Half-integral vertex covers on bipartite bidirected graphs: total dual integrality and cut-rank

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May 2009

Abstract

In this paper we study systems of the form $b \le Mx \le d$, $l \le x \le u$, where M is obtained from a totally unimodular matrix with two nonzero elements per row by multiplying by 2 some of its columns, and b, d, l, u are integral vectors. We give an explicit description of a totally dual integral system that describes the integer hull of the polyhedron P defined by the above inequalities. Since the inequalities of such totally dual integral system are Chvátal inequalities for P, our result implies that the matrix M has cut-rank 1. We also derive a strongly polynomial time algorithm to find an integral optimal solution for the dual of the problem of minimizing a linear function with integer coefficients over the aforementioned totally dual integral system.

Keywords: Integral polyhedra, totally unimodular matrices, total dual integrality, cut-rank.

1 Introduction

A matrix is *totally unimodular* if all its sub-determinants are equal to +1, −1, or 0 (in particular, all its entries are +1, −1, or 0). Given a matrix M

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and a subset I of its columns, we denote with M_I the matrix obtained from M by multiplying by 2 the columns of M in I.

Let M be a totally unimodular matrix with exactly two nonzero elements in every row, let I be a subset of its columns, and let b, c, l, u be integral vectors of appropriate dimension. We study the set of integral solutions of systems of the form

$$\begin{aligned} M_I x &\geq b \\ l \leq \quad x &\leq u, \end{aligned} \tag{1}$$

and the corresponding optimization problem

$$\min\{c^\top x \,:\, x \text{ satisfies } (1), x \text{ integral}\}. \tag{2}$$

We explain how the above problem is related to mixed-integer programming. Indeed, given a problem of the form $\min\{c^\top x \,:\, Ax \geq b,\, x_i \in \mathbb{Z},\, i \in I\}$, where A is a totally unimodular matrix, b is an half-integral vector (i.e. a vector such that $2b$ is integral) and I is the set of integer variables, then it is not difficult to show that there is an optimal solution that is half-integral. Thus, with the change of variables $y_j = x_j,\ j \in I,\ y_j = 2x_j,\ j \notin I$, the original problem is equivalent to $\min\{c^\top y \,:\, A_I y \geq 2b,\ y \text{ integral}\}$. We point out that the latter problem is NP-hard when A is a general totally unimodular matrix. The complexity status of solving mixed-integer programming problems where the constraint matrix is a totally unimodular matrix with at most two nonzero entries per row and with arbitrary right-hand-side is still open (see [2] and [4]), although [2] shows that it is polynomial when the size of the constraint coefficients is bounded by a constant.

Problem (2) where M is the transpose of the incidence matrix of a bipartite graph, and where the variables are restricted to be nonnegative, was studied by Conforti et al. in [3]. In this case, they derived an explicit characterization of the inequalities defining the integer hull. This was accomplished by expressing the integer hull of the system as the projection of some polyhedron in a higher dimensional space. In this paper we show, with a similar construction, how problem (2) can in fact be reduced to a weighted vertex covering problem on a certain extended graph. Using this construction, we describe a totally dual integral system defining the integer hull of the polyhedron defined by the constraint system of (2). We recall that a system $Ax \geq b$ is *totally dual integral* if the dual of $\min\{c^\top x \,:\, Ax \geq b\}$ has an integral optimal solution for every integral vector c for which the primal has a finite optimum (see [10] for an extensive treatment on total dual integrality).

A *bidirected graph* is a triple $D = (N, A, \sigma)$, where (N, A) is an undirected graph and σ is a map that assigns to each $e \in A$ and $v \in e$ a *sign* $\sigma_{e,v} \in \{+1, -1\}$. For convenience, we define $\sigma_{e,v} := 0$ if $v \notin e$. The *edge-node incidence matrix* of a bidirected graph D is the $|A| \times |N|$ matrix $(\sigma_{e,v})$. We call a bidirected graph *bipartite* if its edge-node incidence matrix is totally unimodular.

Let $D = (N, A, \sigma)$ be the bipartite bidirected graph whose edge-node incidence matrix is M. The subset I of the columns of M corresponds to a subset of the nodes in N. Let $L = N \setminus I$.

For every bidirected edge $e \in A$ we call b_e the *requirement* of e. A *trail* in an undirected graph is a walk with no repeated edges. An I*-trail* in D is a trail $T = (v_1, \ldots, v_k)$ in the undirected graph underlying D such that $v_1 \in I$, $v_2, \ldots, v_{k-1} \notin I$. An I*-path* in D is an I-trail in D which is a path in the undirected graph underlying D. For any such I-trail in D we define

$$\begin{aligned}
\gamma_1^T &= \sigma_{v_1 v_2, v_1}, \\
\gamma_i^T &= (\sigma_{v_{i-1} v_i, v_i} + \sigma_{v_i v_{i+1}, v_i})/2, \quad i = 2, \ldots, k-1; \\
\gamma_k^T &= \sigma_{v_{k-1} v_k, v_k}.
\end{aligned}$$

Notice that $\gamma_i^T \in \{0, \pm 1\}$ for every $i = 1, \ldots, k$, and that it is possible that $v_s = v_t$ and $\gamma_s^T \neq \gamma_t^T$ for two indices $s \neq t$. Given an I-trail $P = (v_1, \ldots, v_k)$ of D, the following inequalities are Gomory-Chvátal inequalities for (1), thus they are valid for its integer hull:

$$\begin{array}{lcll}
\sum_{i=1}^{k} \gamma_i^P x_{v_i} & \geq & \left\lceil \frac{\sum_{e \in P} b_e}{2} \right\rceil & \text{if } v_1, v_k \in I, \\
\sum_{i=1}^{k} \gamma_i^P x_{v_i} & \geq & \left\lceil \frac{\sum_{e \in P} b_e + l_{v_k}}{2} \right\rceil & \multirow{2}{*}{\text{if } v_k \notin I, \gamma_k^P = 1,} \\
\sum_{i=1}^{k-1} \gamma_i^P x_{v_i} & \geq & \left\lceil \frac{\sum_{e \in P} b_e - u_{v_k}}{2} \right\rceil & \\
\sum_{i=1}^{k-1} \gamma_i^P x_{v_i} & \geq & \left\lceil \frac{\sum_{e \in P} b_e + l_{v_k}}{2} \right\rceil & \multirow{2}{*}{\text{if } v_k \notin I, \gamma_k^P = -1.} \\
\sum_{i=1}^{k} \gamma_i^P x_{v_i} & \geq & \left\lceil \frac{\sum_{e \in P} b_e - u_{v_k}}{2} \right\rceil &
\end{array} \qquad (3)$$

In fact, all such inequalities are obtained by summing up the inequalities of $M_I x \geq b$ corresponding to the edges of the I-trail, plus or minus the lower or upper bound on the variable corresponding to endnodes not in I, dividing the inequality thus obtained by 2, and rounding up the right-hand-side. We observe that the inequalities (3) are the analogue of the ones considered in [3] in the undirected case.

Example 1. Consider the bidirected graph D in Figure 1, with requirements on the edges and bounds on the nodes. It is immediate to verify that D is bipartite. Let $I = \{a\}$. If we consider the I-path $P = (a, b, c, d)$, we obtain two I-path inequalities (3), namely $x_a - x_b \geq 4$ (since $b_{ab} + b_{bc} + b_{cd} + l_d = 7$) and $x_a - x_b - x_d \geq 3$ (since $b_{ab} + b_{bc} + b_{cd} - u_d = 5$).

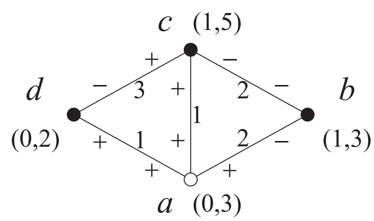

Figure 1: A bipartite bidirected graph. Numbers on the edges represent the requirements, while the pair on each node i represents the bounds (l_i, u_i).

Theorem 1. *Let M be a totally unimodular matrix with two nonzero elements per row, and let I be a subset of its columns. Then the system defined by (1) and (3), for every I-path P, is totally dual integral.*

Notice that in Theorem 1 we only need inequalities of the form (3) when P is an I-path, rather than a general I-trail. We postpone the proof to Section 2.2. Our proof yields a strongly polynomial time algorithm that, given an integral cost vector c, finds an integral optimal solution for the dual of $\min\{c^\top x : x \text{ satisfies } (1), (3) \text{ for every } I\text{-path } P\}$ whenever such problem has a finite optimum. Deriving a polynomial algorithm from the proof, however, is non-trivial, and it is accomplished in Section 4.

Edmonds and Giles [5] showed that if a system of linear inequalities with integer coefficients is totally dual integral, then the polyhedron defined by such a system is integral. Thus the above theorem implies the following.

Corollary 1. *Let M be a totally unimodular matrix with two nonzero elements per row, and let I be a subset of its columns. The polyhedron defined by (1) and (3), for every I-path P, is integral.*

Corollary 1 was proven in [3] in the case where M is nonnegative, $l = 0$ and there is no upper-bound on the variables.

We recall that the *cut-rank* (or *strong Chvátal rank*) of a rational matrix M is the smallest number t such that the polyhedron defined by the system

$b \leq M \leq d$, $l \leq x \leq u$ has Chvátal rank at most t for all integral vectors b, d, l, u. Matrices with cut-rank 1 are also said to have the *Edmonds-Johnson Property.*

Note that, since $\begin{pmatrix} M \\ -M \end{pmatrix}$ is also a totally unimodular matrix with exactly two nonzero elements in every row, then set of the form $\{x : b \leq M_I x \leq d,\ l \leq x \leq u, x \text{ integral}\}$, for integral vectors b, d, l, u, can be written in the form (2). Since the inequalities in (3) are rank-1 Chvátal inequalities for (1), Theorem 1 implies the following.

Corollary 2. *Let M be a totally unimodular matrix with two nonzero elements per row, and let I be a subset of its columns. Then M_I has the Edmonds-Johnson Property.*

When M is the edge-node incidence matrix of a bipartite undirected graph the above corollary follows from the result in [3]. This is one of the few known non-trivial classes of matrices with the Edmonds-Johnson property. To the best of our knowledge, the other known classes of matrices are i) matrices $A = (a_{ij})$ such that, for each column j, $\sum_i |a_{ij}| \leq 2$ (Edmonds and Johnson [6]), ii) matrices $A = (a_{ij})$ such that, for each row i, $\sum_j |a_{ij}| \leq 2$, and such that A does not have an odd-K_4 minor (Gerards and Schrijver [7], the reader is referred to the paper for definitions of the terms), iii) integral binet matrices (Appa et al. [1]).

In the paper we will need the following result.

Theorem 2 (Ghouila-Houri [8])**.** *A $\{0, \pm 1\}$-matrix A is totally unimodular if and only if, for every column submatrix B of A, the columns of B can be partitioned into two classes such that in every row the sum of the entries in one class differs by at most* 1 *from the sum of the entries in the other class.*

Notice that, if A is a $\{0, \pm 1\}$-matrix with exactly two nonzeros per row, the above reduces to the theorem of Heller and Tompkins [9] stating that A is totally unimodular if and only if the columns of A can be partitioned into two classes such that, for each row, if the two nonzeros in the row have the same sign then they are in different classes, and if they have opposite sign then they are both in the same class. In particular, this implies that the edge-node incidence matrix of an undirected graph G is totally unimodular if and only if G is bipartite.

In Section 2 we prove Theorem 1 in the special case where M is nonnegative, $l = 0$ and $u = +\infty$. In Section 3 we show how to prove the general case of Theorem 1 from the case in Section 2.

2 Bipartite case

In this section we study a special case of problem (2), namely the case where M is a *nonnegative* totally unimodular matrix with two nonzero elements per row, and where all the variables are required to be nonnegative (i.e., $l = 0, u = +\infty$). Notice that this is exactly the special case considered in [3]. Let I be a subset of the columns of M.

In this case the matrix M is the edge-node incidence matrix of an undirected graph $G = (N, E)$, and $I \subseteq N$. We define $L = N \setminus I$. As mentioned above, by the result of Heller and Tompkins [9] G is bipartite. Let U, V be the bipartite classes of G.

For every $i \in U \cup V$ we define $a_i = 2$ if $i \in I$, $a_i = 1$ if $i \in L$. Hence, given a cost vector $c \in \mathbb{Z}^{U \cup V}$, in this case problem (2) becomes

$$\begin{array}{lll} \min & \sum_{i \in U \cup V} c_i x_i & \\ \text{s.t.} & a_i x_i + a_j x_j \geq b_{ij} & ij \in E \\ & x_i \geq 0 & i \in U \cup V \\ & x_i \in \mathbb{Z} & i \in U \cup V. \end{array} \tag{4}$$

Throughout the rest of the paper, whenever Z is a set, z is a vector in $\mathbb{R}^Z$, and $Y \subseteq Z$, we denote with $z(Y) = \sum_{i \in Y} z_i$. We will show the following.

Theorem 3. *The system of linear inequalities*

$$\begin{array}{rcll} a_i x_i + a_j x_j & \geq & b_{ij} & ij \in E \\ \sum_{i \in P} x_i & \geq & \left\lceil \frac{b(P)}{2} \right\rceil & P\ I\text{-path} \\ x_i & \geq & 0 & i \in U \cup V \end{array} \tag{5}$$

is totally dual integral.

In [3] it was shown that (5) defines an integral polyhedron. To show Theorem 3 we need to show that the problem

$$\begin{array}{lll} \max & \sum_{e \in E} b_e y_e + \sum_P \left\lceil \frac{b(P)}{2} \right\rceil y_P & \\ \text{s.t.} & \sum_{e \ni i} a_i y_e + \sum_{P \ni i} y_P \leq c_i & i \in U \cup V \\ & y_e \geq 0 & e \in E \\ & y_P \geq 0 & P\ I\text{-path,} \end{array} \tag{6}$$

has an integral optimal solution y for each vector $c \in \mathbb{Z}^{U \cup V}$ for which $\min\{c^\top x : x \text{ satisfies } (5)\}$ has a finite optimum. Since the latter problem is unbounded whenever c has a negative component, throughout this section we will assume that c is nonnegative.

We show how problem (4) can be reduced to another problem where the constraint matrix is the edge-node incidence matrix of some extended bipartite graph.

2.1 The extended graph

Given a bipartite graph $G = (U \cup V, E)$, and a requirement vector $b \in \mathbb{Z}^E$, we define $\tilde{G}_\emptyset = (\tilde{U}_\emptyset \cup \tilde{V}_\emptyset, \tilde{E})$ as follows.

Let U', V' be copies of U, V, respectively, such that U, V, U', V' are pairwise disjoint. For every $i \in U \cup V$, we denote by i' the copy of i in $U' \cup V'$. Let $\tilde{U}_\emptyset = U \cup U'$, $\tilde{V}_\emptyset = V \cup V'$. $\tilde{E}$ contains the edges ij and $i'j'$ for every $ij \in E$ such that b_{ij} is odd, and the edges $i'j$ and ij' for every $ij \in E$ such that b_{ij} is even.

For $I \subseteq U \cup V$, the *extended graph* $\tilde{G}_I = (\tilde{U}_I \cup \tilde{V}_I, \tilde{E})$ is obtained from $\tilde{G}_\emptyset$ by identifying the two copies i, i' of every node $i \in I$, where $\tilde{U}_I$ and $\tilde{V}_I$ correspond to $\tilde{U}_\emptyset$ and $\tilde{V}_\emptyset$. (Notice that we identify the set of edges of $\tilde{G}_\emptyset$ with that of $\tilde{G}_I$.) For an example of a graph and its extended graph see Figure 2.

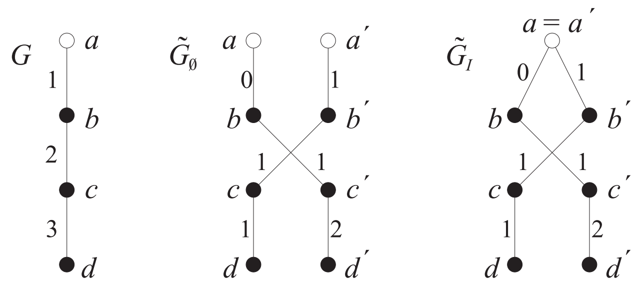

Figure 2: An example of a graph G, the graph $G_\emptyset$, and the extended graph $\tilde{G}_I$. White nodes represent nodes in I. The numbers are the requirements on the edges.

For each node $i \in U \cup V$, the *images* of i in $\tilde{G}_I$ are the nodes i, i' (where $i = i'$ if $i \in I$). For each edge $ij \in E$, the *images* of ij in $\tilde{G}_I$ are the edges ij and $i'j'$ if b_{ij} is odd, the edges ij' and $i'j$ if b_{ij} is even. We say that a node $i \in \tilde{U}_I \cup \tilde{V}_I$ is the *symmetric* of another node $j \in \tilde{U}_I \cup \tilde{V}_I$, and we write $i = sym(j)$, when i, j are the images (possibly coincident) of the same element of $U \cup V$. We say that an edge $e_1 \in \tilde{E}$ is the *symmetric* of another edge $e_2 \in \tilde{E}$, and we write $e_1 = sym(e_2)$, when the two edges e_1 and e_2 are distinct images of the same edge $e \in E$.

To each edge e in $\tilde{E}$ we assign a requirement $\tilde{b}_e$ as follows. For each edge $ij \in E$ with b_{ij} odd we define $\tilde{b}_{ij} = \lfloor \frac{b_{ij}}{2} \rfloor$ and $\tilde{b}_{i'j'} = \lceil \frac{b_{ij}}{2} \rceil$, while for every edge $ij \in E$ such that b_{ij} is even we define $\tilde{b}_{i'j} = \tilde{b}_{ij'} = \frac{b_{ij}}{2}$.

To each node w of $\tilde{U}_I \cup \tilde{V}_I$ we assign a cost $\tilde{c}_w$ equal to the cost of its corresponding node in $U \cup V$.

Now consider the following problem on $\tilde{G}_I$

$$\begin{array}{lll} \min & \sum_{i \in \tilde{U}_I \cup \tilde{V}_I} \tilde{c}_i \tilde{x}_i & \\ \text{s.t.} & \tilde{x}_i + \tilde{x}_j \geq \tilde{b}_{ij} & ij \in \tilde{E} \\ & \tilde{x}_i \geq 0 & i \in \tilde{U}_I \cup \tilde{V}_I \end{array} \tag{7}$$

and its dual problem

$$\begin{array}{lll} \max & \sum_{e \in \tilde{E}} \tilde{b}_e \tilde{y}_e & \\ \text{s.t.} & \sum_{e \ni i} \tilde{y}_e \leq \tilde{c}_i & i \in \tilde{U}_I \cup \tilde{V}_I \\ & \tilde{y}_e \geq 0 & e \in \tilde{E}. \end{array} \tag{8}$$

Note that the constraint matrix of (8) is the incidence matrix of a bipartite graph, and thus it is totally unimodular (see for example [10]). Thus, if problems (7) and (8) admit optimal solutions, then they admit optimal solutions that are integral, provided that $\tilde{b}$ and $\tilde{c}$ are integral. Note that, by construction of $\tilde{G}_I$, if x is a feasible solution for (4) then

$$\begin{array}{rcll} \tilde{x}_i & = & x_i & i \in I \\ \tilde{x}_i & = & \lfloor \frac{x_i}{2} \rfloor & i \in (U \cup V) \setminus I \\ \tilde{x}_{i'} & = & \lceil \frac{x_i}{2} \rceil & i' \in (U' \cup V') \setminus I \end{array}$$

is a feasible integral solution for (7) with the same objective value. If $\tilde{x}$ is an integral feasible solution for (7) then

$$\begin{array}{rcll} x_i & = & \tilde{x}_i & i \in I \\ x_i & = & \tilde{x}_i + \tilde{x}_{i'} & i \in L \end{array} \tag{9}$$

is a feasible solution for (4) with the same objective value. Hence (4), (7) and (8) have the same optimal value. Also, since any feasible solution of (4) is feasible for (5), then by weak duality the optimal value of (6) is at most the optimal value of (4), and therefore of (8).

The above argument shows that the system obtained by juxtaposing the constraints (7) and (9) yields an extended formulation for the convex hull P of feasible solutions of (4), meaning that P is the projection onto the space of x-variables of the points $(x, \tilde{x})$ satisfying (7),(9). This is similar to an extended formulation introduced in [2] and used also in [3]. The difference is that, for $i \in L$, instead of the constraints $\tilde{x}_i \geq 0$ and $\tilde{x}_{i'} \geq 0$, in [2] and [3] there are the two (stronger) constraints $\tilde{x}_i \geq 0$ and $\tilde{x}_{i'} - \tilde{x}_i \geq 0$. For our purposes it seems more convenient to work with the inequalities (7). In [3] it was shown that, projecting down onto the x-variables the inequalities of the extended formulation, the only new inequalities arising are the I-path inequalities, thus implying that the system (5) defines an integral polyhedron. It is not clear if this latter result or the proof of it given in [3] can be used to easily derive Theorem 3, in fact the proof we give next is independent of the one in [3].

2.2 Proof of Theorem 3

We prove Theorem 3 by showing how to derive an integral optimal solution for (6) from an integral optimal solution for (8). First, we need to prove a lemma.

We say that a digraph $\tilde{D}$ is an *antisymmetric* orientation of $\tilde{G}_I$ if $\tilde{D}$ is obtained from $\tilde{G}_I$ by orienting its edges so that, for any pair of symmetric edges e_1, e_2 of $\tilde{G}_I$, one of the two is oriented from $\tilde{U}_I$ to $\tilde{V}_I$, and the other from $\tilde{V}_I$ to $\tilde{U}_I$. For ease of notation, in the remainder, whenever we refer to an edge e of $\tilde{G}_I$, we also denote by e the arc of $\tilde{D}$ obtained by orienting e.

We define the *cost* of each arc (u, v) from $\tilde{U}_I$ to $\tilde{V}_I$ as $\beta_{(u,v)} = \tilde{b}_{uv}$ and the *cost* of each arc (v, u) from $\tilde{V}_I$ to $\tilde{U}_I$ as $\beta_{(v,u)} = -\tilde{b}_{vu}$. Given a directed path or cycle S in $\tilde{D}$, we define the *cost of* S as $\beta(S)$.

Given a directed path (resp. a directed cycle) $S = (v_1, v_2, \ldots, v_n)$ in $\tilde{D}$, $sym(S) = (sym(v_n), sym(v_{n-1}), \ldots, sym(v_2), sym(v_1))$ is a directed path (resp. a directed cycle), which we refer to as the *symmetric of* S in $\tilde{D}$. We say that a directed path or a directed cycle in $\tilde{D}$ is *symmetric* if it coincides with its symmetric.

In the remainder, given two walks P and Q, if they have exactly one endnode i in common we denote with (P, i, Q) the walk obtained by concatenating P and Q, while if they share both endnodes i and j, we denote by (i,P,j,Q,i) the closed walk obtained by concatenating P and Q.

Observation 1. *A directed path S in $\tilde{D}$ is symmetric if and only if S contains exactly one node $i \in I$ and $S = (Q, i, sym(Q))$ for some directed path Q in $\tilde{D}$ that ends in i.*
A directed cycle S is symmetric if and only if S contains exactly two distinct nodes $i, j \in I$ and $S = (i, Q, j, sym(Q), i)$ for some directed path Q in $\tilde{D}$ from i to j.

Proof. If $S = (v_1, v_2, \ldots, v_n)$ is a symmetric directed path in $\tilde{D}$, then $S' = (v_2, \ldots, v_{n-1})$ is also symmetric, so by induction on the length of S we may assume that S' contains exactly one node $i \in I$ and $S' = (Q', i, sym(Q'))$ for some directed path Q' in $\tilde{D}$ that ends in i. Since $(v_{n-1}, v_n) = sym(v_1, v_2)$, if we define $Q = (v_1, v_2, Q')$, then $S = (Q, i, sym(Q))$.
Let S be a symmetric directed cycle. If S does not contain any node in L, then S consists of two distinct nodes $i, j \in I$ and of the two symmetric edges (i,j) and (j,i). So we may assume that S contains a node $w \notin I$ and its symmetric. Since the two distinct paths in S with endnodes w and $sym(w)$ are both symmetric, the statement follows from the case of the symmetric directed path. □

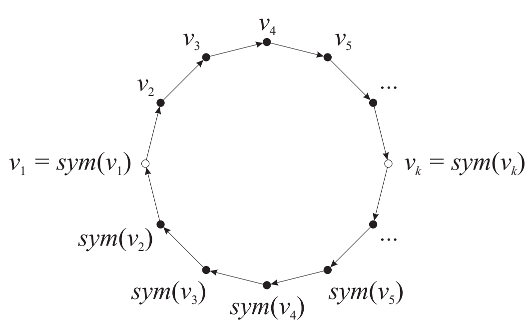

Figure 3: a symmetric directed cycle.

Lemma 1. *Let $\tilde{D}$ be an antisymmetric orientation of $\tilde{G}_\emptyset$. Given a directed path S in $\tilde{D}$, then $-1 \le \beta(S) + \beta(sym(S)) \le 1$, where $\beta(S) + \beta(sym(S)) = 0$ if and only if the endnodes of S are either both in $U \cup V'$ or both in $U' \cup V$. Given a directed cycle S in $\tilde{D}$, then $\beta(S) + \beta(sym(S)) = 0$.*

Proof. Let S be a directed path or a directed cycle in $\tilde{D}$. One can readily verify that

$$\begin{aligned}\beta(S)+\beta(sym(S)) &= -|\{(u,v)\in S \,:\, u\in U, v\in V\}| \\ &\quad +|\{(v,u)\in S \,:\, v\in V, u\in U\}| \\ &\quad +|\{(u',v')\in S \,:\, u'\in U', v'\in V'\}| \\ &\quad -|\{(v',u')\in S \,:\, v'\in V', u'\in U'\}|.\end{aligned}$$

The arcs leaving $U\cup V'$ are either arcs from U to V or from V' to U', while the arcs entering $U\cup V'$ are either arcs from V to U or from U' to V'. Therefore, by the above equation, $\beta(S)+\beta(sym(S))$ is the difference between the number of arcs in S entering $U\cup V'$ and the number of arcs in S leaving $U\cup V'$. Since S is a directed path or a directed cycle, the absolute value of this difference is at most 1, and it is 0 if and only if S is a directed path with endnodes either both in $U\cup V'$ or both in $U'\cup V$, or if S is a directed cycle. □

Proof of Theorem 3.

Through this proof, we denote $\tilde{G}_I=(\tilde{U}_I\cup\tilde{V}_I,\tilde{E})$ simply by $\tilde{G}=(\tilde{U}\cup\tilde{V},\tilde{E})$. Let $\tilde{y}$ be an integral optimal solution of (8). We will show how to derive from $\tilde{y}$ an integral solution y for (6) with value $\tilde{b}^\top\tilde{y}$, thus showing that y is optimal for (6), since we have already observed that the optimal value of (6) is at most that of (8). We say that an edge $e\in\tilde{E}$ is *loaded (for $\tilde{y}$)* if $\tilde{y}_e>0$, *unloaded (for $\tilde{y}$)* if $\tilde{y}_e=0$.

We prove the theorem by induction on $\sum_{i\in U\cup V}c_i$. If E contains an edge e such that both its images in $\tilde{E}$ are unloaded, and if y' is an integral optimal solution for the instance of (6) on the graph $G'=(U\cup V,E\setminus\{e\})$ with value $\tilde{b}^\top\tilde{y}$, then the vector y, obtained by completing y' with $y_e=0$ and with $y_P=0$ for every I-path P in G that contains e, is an integral optimal solution for the problem (6) with value $\tilde{b}^\top\tilde{y}$. Thus from now on we will assume that for every $e\in E$, at least one of its two images in $\tilde{E}$ is loaded.

We will show that one of the following holds: a) $\tilde{E}$ contains two loaded symmetric edges, b) there is a symmetric cycle or path of non-positive cost (in some suitable orientation of $\tilde{G}$ to be defined shortly), c) $\tilde{y}$ can be transformed to another integral optimum solution satisfying a) or b). If $\tilde{y}$ satisfies a), the proof can be concluded by induction by applying one of the following two reductions.

Reduction (with respect to $\tilde{y}$) on the symmetric edges.

Suppose $\tilde{G}$ contains a pair of symmetric edges e_1 and e_2 that are both loaded. For any edge $e \in E$ with images $e_1, e_2 \in \tilde{E}$, let $\gamma_e = \min\{\tilde{y}_{e_1}, \tilde{y}_{e_2}\}$, and define a new cost vector c' on the nodes of G by

$$c'_i = c_i - a_i \sum_{ij \in E} \gamma_{ij}, \quad i \in U \cup V.$$

We call *reduced problem (w.r.t. $\tilde{y}$)* the instance of (6) on the graph G with costs on the nodes c', and *extended reduced problem (w.r.t. $\tilde{y}$)* the corresponding instance of (8). Notice that the vector $\tilde{y}'$ defined by

$$\tilde{y}'_{e_1} = \tilde{y}_{e_1} - \gamma_e, \quad \tilde{y}'_{e_2} = \tilde{y}_{e_2} - \gamma_e; \quad e \in E$$

is an integral optimal solution to the extended reduced problem, with value $\tilde{b}^\top \tilde{y} - \sum_{e \in E} b_e \gamma_e$.

Since $\sum_{i \in U \cup V} c'_i < \sum_{i \in U \cup V} c_i$, by induction the reduced problem has an integral optimal solution y' with value $\tilde{b}^\top \tilde{y}'$. Hence the vector y defined by

$$\begin{array}{rcll} y_e &=& y'_e + \gamma_e & e \in E \\ y_P &=& y'_P & P \text{ } I\text{-path,} \end{array}$$

is a feasible integral solution for problem (6) with value $\tilde{b}^\top \tilde{y}$, thus y is optimal.

Thus we may assume that for every edge $e \in \tilde{E}$ exactly one among e and $sym(e)$ is loaded. Let $\tilde{D}$ be the digraph obtained from $\tilde{G}$ by orienting from $\tilde{U}$ to $\tilde{V}$ the unloaded edges, and from $\tilde{V}$ to $\tilde{U}$ the loaded edges. Note that $\tilde{D}$ is an antisymmetric orientation of $\tilde{G}$. We denote by $\tilde{D}_\emptyset$ the digraph obtained from $\tilde{G}_\emptyset$ by orienting the unloaded edges of $\tilde{E}$ from $U \cup U'$ to $V \cup V'$ and the loaded edges of $\tilde{E}$ from $V \cup V'$ to $U \cup U'$. Notice that $\tilde{D}_\emptyset$ is an antisymmetric orientation of $\tilde{G}_\emptyset$, and that $\tilde{D}$ can be obtained from $\tilde{D}_\emptyset$ by identifying the images of nodes in I.

Next we define the second type of reduction.

Reduction on a symmetric path or symmetric cycle of non-positive cost.

Let S be a symmetric directed cycle of $\tilde{D}$ of non-positive cost, or a symmetric directed path of $\tilde{D}$ of non-positive cost with $\tilde{c}_k - \sum_{e \ni k} \tilde{y}_e > 0$, where k is the only endnode of S incident to an unloaded arc of S. In the first case let $\gamma = \min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$, in the second one let γ be the minimum among $\tilde{c}_k - \sum_{e \ni k} \tilde{y}_e$, and $\min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$. Let c' be the cost vector on the nodes of G obtained from c by setting $c'_i = c_i$ for any node i whose images are not in S, and $c'_i = c_i - \gamma$ for any node i whose images are in S. We call *reduced problem* the instance of (6) on the graph G with costs c', and *extended reduced problem* the corresponding instance of (8). Then

$$\begin{array}{ll} \tilde{y}'_e = \tilde{y}_e - \gamma & e \in S, \tilde{y}_e > 0 \\ \tilde{y}'_e = \tilde{y}_e & \text{otherwise} \end{array}$$

is an integral optimal solution for the extended reduced problem.

By Observation 1, S is the union of a path Q and its symmetric $sym(Q)$, and the endnodes in common among Q and $sym(Q)$ are the only nodes of S in I. Note that the set of edges of G whose images are in S define an I-path R in G. If S is a path, then the unique node of S in I is also the unique endnode of R in I, while if S is a cycle, then the two distinct nodes of S in I are the endnodes of R. Notice that

$$\sum_{e \in R} b_e = \sum_{e \in S} \tilde{b}_e = \sum_{\substack{e \in S \\ e \text{ loaded}}} \tilde{b}_e + \sum_{\substack{e \in S \\ e \text{ unloaded}}} \tilde{b}_e = \beta(S) + 2 \sum_{\substack{e \in S \\ e \text{ loaded}}} \tilde{b}_e. \tag{10}$$

The arcs of Q and of $sym(Q)$ induce directed paths Q' and $sym(Q')$ in $\tilde{D}_\emptyset$, respectively. Furthermore $\beta(S) = \beta(Q') + \beta(sym(Q'))$. By Lemma 1 $|\beta(Q') + \beta(sym(Q'))| \leq 1$. Since S has non-positive cost, $\beta(S) \in \{0, -1\}$. Thus, by (10),

$$\sum_{\substack{e \in S \\ e \text{ loaded}}} \tilde{b}_e = \left\lceil \frac{b(R)}{2} \right\rceil.$$

Hence the optimal value of the extended reduced problem is $\tilde{b}^\top \tilde{y} - \left\lceil \frac{b(R)}{2} \right\rceil \gamma$. Since $\sum_{i \in U \cup V} c'_i < \sum_{i \in U \cup V} c_i$, by induction there exists an integral solution y' for the reduced problem with value $\tilde{b}^\top \tilde{y}'$, thus the vector y defined by

$$\begin{array}{rcll} y_e & = & y'_e & e \in E \\ y_R & = & y'_R + \gamma & \\ y_P & = & y'_P & P \ I\text{-path}, P \neq R \end{array}$$

is an integral feasible solution for problem (6) with value $\tilde{b}^\top \tilde{y}$, hence y is optimal.

We define the *sources* of $\tilde{D}$ as the elements of $\{u \in \tilde{U} : \sum_{e \ni u} \tilde{y}_e < \tilde{c}_u\} \cup \{v \in \tilde{V} : \sum_{e \ni v} \tilde{y}_e > 0\}$ and the *sinks* of $\tilde{D}$ as the elements of $\{u \in \tilde{U} :$

$\sum_{e \ni u} \tilde{y}_e > 0\} \cup \{v \in \tilde{V} : \sum_{e \ni v} \tilde{y}_e < \tilde{c}_v\}$. Let S be either a directed path in $\tilde{D}$ from a source to a sink or a directed cycle, and ε be a positive number. We say that the solution $\tilde{y}'$ is obtained by *augmenting* $\tilde{y}$ *by* ε *on* S if $\tilde{y}'_e = \tilde{y}_e + \varepsilon$ for every unloaded edge $e \in S$, $\tilde{y}'_e = \tilde{y}_e - \varepsilon$ for every loaded edge $e \in S$, and $\tilde{y}'_e = \tilde{y}_e$ for every edge $e \in \tilde{E} \setminus S$. If ε is small enough, then $\tilde{y}'$ is also a feasible solution for problem (8), with value $\tilde{b}^\top \tilde{y} + \varepsilon\beta(S)$ (notice that this is the standard notion of augmentation in flow theory, see for example [11]). Therefore, since $\tilde{y}$ is an optimal solution, we have the following.

Observation 2. *If S is a directed path from a source to a sink or a directed cycle in $\tilde{D}$, then $\beta(S) \leq 0$. Furthermore, if $\beta(S) = 0$, then for $\varepsilon > 0$ small enough the solution obtained by augmenting $\tilde{y}$ by ε on S is optimal for (8).*

Suppose now that $\tilde{D}$ contains a directed cycle S. We show that in this case $\tilde{D}$ contains a directed cycle C that either is symmetric or has at most one node in I. In fact, if S contains two or more nodes in I, let Q be a minimal directed path contained in S with endnodes in I and with no intermediate node in I. The directed graph induced by the arcs of $Q \cup sym(Q)$ is the union of arc-disjoint directed cycles, so it either contains a directed cycle with at most one node in I, or it is a symmetric directed cycle.

Case 1: C is a symmetric directed cycle in $\tilde{D}$.

By Observation 2, $\beta(C) \leq 0$, thus we can apply the reduction on the symmetric directed cycle C of non-positive cost, and we are done.

Case 2: C has at most one node in I.

The arcs in C form a directed cycle or a directed path in the digraph $\tilde{D}_\emptyset$, thus, by Lemma 1, $-1 \leq \beta(C) + \beta(sym(C)) \leq 1$, while by Observation 2 $\beta(C) \leq 0$ and $\beta(sym(C)) \leq 0$. Hence at least one among C and $sym(C)$ has cost zero, and we assume $\beta(C) = 0$. Note that C must cross an unloaded arc $\bar{e}$ whose symmetric is not in C, otherwise all the unloaded arcs of C have their symmetric in C, thus C is symmetric. So we can augment $\tilde{y}$ by $\min\{\tilde{y}_e : e \in C, \tilde{y}_e > 0\}$ on C thus getting another integral optimal solution $\tilde{y}'$ where both $\bar{e}$ and its symmetric have strictly positive value. Thus we can now apply the reduction w.r.t. $\tilde{y}'$ on the symmetric edges, and we are done.

Hence we can assume that the digraph $\tilde{D}$ is acyclic. Notice that every node not isolated in $\tilde{D}$ with in-degree 0 is a source. In fact if j has in-degree 0 and strictly positive out-degree, then $sym(j)$ has out-degree 0 and strictly positive in-degree. So, if $j \in \tilde{U}$, then $\sum_{e \ni j} \tilde{y}_e = 0$ and $\sum_{e \ni sym(j)} \tilde{y}_e > 0$, if $j \in \tilde{V}$, then $\sum_{e \ni j} \tilde{y}_e > 0$. In the same way notice that every node not isolated in $\tilde{D}$ with out-degree 0 is a sink.

Suppose that there exists a node i in I that is not isolated in $\tilde{D}$. Since $\tilde{D}$ is acyclic, there exists a path Q from i to a node j of out-degree 0 in $\tilde{D}$ and, since j has out-degree 0, $j \notin I$. Consider the directed walk $S = (sym(Q), i, Q)$ from $sym(j)$ to j. Notice that, since $\tilde{D}$ is acyclic, S must be a directed path. Since S is a directed path in $\tilde{D}$ from a source to a sink, by Observation 2 $\beta(S) \leq 0$. Moreover, if k is the only endnode of S incident to an unloaded arc of S, then $\tilde{c}_k - \sum_{e \ni k} \tilde{y}_e > 0$, thus we may apply the reduction on the symmetric directed path of non-positive cost S, and we are done.

So we can assume that all the nodes in I are isolated in $\tilde{D}$.

Therefore there exists a directed path S in $\tilde{D}$ from a node with in-degree 0 to a node with out-degree 0. Since both S and $sym(S)$ are directed paths in $\tilde{D}$ from a source to a sink, by Observation 2 $\beta(S) \leq 0$ and $\beta(sym(S)) \leq 0$. By Lemma 1, $-1 \leq \beta(S) + \beta(sym(S)) \leq 1$. Hence at least one among S and $sym(S)$ has cost zero, and we assume it is $S = (v_1, \ldots, v_k)$. Notice that S crosses an unloaded arc $\bar{e}$ whose symmetric is not in S. In fact, if all the unloaded arcs of S have their symmetric in S, it must be $|e \in S : e \text{ loaded}| = |e \in S : e \text{ unloaded}| + 1$, since in S unloaded and loaded arcs alternate and since S is not symmetric. But then (v_1, v_2) and (v_{k-1}, v_k) are both loaded and at least one of them is the symmetric of an unloaded arc in S. By symmetry we may assume it is (v_1, v_2), thus $sym(v_1)$ has out-degree 0, hence $(sym(v_2), sym(v_1))$ is the last arc of S. A contradiction as it is unloaded. So we can augment $\tilde{y}$ on S by the minimum among $\tilde{c}_j$ for every endnode j of S incident to an unloaded arc of S, and $\min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$. Thus we get another integral optimal solution $\tilde{y}'$ where both $\bar{e}$ and its symmetric have strictly positive value. Hence we can now apply the reduction w.r.t. $\tilde{y}'$ on the symmetric edges, and we are done. □

We conclude the section with the following corollary, which will be used in the proof of Theorem 1.

Corollary 3. *Let $l \in \mathbb{Z}^{U \cup V}$. The system*

$$\begin{array}{rcll} a_i x_i + a_j x_j & \geq & b_{ij} & ij \in E \\ x_i & \geq & l_i & i \in U \cup V \\ \sum_{i \in P} x_i & \geq & \left\lceil \frac{b(P)}{2} \right\rceil & P = (v_1, \ldots, v_k) \ I\text{-}path, v_1, v_k \in I \\ \sum_{i \in P} x_i & \geq & \left\lceil \frac{b(P) + l_{v_k}}{2} \right\rceil & P = (v_1, \ldots, v_k) \ I\text{-}path, v_k \notin I. \end{array} \tag{11}$$

is totally dual integral.

Proof. Let c be a vector in $\mathbb{Z}^{U \cup V}$. By Theorem 3 we know that the dual of the problem $\min\{c^\top x : x \text{ satisfies } (5)\}$ with integer requirements $b_{ij} - a_i l_i - a_j l_j$ has an integral optimal solution y^*. It is straightforward to check that the integral solution $\bar{y}$ defined by $\bar{y}_e = y^*_e$, $e \in E$, $\bar{y}_P = y^*_P$, P I-path, $\bar{y}_i = c_i - \sum_{e \ni i} a_i y^*_e - \sum_{P \ni i} y^*_P$, $i \in U \cup V$, is optimal for $\min\{c^\top x : x \text{ satisfies } (11)\}$. □

3 General case

In this section we prove Theorem 1 by reducing the general problem to the bipartite case studied in Section 2.

Proof of Theorem 1. First we show the following.

Claim. *The system defined by (1) and (3), for every I-trail P, is totally dual integral.*

Proof of Claim. We show how to reduce this problem to the previous case. We define the undirected graph $G' = (N \cup N', E')$ as follows. Let N' be a copy of N such that $N \cap N' = \emptyset$. For every $i \in N$ we denote by i' the copy of i in N' and for every $X \subseteq N$ we denote by X' the subset of N' that contains only the copies of the nodes in X. E' contains the edge ii' for every $i \in N$, with requirement 0, and the edge ij (resp. ij', $i'j$, $i'j'$) for every edge $ij \in A$ with $\sigma_{ij,i} = \sigma_{ij,j} = +1$ (resp. $\sigma_{ij,i} = +1$ and $\sigma_{ij,j} = -1$, $\sigma_{ij,i} = -1$ and $\sigma_{ij,j} = +1$, $\sigma_{ij,i} = \sigma_{ij,j} = -1$), with the same requirement of the original bidirected edge ij. Let $b' \in \mathbb{Z}^{E'}$ be the vector of requirements on the edges in E'. Since the edge-node incidence matrix of D is totally unimodular and has two nonzero elements per row, it follows from Theorem 2 that N can be partitioned into two sets R, B such that every edge of D with the same sign in both its endnodes has one endnode in R and the other in B, while every edge with different signs in its endnodes is contained in R or B. Therefore every edge of G' has exactly one endnode in $R \cup B'$ and the other in $R' \cup B$, thus G' is bipartite.

If we define $a'_i = 2$ for $i \in I \cup I'$ and $a'_i = 1$ for $i \in L \cup L'$ then one can verify that a vector x satisfies $M_I x \geq b$, $l \leq x \leq u$, if and only if the vector x' defined by $x'_i = -x'_{i'} = x_i$ for all $i \in N$ satisfies $a'_i x'_i + a'_j x'_j \geq b'_{ij}$ for all $ij \in E'$, $x'_i \geq l_i$ and $x'_{i'} \geq -u_i$ for all $i \in N$. Since the inequalities $x'_i \geq -x'_{i'}$, $i \in N$, are valid for the latter system, as they are the inequalities $a'_i x'_i + a'_{i'} x'_{i'} \geq b'_{ii'}$ for the edges ii' of G', then the polyhedron defined by $M_I x \geq b$, $l \leq x \leq u$ corresponds to the face of the polyhedron defined by $a'_i x'_i + a'_j x'_j \geq b'_{ij}$, $ij \in E'$, $x'_i \geq l_i$, $x'_{i'} \geq -u_i$, $i \in N$ given by $x'_i = -x'_{i'}$, $i \in N$.

Given an $I \cup I'$-path P in G', this determines an inequality as in (11) for the instance given by G', b' and $I \cup I'$. Substituting x'_i for $-x'_{i'}$, for every $i \in N$, into such inequality, we obtain the inequality of (3) relative to the I-trail T obtained from P by identifying the pairs of nodes i, i' for every $i \in N$ such that i, i' are in P.

Since, by Corollary 3, the system obtained from $a'_i x'_i + a'_j x'_j \geq b'_{ij}$, $ij \in E'$, $x'_i \geq l_i$, $x'_{i'} \geq -u_i$, $i \in N$ by juxtaposing the inequalities of the form (11) relative to $I \cup I'$-paths of G' is totally dual integral, and since setting to equality some inequalities of a system preserves total dual integrality (see Theorem 22.2 of [10]), then the system obtained from the above by setting $x'_i = -x'_{i'}$, $i \in N$, is totally dual integral, therefore also the system defined by (1) and (3) for every I-trail P is totally dual integral. This concludes the proof of the claim.

We conclude the proof of Theorem 1. Given a vector $c \in \mathbb{Z}^N$, we show how to get an integral optimal solution for the dual of $\min\{c^\top x : x \text{ satisfies } (1), (3) \text{ for every } I\text{-path } P\}$ from an integral optimal solution y for the dual of $\min\{c^\top x : x \text{ satisfies } (1), (3) \text{ for every } I\text{-trail } P\}$. In fact, if T is an I-trail that is not an I-path such that $y_T > 0$, then there exists a cycle C, a node j and two trails Q, R such that $T = (Q, j, C, j, R)$. Note that $S = (Q, j, R)$ is an I-trail with the same endnodes of T but with less cycles than T. Since the edge-node incidence matrix of D is totally unimodular, by Theorem 2 the edges of C can be partitioned in two subsets C^1 and C^2 such that any two adjacent edges of C are contained in the same subset if and only if one of them has a -1 and the other has a $+1$ in their common endnode. Moreover, we may assume that C^1 has cost at least $\lceil b(C)/2 \rceil$. One can verify that, by our choice of the partition C^1, C^2, the integral vector y' that is identical to y except for $y'_S = y_S + y_T, y'_e = y_e + y_T, \forall e \in C^1$, $y'_T = 0$, is feasible for the dual of $\min\{c^\top x : x$ satisfies (1), (3) for every I-trail $P\}$. Furthermore its objective value is at least that of y plus $y_T(\lceil b(S)/2\rceil + \lceil b(C)/2\rceil - \lceil (b(S)+b(C))/2 \rceil) \geq 0$, thus y' is also optimal. Since the total number of cycles contained in I-trails whose associated dual variables are positive strictly decreases, by repeating the argument we obtain an integral optimal solution for the dual of $\min\{c^\top x : x$ satisfies (1), (3) for every I-path $P\}$. □

Example 2. Figure 4 depicts the construction described in the proof of Theorem 1 applied to the bidirected graph D of Figure 1. Notice that the Heller-Tompkins bipartition on the nodes of D is $\{a, b\}$, $\{c, d\}$, which corresponds to the bipartition $\{a, b, c', d'\}$, $\{a', b', c, d\}$ of the graph G' in Figure 4.
Also, notice that the $I \cup I'$-path (a, b', c', c, d') gives the I-path inequality $x'_a + x'_{b'} + x'_{c'} + x'_c + x'_{d'} \geq 3$, while the $I \cup I'$-path (a, b', c', c, d', d) gives the I-path inequality $x'_a + x'_{b'} + x'_{c'} + x'_c + x'_{d'} + x'_d \geq 4$. Once we substitute $x_i = x'_i = -x'_i$, $i = a, b, c, d$, we obtain exactly the two inequalities given in Example 1, relative to the I-path (a, b, c, d) in D.

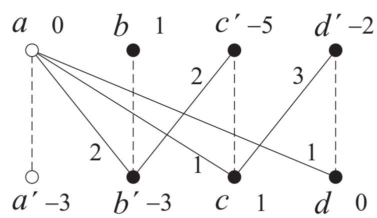

Figure 4: The bipartite graph G' associated with the bidirected graph in Figure 1, with the corresponding requirements on the edges and lower bounds on the nodes. Continuous edges correspond to the original ones. Dashed edges have requirement zero.

4 Polynomial time solvability

4.1 Bipartite case

The proof of Theorem 3 gives an algorithm (albeit not a polynomial time one) to derive an integral optimal solution y^* for (6) from an integral optimal solution $\tilde{y}$ for (8), as follows. Initially we set $y^* := 0$. Each time we apply a reduction, we update the value of y^* and then apply our algorithm recursively on the reduced problem as long as the current vector c is not the all zero vector. Since each time we apply a reduction the value of some entry of c decreases, the total number of iterations is bounded by $\sum_{i \in U \cup V} c_i$, which is not a polynomial bound on the size of the problem.

More in detail: if $\tilde{G}$ contains a pair of symmetric loaded edges, then for each $e \in E$ we update $y^*_e := y^*_e + \min\{\tilde{y}_{e_1}, \tilde{y}_{e_2}\}$, where e_1 and e_2 are the images of e in $\tilde{G}$, apply the reduction on the symmetric edges, and proceed recursively on the reduced problem.

If $\tilde{D}$ has a directed cycle, we can find in polynomial time a directed cycle C that either is symmetric or has at most one node in I. If C is symmetric, then it has non-positive cost, thus we apply the reduction on C, update $y^*_R := y^*_R + \gamma$, where R is the I-path defined by the edges with images in C and γ is the minimum value of $\tilde{y}$ on the loaded edges of C, and proceed recursively on the reduced problem.

Otherwise, we augment on the cycle among C and $sym(C)$ with cost zero by the smallest load on its edges, and apply the reduction on the symmetric edges.

If $\tilde{D}$ is acyclic and there exists a non-isolated node in I, then we can find in polynomial time a symmetric directed path of non-positive cost S in $\tilde{D}$ starting from some node of in-degree 0, we apply the reduction on S, update $y^*_R := y^*_R + \gamma$, where R is the I-path defined by the edges with images in S and γ is defined as in the proof, and proceed recursively on the reduced problem.

If all nodes of I are isolated, we can find in polynomial time a directed path S of cost zero from a node of in-degree zero to a node of out-degree zero. We augment on S by the minimum among $\tilde{c}_j$, for every endnode j of S incident to an unloaded arc of S, and $\min\{\tilde{y}_e : e \in S, \tilde{y}_e > 0\}$, and we apply the reduction on the symmetric edges.

Notice that each iteration can be performed in strongly polynomial time. While we cannot give a polynomial bound on the number of iterations of the algorithm described above, we can prove that the number of iterations in which we apply a reduction on a symmetric path or symmetric cycle of non-positive cost is bounded by the number of edges of G. In fact, each time we apply a reduction on a symmetric cycle, the number of loaded edges decreases by at least one. We apply the reduction on a symmetric path of non-positive cost S only when S starts in a node with in-degree 0 and ends in a node with out-degree 0. In this case, if k is the only endnode of S incident to an unloaded arc of S, we have

$$\tilde{c}_k - \sum_{e \ni k} \tilde{y}_e = \tilde{c}_k = \tilde{c}_{sym(k)} \geq \min\{\tilde{y}_e \,:\, e \in S, \tilde{y}_e > 0\}$$

since k is incident only to unloaded arcs, and $sym(k)$ is incident to a loaded arc of S. Thus, each time we apply a reduction on a symmetric path, the number of loaded edges decreases by at least one.

So, if on a given instance the algorithm described above does not perform any reduction on the symmetric edges, then it performs at most $|E|$ iterations. In particular, this happens if and only if the optimal solution y^* for (6) produced by the algorithm satisfies $y^*_e = 0$ for every $e \in E$. We will show next that we can reduce to this case, thus proving the following.

Theorem 4. *There is a strongly polynomial-time algorithm to compute an integral optimal solution for (6) for each integral vector c for which it has a finite optimum.*

Proof. Let x^* be an integral optimal solution for the problem $\min\{c^\top x \,:\, x \text{ satisfies } (5)\}$ and let $\tilde{x}$ be the solution for (7) defined by $\tilde{x}_i = x^*_i$, $i \in I$, $\tilde{x}_i = \lfloor \frac{x^*_i}{2} \rfloor$, $\tilde{x}_{i'} = \lceil \frac{x^*_i}{2} \rceil$, $i \in L$. By Theorem 3, x^* is optimal if and only if $\tilde{x}$ is optimal for (7), and $c^\top x^* = \tilde{c}^\top \tilde{x}$. Notice that this remains true even if c is not an integral vector. Given $e = ij \in E$, let $\alpha^e \in \mathbb{R}^{U \cup V}$ be the coefficient vector of the constraint of (4) relative to e, that is $\alpha^e_i = a_i$, $\alpha^e_j = a_j$, $\alpha^e_k = 0$ for $k \in (U \cup V) \setminus \{i, j\}$.

Claim. *Given $\bar{e} \in E$ such that $(\alpha^{\bar{e}})^\top x^* = b_{\bar{e}}$, one can compute in strongly polynomial time the maximum γ such that x^* remains optimal for the problem $\min\{(c - \gamma\alpha^{\bar{e}})^\top x \,:\, x$ satisfies $(5)\}$.*

Proof of Claim. Let $J = \{i \in \tilde{U}_I \cup \tilde{V}_I \,:\, \tilde{x}_i > 0\}$, $F = \{ij \in \tilde{E} \,:\, \tilde{x}_i + \tilde{x}_j > b_{ij}\}$. By complementary slackness, a vector $\tilde{y} \in \mathbb{R}^{\tilde{U}_I \cup \tilde{V}_I}$ is optimal for (8) if and only if $\tilde{y}$ satisfies

$$\begin{array}{ll} \sum_{e\ni i}\tilde{y}_e=\tilde{c}_i & i\in J\\ \sum_{e\ni i}\tilde{y}_e\le\tilde{c}_i & i\in(\tilde{U}\cup\tilde{V})\setminus J\\ \tilde{y}_e=0 & e\in F\\ \tilde{y}_e\ge 0 & e\in\tilde{E}\setminus F. \end{array} \tag{12}$$

Let $e_1, e_2 \in \tilde{E}$ be the images of $\bar{e}$ in $\tilde{E}$. Let $\mu = \max\{s : s \le \tilde{y}_{e_1}, s \le \tilde{y}_{e_2}, \tilde{y} \text{ satisfies (12)}\}$. We show that $\gamma = \mu$.

We first show $\gamma \le \mu$. Let $c' = c - \gamma\alpha^{\bar{e}}$, and $\tilde{c}' \in \mathbb{R}^{\tilde{U}_I \cup \tilde{V}_I}$ be the corresponding cost vector on the nodes of $\tilde{G}$. Since x^* is optimal for $\min\{(c-\gamma\alpha^{\bar{e}})^\top x : x \text{ satisfies (5)}\}$, then $\tilde{x}$ is optimal for the problem (7) with respect to the cost vector $\tilde{c}'$. Hence there exists a vector $\tilde{y}'$ that satisfies

$$\begin{array}{ll} \sum_{e\ni i}\tilde{y}'_e=\tilde{c}'_i & i\in J\\ \sum_{e\ni i}\tilde{y}'_e\le\tilde{c}'_i & i\in(\tilde{U}\cup\tilde{V})\setminus J\\ \tilde{y}'_e=0 & e\in F\\ \tilde{y}'_e\ge 0 & e\in\tilde{E}\setminus F. \end{array}$$

Now the vector defined by $\tilde{y}_e = \tilde{y}'_e + \gamma$ if $e \in \{e_1, e_2\}$, $\tilde{y}_e = \tilde{y}'_e$ otherwise, satisfies (12) and $\gamma \le \tilde{y}_{e_1}$, $\gamma \le \tilde{y}_{e_2}$.

Now we show that $\gamma \ge \mu$. Let $c' = c - \mu\alpha^{\bar{e}}$, and $\tilde{c}' \in \mathbb{R}^{\tilde{U}_I \cup \tilde{V}_I}$ be the corresponding cost vector on the nodes of $\tilde{G}$. If $\tilde{y}$ is the solution that satisfies (12) and maximizes s, then the vector defined by $\tilde{y}'_e = \tilde{y}_e - \mu$ if $e \in \{e_1, e_2\}$, $\tilde{y}'_e = \tilde{y}_e$ otherwise, satisfies (12) with respect to the cost vector $\tilde{c}'$. Hence $\tilde{x}$ is optimal for the problem (7) with respect to the cost vector $\tilde{c}'$, and x^* is optimal for the problem $\min\{(c-\mu\alpha^{\bar{e}})^\top x : x \text{ satisfies (5)}\}$.

Finally, since the coefficients of the variables in (12) and in $s \le \tilde{y}_{e_1}, s \le \tilde{y}_{e_2}$, are $0, \pm 1$, γ can be computed in strongly polynomial time using an algorithm of Tardos [12].

Let $e^1, \ldots, e^m$ be the edges in E such that $(\alpha^e)^\top x^* = b_e, e \in \{e^1, \ldots, e^m\}$. Set $c^0 = c$ and, for $k = 1, \ldots, m$, let $c^k = c^{k-1} - \lfloor\gamma_k\rfloor\alpha^{e^k}$, where γ_k is the maximum γ such that x^* remains optimal for the problem $\max\{(c^{k-1} - \gamma\alpha^{e^k})^\top x : x \text{ satisfies (5)}\}$. By the previous claim, we can compute $c^1, \ldots, c^m$ in strongly polynomial time.

Given any integral optimal solution y^* for the dual of $\min\{c^{m\top} x : x \text{ satisfies (5)}\}$, then the vector $\bar{y}$, defined by $\bar{y}_{e^k} = y^*_{e^k} + \lfloor\gamma_k\rfloor$ for every $k = 1, \ldots, m$, $\bar{y}_P = y^*_P$ for every I-path P, is an integral optimal solution for (6). By definition of

$\gamma_1, \ldots, \gamma_m$, $c^1, \ldots, c^m$, for every $e \in E$ we must have $y_e^* < 1$, thus $y_e^* = 0$. This concludes our proof, since we have shown above that in this case the algorithm given by the proof of Theorem 3 finds an integral optimal solution for $\min\{c^{m\top} x : x \text{ satisfies } (5)\}$ in strongly polynomial time. This completes the proof of the claim. □

4.2 General case

Theorem 5. *There is a strongly polynomial-time algorithm to compute an integral optimal solution for the dual of* $\min\{c^\top x : x$ *satisfies* $(1), (3)$ *for every* I*-path* $P\}$ *whenever the problem has a finite optimum.*

Proof. We showed in Theorem 4 that an integral optimal solution for the dual of any problem of the form $\min\{c^\top x : x \text{ satisfies } (5)\}$ can be computed in strongly polynomial time for each integral vector c for which it has a finite optimum.

The proof of Corollary 3 shows how to obtain, in strongly polynomial time, an integral optimal solution for the dual of any problem of the form $\min\{c^\top x : x \text{ satisfies } (11)\}$ from an integral optimal solution for the dual of a problem of the form $\min\{c^\top x : x \text{ satisfies } (5)\}$ with integer requirements $b_{ij} - a_i l_i - a_j l_j$.

The proof of the Claim in the proof Theorem 1 shows how to reduce, in strongly polynomial time, any problem of the form $\min\{c^\top x : x$ satisfies (1), (3) for every I-trail $P\}$ to a problem of the form $\min\{\bar{c}^\top \bar{x} : \bar{x} \text{ satisfies } (11)\}$ in some auxiliary graph G', but with a polynomial number of inequalities (of the form $x'_i + x'_{i'} \geq 0$) set to equality. The next claim shows that an integral optimal solution of the dual of any problem in the latter form can be computed in strongly polynomial time.

Finally, in the last part of the proof of Theorem 1, we showed how to get an integral optimal solution for the dual of $\min\{c^\top x : x$ satisfies $(1), (3)$ for every I-path $P\}$ from an integral optimal solution for the dual of $\min\{c^\top x : x$ satisfies $(1), (3)$ for every I-trail $P\}$. Notice that the procedure described terminates in strongly polynomial time.

Claim. *Let* $Ax \geq b, Cx \geq d$ *be a totally dual integral system of linear inequalities, where* $A \in \mathbb{Z}^{p \times n}$ *and* $C \in \mathbb{Z}^{q \times n}$*. Let* $\alpha = \max\{\|A\|_\infty, \|C\|_\infty\}$*. Given* $c \in \mathbb{Z}^n$*, let* γ *be the* q*-dimensional vector with all entries equal to* $n!\alpha^n \|c\|_\infty$ *and* $\bar{c} = c + C^\top \gamma$*. If* (y^*, u^*) *is an integral optimal solution for the* *dual of*

$$\min\{\bar{c}^\top x \mid Ax \geq b, Cx \geq d\}, \tag{13}$$

(where y and u are relative to the rows of A and C, respectively) then $(y^*, u^* - \gamma)$ is an integral optimal solution for the dual of

$$\min\{c^\top x \mid Ax \geq b, Cx = d\}, \tag{14}$$

provided that the latter has a finite optimum.

Proof of Claim. Clearly $(y^*, u^* - \gamma)$ is integral and feasible for the dual of (14). We show it is optimal. Let $\bar{x}$ be an optimal solution of (14), and $(\bar{y}, \bar{u})$ be an optimal basic solution for the dual of (14). Since $(\bar{y}, \bar{u})$ is basic, then the absolute values of its components are bounded above by $\|c\|_\infty$ times the maximum among the absolute values of the determinants of the square submatrices of $(A^\top, C^\top)$, which is at most $\alpha^n n!$. Therefore $\bar{u} \geq -\gamma$. Thus $(\bar{y}, \bar{u} + \gamma)$ is feasible for the dual of (13), $\bar{x}$ is feasible for (13), and $\bar{c}\bar{x} = b^\top \bar{y} + d^\top \bar{u} + \gamma^\top C\bar{x} = b^\top \bar{y} + d^\top(\bar{u} + \gamma)$, thus $\bar{x}$ and $(\bar{y}, \bar{u} + \gamma)$ are optimal for (13) and its dual, respectively. Thus $b^\top \bar{y} + \beta\bar{u} = b^\top y^* + \beta(u^* - \gamma)$, so $(y^*, u^* - \gamma)$ is an integral optimal solution for the dual of (14). This concludes the proof of the claim.

In particular, if the system $Ax \geq b$, $Cx \geq d$ is of the form (11), and the number of rows of C is bounded by some polynomial in n, then, for any $c \in \mathbb{Z}^n$, the problem of finding an integral dual solution of (14) can be reduced in strongly polynomial time to the problem of finding an integral dual solution of (13), which by Theorem 4 can be solved in strongly polynomial time.

□

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