connects $\underline{L_S \vee L_Y}$, we have $x_2 \in L_S^+ \cap L_Y^+$, and hence by $L_Y^+ \subseteq L_X^+$ it follows that $x_2 \in L_S^+ \cap L_X^+ \cap L_K^-$. Notice also $y_2 \in L_S^- \cap L_X^+ \cap L_K^+$.

(II) Next, assume $x_2 y_2$ connects S. (The above argument yields that this is indeed always the case.) The same argument applies by exchanging $\vee$ and $\wedge$, X and Y, "minimal" and "maximal" everywhere and applying Lemma 3.14(ii) instead of (i).

Namely, choose an orientation H_S of S with $x_2 \in H_S^-$, $y_2 \in H_S^+$ (it will turn out that $H_S = \overleftarrow{L_S}$). The dependence of H_S and L_K may be proved via Lemma 3.14 (ii): in case they were independent, we could apply the lemma for $L_1 = L_K$, $L_2 = L_X$, $L_3 = H_S$.

$H_S \wedge L_X \prec L_X$ follows, as otherwise $x_1 y_2$ would connect S. By the minimal choice of L_X, an edge $f \in S$ connects $H_S \wedge L_X$ and thus H_S. $f = x_2 y_2$ is a contradiction since $x_2 \notin L_X^-$, implying $f = x_1 y_1$. Again we may conclude $x_1 \in H_S^+$, $y_1 \in H_S^-$, which verifies $H_S = \overleftarrow{L_S}$. As above, it follows that $x_1 \in L_S^- \cap L_X^- \cap L_K^-$ and $y_1 \in L_S^+ \cap L_X^- \cap L_K^+$.

In summary, it is always the case that both $x_1 y_1$ and $x_2 y_2$ cover S, and thus the argument of both cases is applicable. Now x_1, y_2, y_1, and x_2 witness that the clumps S, X, and K satisfy the condition in Lemma 3.13, contradicting the assumption that K was a small clump. $\square$

4. The algorithm. As outlined in the introduction, our algorithm is a simple iterative application of a subroutine determining the dual optimum $\nu(G)$. Theorem 3.4 shows that $\nu(G) = \nu(\mathcal{K})$ for an arbitrary skeleton $\mathcal{K}$. Given a skeleton $\mathcal{K}$, $\nu(\mathcal{K})$ can be determined based on Fleiner's theorem: [5] gives a proof of Theorem 3.5 based on a (linear time) reduction to maximum matching in general graphs, as described in section 4.2. Hence the only nontrivial question is how a skeleton can be found. A naive approach is choosing clumps greedily so that they do not cross the previously selected ones. The difficulty arises from the fact that the number of the clumps might be exponentially large, forbidding us from checking all clumps one by one. In fact, it is not even clear how to decide whether a given cross-free system is a skeleton. To overcome these difficulties, we restrict ourselves to a special class of cross-free systems as described in the next subsection.

4.1. Constructing a skeleton. The following property characterizes the special cross-free systems we wish to use.

Definition 4.1. *A cross-free set $\mathcal{H} \subseteq \mathcal{C}$ is* stable *if it fulfills the following condition. For any $U \in \mathcal{C} - H$, if $\mathcal{C} + U$ is cross-free, then there exist no clumps K, $K' \in \mathcal{H}$ so that K, U, K' form a chain.*

Observe that K, U, K' form a chain if and only if K has a piece K_1, K' has a piece K_1', and U has two different pieces U_1 and U_2 with $K_1 \subseteq U_1$, $K_1' \subseteq U_2$.

Let us now introduce some new notation concerning pieces. If the set $B \subseteq V$ is a piece of the basic clump X, then let $B^\sharp$ denote X. Let $\mathcal{Q}$ be the set of all (connected) pieces of all basic clumps, whereas $\mathcal{Q}_1$ is the set of all (not necessarily connected) pieces of all clumps. For a subset $\mathcal{A} \subseteq \mathcal{Q}$, $\mathcal{A}^\sharp$ is the set of corresponding basic clumps (e.g., $\mathcal{Q}^\sharp = \mathcal{C}$). For a set $\mathcal{H} \subseteq \mathcal{C}$, by $\bigcup \mathcal{H}$ we denote the set of all pieces of clumps in $\mathcal{H}$. The following simple claim will be used to handle chains of length three.

CLAIM 4.2. *For pieces B_1, B_2, $B_3 \in \mathcal{Q}_1$, if* (i) $B_1 \subseteq B_2 \subseteq B_3$ *or* (ii) $B_1 \subseteq B_2$ *and* $B_3 \subseteq B_2^*$, *then the corresponding clumps $B_1^\sharp$, $B_2^\sharp$, $B_3^\sharp$ form a chain.*

Proof. Recall that an orientation of a possibly large clump X was defined as an orientation of a small clump in $D_2(X)$, and a sequence of clumps form a chain if they admit orientations forming a chain for the partial order $\preccurlyeq$. In case (i), we have $(B_1, B_1^*) \preccurlyeq (B_2, B_2^*) \preccurlyeq (B_3, B_3)^*$, whereas in case (ii) we have $(B_1, B_1^*) \preccurlyeq (B_2, B_2^*) \preccurlyeq (B_3^*, B_3)$. ◻

Clearly, $\mathcal{H} = \varnothing$ is stable, and every skeleton is stable as well. Let $\mathcal{M} \subseteq \mathcal{Q}$ denote the set of all pieces minimal for inclusion. Based on the following claim, we will be able to determine whether a stable cross-free system is a skeleton. The subroutine for finding the elements of $\mathcal{M}$ will be given in the appendix among other technical details of the algorithm.

CLAIM 4.3. *The stable cross-free system $\mathcal{H} \subseteq \mathcal{C}$ is a skeleton if and only if $\mathcal{M}^\sharp \subseteq \mathcal{H}$.*

Proof. On the one hand, every skeleton should contain $\mathcal{M}^\sharp$. Indeed, consider an $M \in \mathcal{M}$. $\mathcal{M}^\sharp$ cannot cross any $X \in \mathcal{C}$, as $\Upsilon(X, M^\sharp)$ would contain a clump with a piece being a proper subset of M.

On the other hand, assume $\mathcal{H}$ is not a skeleton even though $\mathcal{M}^\sharp \subseteq \mathcal{H}$. Hence there exists a clump $U = (U_1, \dots U_t) \in \mathcal{C} - \mathcal{H}$, not crossing any element of $\mathcal{H}$. Consider minimal pieces $M_1 \subseteq U_1$, $M_2 \subseteq U_2$. Then $M_1^\sharp$, U, $M_2^\sharp$ form a chain by Claim 4.2(ii), contradicting stability. ◻

Assume $\mathcal{H}$ is a stable cross-free system, but not a skeleton. In the following, we show how $\mathcal{H}$ can be extended to a stable cross-free system larger by one. By the above claim, there is an $M \in \mathcal{M}$ with $M^\sharp \in \mathcal{C} - \mathcal{H}$. Let

$$\mathcal{L}_1 := \{X \in \mathcal{H}\colon X \text{ and } M^\sharp \text{ are nested}\},$$
$$\mathcal{L}_2 := \{X \in \mathcal{H}\colon X \text{ and } M^\sharp \text{ are independent}\}. \tag{4.1}$$

CLAIM 4.4. *If $\mathcal{L}_1 = \varnothing$, then $\mathcal{H} + M^\sharp$ is a stable cross-free system.*

Proof. It is clear that $\mathcal{H}' = \mathcal{H} + M^\sharp$ is cross-free. For a contradiction, assume that for some $U \in \mathcal{C} - \mathcal{H}'$ and $K, K' \in \mathcal{H}$, $\mathcal{H}' + U$ is cross-free, although K, U, K' form a chain. $\mathcal{H}$ is stable, and hence $M^\sharp \in \{K, K'\}$; without loss of generality assume $M^\sharp = K'$. Now K and M are dependent and thus nested, a contradiction. ◻

In what follows we assume $\mathcal{L}_1 \neq \varnothing$. The minimality of M implies that for any $X \in \mathcal{L}_1$, the dominant piece of $M^\sharp$ w.r.t. X is a connected component of M^*. The central concept and key lemma of the algorithm are as follows.

DEFINITION 4.5. *The piece $C \in \mathcal{Q}$* fits *the pair $(\mathcal{H}, M)$ if*

(a) $C^\sharp \in \mathcal{C} - \mathcal{H}$, $C \subseteq M^*$.

(b) *there exists a $W \in \mathcal{H}$ with a piece $W_1 \subseteq C$.*

(c) *one can take any clump $X \in \mathcal{L}_1$ with dominant piece X_a w.r.t. $M^\sharp$ and an arbitrary other piece X_i with $i \neq a$. Then either $X_i \nsubseteq C$ or $X_i \cap C = \varnothing$, and if $X_a \cap C \neq \varnothing$, then $X_i \cap C^* = \varnothing$.*

(d) *$C^\sharp$ is independent from every $X \in \mathcal{L}_2$.*

LEMMA 4.6. *Let C be an inclusionwise minimal member of $\mathcal{Q} - \bigcup \mathcal{H}$ fitting $(\mathcal{H}, M)$. Then $\mathcal{H} + C^\sharp$ is a stable cross-free system.*

There exists a C satisfying the conditions of this lemma, as according to the definition, the pieces of $M^\sharp$ different from M (that is, the connected components of M^*) fit

$(\mathcal{H}, M)$. A minimal C can be found using standard bipartite matching theory similarly as in [11]; the technical details are postponed until the appendix. The proof of Lemma 4.6 is based on the following claim.

CLAIM 4.7. *Let $C \in \mathcal{Q} - \bigcup \mathcal{H}$, $C \subseteq M^*$. Assume there exists a $W \in \mathcal{H}$ with a piece $W_1 \subseteq C$. The following two properties are equivalent:* (i) *C fits $(\mathcal{H}, M)$;* (ii) *$\mathcal{H} + C^\sharp$ is cross-free.*

Proof. First we show that (i) implies (ii). $C^\sharp$ is independent from all pairs in $\mathcal{L}_2$. Consider an $X \in \mathcal{L}_1$. $C^\sharp$ and X cannot cross by Lemma 2.3 whenever X or $C^\sharp$ is large. Thus we may assume that both are small basic clumps, $X = (X_1, X_2)$ with X_2 being the dominant piece of X w.r.t. $M^\sharp$. If X and $C^\sharp$ are dependent, then $X_1 \cap C \neq \varnothing$ or $X_2 \cap C \neq \varnothing$. In the first case, (c) implies $X_1 \nsubseteq C$, and hence nestedness follows by Claim 2.5. So let us assume $X_1 \cap C = \varnothing$. Hence $X_2 \cap C \neq \varnothing$, and thus $X_1 \cap C^* = \varnothing$ by the second part of (c). Now X and $C^\sharp$ cannot be dependent since X_1 does not intersect any of C and C^*, a contradiction.

Next we show that (ii) implies (i). (a) and (b) are included among the conditions. For (c), consider an $X \in \mathcal{L}_1$ with dominant piece X_a w.r.t. $M^\sharp$ and another piece X_i with $i \neq a$. Notice that $X_i \subseteq M^*$. If X and $C^\sharp$ are independent, then $X_i \cap C = \varnothing$, as otherwise an edge between $X_i \cap C$ and M would connect both. If they are dependent so that the dominant piece of X w.r.t. $C^\sharp$ is different from X_i, then $X_i \nsubseteq C$ or $X_i \cap C = \varnothing$ follows. Next, assume that the dominant piece is X_i w.r.t. $C^\sharp$. If C were the dominant piece of $C^\sharp$ w.r.t. X, then $M \subseteq X_a \subseteq C$ would give a contradiction. Hence the dominant piece of $C^\sharp$ is different from C, and thus $C \subseteq X_i$. Now $W_1 \subseteq C \subseteq X_i$; hence W, $C^\sharp$, X form a chain by Claim 4.2(i), a contradiction to the stability of $\mathcal{H}$.

Assume next $X_a \cap C \neq \varnothing$ and $X_i \cap C^* \neq \varnothing$. X and $C^\sharp$ are dependent and thus nested, and as above, the dominant piece of X cannot be X_i. C cannot be the dominant piece of $C^\sharp$ as $X_i \subseteq C$ would contradict $X_i \cap C^* \neq \varnothing$. Hence $C \subseteq X_i^*$. We get a contradiction again because of the chain W, $C^\sharp$, X.

Finally, for (d) assume $C^\sharp$ and $X \in \mathcal{L}_2$ are dependent. C cannot be the dominant piece of $C^\sharp$ w.r.t. X as it would yield $X \in \mathcal{L}_1$. Consequently, $X_i \subseteq C^*$ for a nondominant piece X_i of X w.r.t. $C^\sharp$, and thus by Claim 4.2(ii), W, $C^\sharp$, X form a chain, a contradiction again to stability. ∎

Proof of Lemma 4.6. Using Claim 4.7, it is left to show that there exists no $U \in \mathcal{C} - (\mathcal{H} + C^\sharp)$ and $K \in \mathcal{H}$ so that $\mathcal{H} + C^\sharp + U$ is cross-free and $C^\sharp$, U, K form a chain. Indeed, in such a situation $C^\sharp$ and K would be dependent and thus nested. Assume first that the dominant piece of $C^\sharp$ w.r.t. K is different from C. Then for some piece K_1 of K, we have $K_1 \subseteq C^*$, and by Claim 4.2(ii), W, $C^\sharp$, K is a chain, contradicting the stability of $\mathcal{H}$.

If C is the dominant piece of C^* w.r.t. K, then for some pieces U_1 of U and K_1 of K, $K_1 \nsubseteq U_1 \nsubseteq C$. Now $U_1 \in \mathcal{Q} - \bigcup \mathcal{H}$, $U_1 \subseteq M^*$, and $K_1 \subseteq U_1$. By making use of Claim 4.7, U_1 fits $(\mathcal{H}, M)$, a contradiction to the minimal choice of C. ∎

4.2. Description of the dual oracle. To determine the value $\nu(G)$, we first construct a skeleton $\mathcal{K}$ as described above. For $\mathcal{K}$, we apply the reduction to Theorem 3.5 as in section 3.1. As already mentioned, a minimal chain decomposition along with maximal legal subpartition of a symmetric poset $P = (U, \preccurlyeq, M)$ can be found via a reduction to finding a maximum matching. For the sake of completeness and also because it will be needed for the minimum node-induced cost version, we include this reduction. Define the graph $C = (U, H)$ with $uv' \in H$ if and only if $u \prec v$ and $vv' \in M$ for some $v \in U$.

It is easy to see that the set $\{m_1, m_2, \ldots, m_\ell\} \subseteq M$ is a symmetric chain if and only if there exists edges $e_1, \ldots, e_{\ell-1} \in H$ such that $m_1 e_1 m_2 e_2 \ldots m_{k-1} e_{k-1} m_k$ is a path, called an M-alternating path. The transitivity of $\preccurlyeq$ ensures that $M \cup H$ contains no M-alternating cycles. Let $N \subseteq H$ be a matching in C. Then the components of $M \cup N$ are M-alternating paths, each containing exactly two nodes not covered by N. Hence finding a maximum matching in H is equivalent to finding a minimum chain cover in P. The running time of the most efficient maximum matching algorithm for a graph on n_1 nodes with m_1 edges is $O(\sqrt{n_1}m_1)$ [20], Vol. I, p. 423].

Let us now give upper bounds on $|\mathcal{K}|$ and on $|U|$. Jordán [15], [16] showed that the size of the optimal augmenting edge set is at most $\max(b(G)-1, \lceil\frac{t(G)}{2}\rceil) + \lceil\frac{k-2}{2}\rceil$. Here $b(G)$ is the maximum size of a clump, while $t(G)$ is the maximum number of pairwise disjoint sets in $\mathcal{Q}$. Since $b(G) \leq n-(k-1)$, $t(G) \leq n$, it follows that n is an upper bound on the size of an augmenting edge set. In a skeleton $\mathcal{K}$, the set of clumps connected by an edge xy form a chain. Since the size of a chain can also be bounded by n, we may conclude $\sum_{X\in\mathcal{K}}(|X|-1) \leq n^2$ and thus $|\mathcal{K}| \leq n^2$. Using the running time estimation in the appendix, this gives a bound $O(kn^5)$ on finding $\mathcal{K}$.

In section 3.1 the minimum pseudocover of $\mathcal{K}$ is reduced to a minimum symmetric chain cover of a poset $P = (U, \preccurlyeq, M)$ with $|U| = O(n^2)$, since there are $2|X|$ nodes in U corresponding the clump $|X|$. Hence the running time of the matching algorithm can be bounded by $O(n^5)$. As indicated in the introduction, at most $\binom{n}{2}$ calls of the dual oracle enable us to compute an optimal augmentation. This gives a total running time $O(kn^7)$.

As in [11], another algorithm can be constructed which calls the dual oracle only once. First, let us find a skeleton $\mathcal{K} = \{K_1, \dots, K_\ell\}$ with a cover F and a grove Π of $\mathcal{K}$ with $\operatorname{def}(\Pi) = |F|$. Then we iteratively apply sequences of flipping operations as in Lemma 3.3 for $\mathcal{F}_{i-1} = \mathcal{C} \div \{K_1, \dots, K_{i-1}\}$ and K_i for $i = \ell, \ell-1, \dots, 1$, resulting finally in a cover F' of $\mathcal{C}$ with $|F| = |F'|$. For each i it can be easily seen that after $O(n^2)$ flippings we get a cover of $\mathcal{F}_{i-1}$; thus $O(n^4)$ improving flippings suffice. The realization of a flipping step can be done using similar techniques as found in the appendix. We omit this analysis as it is highly technical, and we could not get a better running time estimation as for the previous algorithm.

4.3. Node-induced cost functions. The problem of finding a minimum-cost edge set whose addition makes a $(k-1)$-connected graph k-connected is NP-complete, as already making the graph $G = (V, \varnothing)$ connected by adding a minimum-cost edge set generalizes the Hamiltonian circuit problem, even for 0-1-valued cost functions.

However, for node-induced cost functions the other three basic problem—directed node-connectivity and both directed and undirected edge-connectivity augmentation—are solvable. We show that augmenting undirected node-connectivity by one is also tractable.

A cost function $c'\colon E \to \mathbb{R}$ is *node-induced* if there exists a $c\colon V \to \mathbb{R}$ so that $c'(uv) = c(u) + c(v)$ for every $uv \in E$. By the second part of Theorem 3.4, for a skeleton $\mathcal{K}$ and a node-induced cost function c', the minimum c'-cost of a cover of $\mathcal{C}$ is the same as that of $\mathcal{K}$. Hence it is enough to construct a subroutine for determining the minimum-cost $\nu_{c'}(\mathcal{K})$ of a cover of $\mathcal{K}$. A minimum-cost augmenting edge set can be found by iteratively calling this dual oracle.

Furthermore, by Lemma 3.7, $\nu'_c(\mathcal{K})$ equals the minimum-cost of a pseudocover of $\mathcal{K}$. Finding a minimum-cost pseudocover can be easily done based on the following weighted version of Fleiner's theorem, which reduces to maximum-cost matching in general graphs.

Given a symmetric poset $P = (U, \preccurlyeq, M)$ and a cost function $w\colon U \to \mathbb{R}$, let us define the cost of the symmetric chain $S = \{u_1v_1, \dots, u_\ell v_\ell\} \subseteq M$ with $u_1 \preccurlyeq \dots \preccurlyeq u_\ell$,

$v_1 \succcurlyeq \ldots \succcurlyeq v_\ell$ by $w(S) = w(u_\ell) + w(v_1)$. Our aim is to find a chain cover of minimum total cost.

Consider the reduction to the matching problem in section 4.2. For a matching $N \subseteq H$ of C, the components of $M \cup N$ are M-alternating paths each corresponding to a symmetric chain. The alternating path corresponding to the chain S is $v_1 u_1 v_2 u_2 \ldots v_\ell u_\ell$; hence the cost of the two nodes not covered by N equals the cost of the chain. Consequently, the cost of a symmetric chain cover equals the total cost of the nodes not covered by N. Hence minimizing the cost of a symmetric chain cover is equivalent to finding a maximum-cost matching. Note that here we need a maximum-cost matching only for node-induced cost functions, although this problem is tractable for arbitrary cost functions as well.

To find a minimum-cost pseudocover of $\mathcal{K}$, we construct the symmetric poset $P = (U, \preccurlyeq', M)$ as in section 3.1. For a one-way pair $u = (u^-, u^+) \in U$, let $w(u) = \min_{x \in u^+} c(x)$. We claim that finding a minimum-cost symmetric chain cover for this w is equivalent to finding a minimum-cost pseudocover of $\mathcal{K}$.

Indeed, there is a one-to-one correspondence between chains consisting of clumps of the form (X_i, X_i^*) and the symmetric chains of U (with the restriction that a chain may not contain both (X_i, X_i^*), (X_j, X_j^*) for $i, j > 1$). A chain $K_1, K_2, \ldots, K_\ell$ of clumps with orientations $L_1 \preccurlyeq L_2 \preccurlyeq \ldots \preccurlyeq L_\ell$ can be covered by any edge between L_1^- and L_ℓ^+; thus the minimum cost of an edge covering it is $w(L_\ell) + w(\overleftarrow{L_1})$ with w defined as above. Hence the minimum c-cost of a pseudocover in $\mathcal{K}$ equals the minimum w-cost of a symmetric chain cover of P.

5. Further remarks.

5.1. Degree sequences. What can we say about the degree sequences of the augmenting edge sets? It is well known that, in a graph G with arbitrary cost function on the edges, the sets of nodes covered by a minimum-cost matching form the bases of a matroid. A natural generalization of matroid bases are base polytopes (see, e.g., [20], Vol. II, p. 767]).

For undirected edge-connectivity augmentation, the degree sequences of the optimal augmenting edge sets form a base polytope, and the same holds for the in- and out-degree sequences for directed edge-connectivity augmentation (see, e.g., [6]). This is also true in the case of directed node-connectivity augmentation [9]. Moreover, all these results can be generalized for node-induced cost functions: the degree (resp., in- and out-degree) sequences of minimum-cost augmenting edge sets form a base polytope. Hence a natural conjecture is as follows.

Conjucture 5.1. *Given a $(k-1)$-connected graph G and a node-induced cost function, the degree sequences of the minimum-cost augmenting edge sets form a base polytope.*

This was essentially proved by Szabó in his master's thesis [21] for $k = n - 2$. His result holds even without the assumption that the graph is $(k-1)$-connected, indicating that the conjecture might hold for arbitrary graphs as well.

5.2. Abstract generalizations. In this section, we discuss possible generalizations and extension of our results. A natural question is whether it is possible to give a generalization of Theorem 1.3 for abstract structures. For directed connectivity augmentation, Theorem 1.4 is only a special case of covering crossing families of set pairs [9], Theorem 2.5], which is still only a special case of the general theorem on covering positively crossing bisupermodular functions [9], Theorem 2.3].

It would be possible to formulate an abstract theorem for describing coverings of a system $\mathcal{C}$ of "basic clumps," where under basic clump we simply mean a subpartition of a set satisfying certain properties. However, it is not easy to extract the abstract properties $\mathcal{C}$ needs to fulfill so that the argument can carry over. In particular, we need to ensure Claim 2.1, Lemma 2.3, Claims 2.4 and 2.5, and Lemmas 3.13 and 3.14 (for set pairs arising from orientations of clumps). It may be verified that whenever $\mathcal{C}$ satisfies these, all other proofs carry over; for the algorithm we also need a good representation of $\mathcal{C}$.

Since the argument is already quite abstract and complicated, and we could not find an elegant list of properties that ensure all these claims, we did not formulate such an abstract theorem in order to avoid an additional level of complexity. Furthermore, we believe that there should be a relatively simple abstract generalization of Theorem 1.3, which does not rely on all claims listed above. For comparison, the argument given in [11] for proving Theorem 1.4 strongly relies on properties of one-way pairs in a $(k-1)$-connected digraph. Nevertheless, these are not needed (and in fact, not necessarily true) for the general theorem for crossing families, which admits a much simpler proof.

A natural application of such an abstract theorem would be rooted connectivity augmentation. Given a graph or digraph with designated node $r_0 \in V$, it is called *rooted k-connected* if there are at least k internally disjoint (directed) paths between r_0 and any other node. Similarly, a digraph is *rooted k-edge-connected* with root r_0 if there are at most $k-1$ edge-disjoint directed paths from r_0 to any other node. One might ask the augmentation questions for rooted connectivity as well. It turns out that for digraphs the minimum-cost versions of rooted k-connectivity and rooted k-edge-connectivity augmentation are both solvable in polynomial time (see Frank and Tardos [10] and Frank [7]): both problems can be formulated via matroid intersection (although the reduction of the node-connectivity version is far from trivial).

In contrast, for undirected graphs the minimum-cost version of rooted k-connectivity augmentation is NP-complete: Hamiltonian cycle reduces to it even for $k = 2$ and 0-1 costs. The minimum cardinality version of augmenting rooted connectivity by one was studied by Nutov [19], who gave an algorithm finding an augmenting edge set of size at most $\mathrm{opt} + \min(\mathrm{opt}, k)/2$.

An important difference between minimum cardinality directed and undirected rooted connectivity augmentation is that while in the directed case there is an optimal augmenting edge set consisting only of edges outgoing from r_0, in the undirected case it may contain edges not incident to r_0. An example is $V = \{r_0, x, y, a\}$, $E = \{r_0x, r_0y, xa, ya\}$ (a rectangle). For $k = 3$, $F = \{xy, r_0a\}$ is an optimal augmenting set, but there is no augmenting set of size 2 of edges incident to r_0.

We believe that a min-max formula and a polynomial time algorithm for finding an optimal solution could be given by extending the method of the paper. However, it is not completely straightforward how clumps should be defined in this setting. At this point, we leave this question open, since we believe that it will be an easy consequence of a later general abstract theorem.

5.3. General connectivity augmentation. In what follows, we give an argument showing that there is no straigthforward way of generalizing Theorem 1.3 for general connectivity augmentation. For this, let us study directed connectivity augmentation first. In case of augmentation by one, Theorem 1.4 states that the minimum size of an augmenting arc set equals the maximum number of pairwise independent one-way pairs. The min-max formula is quite similar if $(k-1)$-connectivity is not assumed. In this case, we need to consider a broader class of one-way pairs: for a digraph $D = (V, A)$

and nonempty disjoint $X^-, X^+ \subseteq V$, $X = (X^-, X^+)$ is a one-way pair if there is no arc in A from X^- to X^+ (notice that $|V - (X^- \cup X^+)| = k - 1$ is not assumed). Let us define $p(X) = \max(0, k - |V - (X^- \cup X^+)|)$. Clearly, an augmenting arc set should contain at least $p(X)$ arcs covering X. Then the minimum size augmenting edge set equals the maximum of $\sum_i p(X_i)$ over pairwise independent one-way pairs X_i. Actually, this is still only a special case of [9], Theorem 2.3] where minimum coverings of positively crossing bisupermodular functions are considered.

Hence a possible approach for general undirected connectivity augmentation would be the following. Let a clump be a subpartition $X = (X_1, \ldots, X_\ell)$ of V with $d(X_i, X_j) = 0$ (we do not assume $|N_X| = k - 1$), and let $p(X)$ be a lower bound on the number of edges needed to cover X. There are multiple possible candidates for $p(X)$, and we do not commit to any of them; we work only with the mild assumption that $(\star)$ $p(X) = \max(0, k - |N_X|)$ whenever $|X| = 2$, and $p(X) = 0$ whenever $|N_X| \geq k$. A natural conjecture would be the following: the minimum size of an augmenting edge set equals the maximum deficiency of a grove. Let a grove now mean a subpartition $\Pi = \{\mathcal{B}_1, \ldots, \mathcal{B}_\ell\}$ of clumps so that each $\mathcal{B}_i$ is semi-independent, and clumps belonging to different $\mathcal{B}_i$'s are independent. The deficiency of this grove is defined as

$$\operatorname{def}(\Pi) = \sum_{i=1}^{\ell} \left\lceil \frac{\sum_{X \in \mathcal{B}_i} p(X)}{2} \right\rceil.$$

We show by an example that this conjecture fails even if $(\star)$ is the only assumption on $p(X)$. Let $G = (V, E)$ be the complement of the graph in Figure 5.1, and let $k = 9$. For a node $z \in V$, let $Z_z = (\{z\}, \{z\}^*)$. The only basic clumps in G with $|N_X| < 9$ are Z_a, Z_b, Z_{u_1}, Z_{u_2}, Z_{v_1}, Z_{v_2}, $(\{u_1, u_2\}, \{u_3\}, \{u_4\})$, $(\{v_1, v_2\}, \{v_3\}, \{v_4\})$, and $(\{a, c\}, \{b, d\})$. $\{u_1u_4, u_2u_3, v_1v_4, v_2v_3, ab, ad, bc\}$ is an augmenting edge set of size 7, while a grove of value 6 is $\{\mathcal{B}_1, \mathcal{B}_2\}$ with $\mathcal{B}_1 = \{Z_{u_1}, Z_{u_2}, Z_{u_3}, Z_{u_4}, (\{a\}, \{u_1, u_2, d\})\}$ and $\mathcal{B}_2 = \{Z_{v_1}, Z_{v_2}, Z_{v_3}, Z_{v_4}, (\{b\}, \{v_1, v_2, c\})\}$.

We show that neither an augmenting edge set of size 6, nor a grove of value 7 exists. On the one hand, assume there were an augmenting edge set F with $|F| = 6$. Then F

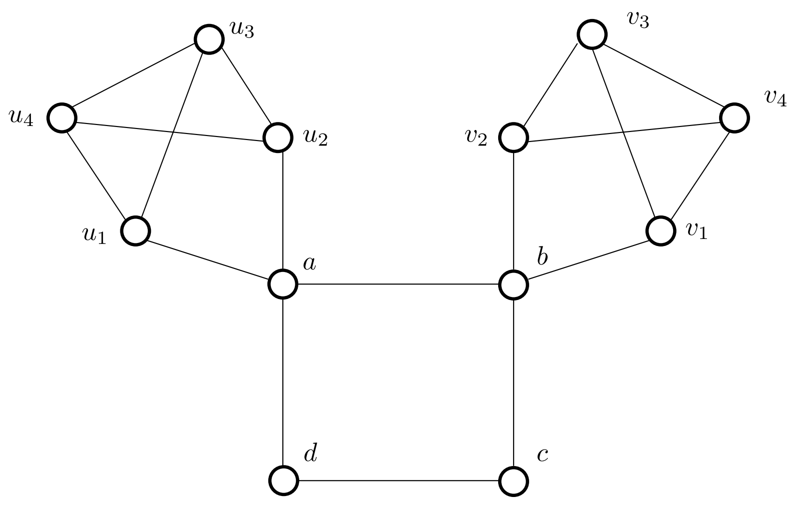


FIG. 5.1. *Example concerning general connectivity augmentation.*

could be partitioned into $F = F_1 \cup F_2$ with $|F_1| = |F_2| = 3$, F_1 covering $\mathcal{B}_1$ and F_2 covering $\mathcal{B}_2$. However, we need at least two edges to cover Z_a and two to cover Z_b, and these can only be contained in F_1 and F_2, respectively. If $ad \in F_1$, then F_1 cannot contain any of au_1 and au_2, as otherwise at least one of Z_{u_3} and Z_{u_4} would remain uncovered. Hence $ad \notin F_1$, and similarly $bc \notin F_2$. ab, $cd \notin F$, as they do not cover any clump in $\mathcal{B}_1$ or $\mathcal{B}_2$; thus $(\{a, c\}, \{b, d\})$ remains uncovered.

On the other hand, assume a grove of value 7 exists. We claim that it should contain $(\{a, c\}, \{b, d\})$, and two clumps of the form $(\{a\}, A)$ and $(\{b\}, B)$ with $b \in A$ and $a \in B$. This is clearly a contradiction as they cannot be simultaneously contained in a grove, since the edge ab connects all three of them. It can easily be checked that if we do not require $(\{a, c\}, \{b, d\})$ to be covered, then the remaining clumps may all be covered by six edges. The same holds unless we require all clumps of the form $(\{a\}, A)$ with $b \in A$, $|A| \geq 3$ and all clumps of $(\{b\}, B)$ with $a \in B$, $|B| \geq 3$ to be covered. Consequently, every grove of value 7 should contain such clumps.

Appendix. In this appendix we present how the subroutine for constructing a skeleton may be implemented using bipartite matching theory. The argument follows the same lines as the one in the appendix of [11]. Let us start with a simple claim concerning pieces.

CLAIM A.1. *For a piece $Y \in \mathcal{Q}_1$ and an arbitrary set $X \subseteq V$, if $X^* \supseteq Y^*$, then $X \subseteq Y$.*

Proof. Indeed, assume X is not a subset of Y; thus $|X \cup Y| > |Y|$. The condition gives $(X \cup Y)^* = Y^*$, and hence $|N(X \cup Y)| < |N(Y)| = k - 1$, contradicting that G is $(k-1)$-connected. ∎

Given the $(k-1)$-connected graph $G = (V, E)$, let us construct the bipartite graph $B = (V', V''; H)$ as follows. With each node $v \in V$ associate nodes $v' \in V'$ and $v'' \in V''$ and an edge $v'v'' \in H$. With each edge $uv \in E$ associate two edges $v'u''$, $u'v'' \in H$. For a set $X \subseteq V$, we denote by X' and X'' its images in V' and V'', respectively. The $(k-1)$-connectivity of G implies that B is *$(k-1)$-elementary bipartite*; that is, for each $\varnothing \neq X' \subseteq V'$, either $N(X') = V''$ or $|N(X')| \geq |X'| + k - 1$. We say that $X' \subseteq V'$ is *tight* if $|N(X')| = |X'| + k - 1$ and $N(X') \neq V''$. Observe that X' is tight if and only if $X \in \mathcal{Q}_1$.

Given a function $f\colon V' \cup V'' \to \mathbb{N}$ we call the set $F \subseteq H$ an *f-factor* if $d_F(x) = f(x)$ for every $x \in V' \cup V''$. Let $f(Z) = \sum_{x \in Z} f(x)$ for $Z \subseteq V' \cup V''$.

CLAIM A.2. *Consider a bipartite graph $G = (V', V''; H)$ and a function $f\colon V' \cup V'' \to \mathbb{N}$ so that $f(V') = f(V'')$ and $f(x) = 1$ or $f(y) = 1$ for every $xy \in H$. An f-factor exists if and only if $f(X) \leq f(N(X))$ for every $X \subseteq V'$.*

Proof. An easy consequence of Hall's theorem involves replacing each $x \in V' \cup V''$ by $f(x)$ copies. Note that by the condition $f(x) = 1$ or $f(y) = 1$ for every $xy \in H$, at most one copy of the same edge may be used. ∎

First we need to find the set $\mathcal{M}$ of minimal pieces. Let us consider nodes $u, v \in V$ with $uv \notin E$. A piece $X \in \mathcal{Q}_1$ is called a *uv-piece* if $u \in X$ and $v \in X^*$. For a $uv \notin E$, consider the following f. Let $f(u') = f(v'') = k + 1$ and for $z \in (V' - u') \cup (V'' - v'')$, let $f(z) = 1$. An f-factor for this f is called a *k-uv-factor.* If G is $(k-1)$-connected and thus B a $(k-1)$-elementary bipartite graph, then Claim A.2 implies the existence of a $(k-1)$-uv-factor. Let F_{uv} denote one of them.

CLAIM A.3. *If there is a k-uv-factor, then there exists no uv-piece.*

Proof. Assume X is a uv-piece. As $X \in \mathcal{Q}_1$, $|N(X')| = |X'| + k - 1$. Since $u' \in X'$, $v'' \notin N(X')$, we have $f(X') = |X'| + k$, $f(N(X')) = |X'| + k - 1$; thus by Claim A.2, no k-uv-factor exists. ∎

It is easy to see that any two uv-pieces are dependent and the union and intersection of two uv-pieces are uv-pieces as well. Thus if the set of uv-pieces is nonempty, then it contains a unique minimal element. In what follows we show how this can be found algorithmically. For an edge set $F \subseteq H$, we say that the path $U = x_0 y_0 x_1 y_1 \dots x_t y_t$ is an *alternating path* for F from x_0 to y_t if $x_i \in V'$, $y_i \in V''$, $x_i y_i \in H - F$ for $i = 0, \dots, t$, and $y_i x_{i+1} \in F$ for $i = 0, \dots, t-1$. Under the same conditions we also say that $x_0 y_0 x_1 y_1 \dots x_t$ is an alternating path for F from x_0 to x_t.

Claim A.4. (a) *If there exists an alternating path for F_{uv} between u' and v'', then there exists no uv-piece.* (b) *Assume there is no alternating path for F_{uv} from u' to v''; let S denote the set of nodes $z \in V$ having an alternating path for F_{uv} from u' to z'. Then S is the unique minimal uv-piece, and S is connected.*

Proof. (a) Let U be an alternating path for F_{uv} from u' to v''. Then $F_{uv} \Delta U$, the symmetric difference of the edge sets F_{uv} and U, is a k-uv-factor so, by Claim A.3, no uv-piece exists. (b) Let Z be an arbitrary uv-piece. For every $x \in Z - u$, $N(Z')$ contains a unique y'' with $x'y'' \in F_{uv}$. The number of $y \in V$ with $u'y'' \in F_{uv}$ is exactly k, and all of them are contained in $N(Z')$. These are $|Z'| + k - 1$ different elements of $N(Z')$, and since $Z \in \mathcal{Q}_1$, $N(Z')$ has no elements other than these. This easily implies that Z' contains every $x' \in V$ for which there is an alternating path for F_{uv} from u' to x', showing $S \subseteq Z$. It is left to prove that $S \in \mathcal{Q}_1$. From the definition of S, it follows that for every $y'' \in N(S')$, there exists an $x \in S$ with $x'y'' \in F_{uv}$, proving $|N(S')| = |S'| + k - 1$. The connectivity of S follows since otherwise the connected component containing u would be a smaller uv-piece. $\square$

For the initialization of the algorithm, we determine the edge sets F_{uv} by a single max-flow computation for every $u, v \in V$, $uv \notin E$. By Claim A.4 the minimal uv-pieces can be found by a breadth-first search. The minimal ones among these will give the elements of $\mathcal{M}$ (note that the minimal $u_i v_i$-set might be contained in some other $u_j v_j$-set). We will use the sets F_{uv} also in the later steps of the algorithm.

Consider now a stable cross-free $\mathcal{H}$ which is not a skeleton, a minimal element $M \in \mathcal{M} - \bigcup \mathcal{H}$, and $\mathcal{L}_1, \mathcal{L}_2$ as defined by (4.1). If $\mathcal{L}_1 = \varnothing$, then we are done by Claim 4.4; hence in what follows we assume $\mathcal{L}_1 \neq \varnothing$.

By Lemma 4.6, our task is to find a minimal C fitting $(\mathcal{H}, M)$. Let $\mathcal{T}$ be the set of the maximal ones among those pieces of the clumps in $\mathcal{L}_1$ which are subsets of M^*.

Claim A.5. *$\mathcal{T}$ consists of pairwise disjoint sets, and all of them are subsets of the same piece $\hat{M} \neq M$ of $M^\sharp$.*

Proof. Consider clumps $X, Y \in \mathcal{L}_1$ with pieces $X_1, Y_1 \in \mathcal{T}$. If X and Y are independent, then $X_1 \cap Y_1 = \varnothing$, as otherwise an edge between $X_1 \cap Y_1$ and M would connect both. If they are dependent, then we show that the dominant piece X_a of X w.r.t. Y is different from X_1. Indeed, if $X_a = X_1$, then the dominant piece of Y w.r.t. X should be $Y_b \neq Y_1$ as otherwise $M \subseteq Y_1$ would follow. Hence $Y_1 \not\subseteq X_1$, a contradiction to the maximality of Y_1. Similarly, the dominant piece of Y w.r.t. X may not be Y_1. Hence $Y_1 \subseteq X^*$; thus $X_1 \cap Y_1 = \varnothing$.

Finally, assume that $X_1 \subseteq \hat{M}$ and $Y_1 \subseteq \tilde{M}$ for pieces $\hat{M}, \tilde{M}$ of $M^\sharp$. Then $X, M^\sharp, Y$ form a chain by Claim 4.2(ii), a contradiction to stability. $\square$

Let us construct the bipartite graph $B_1 = (V', V''; H_1)$ from B by adding some new edges as follows.

(1) For each $X \in \mathcal{L}_2$, let $x'y'', y'x'' \in H_1$ for every xy connecting X.
(2) Let $x'y'' \in H_1$ whenever $T \in \mathcal{T}$, $x \in T$ and $y \in T \cup N(T)$.

(3) For each $X \in \mathcal{L}_1$ with dominant piece X_a w.r.t. $M^\sharp$, let $x'y'' \in H_1$ for every $x \in X_a$, $y \in X_a^*$.

Claim A.6. *Let $C \in \mathcal{Q} - \bigcup\mathcal{H}$, $C \subseteq \hat{M}$, supported by some $W \in \mathcal{H}$. C fits $(\mathcal{H}, M)$ if and only if C' is tight in B_1.*

Proof. $C' \subseteq V'$ is tight in B_1 if and only if it is tight in B and there is no edge in $x'y'' \in H_1 - H$ with $x' \in C'$, $y' \in V'' - N(C')$. In such a configuration, we say that the edge $x'y''$ *blocks* the set C'. (This is equivalent to that xy connects the clump (C, C^*).)

Assume C fits $(\mathcal{H}, M)$. Property (d) forbids any $x'y'' \in H_1 - H$ of the first type block C', while property (c) forbids any $x'y''$ of the second or third type to block C'. For the other direction, properties (a) and (b) follow by the conditions. For (d), if C were dependent with some $X \in \mathcal{L}_2$, then a new edge of the first type would block C'. For (c), if $C \cap X_i \neq \varnothing$, $X_i - C \neq \varnothing$ for some $X \in \mathcal{L}_1$ with a piece $X_i \nsubseteq M^*$, then consider a $T \in \mathcal{T}$ with $X_i \subseteq T$. $C - T \neq \varnothing$, as otherwise W, $C^\sharp$, $T^\sharp$ would contradict stability. By Claim A.1, $C^* \cap (T \cup N(T)) \neq \varnothing$; hence a new edge of the second type blocks C'. Finally, if X_a is the dominant piece of X w.r.t. $M^\sharp$, $X_a \cap C \neq \varnothing$, and $X_i \cap C^* \neq \varnothing$, then there is a new edge of the third type blocking C'. ∎

To find a C as in Lemma 4.6, we need to add some further edges to B_1. Indeed, we need to ensure that $C \in \mathcal{Q} - \bigcup\mathcal{H}$ and furthermore that C is supported by some $W \in \mathcal{L}_1$. Consider now a $W \in \mathcal{L}_1$ with a piece $W_1 \in \mathcal{T}$ and a connected set Q with $W_1 \nsubseteq Q \subseteq \hat{M}$. Let $Z(Q)$ denote the unique minimal X satisfying the following property:

$$X \in \mathcal{Q}, \quad Q \subseteq X, \quad \text{and} \quad X \text{ fits } (\mathcal{H}, M). \tag{A.1}$$

We will determine $Z(Q)$ for different sets Q in order to find C. $Z(Q)$ is well defined since it is easy to see the following: (i) $\hat{M}$ satisfies (A.1); (ii) if X and X' satisfy (A.1), then X and X' are dependent and $X \cap X'$ also satisfies (A.1); (iii) $Z(Q)$ is connected. The next claim gives a simple algorithm for finding $Z(Q)$ for a given Q.

Claim A.7. *Fix some $u \in Q$, $v \in M$. Let B_2 denote the graph obtained from B_1 by adding all edges $u'y''$ with $y \in Q \cup N(Q)$. Let S denote the set of nodes z for which there exists an alternating path for F_{uv} from u' to z'. Then $Z(Q) = S$.*

Proof. As M^* is a uv-set in B_2, applying Claim A.4(a) for B_2 instead of B, we get that B_2 contains no alternating path for F_{uv} between u' and $v^{\boxtimes}$. By Claim A.4(b), S is the unique minimal uv-piece in B_2. $N(S' \cup Q') = N(S')$, and thus $Q \cup N(Q) \subseteq S \cup N(S)$ because of the new edges in B_2; hence by Claim A.1, $Q \subseteq S$. By making use of Claim A.6, S is the unique minimal set satisfying (A.1); thus $Z(Q) = S$. ∎

Consider now a clump $W = (W_1, W_2, \ldots, W_h) \in \mathcal{L}_1$ with $W_1 \in \mathcal{T}$. We want to find a C_W fitting $(\mathcal{H}, M)$ supported by W_1. For each $q \in N_W \cap \hat{M}$, let us compute $Z(Q)$ for $Q = W + q$. Let C_W denote a minimal set among these. A $Z(Q)$ can be found by a single breadth-first search; thus we need at most $k - 1$ breadth-first searches. We can compute such a C_W for all possible choices of W, and a minimal among these gives a minimal C fitting $(\mathcal{H}, M)$. Therefore, the running time may be bounded by $(k-1)n$ breadth-first searches, since by Claim A.5, $|\mathcal{T}| \leq n$.

Complexity. To find a skeleton system first we need n^2 max-flow computations to determine the minimal pieces and the auxiliary graphs. The running time for extending the stable cross-free system by one member is dominated by $(k-1)n$ breadth-first searches. Thus if s is an upper bound on the size of a skeleton, then we can determine one in $O(n^5 + skn^3)$ running time by using an $O(n^3)$ maximum-flow algorithm and an $O(n^2)$ breadth-first search algorithm.

Acknowledgments. The author is grateful to Kristóf Bérczi and András Frank for several fruitful discussions on this topic and for their valuable ideas and suggestions. The author also thanks the anonymous referees for their valuable remarks which helped to improve the presentation.

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