On Perfect Graphs and Balanced Matrices

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Chapter 1

Introduction

A graph G is *perfect* if for every induced subgraph G' of G, the *chromatic number* of G' equals the cardinality of a largest clique in G'. The definition of perfect graph is due to Berge [2]. Berge proposed two conjectures on perfect graphs, that became known as the *weak perfect graph conjecture* (WPGC) and the *strong perfect graph conjecture* (SPGC). The weak perfect graph conjecture states that a graph is perfect if and only if its complement is, and was settled by Lovász [61]. The strong perfect graph conjecture characterizes perfect graphs in terms of minimal obstructions, by stating that a graph is perfect if and only if it does not contain a chordless cycle of odd length at least 5 (*odd hole*), or the complement of one such graph (*odd antihole*). Since this property is closed under going to the complement, the SPGC implies the WPGT. In May 2002, Chudnovsky, Robertson, Seymour and Thomas [13] proved the strong perfect graph conjecture. The idea of the proof is showing that every graph with no odd hole and no odd antihole either belongs to some simple *basic class* of graphs that are known to be perfect, or it has some structural fault, a *decomposition*, that cannot occur in a minimally imperfect graph. The existence of a similar decomposition theorem had been conjectured by Conforti, Cornuéjols and Vušković [28]. Special cases of this decomposition theorem have been proved by Conforti, Cornuéjols, Vušković and Zambelli [31][33], and they will be presented in Chapters 3 and 4. In Chapter 2 we will survey some of the main results on perfect graphs and give a brief outline of the proof of the SPGC by Chudnovsky et al. [13].

Perfect graphs are of particular interest in combinatorial optimization due to their relation with the *set packing problem*

$$\max\{wx \mid Ax \leq \mathbf{1},\ x \in \{0,1\}^n\} \tag{1.1}$$

where A is an $m \times n$ $0,1$ matrix, w an n-dimensional row vector, $\mathbf{1}$ is the m-dimensional column vector of all ones, while x is an n-dimensional column vector of variables of unknowns. This problem is NP-hard in general, but it can be solved in polynomial time when the *set packing polytope* $P(A) = \{x \in \mathbb{R}^n \mid Ax \leq \mathbf{1},\ x \geq \mathbf{0}\}$ has only integral vertices, in which case the matrix A is said to be *perfect*. Results of Fulkerson [49], Lovász [61] and Chvátal [16] imply that A is perfect if and only if the undominated rows of A are the incidence vectors of the maximal cliques of a perfect graph. Thus, characterizing perfect graphs is equivalent to characterizing when the set packing polytope is integral.

An important class of perfect matrices is the class of *balanced* matrices, introduced by Berge [4], which consists of the 0,1 matrices that do not contain any square submatrix of odd order with exactly two nonzero elements per row and per column. This definition was later extended to $0, \pm 1$ matrices by Truemper [75]. As mentioned above, $0,1$ balanced matrices are perfect, as shown by Berge [4], who also showed that several other polytopes are integral when the constraint matrix is balanced. Such polyhedral properties extend to the $0, \pm 1$ case: particularly important examples of integral polytopes associated with balanced matrices are the *generalized set packing, set covering* and *set partitioning* polytopes, which are defined, respectively, by the systems $\{x \in \mathbb{R}^n \mid Ax \leq \mathbf{1} - n(A), \mathbf{0} \leq x \leq \mathbf{1}\}$, $\{x \in \mathbb{R}^n \mid Ax \geq \mathbf{1} - n(A), \mathbf{0} \leq x \leq \mathbf{1}\}$ and $\{x \in \mathbb{R}^n \mid Ax = \mathbf{1} - n(A), \mathbf{0} \leq x \leq \mathbf{1}\}$, where A is an $m \times n$ $0, \pm 1$ matrix, and $n(A)$ is the m-dimensional vector whose ith entry is the number of -1s in the ith row of A. All these polytopes are integral when A is balanced, as proven by Berge [4] for the $0,1$ case, and by Conforti and Cornuéjols [20] for $0, \pm 1$ matrices. More generally, Fulkerson, Hoffman and Oppenheim [50], showed that the inequalities defining such polytopes form totally dual integral systems whenever A is a $0,1$ balanced matrix, and an analogous statement was proven by Conforti and Cornuéjols [20] for the $0, \pm 1$ case.

The problem of deciding whether or not a given matrix is balanced can be solved in polynomial time, as shown by Conforti, Cornuéjols and Rao [26] for $0,1$ matrices, and by Conforti, Cornuéjols, Kapoor and Vušković [23] for the $0, \pm 1$ case. In Chapter 5 we will provide a new, simpler proof of this theorem [80], by giving a new polynomial time algorithm. We will also give a similar algorithm to solve the following problem. A $0, 1$ matrix is *balanceable* if it is the support matrix of a balanced matrix. Deciding if a given matrix A is balanceable can be reduced to the problem of deciding if a certain $0, \pm 1$ matrix B obtained by A through a signing algorithm due to Camion [9] is balanced. So far, this was the only known approach to solve the recognition problem for balanceable matrices. On the other hand, Truemper [76] gave a co-NP characterization of the class of balanceable matrices in terms of certain well understood forbidden submatrices. We will present an algorithm, due to Conforti and Zambelli [38], that uses Truemper's characterization, rather than relying on Camion's algorithm.

Berge [3] and Conforti and Cornuéjols [20] characterized balanced matrices in terms of bicolorings (special types of partitions) of the column sets of their submatrices. This result is in the same spirit as Ghouila-Houri's classical characterization of totally unimodular matrices in terms of equitable bicolorings [52]. In Chapter 6 we will give a theorem, due to Conforti, Cornuéjols and Zambelli [32], that generalizes both results mentioned above. The theorem offers a characterization of a certain class of $0, \pm 1$ matrices, named *k-balanced matrices*, introduced by Truemper and Chandrasekaran [77] and Conforti, Cornuéjols and Truemper [27], that generalize balanced and totally unimodular matrices. Based on such a characterization, we show that certain polyhedra arising from the $0, 1$ matrices in this class have the *integer decomposition property* [81].

In the remainder of this chapter we provide some definitions and notations that will be needed later.

1.1 Notations and definitions

1.1.1 Graph Theory

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of *nodes* of G and $E(G)$ the set of *edges* of G, and both $V(G)$ and $E(G)$ are finite. For our purposes, all graphs are undirected and simple, hence we will assume $E(G) \subseteq \{\{u, v\} \,|\, u, v \in V, u \neq v\}$. We will denote every element $\{u, v\} \in E(G)$ by uv. We refer to [79] for a more general definition of graph and for standard definitions and terminology. Given $u, v \in V(G)$, u and v are *adjacent* if there exists $e \in E$ such that $e = uv$, we say that u and v are the *endnodes* of e and that v is a *neighbor* of u. Given $u \in V(G)$, we denote by $N(u)$ the set of neighbors of u. The degree of a node is the number of its neighbors. Two edges are *adjacent* if the have a common endnode. The *complement* $\bar{G}$ of G is defined by $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{uv \,|\, u, v \in V(G),\ uv \notin E(G),\ u \neq v\}$.

Two graphs G and G' are *isomorphic* if there exists a *graph isomorphism* between them, that is a bijection φ between $V(G)$ and $V(G')$ with the property that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(G')$. With a slight abuse of terminology, we will say that G is (equal to) G'.

A *subgraph* of G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. G' is an *induced subgraph* of G if G' is a subgraph of G and $E(G') = \{uv \in E(G) \,|\, u, v \in V(G')\}$, and we say that G' is *induced* by $V(G')$. Given $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and by $G \setminus X$ the graph $G[V(G) \setminus X]$. If G' is an induced subgraph of G, often we write $G \setminus G'$ instead of $G \setminus V(G')$. Given a graph G', when we say that G *contains* G' (or G' is contained in G), we always mean that G' is isomorphic to some induced subgraph of G. Given $X \subseteq V(G)$ and a graph G', we say that X *induces* G' if $G[X]$ is isomorphic to G'. Given an induced subgraph G' of G and a node $x \in V(G)$, we denote by $N_{G'}(x)$ the set of neighbors of x contained in $V(G')$. Given $x, x' \in V(G)$ and an induced subgraph G' of G, we say that x and x' are *twins* w.r.t. G' if $N_{G'}(x) = N_{G'}(x')$, where x and x' are *true twins* if $xx' \in E(G)$, *false twins* otherwise.

Given an edge $e \in E(G)$, we denote by $G \setminus e$ the graph obtained by *deleting* e, where $V(G \setminus e) = V(G)$, $E(G \setminus e) = E(G) \setminus \{e\}$.

A *cut* of G is a subset of $E(G)$ of the form $(S, \bar{S}) = \{uv \in E(G) \,|\, u \in S, v \notin S\}$, where S is a nonempty proper subset of $V(G)$ and $\bar{S} = V(G) \setminus S$. A graph G is *connected* if every cut of G is nonempty. The *connected components* of G are the maximal subsets of G inducing a connected graph. A *(node) cutset* of G is a set $X \subseteq V(G)$ such that $G \setminus X$ is not connected; if u and v belong to distinct connected components of $G \setminus X$, we say that X *separates* u and v. A graph G is *k-connected* if it does not have a cutset of size strictly less than k.

We say that a graph G is *anticonnected* if $\bar{G}$ is connected, and the *anticonnected components* of G are the connected components of $\bar{G}$. A set $X \subseteq V(G)$ is *connected* if $G[X]$ is connected, and X is *anticonnected* if $G[X]$ is anticonnected.

A *path* P is a connected graph such that every node has degree at most 2 and there exists a node of degree less than 2. It is immediate to check that the elements of $V(P) = \{v_1, \ldots, v_k\}$ can be ordered so that the only edges of P are of the form v_iv_{i+1} for $1 \leq i \leq k-1$. We will denote P by the sequence of such nodes, that is $P = v_1, \ldots, v_k$. Obviously, the only two (possibly identical) nodes of P of degree less than 2 are v_1 and v_k, and they are the *endnodes* of P. P is said to be a *path between* v_1 and v_k. The *interior* of P is the set of nodes of degree 2 in P, which are called the *intermediate nodes* of P. Given two nodes a and b of P, there exists a unique subpath of P between a and b, and we denote such path by P_{ab}.

A *cycle* C is a connected graph such that every node has degree exactly 2. It is immediate to check that $V(C) = \{v_1, \ldots, v_k\}$ can be ordered so that the only edges of C are of the form $v_i v_{i+1}$ for $1 \leq i \leq k$, where $v_1 = v_{k+1}$. We will use the notation $C = v_1, \ldots, v_k, v_1$ to specify the set of nodes and edges of C. If Q is a path or a cycle, the *length* of Q, denoted by $|Q|$, is the number of edges of Q. The *parity* of Q (odd or even) is defined as the parity of $|Q|$.

If G is a graph and Q is a subgraph of G that is either a path or a cycle, we say that Q is *chordless* if Q is induced in G. If Q is not chordless, the edges of G in $E(G) \setminus E(Q)$ with both endnodes in $V(Q)$ are the *chords* of Q. A chordless cycle of length at least 4 is called a *hole*. Given two chordless paths P and Q with endnodes a, b and b, c, respectively, such that no node in $P \setminus b$ belongs to or has a neighbor in $Q \setminus b$, we denote by a, P, b, Q, c the chordless path between a and c induced by $V(P) \cup V(Q)$.

Given two disjoint sets $A, B \subset V(G)$, a *direct connection between A and B* is a minimal chordless path (in term of its node set) $P = x_1, ..., x_n$ such that x_1 has a neighbor in A and x_n has a neighbor in B.

The complement of a path is an *antipath*, while the complement of a hole is an *antihole*. If Q is an antipath or an antihole that is a subgraph of G, Q is *chordless* if $\bar{Q}$ is chordless in $\bar{G}$.

A set $K \subseteq V(G)$ is a *clique* if the nodes in K are pairwise adjacent. We denote by K_n the graph on n nodes such that $V(K_n)$ is a clique. A set $S \subseteq V(G)$ is a *stable set* if the nodes in S are pairwise nonadjacent.

A graph G is *bipartite* if $V(G)$ can be partitioned into two stable sets A, B. A, B is said the *bipartition* of G and A and B are the *sides of the bipartition* of G. If G is bipartite, sometimes we use the notation $G = (A, B; E)$ to indicate that A, B is the bipartition of G, and E is the edge-set of G. $G = (A, B; E)$ is *complete* if every node in A is adjacent to every node in B. We denote by $K_{n,m}$ the complete bipartite graph $G = (A, B; E)$ with $|A| = n$ and $B = |m|$.

Given a graph H, the *line graph* $L(H)$ of H is the graph G where $V(G) = E(H)$ and there is an edge between $e, e' \in V(G)$ if and only if e and e' have an endnode in common in H.

Given a graph G and $e = uv \in E(G)$, the graph G' obtained by *subdividing* e is the graph defined by $V(G') = V(G) \cup \{w\}$, where $w \notin V(G)$ is a new node, and $E(G') = E(G) \setminus e \cup \{uw, wv\}$. Given two graphs G and G', G' is a *subdivision* of G if G' can be obtained from G by iteratively subdividing edges. We say that G' is a *bipartite subdivision* of G if G' is a bipartite graph that is a subdivision of G.

Given a set $S \subset V(G)$ and a node $x \notin S$, we say that x is *universal for* S if x is adjacent to every node of S. We say that an edge $e = yz$ such that $y, z \notin S$ *sees* S if both y and z are universal for S. Given a chordless path or a hole P in $G \setminus S$, we denote by $E_S(P)$ the set of edges in P that see S.

1.1.2 Polyhedra and Linear Programming

If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots y_n)$ are vectors, we write $x \leq y$ if $x_i \leq y_i$ for every $i \in [n]$. We denote by xy the scalar product of x and y. Also, whenever we have expressions involving matrix multiplications, scalar products, identities or inequalities among vectors or matrices, we always assume that the sizes of the matrices and vectors involved are compatible.

A matrix or a vector is *integral* if all its entries are integers. We use the expression $0, 1$ *matrix* (resp. *vector*) or $0, \pm 1$ *matrix* (resp. *vector*) to indicate that all the entries of the matrix (resp. *vector*) are in $\{0, 1\}$ or in $\{-1, 0, +1\}$, respectively. Given a matrix A, the *support matrix* of A is the $0, 1$ matrix, with the same number of rows and columns, such that an entry is 1 if and only if the corresponding entry of A is nonzero. Given a real number k, we denote by $\mathbf{k}$ the vector (of appropriate dimension, depending on the context) whose components are all equal to k.

Given a matrix A, a column vector b and a column vector of unknowns x, the expression

$$Ax \leq b$$

is a *system of linear inequalities*. The matrix A is called the *constraint matrix* of the system.

A set P of vectors in $\mathbb{R}^n$ is a *(convex) polyhedron* if

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \tag{1.2}$$

for some matrix A and vector b. We say that P is *rational* if one can choose A and b in (1.2) such that every entry of A and b is a rational number.

Given sets X and Y of vectors in $\mathbb{R}^n$, the *convex hull* of X is the minimal convex set $conv(X)$ containing X, while the *cone generated by* Y is the set $cone(Y) = \{\lambda_1 y_1 + \ldots + \lambda_k y_k \,|\, k \in \mathbb{N},\ \lambda_1 \ldots, \lambda_k \in \mathbb{R}_+,\ \text{and } y_1, \ldots, y_k \in Y\}$.

Theorem 1.1 (Minkowski's Theorem) *$P \subseteq \mathbb{R}^n$ is a convex polyhedron if and only if there exist finite sets $X, Y \subset \mathbb{R}^n$ such that $P = conv(X) + cone(Y)$.*

If P is a polyhedron and X and Y are the (unique) minimal sets such that $P = conv(X) + cone(Y)$, the vectors in X are the *vertices* of P, while the vectors in Y are the *extreme rays* of P. A vertex of P is also called a *basic feasible solution.*

Theorem 1.2 *Given an $m \times n$ matrix A with rows a^i, $i \in [m]$, and a vector $b \in \mathbb{R}^m$, a vector $\bar{x}$ in $P = \{x \in \mathbb{R}^n \,|\, Ax \leq b\}$ is a vertex of P if and only if there are n linearly independent rows $a^{i_1}, \ldots, a^{i_n}$ of A such that $a^{i_j}\bar{x} = b_{i_j}$ for every $j \in [n]$.*

A subset P of $\mathbb{R}^n$ is a *polytope* if and only if $P = conv(X)$ for some finite $X \subset \mathbb{R}^n$. One can show that P is a polytope if and only if P is a bounded polyhedron, i.e. if there exists $M \in \mathbb{R}$ such that $||x|| \leq M$ for every $x \in P$. A rational polyhedron P (resp. polytope) is *integral* if all the vertices of P are integral.

A *linear program* is the problem of maximizing a linear function over a polyhedron. Given a linear program

$$\max\{cx \,|\, Ax \leq b\}, \tag{1.3}$$

the *dual* of (1.3) is the linear program

$$\min\{yb \,|\, yA = c, y \geq 0\}. \tag{1.4}$$

A vector $\tilde{x}$ satisfying $A\tilde{x} \leq b$ is a *feasible solution* for (1.3), and is said to be an *optimal feasible solution* if the maximum in (1.3) is attained by $c\tilde{x}$.

Theorem 1.3 (Strong Duality Theorem) *Given a matrix A and vectors b and c,*

$$\max\{cx \,|\, Ax \leq b\} = \min\{yb \,|\, yA = c, y \geq 0\} \tag{1.5}$$

provided that both sets in (1.5) *are nonempty.*

Theorem 1.4 *If* (1.3) *has a finite optimum, then there exists an optimal solution* $\tilde{x}$ *that is a vertex of* $P = \{x \mid Ax \leq b\}$.

If $\tilde{x}$ is a vertex of $P = \{x \mid Ax \leq b\}$ that is optimal for (1.3), we say that $\tilde{x}$ is a *basic optimum solution.*

Chapter 2

Perfect Graphs

2.1 Introduction

Let $G = (V, E)$ be an undirected, simple graph. Given a positive integer k, a *k-coloring* of G is a function

$$\varphi : V(G) \to [k]$$

such that $\varphi(u) \neq \varphi(v)$ for every $uv \in E(G)$. The elements of $[k]$ are referred to as *colors*, and the maximal sets of nodes of G with the same color are the *color classes* of φ. Equivalently, G has a k-coloring if its node-set can be partitioned into k stable sets. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k such that G has a k-coloring.

We denote by $\omega(G)$ the *clique number* of G, that is the cardinality of a maximum-size clique in G. Obviously, chromatic number and clique number are related by the following inequality

$$\chi(G) \geq \omega(G).$$

We will denote with $\alpha(G)$ the *stability number* of G, that is the cardinality of a maximum-size stable set of G.

Definition 2.1 *A graph G is* perfect *if $\chi(G') = \omega(G')$ for every induced subgraph G' of G.*

Berge [2] made two conjectures about perfect graphs. The first, known as the weak perfect graph conjecture, states that a graph is perfect if and only if its complement is, and was proven by Lovász [61]. The other, known as the strong perfect graph conjecture, characterizes the class of perfect graphs in terms of minimally imperfect graphs. A graph G is *minimally imperfect* if G is not perfect but every proper induced subgraph of G is perfect. Berge observed that odd holes and their complements were both minimally imperfect and conjectured that, in fact, odd holes and odd antiholes are the only minimally imperfect graphs. Recently, Chudnovsky, Robertson, Seymour and Thomas [13] proved the conjecture. In Section 2.5 we will briefly describe some of the main ideas involved in this long and difficult proof.

Next, we make a simple, yet useful, observation. Let G be a minimally imperfect graph and S a nonempty stable set. Obviously, $\omega(G \setminus S) \leq \omega(G)$. On the other hand, if $\omega(G \setminus S) \leq \omega(G)-1$, then $G \setminus S$ is $(\omega(G)-1)$-colorable, since $G \setminus S$ is perfect, thus G is $\omega(G)$-colorable, as we can assign a single new color to all the nodes in S. This contradicts the assumption that G is minimally imperfect. We summarize this in the following.

Remark 2.2 *If G is a minimally imperfect graph, and S is a stable set of G, then S does not intersect all the cliques of size $\omega(G)$ in G.*

2.2 The Perfect Graph Theorem

Lovász [60] proved also a theorem stronger than the weak perfect graph theorem, which is commonly known as the perfect graph theorem. Here we present a beautiful, simple proof of this result, due to Gasparyan [51].

Theorem 2.3 (Perfect Graph Theorem) (Lovász) *A graph G is perfect if and only if $\omega(G')\alpha(G') \geq |V(G')|$ for every induced subgraph G' of G.*

Proof: One direction is trivial: if G is perfect, then, for every induced subgraph G', $V(G')$ can be partitioned by $\omega(G')$ stable sets of size at most $\alpha(G')$, hence $\omega(G')\alpha(G') \geq |V(G')|$. For the converse, since every graph that is not perfect contains a minimally imperfect graph, it suffices to prove that, if G is minimally imperfect, than $|V(G)| \geq \omega(G)\alpha(G) + 1$. Let $\alpha = \alpha(G)$ and $\omega = \omega(G)$. W.l.o.g., $V(G) = [n]$, and $S_0 = [\alpha]$ is a maximum-size stable set. For every $i \in S_0$, $G \setminus i$ is perfect and, by Remark 2.2, $\omega(G \setminus i) = \omega$, hence $V(G) \setminus \{i\}$ can be partitioned into ω stable sets $S_{\omega(i-1)+1}, \ldots, S_{i\omega}$. Clearly, each node of G is contained in exactly α of the sets $S_0, \ldots, S_{\alpha\omega}$ so defined. By Remark 2.2, for every j, $0 \leq j \leq \alpha\omega$, there exists a clique C_j of size $\omega(G)$

not intersecting S_j. Since, for every i, j, $0 \le i, j \le \alpha\omega$, $|S_i \cap C_j| \le 1$ and every node of C_j is contained in exactly α of the S_i's, then $|S_i \cap C_j| = 1$ if $i \ne j$. Thus, if $\mathbf{S}$ is the $(\alpha\omega + 1) \times n$ $0, 1$ matrix where $\mathbf{S}_{ij} = 1$ if and only if $j \in S_i$, and $\mathbf{C}$ is the $(\alpha\omega + 1) \times n$ $0, 1$ matrix where $\mathbf{C}_{ij} = 1$ if and only if $j \in C_i$, then $\mathbf{SC}^\intercal = J - I$, where J is the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ matrix with all 1 entries and I is the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ identity. Since $J - I$ is nonsingular, then both $\mathbf{S}$ and $\mathbf{C}$ have rank $\alpha\omega + 1$, thus $n \ge \alpha\omega + 1$. □

Clearly, since $\alpha(G) = \omega(\bar{G})$ and $\omega(G) = \alpha(\bar{G})$, the perfect graph theorem implies the weak perfect graph theorem. Lovász's original proof of the weak theorem was polyhedral, and it relied on previous work by Fulkerson [49], who was able to show that the weak perfect graph theorem was equivalent to the *Replication Lemma* (Lemma 2.4). Next we also report this proof, as it bears important consequences in relating perfect graphs to certain integral polyhedra.

Given a graph G and a node $v \in V(G)$, a graph H is obtained from G by *replicating* v, if H is obtained by adding a new node v' to G, which is adjacent to v and to all, and only, the neighbors of v.

Lemma 2.4 (Replication Lemma) (Lovász [61]) *If G is a perfect graph and H is obtained from G by replicating $v \in V(G)$, then H is perfect.*

Proof: By contradiction, suppose H' is an induced subgraph of H that is minimally imperfect. Clearly, if H' does not contain both v and v', then H' is isomorphic to an induced subgraph of G, hence it is perfect. So we may assume $v, v' \in V(H')$. Let $G' = H' \setminus v'$ and, given an $\omega(G')$-coloring of G', let S be the color class containing v. Note that either $\omega(H') = \omega(G') + 1$ and v and v' belong to all the maximum-size cliques in H', or $\omega(H') = \omega(G')$ and the maximum-size cliques of H' are contained in $G' \setminus v$. In both cases, S intersects all the maximum-size cliques of H', contradicting Remark 2.2. □

Given a graph G, the *stable set polytope* of G, $STAB(G)$, is the convex hull in $\mathbb{R}^{|V(G)|}$ of the node-incidence vectors of the stable sets of G. One can readily verify that a vector in $\{0, 1\}^{|V(G)|}$ is the incidence vector of a stable set if and only if it satisfies the following constraints:

$$\sum_{v \in K} x_v \le 1 \qquad \text{for every clique } K \text{ in } G \tag{2.1}$$

$$x_v \ge 0 \qquad \text{for every } v \in V(G) \tag{2.2}$$

Theorem 2.5 *Let G be a graph. The following are equivalent.*

(*i*) *G is perfect,*

(*ii*) *$STAB(G)$ is determined by (2.1) and (2.2),*

(*iii*) *$\bar{G}$ is perfect.*

Proof: (i)⇒(ii) Suppose G is perfect. Let $x \in \mathbb{R}^{|V(G)|}$ be a vector satisfying (2.1) and (2.2), and N be a positive integer such that $y = Nx$ is integral. Let Y_v, $v \in V(G)$, be pairwise disjoint sets such that $|Y_v| = y_v$, and let H be the graph such that $V(H) = \cup_{v\in V(G)} Y_v$ and such that a node in Y_u is adjacent to a node in Y_v if and only if $u = v$ or $uv \in E(G)$. Thus H is obtained from G by iteratively replicating nodes of an induced subgraph of G. So H is perfect. Given a maximum-size clique K' of H, clearly $K = \{v \in V(G) \,|\, K' \cap Y_v \neq \emptyset\}$ is a clique, thus $\omega(H) = |K'| \leq \sum_{v\in K} |Y_v| = N \sum_{v\in K} x_v \leq N$. Then $V(H)$ can be partitioned into N stable sets. Let $s_1, \ldots, s_N$ be the incidence vectors of such stable sets, then $y = \sum_{i=1}^{N} s_i$. Thus $x = \frac{1}{N}\sum_{i=1}^{N} s_i$, so x is a convex combination of incidence vectors of stable sets.
(ii)⇒(iii) One can verify that property (ii) is inherited by induced subgraphs, therefore, by remark 2.2, one has only to verify that, if (ii) holds, then $\bar{G}$ contains a stable set that intersects all maximum-size cliques. Consider the face F of $STAB(G)$ defined by $\sum_{v\in V(G)} x_v \leq \alpha(G)$. Then F is contained in a facet of $STAB(G)$ of the form $\sum_{v\in K} x_v \leq 1$ for some clique K of G. But then K intersects all the maximum-size stable sets of G, therefore K is a stable set of $\bar{G}$ intersecting every maximum-size clique of $\bar{G}$.
(iii)⇒(i) Just apply (i)⇒(iii) to $\bar{G}$. □

2.3 Perfect graphs and set packing

Given an $m \times n$ 0,1 matrix $A = (a_{ij})$ with at least a 1 in each column, the *set packing polytope* is defined as

$$P(A) = \{x \in \mathbb{R}^n \,|\, Ax \leq \mathbf{1}, x \geq \mathbf{0}\}. \tag{2.3}$$

A 0,1 matrix A is *perfect* if $P(A)$ has integral vertices only. A row of A, say row i, is said to be *dominated* if there exists another row, say row j, such that $a_{ik} \leq a_{jk}$ for every $k \in [n]$. Obviously, constraints of (2.3) corresponding to dominated rows of A are redundant, hence we may assume that A has no dominated rows. Chvátal [16] showed that $P(A)$ is an integral polytope if and only if A is the clique-node incidence matrix of a perfect graph, where the *clique-node incidence matrix* of a graph G is the $0,1$ matrix whose columns are indexed by the nodes of G and whose rows are the incidence vectors of the maximal cliques of G.

Theorem 2.6 *Let A be a $0,1$ matrix with at least a 1 in each column and containing no dominated rows. Then $P(A)$ is integral if and only if A is the clique-node incidence matrix of a perfect graph.*

Proof: Let $A = (a_{ij})$ ba an $m \times n$ $0,1$ matrix. By Theorem 2.5, if A is the clique-node incidence matrix of a graph G, then $P(A)$ is integral if and only if G is perfect. Hence we only need to prove that, if $P(A)$ is integral, then A is the clique-node incidence matrix of some graph. Let G be the graph with node-set $V(G) = [n]$ such that $ij \in E(G)$ if and only if there exists $h \in [m]$ such that $a_{hi} = a_{hj} = 1$. Suppose, by contradiction, that A is not the clique-node incidence matrix of G. Then there exists a clique K of G such that, for every $h \in [m]$, there exists $k \in K$ such that $a_{hk} = 0$. Let $q = |K|$, then, by construction, $q \geq 3$. Let $x \in \mathbb{R}^n$ be defined by $x_i = \frac{1}{q-1}$ if $i \in K$, and $x_i = 0$ otherwise. Clearly $x \in P(A)$. Let c be the incidence vector of K, then $cx = \frac{q}{q-1} > 1$, while $cy \leq 1$ for every integral point in $P(A)$, contradicting the fact that $P(A)$ is integral. □

2.4 Algorithmic aspects of perfect graphs

Several NP-hard problems, such as determining the stability number of a graph, or finding an optimal coloring, can be solved in polynomial time when the graph is perfect.

Lovász's Theta function and the maximum stable set

There are several equivalent definitions of Lovász's Theta function. Given a graph G, define $\mathcal{M}_G$ to be the family of symmetric $|V(G)| \times |V(G)|$ matrices such that, for every element $M = (m_{ij})$, $m_{ij} = 0$ for every $ij \in E(G)$ and the trace of M (i.e. the sum of the diagonal elements) is 1. Lovász's Theta function can be defined as

$$\vartheta(G) = \max\{\mathbf{1}^{\intercal} M \mathbf{1} \mid M \in \mathcal{M}_G \text{ positive semidefinite}\}. \qquad (2.4)$$

Lovász [62] showed that, for any graph G,

$$\alpha(G) \leq \vartheta(G) \leq \chi(\bar{G}),$$

while Grötschel, Lovász and Schrijver [53][54] showed that $\vartheta(G)$ can be computed in polynomial time using semidefinite programming. Hence, if G is a perfect graph, then, by the Perfect Graph Theorem, $\alpha(G) = \chi(\bar{G})$, hence $\alpha(G) = \vartheta(G)$. These facts immediately imply the following:

Theorem 2.7 *There exists a polynomial time algorithm to compute the stability number of any perfect graph.*

The previous theorem also provides a way to compute a maximum-size stable set in a perfect graph as follows: let $V(G) = \{v_1, \ldots, v_n\}$ and let $G_0 := G$. For $0 \leq i \leq n-1$, if $\alpha(G_i \setminus \{v_{i+1}\}) = \alpha(G)$ then let $G_{i+1} := G_i \setminus \{v_{i+1}\}$, else let $G_{i+1} := G_i$. Obviously, $S = V(G_n)$ is a maximum-size stable set.

Coloring perfect graphs

Theorem 2.7 implies that, given a perfect graph G, it is possible to find an $\omega(G)$-coloring of G.

Theorem 2.8 (Grötschel et Al. [54]) *There exists a polynomial time algorithm to find a minimum coloring in a perfect graph.*

Proof: Let G be a perfect graph. We only need to show how to find a stable set S intersecting all maximum-size cliques in G, since we can apply recursion to $G \setminus S$.

Start with $t = 0$. At each iteration, we have a list of t maximum-size cliques $K_1, \ldots, K_t$ and we compute a set S that intersect every K_i, $i \in [t]$. If S intersects all the maximum-size cliques then we are done, else we compute a maximum-size clique K_{t+1} that does not intersect S. The polynomiality of the algorithm will follow from bounding the number of iterations by $|V(G)|$.

To compute a stable set intersecting every K_i, construct the following graph H. For every $v \in V(G)$, let $y_v = |\{i \,|\, v \in K_i\}|$, and let Y_v, $v \in V(G)$, be pairwise disjoint sets such that $|Y_v| = y_v$. Let H be the graph such that $V(H) = \cup_{v \in V(G)} Y_v$ and such that a node in Y_u is adjacent to a node in Y_v if and only if $u \neq v$ and $uv \in E(G)$. Since G is perfect, H is perfect and so is $\bar{H}$. Compute a maximum-size stable set S' of H and let $S = \{v \in V \,|\, Y_v \cap S' \neq \emptyset\}$. By construction, $V(H)$ can be partitioned into t cliques of size $\omega(G)$. Since

$\omega(H) = \omega(G)$, then such a partition is optimal and, since $\bar{H}$ is perfect, $|S'| = t$ and S intersects every K_i, $i \in [t]$.

If $\omega(G \backslash S) < \omega(G)$ then S intersects every maximum-size clique, otherwise we can compute a maximum-size clique K_{t+1} in $G \setminus S$. The number of iterations is bounded by $|V(G)|$, since the dimension of the vector space L_t, defined by the equations $\sum_{v \in K_i} x_v = 1$, $i \in [t]$, drops at each iteration, as the incidence vector of S belongs to L_t but not to L_{t+1}. □

Theorems 2.7 and 2.8 together provide a polynomial algorithm that, given a graph G, either finds an optimal coloring of G and a proof of the optimality of such coloring, that is a clique of same cardinality as the number of colors, or the proof that the graph is not perfect.

Recognizing perfect graphs

The problem of deciding in polynomial time whether a graph is perfect or not has been solved recently in a series of three papers, providing two different algorithms, by Chudnovsky, Cornuéjols, Liu, Seymour and Vušković. The first paper, joint by Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [12], describes a preprocessing technique that is needed in both algorithms. The two papers describing the recognition algorithms are by Chudnovsky and Seymour [15] and Cornuéjols, Liu and Vušković [41]. Interestingly, none of the two algorithms uses the decomposition theorem for perfect graphs, which will be discussed in Section 2.5.2 (but they do rely on the validity of the Strong Perfect Graph Theorem). Prior to these papers, it was not known whether the problem of deciding if a graph is perfect was in NP. While, by the Strong Perfect Graph Theorem, deciding if a graph is perfect amounts to determine if the graph contains an odd hole or an odd antihole, the problem of deciding in polynomial time if a graph contains an odd hole is still open. In this regard, Bienstock [6] showed that deciding if a graph contains an odd hole going through a prescribed node is NP-complete.

2.5 The Strong Perfect Graph Theorem

Let us restate the strong perfect graph theorem, recently proved by Chudnovsky, Robertson, Seymour and Thomas [13], in a more convenient form.

Definition 2.9 *A graph G is* Berge *if G does not contain an odd hole or an odd antihole as an induced subgraph.*

Theorem 2.10 (Strong Perfect Graph Theorem) *Let G be a graph. G is perfect if and only if G is Berge.*

As mentioned in the introduction, the approach used to prove Theorem 2.10 consists in showing that every Berge graph either belongs to some *basic class* of perfect graphs, or has a *decomposition* that cannot occur in a minimally imperfect graph. Next we define what these basic classes and decompositions are.

2.5.1 Basic classes

A graph G is *basic* if and only if it belongs to one of the following five families of graphs:

1. Bipartite graphs,
2. Line graphs of bipartite graphs,
3. Complements of bipartite graphs,
4. Complements of line graphs of bipartite graphs,
5. Doubled-split graphs,

where G is a *doubled-split graph* if $V(G)$ can be partitioned into sets A and B, each of cardinality at least 4, such that every node of A has degree 1 in $G[A]$, every node of B has degree 1 in $\bar{G}[B]$, and for every pair of adjacent nodes $a, a' \in A$, and for every pair of nonadjacent nodes $b, b' \in B$, $\{a, a', b, b'\}$ induces a chordless path of length 3.

Proposition 2.11 *If G or $\overline{G}$ is bipartite or the line graph of a bipartite graph, then G is perfect.*

Proof: Note that every induced subgraph of G still belongs to the same basic class as G, hence we only need to show that the chromatic number of G equals its clique number. The statement is trivial if G is bipartite, so by the weak perfect graph theorem also the complements of bipartite graphs are perfect. If G is the line graph of some bipartite graph H, then $\omega(G) = \Delta(H)$, where $\Delta(H)$ is the maximum degree of a node in H and $\chi(G) = \chi'(H)$, where $\chi'(H)$ is the minimum number of colors $\chi'(H)$ to be assigned to the edges of H so that no pair of adjacent edges have the same color. A theorem of König [59] implies that $\chi'(H) = \Delta(H)$, hence $\omega(G) = \chi(G)$. As above, this implies that also the complements of line graphs of bipartite graphs are perfect. □

One can show that doubled-split graphs are perfect as well, but the class is not closed under taking induced subgraphs. However, it is sufficient, for proving the SPGC, to show the following weaker statement.

Proposition 2.12 *No minimal imperfect graph is a doubled-split graph.*

Proof: We only need to show that the clique number and the chromatic number of G are equal. If G is a doubled-split graph and $V(G)$ is partitioned into sets A and B as in the definition, then $\omega(G) = |B|/2 + 1$ and G can be partitioned into $\omega(G)$ stable set as follows: given two nonadjacent nodes $b, b' \in B$, $|B|/2 - 1$ stable sets are of the form $\{\beta, \beta'\} \subset B$ where $\{\beta, \beta'\} \neq \{b, b'\}$, and the remaining two stable sets are $b \cup (A \setminus N(b))$ and $b' \cup (A \setminus N(b'))$.
□

2.5.2 Decompositions

We consider three types of decompositions: 2-joins, homogeneous pairs and skew partitions.

2-Join

A graph G has a *2-join* if $V(G)$ can be partitioned into nonempty subsets V_1 and V_2 each of cardinality at least 3, with nonempty pairwise disjoint subsets $A_1, B_1 \subset V_1$ and $A_2, B_2 \subset V_2$ such that every node in A_1 is adjacent to every node in A_2, every node in B_1 is adjacent to every node in B_2, and there is no other edge between V_1 and V_2.
The concept of 2-Join was introduced by Cornuéjols and Cunningham [40], who proved the following.

Theorem 2.13 *If a minimally imperfect graph G has a 2-join, then G is an odd hole.*

Homogeneous Pair

Given a graph G, an *homogeneous pair* consists of two disjoint sets A_1 and A_2, each nonempty, such that $3 \leq |A_1| + |A_2| \leq |V(G)| - 2$ and, for every $v \in V(G) \setminus (A_1 \cup A_2)$, if v is adjacent to a node in A_i, then it is adjacent to every node in A_i, for $i \in [2]$.

Homogeneous pairs where introduced by Chvátal and Sbihi [19], who proved the following.

Theorem 2.14 *No minimally imperfect graph has an homogeneous pair.*

Skew Partition

A graph G has a *skew partition* if $V(G)$ can be partitioned into nonempty sets A, B, C, D such that every node in A is adjacent to every node in B and no node in C has a neighbor in D. Skew partitions were introduced by Chvátal [17], who conjectured that no minimally imperfect graph has a skew partition. It is easy to verify that the strong perfect graph theorem implies the skew partition conjecture, but there is no direct proof of it. However, several weaker results were known in the literature, and we will present some of them. To this purpose, we will provide a useful lemma, due to Hoáng [57].

Given a minimally imperfect graph G with a skew partition A, B, C, D, let $G_1 = G \setminus D$ and $G_2 = G \setminus C$. It is immediate to verify that every maximum-size clique of G is contained in G_1 or G_2. Also, G_1 and G_2 are both perfect.

Lemma 2.15 (Hoáng) *Let φ_1 and φ_2 be, respectively, $\omega(G)$-colorings of G_1 and G_2. Then $|\varphi_1(A)| \neq |\varphi_2(A)|$.*

Proof: By contradiction, suppose that $|\varphi_1(A)| = |\varphi_2(A)| = k$. Then, w.l.o.g. $\varphi_1(A) = \varphi_2(A) = [k]$. Let K be the subgraph of G induced by the nodes of G that get a color in $[k]$ in either of the two colorings and let $H = G \setminus K$. Since every maximum-size clique of G is contained in G_1 or in G_2, $\omega(K) = k$ and $\omega(H) = \omega(G) - k$. Since H and K are both perfect, their nodes can be partitioned into, respectively, $\omega(G) - k$ and k stable sets, hence $V(G)$ can be partitioned into $\omega(G)$ stable sets, a contradiction. □

If $|A| = 1$, then the set $A \cup B$ is called *star cutset.* Star cutsets were introduced by Chvátal [17], who showed that they cannot occur in a minimally imperfect graph. This last fact is obviously implied by the previous lemma.

Another important type of skew partition was introduced by Chudnovsky, Robertson, Seymour and Thomas [13]: a skew partition A, B, C, D is *balanced* if

(i) every chordless path of length at least 2 with endnodes in $A \cup B$ and interior in $C \cup D$ has even length,

(ii) every chordless antipath of length at least 2 with endnodes in $C \cup D$ and interior in $A \cup B$ has even length.

A counterexample (if any) to the Strong Perfect Graph Theorem that has the smallest number of nodes, is said a *Berge minimum imperfect graph*. In other words, a Berge minimum imperfect graph is a Berge graph that is not perfect and has the smallest number of nodes among graphs with this property. Clearly, a Berge minimum imperfect graph is also minimally imperfect.

Theorem 2.16 (Chudnovsky, Robertson, Seymour and Thomas [13]) *No Berge minimum imperfect graph has a balanced skew partition.*

Proof: Let G be a Berge minimum imperfect graph. Suppose that G has a balanced skew partition A, B, C, D. By Lemma 2.15 applied to $\bar{G}$, we may assume $|C|, |D| \geq 2$. Let $G_1 = G \setminus D$ and $G_2 = G \setminus C$. For $i \in [2]$, let G'_i be obtained from G_i by adding a new node x_i such that $N_{G'_i}(x_i) = A$. Because A, B, C, D is balanced, it is immediate to verify that G'_i is Berge. Also, G'_i has fewer nodes than G, hence G'_i is perfect. Let G''_i be the graph obtained from G'_i by replacing x_i with a clique X_i of cardinality $\omega(G) - \omega(G[A])$ whose nodes are adjacent to every node in A and to no node of $G_i \setminus A$. By the Replication Lemma 2.4, G''_i is perfect. By construction, $\omega(G''_i) = \omega(G)$. Let φ'_i be an $\omega(G)$-coloring of G''_i. By construction, $\varphi'_i(A) \cap \varphi'_i(X_i) = \emptyset$ and $|\varphi'_i(A)| + |\varphi'_i(X_i)| \leq \omega(G) = \omega(G[A]) + \omega(G[X_i])$. Thus the restriction φ_i of φ'_i to $V(G_i)$ is an $\omega(G)$-coloring of G_i with $|\varphi_i(A)| = \omega(G[A])$, contradicting Lemma 2.15. □

The next lemma, due to Chudnovsky et Al. [13], provides, in several cases, an easy way to prove that a graph that contains a skew partition contains, in fact, a skew partition that is balanced. A skew partition A, B, C, D is *loose* if either there exists $u \in A \cup B$ such that u has no neighbors in C or in D, or there exists $v \in C \cup D$ such that v is adjacent to every node in A or in B.

Lemma 2.17 *If G is a Berge graph containing a loose skew partition, then G has a balanced skew partition.*

We postpone the proof of Lemma 2.17 to the end of Section 2.5.4, since we will need the Roussel-Rubio Lemma 2.27. Lemma 2.17 generalizes previous results due to Hoàng, who showed that no minimally imperfect graph contains a *T-cutset* (i.e. a skew cutset in which both C and D contain a node universal for A) or a *U-cutset* (i.e. a skew cutset in which C contains a node universal for A and a node universal for B).

Other types of skew-partitions have been considered in the literature. For example, Cornuéjols and Reed [42] considered the case in which $A \cup B$ induces a multi-partite graph, that is a graph in which every anticonnected component is a stable set. In this case, $A \cup B$ is a *multipartite cutset.*

Theorem 2.18 (Cornuéjols and Reed [42]) *No minimally imperfect graph has a multipartite cutset.*

The previous theorem was generalized by Roussel and Rubio [69], who showed that no minimally imperfect graph can contain a skew-partition A, B, C, D in which A is a stable set.

2.5.3 Decomposition of Berge graphs

Chudnovsky, Robertson, Seymour and Thomas [13] showed the following.

Theorem 2.19 (Decomposition Theorem) (Chudnovsky et Al.)
Let G be a Berge graph. Then either G is basic, or either G or $\bar{G}$ has a 2-join, a homogeneous pair or a balanced skew-partition.

The decomposition theorem easily implies the Strong Perfect Graph Theorem. Suppose, indeed, that there exists a Berge graph that is not perfect, and let G be one such graph with the minimum number of nodes. Clearly G is minimally imperfect, and so is $\bar{G}$. By Propositions 2.11 and 2.12, G is not basic, hence either G or $\bar{G}$ has a 2-join, contradicting Theorem 2.13, or a homogeneous pair, contradicting Theorem 2.14, or a balanced skew-partition, contradicting Theorem 2.16.

Conforti, Cornuéjols and Vušković [28] had conjectured that, for every Berge graph G, either G or $\bar{G}$ is bipartite or the line graph of a bipartite graph, or G or $\bar{G}$ has a 2-join or a skew partition. Chudnovsky [11] showed that Theorem 2.19 can be strengthened by using only 2-joins and balanced skew partitions (and it is claimed in [14] that the proof of this fact is as hard as the proof of Theorem 2.19 itself). Clearly, if G is a doubled split graph, then $V(G)$ can be partitioned into sets A, B, $|A|, |B| \geq 4$, where every node in A has degree one in $G[A]$ and every node in B has degree one in $\bar{G}[B]$. Hence $G[A]$ is not connected, so B is a multipartite cutset. We summarize this in the next remark.

Remark 2.20 *Let G be a Berge graph. Then either G or $\bar{G}$ is bipartite, or the line graph of a bipartite graph, or G or $\bar{G}$ has a 2-join, a balanced skew partition or a multipartite cutset.*

Since multipartite cutsets are special types of skew partitions, then the conjecture of Conforti et Al. mentioned above holds. Moreover, since the skew partitions used cannot occur in a Berge minimum imperfect graph, it also implies the strong perfect graph theorem.

Classes of graphs

The Decomposition Theorem 2.19 was known to hold for several classes of Berge graphs. The decompositions used for these classes are all special cases of the decompositions we discussed in Section 2.5.2. Here we list some of them:

- *Chordal* (or *triangulated*) *graphs*: are the graphs that do not contain a hole. Berge [2] in 1960 showed that chordal graphs are perfect, while a result of Dirac [46] in 1961 implies that every chordal graph has a *clique-cutset*, which is a special type of star cutset.

- *Meyniel graphs:* are the graphs such that every cycle of odd length has at least 2 chords. Meyniel graphs are obviously Berge, and they were shown to be perfect by Markosyan and Karapetyan [64] and by Meyniel [66] in 1976. A very elegant variant of the Decomposition Theorem was proven by Burlet and Fonlupt [7] for this class.

- *Claw-free graphs:* are the graphs that do not contain $K_{1,3}$. In 1976, Parthasarathy and Ravindra [68] showed the validity of the strong perfect graph conjecture for claw-free graphs, while Maffray and Reed [63] in 1995 showed that every claw-free Berge graph either has a clique-cutset, or is an *augmentation of a flat edge* (which is a special case both of 2-joins and homogeneous pairs) or belongs to the family of *peculiar graphs*, which in turn can be easily decomposed into the basic classes by using star cutsets.

- *Weakly chordal* (or *weakly triangulated*) *graphs:* graphs that do not contain a hole or an antihole of length greater than 4. In 1985, Hayward [55] showed that every weakly triangulated graph that is not basic has a star cutset.

- *Diamond-free graphs:* are the graphs that do not contain a diamond, that is the graph K_4 minus an edge. The strong perfect graph conjecture was proven to hold for such graphs by Tucker [78] in 1987, while the Decomposition Theorem for Berge graphs in this class was shown by Fonlupt and Zemirline [48] in 1987.

- *Bull-free graphs:* are the graphs that do not contain a *bull*, that is the graph on five nodes a, b, c, d, e and edges ab, bc, bd, cd, ce. The Decomposition Theorem for bull-free Berge graphs was proven in 1987 by Chvátal and Sbihi [19], where they also introduced the concept of homogeneous pair.

- *Dart-free graphs:* are the graphs that do not contain a *dart*, that is the graph on five nodes a, b, c, d, e and edges ab, bc, bd, be, ce, de. The strong perfect graph conjecture was proven by Sun [74] for dart-free Berge graphs in 1991, while the Decomposition Theorem was shown in 2000 by Chvátal, Fonlupt, Sun and Zemirline [18].

- *Square-free graphs:* are the graphs that do not contain a *square*, that is a hole of length 4. The Decomposition Theorem for this class was proven in 2001 by Conforti, Cornuéjols and Vušković [28], where they showed that every Berge square-free graph is either basic or it has a 2-join or a star-cutset.

Outline of the proof of the Decomposition Theorem

We will now provide a brief overview of Chudnovsky, Robertson, Seymour and Thomas' proof of Theorem 2.19 [13]. Proofs of parts of it, discovered independently by Conforti, Cornuéjols, Vuškcović and Zambelli [31] and by Conforti, Cornuéjols, and Zambelli [33] will be given in Chapters 3 and 4.

The proof can be roughly divided into two main parts: the first one considers the case in which a Berge graph G contains some induced subgraph L, that is the line graph of some "large" bipartite graph. The idea of the proof is to show that some structural property of L can be extended to a larger induced subgraph L' of G, until either $G = L'$, in which case G is basic, or one can show that there is a limited number of ways in which the nodes of $G \setminus L'$ can attach to (i.e. have neighbors in) L', in which case one can prove that the graph has one of the decompositions in the statement of Theorem 2.19.
The second part considers the case in which G is a Berge graph that does not contain any "large" line graph of a bipartite graph as an induced subgraph. Unlike the first part, in which the aim was to start from some basic induced subgraph L of G and to maximally extend it into some basic induced subgraph L' with certain properties, in this second part the idea is to show that either G is basic, or it contains some non-basic induced subgraph G' such that G' has a decomposition that can be extended to a decomposition of G itself.

LINE GRAPHS

The first part of the proof of Chudnovsky et Al. [13] involves several steps. The first one is summarized in the next theorem.

Theorem 2.21 *Let G be a Berge graph. Let J be a 3-connected graph and H be a bipartite subdivision of J. If G contains $L(H)$ as an induced subgraph, then either G or $\bar{G}$ is the line graph of a bipartite graph, or a doubled split graph, or G or $\bar{G}$ has a 2-join or a balanced skew partition.*

An example of line graph of a bipartite subdivision of a 3-connected graph is depicted in Figure 2.1.

The technique used in proving Theorem 2.21 is reminiscent of a result in the same spirit due to Conforti and Cornuéjols and contained in [21].

A *long prism* is the graph consisting of two disjoint triangles (cycles of length 3) a_1, a_2, a_3 and b_1, b_2, b_3 and three pairwise node-disjoint chordless paths P^1, P^2, P^3, not all of length one, such that, for every $i \in [3]$, P^i is a path between a_i and b_i, and, for every $1 \leq i < j \leq 3$, the only edges of G with one endpoint in P^i and the other in P^j are $a_i a_j$ and $b_i b_j$. We will denote such graph by $3PC(a_1a_2a_3, b_1b_2b_3)$.

It is easy to verify that Berge long prisms are line graphs of bipartite graphs.

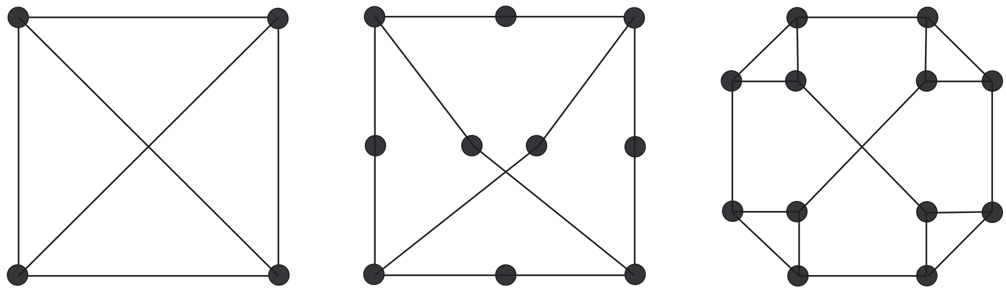

Figure 2.1: A 3-connected graph J, a bipartite subdivision H of J and $L(H)$.

Theorem 2.22 *Let G be a Berge graph such that neither G nor $\bar{G}$ contains the line graph of a bipartite subdivision of a 3-connected graph as an induced subgraph. If G contains a long prism as an induced subgraph, then either G or $\bar{G}$ is the line graph of a bipartite graph, or G or $\bar{G}$ has a 2 join, an homogeneous pair or a balanced skew partition.*

It is known that a subdivision of a 3-connected graph has a subgraph that is a subdivision of K_4. It is easy to verify that, if a line graph of a bipartite subdivision of K_4 does not contain a long prism, then it must contain the line graph of $K_{3,3} \setminus \{e\}$, where $K_{3,3} \setminus \{e\}$ is the graph obtained deleting an edge e from the complete bipartite graph $K_{3,3}$. Hence, in light of Theorems 2.21 and 2.22, one may assume, in the remainder of the proof, that G does not contain a long prism or $L(K_{3,3} \setminus \{e\})$.

A *double diamond* is the graph with node-set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$ such that the a_i's are pairwise adjacent, except a_1, a_2, the b_i's are pairwise adjacent, except b_1, b_2, there is an edge a_ib_i for every $i \in [4]$, and these are the only edges (see Figure 2.2).

Theorem 2.23 *Let G be a Berge graph such that neither G nor $\bar{G}$ contains a long prism or $L(K_{3,3} \setminus \{e\})$. If G contains a double diamond as an induced subgraph, then either G or $\bar{G}$ has a 2 join or a balanced skew partition.*

WHEELS

The second part of the proof deals with the case in which G is a Berge graph such that neither G nor $\bar{G}$ contains an induced subgraph isomorphic to a long

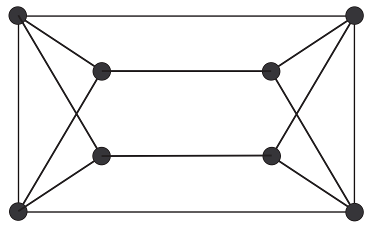

Figure 2.2: A double diamond.

prism, $L(K_{3,3} \setminus \{e\})$ or a double diamond. Chudnovsky et Al. [13] named the graphs with such property *bipartisan graphs*.

A *wheel*, denoted by (H, v), is a graph induced by a hole H and a node $v \notin V(H)$, called *center*, having at least three neighbors in H. A wheel (H, v) is a *twin wheel* if v has exactly three neighbors in H and (H, v) contains exactly two triangles; the neighbor of v in H that is adjacent to all the other neighbors of v in H is said the *twin of* v *in* H. A wheel (H, v) is a *line wheel* if v has exactly four neighbors in H and (H, v) contains exactly two triangles and these two triangles have only the center v in common. A *universal wheel* is a wheel (H, v) where the center v is adjacent to all the nodes of H. A *triangle-free wheel* is a wheel containing no triangle. A *proper wheel* is a wheel that is not any of the above four types.

A *hub*, denoted by (H, S), is the graph induced by a hole H of length at least 6 and by an anticonnected set $S \subseteq V(G) \setminus V(H)$, with the property that there is a positive, even number of edges of H whose endnodes are both universal for S. Given a hub (H, S), we say that (H, S) is *good* if the graph $G[Y]$, induced by the set Y of nodes of H that are universal for S, has a connected component that induces a path of odd length.

Here we should point out that this definition of wheel, first given by Conforti and Cornuéjols in [21], is somewhat different from that of Chudnovsky, Robertson, Seymour and Thomas [13]. In particular, what we call good hub is exactly what Chudnovsky et Al. call "odd wheel".

Chudnovsky, Robertson, Seymour and Thomas [13], and independently Conforti, Cornuéjols, Vušković and Zambelli [31] proved the following.

Theorem 2.24 *Let G be a Berge graph such that neither G nor $\bar{G}$ contains an induced subgraph isomorphic to a long prism or to $L(K_{3,3} \setminus \{e\})$. If G*

contains a good hub, then G has a balanced skew partition.

The proof of Theorem 2.24 will be given in Chapter 3. There are only two cases left to prove: the first one is the case in which G is a bipartisan graph such that neither G nor $\bar{G}$ contains a good hub but G contains a wheel that is not a triangle-free wheel or a twin wheel, and the last case is the case in which G is a bipartisan graph such that G and $\bar{G}$ do not contain any wheel except, possibly, twin wheels and triangle-free wheels. The first of these two cases is disposed of by the next theorem, due to Chudnovsky et Al. [13].

Theorem 2.25 *Let G be a bipartisan graph such that neither G nor $\bar{G}$ contains a good hub as an induced subgraph. If G contains a wheel that is not a triangle-free or a twin wheel, then G has a balanced skew partition.*

Finally, the proof is concluded by the following theorem, proven independently also by Conforti, Cornuéjols and Zambelli [33].

Theorem 2.26 *Let G be a bipartisan graph such that neither G nor $\bar{G}$ contains a wheel that is not a twin wheel or a triangle-free wheel. Then either G or $\bar{G}$ is bipartite, or G has a balanced skew-partition.*

We conclude this section presenting a useful tool in studying the structure of Berge graphs.

2.5.4 The Roussel-Rubio Lemma

The following is a powerful result due to Roussel and Rubio [69] that is used in several places both in [13] and in [31][33]. We remind the reader that, given a set $S \subset V(G)$ and a node $x \notin S$, x is *universal for S* if x is adjacent to every node of S and an edge $e = yz$, $y, z \notin S$, *sees S* if both y and z are universal for S. Given a chordless path (or a hole) P in $G \setminus S$, we denote by $E_S(P)$ the set of edges in P that see S.

Lemma 2.27 (Roussel and Rubio [69]) *Let G be a Berge graph where $V(G)$ can be partitioned into an anticonnected set S and an odd chordless path $P = u, u', \dots, v', v$ of length at least 3 such that both u and v are universal for S. Then one of the following holds:*

(i) An odd number of edges of P see S.

(ii) $|P| = 3$ *and* $S \cup \{u', v'\}$ *contains an odd chordless antipath between* u' *and* v'*.*

(iii) $|P| \geq 5$ *and there exist two nonadjacent nodes* x*,* x' *in* S *such that* $P \setminus \{u, v\} \cup \{x, x'\}$ *is a chordless path between* x *and* x'*.*

Proof: The proof is by induction on $|S| + |P|$. Note that, $\forall x \in S$, there is an odd number of edges of P that see x, otherwise $P \cup x$ contains an odd hole.

Claim 1: *Lemma 2.27 holds if* $|P| = 3$*.*

If $|P| = 3$ and (i) does not hold, then both u' and v' have a nonneighbor in S. Since S is anticonnected, $S \cup \{u', v'\}$ contains a chordless antipath Q between u' and v'. Since u, v, u', Q, v', u is an antihole, Q has odd length.

We may assume, then, that $|P| \geq 5$ and $|S| \geq 2$.

Claim 2: *For any anticonnected subset* $S' \neq \emptyset$ *of* S*, and any odd subpath* $P_{zz'}$ *of* P *such that* z*,* z' *are universal for* S' *and* $|S'| + |P_{zz'}| < |S| + |P|$*, we may assume that* $E_{S'}(P_{zz'})$ *has odd cardinality.*

Otherwise, by induction, either S' contains two nonadjacent nodes x, x' such that $P' = P_{zz'} \setminus \{z, z'\} \cup \{x, x'\}$ is a chordless path , or $P_{zz'} = z, y, y', z'$ for some $y, y' \in V(P)$, and $S' \cup \{y, y'\}$ contains a chordless odd antipath Q between y and y'. In the former case, either $z = u$ and $z' = v$ and we are done, or there exists $w \in \{u, v\}$ with no neighbors in the interior of $P_{zz'}$, so $w, x, P'x', w$ is an odd hole. In the second case, since $|P| \geq 5$, there exists $w \in \{u, v\}$ nonadjacent to both y, y', so w, y, Q, y', w is an odd antihole.

Claim 3: *We may assume that no node in the interior of* P *is universal for* S*.*

Suppose there is a node in the interior of P universal for S. Let $u_1, \ldots, u_{k+1}$ be the nodes of P universal for S in the order they appear in P going from u to v, and, $\forall i \in [k]$, let P_i be the subpath of P between u_i and u_{i+1}. By Claim 2, $\forall i \in [k]$, if P_i is odd then it has length 1. Hence, since P is odd, there is an odd number of edges of P that see S.

Let $Q = s_1, \ldots, s_2$ be a longest chordless antipath contained in S. Let $S_h = S \setminus s_h$, $h = 1, 2$. By the choice of s_1 and s_2, S_1 and S_2 are anticonnected.

Claim 4: *Q has odd length.*

By Claim 2, $E_{S_h}(P)$ has odd cardinality, $h = 1, 2$, and, by Claim 3, no node in the interior of P is universal for both S_1 and S_2. Since $|P| \geq 5$, there exist two nonadjacent nodes z_1 and z_2 in the interior of P such that z_h is universal for S_h, $h = 1, 2$. So $z_1, s_1, Q, s_2, z_2, z_1$ is an antihole and Q is odd.

Let $u_1, \ldots, u_{k+1}$ be the nodes of P universal for S_1 or S_2 in the order they appear in P going from u to v, and let P_i be the subpath of P between u_i and u_{i+1}.

Claim 5: *S is a stable set.*

By Claim 2, $E_{S_h}(P)$ has odd cardinality for $h = 1, 2$. By Claim 3, $E_{S_1}(P) \cap E_{S_2}(P) = \emptyset$, so $E_{S_1}(P) \cup E_{S_2}(P)$ has even cardinality. Thus, since P is odd, there exists $j \in [k]$ such that P_j is odd and $u_j u_{j+1}$ is not an edge that sees S_1 or S_2. If $|P_j| > 1$ and S is not stable, then $|Q| \geq 2$ and, by Claim 2, there is a node z in the interior of P_j that is universal for $V(Q) \setminus \{s_1, s_2\}$. But then, by Claim 4, z, s_1, Q, s_2, z is an odd antihole. Hence we may assume that $|P_j| = 1$. W.l.o.g., s_1 is adjacent to u_j but not u_{j+1} and s_2 is adjacent to u_{j+1} but not u_j. Since $|P| \geq 5$, there exists $w \in \{u, v\}$ not adjacent to any of u_j and u_{j+1}. By Claim 4, $u, u_{j+1}, s_1, Q, s_2, u_j, u$ is an odd antihole, a contradiction.

To conclude, let $z_1, \ldots, z_{m+1}$ be the nodes of P that have a neighbor in S in the order they appear in P going from u to v, and let Z_i, $i \in [m]$, be the subpath of P between z_i and z_{i+1}.

Claim 6: *We may assume that, $\forall i \in [m]$, Z_i has either even length or it is an edge that sees some node in S.*

Suppose Z_i, $i \in [k]$, contradicts the claim. No node $s \in S$ is adjacent to both z_i and z_{i+1}, else Z_i must have odd length at least 3 and s, z_i, Z_i, z_{i+1}, s is an odd hole, hence there exists $s_1, s_2 \in S$ such that s_1 is adjacent to z_i and not z_{i+1} and s_2 is adjacent to z_{i+1} and not z_i. If $z_i = u'$ and $z_{i+1} = v'$ then we are done, else there exists $w \in \{u, v\}$ with no neighbors in Z_i and $w, s_1, z_i, Z_i, z_{i+1}, s_2, w$ is an odd hole, a contradiction.

Let $\delta = |\{i \in [m] \,|\, |Z_i| = 1\}|$. By Claim 6, a simple counting argument implies

$$\delta = \sum_{i=1}^{|S|} (-1)^{i+1} \sum_{A \subseteq S,\, |A| = i} |E_A(P)| \, .$$

Since $|E_A(P)|$ is odd for every proper subset A of S, then the parity of δ equals the parity of $\sum_{i=1}^{|S|-1} \binom{|S|}{i} + |E_S(P)|$ which is equal to the parity of $|E_S(P)|$. By Claim 6 and because P is odd, δ is odd, hence $|E_S(P)|$ is odd and we are done. $\square$

The following is an easy consequence of Lemma 2.27.

Corollary 2.28 *Assume G is a Berge graph containing an anticonnected set S and an odd chordless path $P = u, u', \ldots, v', v$ disjoint from S of length at least 3 such that u, v are both universal for the set S. Furthermore, assume that $G \setminus (S \cup V(P))$ contains a node w universal for S such that no node in the interior of P is adjacent to w. Then an odd number of edges of P see S.*

Proof: Assume not. Then, by Lemma 2.27, either $|P| = 3$ and $S \cup \{u', v'\}$ contains an odd anti-path Q between u' and v', or $|P| \geq 5$ and there exist two nonadjacent nodes x, x' in S such that $x, u', P_{u'v'}, v', x'$ is a chordless path. In the first case, w, u', Q, v', w is an odd anti-hole, and in the other case $w, x, u', P_{u'v'}, v', x', w$ is an odd hole, a contradiction. $\square$

Now we have the necessary tools to prove Lemma 2.17.

Proof of Lemma 2.17:
Assume G has a loose skew partition A, B, C, D. Possibly by going to the complement, we may assume that D contains a node v that is universal for A. Define the *deficiency* of such skew partition as $(|B| + 2|D|) - (|A| + |C|)$. Assume that the deficiency of A, B, C, D is the largest possible. Then A is anticonnected and C is connected, otherwise let A' be an anticonnected component of $G[A]$ and C' be a connected component of $G[C]$, let $B' = A \cup B \setminus A'$ and $D' = C \cup D \setminus C'$, then A', B', C', D' is a skew partition with $v \in D'$ universal for A' and it has larger deficiency, a contradiction. Also, either there is no node $u \in C$ that is universal for A, or $|C| = 1$, otherwise given $C' = C \setminus \{u\}$ and $B' = B \cup \{u\}$, A, B', C', D has larger deficiency. Analogously, either every node in A has a neighbor in C or $|A| = 1$, and either every node in B has a neighbor in C or $|B| = 1$, otherwise we could find another loose skew partition of larger deficiency.

If there exists an odd chordless path $P = x_1, \ldots, x_k$ with endnodes in $A \cup B$ and interior in $C \cup D$, then, clearly, the interior of P is all contained in C or D, while either $x_1, x_k \in A$ or $x_1, x_k \in B$. Assume first that the interior of P is contained in C. If $x_1, x_k \in A$, then $H = v, x_1, P, x_k, v$ is an odd hole, a contradiction. If $x_1, x_k \in B$, then A is an anticonnected set, both endnodes of P are universal for A, v is universal for A and v does not have any neighbor in the interior of P. By corollary 2.28, there must be a node in the interior of P (which is contained in C) that is universal for A, a contradiction. Hence the interior of P is contained in D. Since x_1 and

x_k both have a neighbor in C and C is connected, there exists a path P' between x_1 and x_k with interior in C. By the previous argument, P' must be even, therefore $H = x_1, P, x_k, P', x_1$ is an odd hole, a contradiction.

If there exists an odd chordless antipath $Q = y_1, \ldots, y_k$ with endnodes in $C \cup D$ and interior in $A \cup B$, then the interior of Q is contained either in A or in B, while either $y_1, y_k \in C$ or $y_1, y_k \in D$. Furthermore, $|Q| \geq 5$, since every path of length 3 is also an antipath of length 3. Assume first that $y_1, y_k \in C$. If the interior of Q is contained in A, then $H = v, y_1, Q, y_k, v$ is an odd antihole. Hence the interior of Q is contained in B. Since both y_1 and y_k have a nonneighbor in A and A is anticonnected, there exists a chordless antipath Q' between y_1 and y_k with interior in A, which must be even by the argument above, hence $H = y_1, Q, y_k, Q', y_1$ is an odd antihole. Thus $y_1, y_k \in D$. C is a connected set and y_1, y_k have no neighbors in C, so we can apply Lemma 2.27 in the complement. Since $|Q| \geq 5$, there are two possible outcomes: either there is a node in the interior of Q (which is contained in $A \cup B$) which has no neighbor in C, contradicting our assumptions, or there exist two adjacent nodes $y', y'' \in C$ such that $Q' = y', y_2, Q_{y_2 y_{k-1}}, y_{k-1}, y''$ is an odd chordless antipath, which we already proved is not possible. □

Lemma 2.29 *Let G be a Berge graph. If G has a star-cutset, then G has a balanced skew partition.*

Proof: Trivially, if G has a star cutset then G has a loose skew partition. □

Chapter 3

About Berge Graphs Containing Wheels

3.1 Introduction

Let G be a Berge graph and (H, S) be a hub in G. Observe that (H, S) has a skew partition. Indeed, we can color the nodes of H Red or Blue so that two nodes have distinct colors if and only if the subpaths of H between them contain an odd number of edges with both endpoints universal for S. If we let $A = S$, B be the set of Red nodes of H that are universal for A, then $A \cup B$ is a cutset of (H, S), hence there are two nonempty disjoints sets C and D of nodes of $G \setminus (A \cup B)$ such that there is no edge crossing them, hence A, B, C, D is a skew partition of (H, S). If G is a Berge graph and (H, S) is an induced subgraph of G, we want to investigate when such a skew partition extends to G, that is when there exists a skew partition A', B', C', D' of G with $A \subseteq A'$. Obviously, if the skew partition of (H, S) does not extend, then G contains a minimal obstruction, that is a minimal path in $G \setminus (V(H) \cup A)$ joining the red nodes to the blue nodes but not containing any node universal for A. In Section 3.4 we characterize all such minimal obstructions. This characterization will allow us to identify two cases in which the presence of a hub (H, S) in G with certain characteristics, will imply the existence of a skew partition in the whole graph. The following two theorems are the main results of this Chapter.

Theorem 3.1 *Let G be a Berge graph such that neither G nor $\bar{G}$ contains an induced subgraph isomorphic to a long prism or to $L(K_{3,3} \setminus \{e\})$. If G*

contains a good hub, then G has a balanced skew partition.

The proof of Theorem 3.1 is given in Section 3.6.

Given a hub (H, S) and an edge $ab \in E_S(H)$, an *ear on ab* (with respect to (H, S)) is a chordless path $P = x_1, \dots, x_n$ in $G \setminus (V(H) \cup S)$ such that x_1 is adjacent to a, x_n is adjacent to b, x_i is not adjacent to a if $i \geq 2$, x_i is not adjacent to b if $i \leq n-1$, no node in $V(H) \setminus \{a, b\}$ has a neighbor in P, and no node of P is universal for S. Given edge uv in $E_S(H)$, uv is *isolated* if no other edge in $E_S(H)$ is adjacent to uv. In Section 3.5 we will show the following.

Theorem 3.2 *Let (H, S) be a hub of a Berge graph. If G contains an ear P on an isolated edge uv of $E_S(H)$, then G has a balanced skew partition.*

This last theorem, observed by Seymour [73] for the case in which H has length 6, is not used in the proof of the Decomposition Theorem 2.19, but it is interesting for its own sake. In fact, since no Berge minimum imperfect graph has a balanced skew partition, then no Berge minimum imperfect graph can contain a hub with an ear on an isolated edge. As we saw in the previous chapter, in the literature there are many theorems of the form "if G is a Berge graph that does not contain a graph in $\mathcal{F}$ as an induced subgraph, then G is not minimally imperfect" (where $\mathcal{F}$ is some family of graphs), but, before Theorems 2.21 and 3.2, no family of graphs $\mathcal{F}$ was known for which one could prove that no Berge minimum imperfect graph contains an element of $\mathcal{F}$.

3.2 Finding odd holes and balanced skew partitions

Odd wheels and $3PC(\Delta, .)$: the following two graphs will play an important role in the remainder of this chapter as well as in Chapter 4.

An *odd wheel* is a wheel that contain an odd number of triangles.

A $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths $P^1 = x_1, \dots, y$, $P^2 = x_2, \dots, y$ and $P^3 = x_3, \dots, y$, having no common nodes other than y and such that the only adjacencies between nodes of $P^i \setminus y$ and $P^j \setminus y$, for $i, j \in \{1, 2, 3\}$ distinct, are the edges of the clique of size three induced by $\{x_1, x_2, x_3\}$. Also, at most one of the paths P^1, P^2, P^3 is an edge.

We say that a graph G contains a $3PC(\Delta, .)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$.

The next easy remark will be used several times in the remainder of this chapter and in Chapter 4.

Remark 3.3 *Every odd wheel and every $3PC(\Delta, \cdot)$ contain an odd hole. Therefore, no Berge graph contains one of these graphs as an induced subgraph.*

Finding balanced skew partitions: proving that certain Berge graphs contain a skew partition is not sufficient in order to prove the Strong Perfect Graph Theorem. Given a skew partition in a Berge graph G, one needs to prove that G contains a skew partition that is balanced. In order to do this, Chudnovsky et Al. [13] provided several criteria that one can apply.

Lemma 3.4 *Let G be a Berge graph containing a skew partition A, B, C, D. If $A \cup B$ contains two nonadjacent nodes joined by an even chordless path and by an odd chordless path both with interior in $C \cup D$, then G has a balanced skew partition.*

Proof: By Lemma 2.17, we only need to show that G has loose skew partition. Let u and v be two nonadjacent nodes of $A \cup B$, and P, Q be two paths between u and v with interior in $C \cup D$ and lengths of distinct parity. W.l.o.g., $u, v \in A$ and $V(P) \subseteq A \cup C$. Then the interior of Q is contained in C as well, otherwise $H = u, P, v, Q, u$ is an odd hole. Let D' be a connected component of D. If both u and v have a neighbor in D', then there exists a chordless path R between u and v with interior in D', and either P and R or Q and R have different parities, a contradiction. Hence, w.l.o.g., u has no neighbor in D' and, given $C' = C \cup D \setminus D'$, A, B, C', D' is a loose skew partition. □

Given a skew partition A, B, C, D, a *kernel* for it is an anticonnected set $W \subseteq A \cup B$ such that there exists a connected component of $C \cup D$ that contains no node universal for W.

Lemma 3.5 *Let G be a Berge graph containing a skew partition A, B, C, D and let W be a kernel for it. If there exists a connected component X of $C \cup D$ such that every pair of nonadjacent nodes of W with neighbors in X is joined by an even chordless path with interior in X, and every pair of adjacent nodes of X with nonneighbors in W is joined by an even chordless antipath with interior in W, then G has a balanced skew-partition.*

Proof: Suppose, by contradiction, that G has no balanced skew-partition. Thus by Lemma 2.17 A, B, C, D is not loose, so every node in $C \cup D$ has a nonneighbor in every anticonnected component of $A \cup B$, and every node in $A \cup B$ has a neighbor in every connected component of $C \cup D$. Thus every pair of nonadjacent nodes of W is joined by an even antipath with interior in X. By Lemma 3.4, this implies the following.

Claim 1: *Every chordless path of length greater then one with endnodes in W and interior in $C \cup D$ has even length.*

Claim 2: *Every chordless antipath of length greater then one with endnodes in $C \cup D$ and interior in W has even length.*

Proof of Claim 2: if there is an odd chordless antipath $Q = u, \dots, v$ of length at least 2 with endnodes in $C \cup D$ and interior in W, then $u, v \notin X$, else, by assumption, there is an even chordless antipath between u and v with interior in W, and by Lemma 3.4 G has a balanced skew partition. Thus $u, v \notin X$ have no neighbors in X. Since every node in $A \cup B$ has a neighbor in X, then by Lemma 2.27 applied to Q and X in $\bar{G}$, either there exists an odd chordless antipath with interior in W and endnodes in X, which is not possible by the previous argument, or an odd chordless path with endnodes in W and interior in X, contradicting Claim 1. This proves Claim 2.

Since W is a kernel, then, by Claims 1 and 2, we may choose X to be a connected component of $C \cup D$ such that every node in X has a nonneighbor in W. Thus every pair of adjacent nodes in X is joined by an even chordless antipath with interior in W. Lemma 3.4 implies the following.

Claim 3: *Every chordless antipath of length greater then one with endnodes in X and interior in $A \cup B$ has even length.*

Let U be any anticonnected component of $A \cup B$ not containing W.

Claim 4: *Every pair of nonadjacent nodes of U with neighbors in X is joined by an even chordless path with interior in X.*

Proof of Claim 4: assume $u, v \in U$ are nonadjacent and both have neighbors in X. Since X is connected, there exists a path P between u and v with interior in X. If P is odd, then by Lemma 2.27 applied to P and W, since X has no node universal for W, either there is an odd chordless antipath of length greater then one with endnodes in X and interior in W, or there exists an odd chordless path of length greater then one with endnodes in W and interior in X, contradicting Claim 1 or 2. This proves Claim 4.

Since A, B, C, D is not loose, then U is a kernel. Claims 3 and 4 imply that every pair of nonadjacent nodes of U with neighbors in X is joined by an even chordless path with interior in X and every pair of adjacent nodes of X with nonneighbors in U is joined by an even chordless antipath with interior in U. Let U' be the anticonnected component of $A \cup B$ containing W. By Claims 1 and 2 applied to U instead of W, for every possible choice of U, we conclude that every chordless path between two nonadjacent nodes of $(A \cup B) \setminus U'$ with interior in $C \cup D$ is even, and every chordless antipath between two adjacent nodes of $C \cup D$ with interior in $(A \cup B) \setminus U'$ is even. Since we could have chosen U instead of W at the beginning, then by symmetry we conclude that every chordless path between two nonadjacent nodes of $(A \cup B)$ with interior in $C \cup D$ is even, and every chordless antipath between two adjacent nodes of $C \cup D$ with interior in $(A \cup B)$ is even, thus A, B, C, D is balanced. $\square$

3.3 Hubs

Theorem 3.6 *Let G be a Berge graph consisting of a hole H of length at least 6 and an anticonnected set S of nodes disjoint from $V(H)$. If an odd number of edges of H see S, then S sees exactly one edge uv of H and one of the following holds:*

(1) S contains a node x with exactly two neighbors in H.

(2) S contains nonadjacent nodes x, y such that $V(H) \setminus \{u, v\} \cup \{x, y\}$ induces a chordless path.

Proof: Suppose S sees an od number of edges of H and let $uv \in E_S(H)$. Then u and v are the only nodes of H universal for S. Suppose not, then, since H has even length, H contains an odd chordless subpath P of length at least 3 such that both endnodes of P are universal for S, no node in the interior of P is universal for S and there exists $w \in \{u, v\}$ such that w has no neighbors in the interior of P, but this contradicts Corollary 2.28.

Let $G' = G \setminus uv$. If G' is Berge, then Lemma 2.27 applied to the path $H \setminus uv$ (of length at least 5) and the set S implies that (2) holds. If G' is not Berge, then G' contains an induced subgraph C which is either an odd hole or an odd antihole. In both cases C must contain both u and v. If C is an odd hole, then C contains exactly one node x in S, since every node in

S is adjacent to both u and v, hence $C = H \cup x \setminus uv$, and x satisfies (1). If C is an odd antihole, then C does not contain any other node in H except u, v and at most one of the neighbors of u or v in $H \setminus uv$. But then, since u and v are universal for S, either u or v has only one nonneighbor in C, a contradiction. □

Note that an edge set C of H of even cardinality induces a *Red and Blue bicoloring of the nodes of H*: two nodes of H are colored with distinct colors if and only if the subpaths of H connecting them contain an odd number of edges in C.

Definition 3.7 *A* hub*, denoted by (H,S), is the graph induced by a hole H of length at least* 6 *and by an anticonnected set $S \subseteq V(G) \setminus V(H)$, with the property that there is a positive, even number of edges of H whose endnodes are both universal for S.*
A sector *of a hub (H,S) is a maximal subpath of H containing no edge of $E_S(H)$.*

Remark 3.8 *Let G be a Berge graph and (H,S) a hub of G. Then the endnodes of a sector are endnodes of edges of $E_S(H)$ and every sector of (H,S) has even length.*

Proof: By maximality in the definition of sector, every endnode of a sector must be an endnode of an edge in $E_S(H)$. Assume there exists a sector $P = x_1, \ldots, x_n$ of (H,S) of odd length. Let w be the endnode of some edge in $E_S(H)$ distinct from x_1 and x_n. Since both x_1 and x_n are universal for S and P has length at least 3, then by Corollary 2.28 applied to S, P and w, there is an odd number of edges of P that sees S, a contradiction. □

Corollary 3.9 *Let G be a Berge graph and (H,S) be a hub of G. Let $y \in V(G) \setminus (V(H) \cup S)$ be a node that sees an odd number of edges in a sector of (H,S). Assume $S \cup y$ is anticonnected. Then*

(i) y has exactly two neighbors in H and they are adjacent or

(ii) There exists $x \in S$ not adjacent to y such that (H,x) and (H,y) are twin wheels and exactly one edge of H sees both x and y or

(iii) S contains a node x not adjacent to y such that (H,y) and (H,x) are both line wheels and no edge of H sees both x and y or

(iv) $|H| = 6$, *(H, y) is a line wheel and $S \cup y$ contains an odd chordless anti-path Q of length at least* 3 *between y and a node x such that (H, x) is a line wheel, no edge of H sees both x and y and every intermediate node of Q is adjacent to every node in H.*

Proof: If y has exactly two neighbors in H then conclusion (i) holds. Assume then that y has at least 3 neighbors in H. If $E_{S\cup y}(H)$ has odd cardinality, then, by Theorem 3.6, conclusion (ii) holds. So $E_{S\cup y}(H)$ has even cardinality. Since there is an even number of edges of H that see y, and y sees an odd number of edges in some sector of (H, S), then there are at least 2 sectors $P = x_1, \ldots, x_h$ and $P' = x'_1, \ldots, x'_k$ of (H, S) such that an odd number of edges of P and P', respectively, see y. Let y_1, y_2, (resp. y'_1, y'_2) be the neighbors of y in P (resp. P') closest to x_1 and x_h (resp. x'_1 and x'_k) respectively. Since an odd number of edges of P see y, then $P_{x_1y_1}$ and $P_{y_2x_h}$ have length of distinct parity. We can therefore assume that $P_{x_1y_1}$ has odd length and $P_{y_2x_h}$ has even length. Analogously, assume that $P'_{x'_1y'_1}$ has odd length and $P'_{y'_2x'_k}$ has even length.
If y_1 and y_2 are nonadjacent, then $F = x_1, P_{x_1y_1}, y_1, y, y_2, P_{y_2x_h}, x_h$ is an odd path so, by Corollary 2.28 applied to S, F and x'_1, F has an odd number of edges that see S, contradicting either the definition of sector or the assumption that $S \cup y$ is anticonnected. Hence y_1y_2 is an edge and, analogously, $y'_1y'_2$ is an edge. Let now $F = x_1, P_{x_1y_1}, y_1, y, y'_2, P'_{y'_2x'_k}, x'_k$. If F is a chordless path then F is odd and by Corollary 2.28 applied to S, F and x'_1, F has an odd number of edges that see S, a contradiction. Therefore F is not a chordless path, but then x_1 must be adjacent to x'_k. By symmetry, x_h must be adjacent to x'_1.

Suppose $|H| > 6$. Then, w.l.o.g., $H' = x'_1, P'_{x'_1y'_1}, y'_1, y, y_2, P_{y_2x_h}, x_h, x'_1$ is a hole of length at least 6. Since $E_S(H') = \{x'_1x_h\}$, Theorem 3.6 applies. If conclusion (1) of Theorem 3.6 holds, then there exists a node x in S such that the only neighbors of x in H' are x_h and x'_1. Since x sees an odd number of edges in a sector of (H, y), then, by the previous argument, (H, x) is an L-wheel and (iii) holds. If conclusion (2) of Theorem 3.6 holds, then there exists two nonadjacent nodes x and x' in S such that $F = H' \setminus \{x_h, x'_1\} \cup \{x, x'\}$ is a chordless path. Since F has odd length, x_1, x, F, x', x_1 is an odd hole, a contradiction.

Hence we may assume that $|H| = 6$, therefore $y_2 = x_h$ and $y'_2 = x'_k$. Since y_1 and y'_1 are not universal for S and $S \cup y$ is anticonnected, let Q be a shortest anti-path in $S \cup y$ from y to a node x that is not adjacent to both y_1

and y_1'. Assume, w.l.o.g., that x is not adjacent to y_1, then y, Q, x, y_1, x_1', y is an anti-hole, therefore Q must be an odd anti-path. If x is adjacent to y_1', then $y, Q, x, y_1, y_1', x_1, y$ is an odd anti-hole, a contradiction. Therefore (H, x) is a line wheel. If Q has length 1 then (iii) holds, else (iv) holds. □

3.4 Connections from blue to red sectors of a hub

Let G be a Berge graph and (H, S) be a hub in G. Let P be a connected subgraph of $G \setminus (H \cup S)$. The *attachments* of P to H are the nodes of H adjacent to at least one node of P. In this section we study the minimal connected sets of $G \setminus (H \cup S)$ that contain no nodes universal for S and have attachments of distinct colors.

The following graph will appear in this analysis: *connected diamonds* consist of two node disjoint sets $\{a_1, \ldots, a_4\}$ and $\{b_1, \ldots, b_4\}$ each of which induces a diamond (the graph on four nodes with five edges) such that a_1a_4 and b_1b_4 are not edges, together with four chordless paths $P^1, \ldots, P^4$ such that for $i = 1, \ldots, 4$, P^i is a path between a_i and b_i. Paths $P^1, \ldots, P^4$ are node disjoint and the only adjacencies between them are the edges of the two diamonds.

Let H be a hole and $N \subseteq V(H)$. We say that two nodes of N are *consecutive* if at least one of the two subpaths of H joining them contains no node of N in its interior.

Theorem 3.10 *Let (H, S) be a hub of a Berge graph G. Let $P = x_1, \ldots, x_n$ be a minimal chordless path in $G \setminus (V(H) \cup S)$ containing no node that is universal for S, such that x_1 has a blue neighbor in H and x_n has a red neighbor in H, w.r.t. the bicoloring induced by $E_S(H)$ ($n = 1$ is allowed). If there exist consecutive attachments of P with distinct colors that are not adjacent, then one of the following holds.*

(a) There exists $y \in S$ such that $V(H) \cup V(P) \cup \{y\}$ induces the line graph of a bipartite subdivision of K_4.

(b) $n = 1$, $|H| = 6$, (H, x_1) is a line wheel and $S \cup x_1$ contains a chordless odd anti-path Q of length at least 3 between x_1 and a node $y \in S$ such that (H, y) is a line wheel, no edge of H sees both x_1 and y and every intermediate node of Q is adjacent to every node in H.

(c) *There exists $y \in S$ such that $V(H) \cup V(P) \cup \{y\}$ induces connected diamonds.*

(d) *$n = 1$ and there exists $y \in S$ nonadjacent to x_1 such that (H, x_1) and (H, y) are twin wheels and exactly one edge of H sees both x_1 and y.*

(e) *There exists $y \in S$ such that (H, y) is a twin wheel, no node of P is a neighbor of y, x_1 is adjacent to the twin of y in H and no other node in H while x_n is not adjacent to both red neighbors of y in H.*

(f) *$n = 1$, H contains a subpath u, z, w, z', u' such that $E_S(H) = \{wz, wz'\}$, x_1 is adjacent to u, w and u' but not z and z', $S \cup x_1$ contains a chordless odd anti-path Q of length at least 3 between x_1 and a node $y \in S$ such that y is nonadjacent to u and u' and every intermediate node of Q is adjacent to both u and u'.*

(g) *$n = 1$, H contains a subpath w, z, u, z', w' such that wz and $w'z'$ are edges of $E_S(H)$, x_1 is adjacent to u, w and w' but not z and z', $S \cup x_1$ contains an even anti-path Q between x_1 and a node $y \in S$ such that y is nonadjacent to u and every intermediate node of Q is adjacent to u. Furthermore, every node in $V(H) \setminus \{z, z'\}$ that is universal for S is adjacent to x_1.*

(h) *$n > 1$, H contains a subpath w, z, u, z', w' such that wz and $w'z'$ are edges of $E_S(H)$, x_1 is adjacent w and w' but not u, z and z', while x_n is adjacent to u but not w, z, w' and z'. Furthermore S contains two nonadjacent nodes y and y' such that the only neighbors of y in $V(P) \cup \{w, z, u, z', w'\}$ are u, z, z', w, w' while the only neighbors of y' in $V(P) \cup \{w, z, u, z', w'\}$ are x_1, z, z', w, w'.*

(k) *$n > 1$, $H = v, w, z, u, z', w', v$, $E_S(H) = \{wz, w'z'\}$, x_1 is adjacent only to v in H and x_n is adjacent only to u in H. Furthermore, S contains two nonadjacent nodes y and y' such that y and y' are adjacent to every node in H except v and u, respectively, and no node in P is adjacent to y or y'.*

Proof: Note that, by the minimality assumption on P, no intermediate node of P has a neighbor in H.

Case 1: x_1 or x_n sees an odd number of edges in some sector of (H, S).

Assume, w.l.o.g., that x_1 sees an odd number of edges in some sector of (H, S): then conclusion (i), (ii), (iii) or (iv) of Corollary 3.9 holds. If conclusion (ii) of Corollary 3.9 holds, then (d) holds. If conclusion (iii) of Corollary 3.9 holds, $n = 1$ and there exists y in S nonadjacent to x_1 such that (H, x_1) and (H, y) are line wheels and no edge in H sees both x_1 and y, but then one can verify that $V(H) \cup \{x_1, y\}$ is the line graph of a bipartite subdivision of K_4, so (a) holds. If conclusion (iv) of Corollary 3.9 holds, then (b) holds. Therefore we can assume that conclusion (i) of Corollary 3.9 holds and x_1 has exactly two neighbors u, u' in H, u and u' are adjacent and they are both blue. If x_n has exactly one neighbor t in H, then there is a $3PC(x_1uu', t)$. If x_n has two neighbors in H that are not adjacent, then there is a $3PC(x_1uu', x_n)$. Hence x_n has exactly two neighbors v and v' in H and they are adjacent and both red. Assume that u and v are consecutive attachments of P and u', v' are consecutive attachments of P. W.l.o.g., u and v are nonadjacent. Let H_{uv} and $H_{u'v'}$ be the disjoint paths contained in H between u and v, and between u' and v', respectively. Since u and v are nonadjacent, then $H' = u, H_{uv}, v, x_n, P, x_1, u$ is a hole of length at least 6 and, since u and v have distinct colors and no node in P is universal for S, an odd number of edges of H' see S. Also $H'' = u', H_{u'v'}, v', x_n, P, x_1, u'$ is a hole (possibly of length 4) and an odd number of edges of H'' sees S. By Theorem 3.6, exactly one edge wz of H' and one edge of $w'z'$ of H'' sees S and one of the following cases holds.

Case 1.1: There exists $y \in S$ such that y has only two neighbors in H'.

But then y sees an odd number of edges in $H_{u'v'}$, so y must see exactly one edge in $H_{u'v'}$, otherwise (H'', y) would be an odd wheel. But then (H, y) is a line wheel and one can verify that $V(H) \cup V(P) \cup \{y\}$ induces the line graph of a bipartite subdivision of K_4, hence (a) holds.

Case 1.2: There exist nonadjacent nodes $y, y' \in S$ such that $V(H') \setminus \{w, z\} \cup \{y, y'\}$ induces a chordless path.

Let t and t' be the neighbors of y and y', respectively, in $V(H') \setminus \{w, z\}$. If u' and v' are nonadjacent, then at least one node among w' and z' has no neighbor in P, say w', but then $V(H') \cup \{w', y, y'\} \setminus \{w, z\}$ induces an odd hole, a contradiction. So $u'v'$ is an edge, and $t = u$ and $t' = v$, else (H, y) or (H, y') is an odd wheel. Since H' is even, P must be odd, therefore y, u, x_1, P, x_n, v', y is an odd hole, a contradiction.

Case 2: Both x_1 and x_n see an even number of edges in every sector of (H, S).

Let u and v be two consecutive, nonadjacent attachments of P with distinct colors in the bicoloring of H induced by $E_S(H)$. Assume, w.l.o.g., v is adjacent to x_1 and u to x_n. Let H_{uv} be a subpath of H between u and v containing no attachments of P except u and v. Since u and v have distinct colors, H_{uv} contains an odd number of edges of $E_S(H)$, therefore the hole $H' = x_1, P, x_n, u, H_{uv}, v, x_1$ has an odd number of edges that see S, otherwise P would contain some node universal for S. By Theorem 3.6, H' must contain a unique edge of $E_S(H)$, say edge zw, and no node universal for S except z and w. Assume, w.l.o.g., that z is one endnode of the sector Z containing u, and let z' be the other endnode of Z. Let w' be the neighbor of z' in $V(H) \setminus V(Z)$; hence $z'w' \in E_S(H)$. Since H' is an even hole, H_{uv} has length of the same parity as P. Since u and v are nonadjacent, we may assume, w.l.o.g, that u and z are distinct. Let H_{uz} be the path between u and z in H_{uv} and H_{wv} be the path between w and v in H_{uv}.

Case 2.1: $w = w'$.

Then $w = w' = v$ and $E_S(H) = \{wz, wz'\}$.

Case 2.1.1: There exists a node $y \in S$ whose only neighbors in H' are w and z.

If (H, y) is a twin wheel, then case (e) applies. If (H, y) is not a twin wheel, y has at least a neighbor in $V(H) \setminus \{w, z, z'\}$. If u is the only neighbor of x_n in Z, then G contains a $3PC(zwy, u)$, hence x_n has a neighbor in Z distinct from u. Furthermore, since x_n sees an even number of edges in Z, x_n has a neighbor in Z that is not adjacent to u. If y has a neighbor in Z that is not adjacent to u, then there is a $3PC(zwy, x_n)$, hence y has a unique neighbor t in Z and t is adjacent to u. Furthermore, t is adjacent to x_n, else there is a $3PC(zwy, u)$. Let u' be the neighbor of x_n in Z closest to z', then $u' \neq t$. If u' is not adjacent to t, then there is a $3PC(x_n tu, y)$. So u' is adjacent to t and hence $V(H) \cup V(P) \cup \{y\}$ induces connected diamonds, so conclusion (c) holds.

Case 2.1.2: Every node in S has at least 3 neighbors in H'.

If $|H'| \geq 6$ then, by Theorem 3.6, S contains two nonadjacent nodes y and y' such that (H', y) and (H', y') are twin wheels and wz is the only edge of H' that sees both y and y'. But then $(V(H') \cup \{y, y'\}) \setminus \{w, z\}$ induces an odd path R between y and y' and z', y, R, y', z' is an odd hole unless z' is adjacent to x_n. But then, since x_n sees an even number of edges in Z, H_{zu} must have even length. W.l.o.g. assume that y is not adjacent to x_1, then $(V(H_{uz}) \cup \{y, z', x_n\}) \setminus \{z\}$ induces an odd hole, a contradiction.

Hence $|H'| = 4$, so u is adjacent to z and $n = 1$. Let u' be the neighbor of x_1 in Z closest to z'. Then, since x_1 sees an even number of edges in Z and u is adjacent to z, u' and z' have odd distance in H. By repeating the previous argument on the hole H'' containing w, u' and x_1 in $V(Z) \cup \{x_1, w\}$ instead of H', we argue that u' and z' must be adjacent. Since u and u' are not universal for S, let Q be a shortest possible anti-path in $S \cup x_1$ between x_1 and a node y not adjacent to both u and u'. Assume, w.l.o.g, that y is not adjacent to u. Q must have odd length, or else x_1, Q, y, u, z', x_1 is an odd anti-hole. Moreover, since every node in S has at least 3 neighbors in H', Q has length at least 3. Finally, if u' is adjacent to y, then x_1, Q, y, u, u', z, x_1 is an odd anti-hole, a contradiction. Hence conclusion (f) holds.

Case 2.2: $w \neq w'$.

Note that, since w' is universal for S and distinct from w and z, then w' is not in H_{uv}. Let s be the neighbor of x_n in Z closest to z' and let $H_{sz'}$ be the path between s and z' in Z. Since x_n sees an even number of edges in Z and H_{zu} has length of the same parity as $H_{sz'}$. Let $F = w, H_{wv}, v, x_1, P, s, H_{sz'}, z'$. Since H' is an even hole and H_{zu} has the same length as $H_{sz'}$, F is an odd path between w and z'. If z is not adjacent to s then, by Corollary 2.28 applied to S, F and z, an odd number of edges of F see S, a contradiction. Hence u is the unique neighbor of x_n in Z and it is adjacent to z. Also, given any node t in $V(H) \setminus \{z, z', w\}$ universal for S, if t has no neighbors in the interior of F then, by Corollary 2.28 applied to S, F and t, an odd number of edges of F see S, a contradiction. In particular, w' must be adjacent to x_1 or to v.

If w' is adjacent to v then $F' = w', v, x_1, P, x_n, u, z$ is an odd path, therefore, by a similar argument, z' is adjacent to u and w is also adjacent to v (since x_1 sees an even number of edges in every sector, hence w cannot be adjacent to x_1). Therefore $|H| = 6$ and, since F' must have length at least 5, by Lemma 2.27 there exists two nonadjacent nodes y and y' in S such that y is adjacent to every node in H except v, y' is adjacent to every node in H except u and neither y nor y' has a neighbor in P, hence (k) holds.

If w' is adjacent to x_1 then $F' = w', x_1, P, x_n, u, z$ is an odd path, therefore, by the usual argument, z' is adjacent to u and w is adjacent to x_1. If $|F'| = 3$, then $n = 1$ and, by Lemma 2.27, there exists an odd anti-path x_1, Q, y, u between x_1 and u in $S \cup \{u, x_1\}$, hence case (g) holds. If $|F'| \geq 5$, then by Lemma 2.27 S contains two nonadjacent nodes y and y' such that y is adjacent to x_1, z, z', w, w' an no other node in $V(P) \cup \{w, z, u, z', w'\}$ while

y' is adjacent to u, z, z', w, w' an no other node in $V(P) \cup \{w, z, u, z', w'\}$, hence case (h) holds.

□

Theorem 3.11 *Let (H, S) be a hub of a Berge graph G where S is maximal with the property that (H, S) is a hub. Let $P = x_1, \ldots, x_n$ be a minimal chordless path in $G \setminus (H \cup S)$ containing no node universal for S such that x_1 has a blue neighbor in H and x_n has a red neighbor ($n = 1$ is allowed). If every pair of consecutive attachments of P with distinct colors are adjacent, then one of the following holds.*

(a) P is an ear on some edge of $E_S(H)$.

(b) $n > 1$, there exist two adjacent edges ab, bc of $E_S(H)$ such that b is the only neighbor of x_1 in H and x_n is adjacent to a, c and not to b. Moreover, if $E_S(H) \supsetneq \{ab, bc\}$, then no node of P has a neighbor in $V(H) \setminus \{a, b, c\}$.

(c) $n > 1$, $E_S(H)$ contains at least two nonadjacent edges, x_1 is adjacent to all the blue endnodes of the edges of H that see S (and possibly to other blue nodes of H), x_n is adjacent to all the red endnodes of the edges of H that see S (and possibly to other red nodes of H). If $n > 2$, then there exist nonadjacent $y, z \in S$ such that y is adjacent to x_1 and to no other node of P, and z is adjacent to x_n and to no other node of P. If $n = 2$, then $S \cup \{x_1, x_2\}$ contains an odd anti-path between x_1 and x_2.

Proof: Note that, by the minimality assumption on P, no intermediate node of P has a neighbor in H. Let a and b be two consecutive attachments of P with distinct colors. Then, by assumption, a and b are adjacent and $ab \in E_S(H)$. Assume, w.l.o.g., that a is adjacent to x_n and b is adjacent to x_1. Let c be the neighbor of b in $V(H) \setminus \{a\}$. If P has no neighbor in $V(H) \setminus \{a, b\}$, then P is an ear of ab and (a) occurs. Therefore we may assume, w.l.o.g., that x_n has a neighbor in $V(H) \setminus \{a, b\}$. Note that $n > 1$, otherwise either $S \cup x_1$ sees a positive even number of edges of H, contradicting the maximality of S, or ab is the only edge of H that sees $S \cup x_1$, and by Theorem 3.6 there exists $y \in S$ nonadjacent to x_1 such that (H, x_1) and (H, y) are twin wheels and exactly one edge of H sees both x_1 and y, thus contradicting the assumption that every two consecutive attachments of P with distinct colors are adjacent.

Therefore x_1 has only blue neighbors and x_n has only red neighbors. If x_n sees an odd number of edges in some sector of (H, S) then, by Corollary 3.9, the only neighbors of x_n in H are a and the neighbor d of a in $V(H) \setminus \{b\}$. If x_1 has no neighbor in $V(H) \setminus \{b\}$, then G contains a $3PC(x_n ad, b)$. If x_1 has two nonadjacent neighbors in H, then G contains a $3PC(x_n ad, x_1)$. Therefore x_1 is adjacent to b, c and no other node in H. But then c and d are consecutive, nonadjacent attachments of P with distinct colors in the bicoloring of H induced by $E_S(H)$, a contradiction. Therefore x_n sees an even number of edges in every sector of (H, S) and, by the same argument, also x_1 sees an even number of edges in every sector of (H, S).

We may assume that x_n has at least as many neighbors in H as x_1 does. If $E_S(H) = \{ab, bc\}$ then (b) holds. Next we show that if x_n has no neighbor in $H \setminus \{a, c\}$, then (b) holds. Suppose that x_n has no neighbor in $H \setminus \{a, c\}$. Then x_n is adjacent to c. If x_1 has no neighbors in $H \setminus b$ then (b) holds. Otherwise , x_1 has exactly two neighbors in H, b and say d. Since all pairs of consecutive attachments of P having distinct colors are adjacent, then a, d and c, d are adjacent, hence $|H| = 4$, contradicting the assumption that (H, S) is a hub. Now we may assume that (b) does not hold, hence there exists a red sector $Z = z_1, \ldots, z_k$ of (H, S) such that $\{a, c\} \neq \{z_1, z_k\}$ and such that x_n has a neighbor in $V(Z) \setminus \{a, c\}$. Assume, w.l.o.g, that $z_1 \notin \{a, c\}$ and x_n has a neighbor in $V(Z) \setminus \{z_k\}$. Let z_i be the neighbor of x_n of lowest index in Z, and let $H_{z_1 z_i}$ be the subpath between z_1 and z_i in Z. Note that $i < k$. Since x_n sees an even number of edges in every sector of (H, S) and x_n has only red neighbors in H, then $H_{z_1 z_i}$ has even length (since x_n is adjacent to a) and also z_k and z_i have even distance in Z, hence they are not adjacent. Moreover, $H' = a, b, x_1, P, x_n, a$ is an even hole, therefore P is an odd path. But then $F = b, x_1, P, x_n, z_i, H_{z_1 z_i}, z_1$ is an odd chordless path. If there exists a node w universal for S in $V(H) \setminus \{a, b, z_1\}$ that has no neighbor in the interior of F, then Corollary 2.28 applied to S, F and w implies that there exists an odd number of edges in F that see S, a contradiction. Therefore every node universal for S in $V(H) \setminus \{a, b, z_1\}$ is adjacent either to x_1 or to x_n. Let t be the unique blue neighbor of z_1 in H. Note that t is adjacent to x_1. Since t and z_i are consecutive attachments of P, they must be adjacent. So x_n is adjacent to z_1. Hence every node of H that is universal for S must be adjacent to x_1 or x_n. In particular, x_1 is adjacent to all the blue endnodes of the edges of H that see S, x_n is adjacent to all the red endnodes of the edges of H that see S. If $n > 2$, then F has length at least 5 and by Lemma 2.27 there exist nonadjacent $y, z \in S$ such that y is adjacent to x_1 and to no other node of P, and z is adjacent to x_n and to no other node of P. If $n = 1$, then $|F| = 3$ and, by Lemma 2.27, $S \cup \{x_1, x_2\}$ contains an odd anti-path between x_1 and x_2. So conclusion (c) holds. □

In the bicoloring of H induced by $E_S(H)$, we say that a node u of H is an *inner* blue (resp. red) node if both neighbors of u in H are blue (resp. red).

Theorem 3.12 *Let (H, S) be the hub of a Berge graph G. Assume that S is a maximal set such that (H, S) is a hub with the further property that S does not contain any center of a twin wheel w.r.t. H. Let $P = x_1, \dots, x_n$ be a minimal chordless path in $G \setminus (V(H) \cup S)$ containing no node universal for S such that x_1 has a red neighbor, no other node of P has a red neighbor and x_n has a blue neighbor b in H so that neither of the neighbors of b in H is a red neighbor of x_1. Then one of the following holds:*

(a) P has two consecutive attachments of different colors that are nonadjacent, and P is of one of the types in Theorem 3.10 (a)-(c) or (f)-(k).

(b) There exist two adjacent edges ab_1, ab_2 of $E_S(H)$ such that a is the only red neighbor of x_1 in H and at least one node of P is adjacent to both b_1 and b_2. If $E_S(H) \supsetneq \{ab_1, ab_2\}$ or if S contains a node s with no neighbors in P, then the path $Q = a, x_1, \dots, x_n$ contains an odd number of edges that see both b_1 and b_2.

(c) $n > 1$, $E_S(H)$ contains at least two nonadjacent edges, x_1 is adjacent to all the red endnodes of the edges of H that see S and the node x_j of lowest index adjacent to some blue node is adjacent to all the blue endnodes of the edges of H that see S. If $j > 2$, then S contains two nonadjacent nodes y and z such that y is adjacent to x_1 and to no other node of $P_{x_1x_j}$, and z is adjacent to x_j and to no other node of $P_{x_1x_j}$. If $j = 2$, then $S \cup \{x_1, x_2\}$ contains an odd chordless anti-path between x_1 and x_2.

Note that every path $P = x_1, \dots, x_n$ such that x_1 has a red neighbor and x_n has an inner blue neighbor contains a subpath as in the hypothesis of Theorem 3.12.

Proof: Let x_j be the node of P of lowest index having a blue neighbor. If the path $P_{x_1x_j}$ has consecutive attachments of distinct colors that are not adjacent, then $P_{x_1x_j}$ satisfies the hypothesis of Theorem 3.10, hence one the cases (a)-(c) or (f)-(k) of Theorem 3.10 apply (cases (d) and (e) cannot occur since S does not contain any center of a twin wheel w.r.t. H). Since in any of these cases x_j has a blue neighbor that is not adjacent to any red neighbor of x_1 in H, then $j = n$ and case (a) holds .

Hence we may assume that every pair of consecutive attachments with distinct colors of $P_{x_1x_j}$ are adjacent, so case (a)-(c) of Theorem 3.11 occur. If case (c) occurs, then case (c) of Theorem 3.12 holds and we are done. Hence we may assume that case (a) or (b) of Theorem 3.11 holds. In particular, x_1 has a unique red neighbor, say a and, given b_1 and b_2 the two neighbors of a in H, ab_1 sees S and x_j is adjacent to b_1. Since x_n has a blue neighbor in H neither of whose neighbors in H is a red neighbor of x_1, $n > 1$.

Claim 1 *ab_2 sees S and b_2 has a neighbor in P.*

Let t be the attachment of P in $V(H) \setminus \{a, b_1\}$ that is closest to a in the path induced by $V(H) \setminus \{b_1\}$. Since a is the unique red attachment of P, then t is blue. If $t = b_2$ then ab_2 sees S and we are done. Assume then that $t \neq b_2$, hence no neighbor of t in H is a red neighbor of x_1 so t is adjacent to x_n and no other node in P. Let H_{b_2t} be the path between b_2 and t in the graph induced by $V(H) \setminus \{b_1\}$, and let $H' = a, x_1, P, x_n, t, H_{b_2t}, b_2, a$. Then H' is an hole of length at least 6 and, since a and t have distinct colors in the bicoloring of H induced by $E_S(H)$ and no node in P is universal for S, an odd number of edges of H' sees S, therefore, by Theorem 3.6, exactly one edge of H' sees S and no node of H' is universal for S except the endnodes of such edge. Since a is universal for S, then the unique edge in H' that sees S must be ab_2. Also, by Theorem 3.6, we have two possibilities.

Case 1: There exists a node $y \in S$ such that the only neighbors of y in H' are a and b_2.

Then t is not adjacent to b_1, otherwise (H, y) would be a twin wheel. Let $Z = z_1, \ldots, z_k$ be the path induced by $V(H) \setminus (V(H_{b_2t}) \cup \{a, b_1\})$, where z_1 is adjacent to t and z_k is adjacent to b_1. Since (H, y) is not a twin wheel, then y has a neighbor in Z. If x_n does not have a neighbor in Z, then there is a $3PC(yab_2, t)$. If both y and x_n have a neighbor in Z distinct from z_1, then there is a $3PC(yab_2, x_n)$. Note that b_1 has a neighbor in $V(P) \setminus \{x_1\}$, otherwise $y, b_1, x_1, P, x_n, t, H_{b_2t}, b_2, y$ is an odd hole.

If x_n has no neighbor in Z except z_1, then t and z_1 are the only neighbors of x_n in H, otherwise (H, x_n) is an odd wheel. Since b_1 has a neighbor in $V(P) \setminus \{x_1\}$, then there is a $3PC(x_n t z_1, b_1)$.

Hence x_n has a neighbor in $V(Z) \setminus \{z_1\}$, therefore the only neighbor of y in Z is z_1. Also x_n is adjacent to z_1 otherwise there is a $3PC(yab_2, t)$. Consider now the hole $H'' = z_1, y, a, x_1, P, x_n, z_1$. Since b_1 sees at least one edge in H'' and b_1 has at least one neighbor in $V(P) \setminus \{x_1\}$, then either (H', b_1) or (H'', b_1) is an odd wheel since b_1 sees in H'' exactly one edge more than in H'.

Case 2: *S contains two nonadjacent nodes y and z such that the only neighbors of y in H' are a, b_2 and x_1 and the only neighbors of z in H' are a, b_2 and the node $c \neq a$ adjacent to b_2 in H' .*

Then t is not adjacent to b_1, otherwise (H, y) would be a twin wheel. Let $Z = z_1, \ldots, z_k$ be the path induced by $V(H) \setminus (V(H_{b_2 t}) \cup \{a, b_1\})$, where z_1 is adjacent to t and z_k is adjacent to b_1. Since (H, y) is not a twin wheel, then y has a neighbor in Z. Also, since (H, z) is not an odd wheel, also z has a neighbor in Z. Let p and q be two neighbors in Z of y and z respectively with minimum distance in Z. Let Z_{pq} be the path between p and q in Z. Z_{pq} is an even path, otherwise a, y, p, Z_{pq}, q, z, a would be an odd hole. If b_1 has a neighbor in $P \setminus x_1$, then $(P \setminus x_1) \cup H_{b_2 t} \cup \{y, z, b_1\}$ contains a $3PC(b_2 zc, b_1)$. So x_1 is the unique neighbor of b_1 in P. If x_n has no neighbors in Z, then $H \cup P$ induces a $3PC(x_1 ab_1, t)$. If z_1 is not the unique neighbor of x_n in Z, then $H \cup P$ contains a $3PC(x_1 ab_1, x_n)$. So z_1 is the unique neighbor of x_n in Z. If Z_{pq} contains z_1, then $V(Z_{pq}) \cup V(P) \cup \{y, z, a\}$ induces a $3PC(x_1 ay, z_1)$. Otherwise, $V(P) \cup (V(H_{b_2 t}) \setminus b_2) \cup V(Z_{pq}) \cup \{y, z\}$ induces an odd hole.

Claim 2 *There exists a node in P that is adjacent to both b_1 and b_2.*

Assume not. Let x_k be the node of P of lowest index that is adjacent to b_2. Since we assumed that the node x_j of lowest index in P adjacent to some blue node is adjacent to b_1, then $k > j$.

Case 1: *x_1 is the unique neighbor of b_1 in $P_{x_1 x_k}$.*

Then x_k must be adjacent to the neighbor c of b_2 in $V(H) \setminus \{a\}$ and to no other node in $V(H) \setminus \{b_2, c\}$, or else there is either a $3PC(ab_1 x_1, b_2)$ or a $3PC(ab_1 x_1, x_k)$. Let $F = b_1, x_1, P_{x_1 x_k}, x_k, b_2$. F is an odd path and b_1 and b_2 are universal for S. Since P does not contain any node universal for S, then conclusion (ii) or (iii) of Lemma 2.27 holds.

If conclusion (ii) holds, then F has length 3 and $S \cup \{x_1, x_2\}$ contains an odd anti-path Q between x_1 and x_2. Since no node of $V(H) \setminus \{a, b_1, b_2, c\}$ is adjacent to x_1 or x_2 and a is universal for all intermediate nodes of Q, then we can apply Corollary 2.28 in $\bar{G}$ to the set $V(H) \setminus \{a, b_1, b_2, c\}$, the path Q and the node a. Therefore there must exists an intermediate node y of Q with no neighbors in $V(H) \setminus \{a, b_1, b_2, c\}$. But then the only neighbors of y in H are a, b_1 and b_2 and (H, y) is a twin wheel, a contradiction.

If conclusion (iii) holds, then S contains two nonadjacent nodes y and z such that y is adjacent to x_1 and no other node of $P_{x_1x_k}$ while z is adjacent to x_k and no other node of $P_{x_1x_k}$. Since S does not contain any center of twin wheels w.r.t. H, then y and z must have neighbors in $V(H) \setminus \{a, b_1, b_2, c\}$. Let p and q be two neighbors of y and z, respectively, that are closest possible in $V(H) \setminus \{a, b_1, b_2, c\}$ and let Z be the path between p and q in the graph induced by $V(H) \setminus \{a, b_1, b_2, c\}$. Z must have even length otherwise a, y, p, Z, q, z, a is an odd hole, but then $y, x_1, P_{x_1x_k}, x_k, z, q, Z, p, y$ is an odd hole, a contradiction.

Case 2: b_1 has a neighbor in $P_{x_2x_k}$.

Then $k > 2$ and $H' = a, x_1, P_{x_1x_k}, x_k, b_2, a$ is a hole of length at least 6. The only edge of H' that sees S is ab_2 hence conclusion (2) or (3) of Theorem 3.6 holds.

If conclusion (2) holds, then S contains two nonadjacent nodes y and z such that y is adjacent to x_1 and no other node of $P_{x_1x_k}$ while z is adjacent to x_k and no other node of $P_{x_1x_k}$, but then there exists a $3PC(zb_2x_k, b_1)$.

If conclusion (3) holds, then S contains a node y whose only neighbors in H' are a and b_2. Let P' be the shortest path between x_1 and y in the graph induced by $(V(P) \cup V(H) \cup \{y\}) \setminus \{a, b_1, b_2\}$. Then $H'' = a, x_1, P', y, a$ is a hole. Both b_1 and b_2 see the edge ay of H'', both b_1 and b_2 have a neighbor in $P_{x_1x_j}$ and y is not adjacent to x_k, therefore by Theorem 3.6 b_1 and b_2 see an even number of edges in H'', but then there exists a node of P that is adjacent to both b_1 and b_2.

Claim 3 *If $E_S(H) \supsetneqq \{ab_1, ab_2\}$ then the path $Q = a, x_1, \ldots, x_n$ contains an odd number of edges that see both b_1 and b_2.*

Assume that $E_S(H) \supsetneqq \{ab_1, ab_2\}$. Suppose it is not the case that an odd number of edges of Q see both b_1 and b_2. Let x_ℓ be the node of highest index that is adjacent to both b_1 and b_2. Then $\ell > 1$. Suppose ℓ is odd. Then $F = a, x_1, P_{x_1x_\ell}, x_\ell$ is an odd path and hence by Lemma 2.27 applied to F and set $\{b_1, b_2\}$, b_1 is adjacent to x_1, x_ℓ and no other node in $P_{x_1x_\ell}$ while b_2 is adjacent to $x_{\ell-1}$, x_ℓ and no other node in $P_{x_1x_\ell}$. But then $(V(H) \cup V(P_{x_1x_{\ell-1}})) \setminus \{a\}$ induces an odd hole, a contradiction. Therefore ℓ is even. Let x_h and x_k

be the nodes of highest index adjacent to, respectively, b_1 and b_2. W.l.o.g., $h \leq k$. We want to show that $P_{x_\ell x_h}$ has even length. Assume not, then $\ell < h$, therefore, by definition of ℓ, h and k, $h < k$. Since $P_{x_\ell x_h}$ has odd length, then b_1 must see an odd number of edges of $P_{x_\ell x_h}$. Let $\ell = k_1 \leq \ldots \leq k_m = k$ be all the indexes between ℓ and k such that b_2 is adjacent to x_{k_i}. Then there exists i, $1 \leq i \leq m-1$ such that b_1 sees an odd number of edges in $P_{x_{k_i} x_{k_{i+1}}}$. But then $P_{x_{k_i} x_{k_{i+1}}}$ has length at least 2 and $C = b_2, x_{k_i}, P_{x_{k_i} x_{k_{i+1}}}, x_{k_{i+1}}, b_2$ is an hole, therefore b_1 sees exactly one edge uv in C, and $V(C) \cup \{a, b_1\}$ induces a $3PC(b_1uv, b_2)$, a contradiction. Hence we have proven that $a, x_1, P_{x_1 x_h}, x_h$ has even length.

Case 1: x_n sees an odd number of edges in some sector of (H, S).

Since x_n has only blue neighbors in H, by Corollary 3.9, x_n has exactly two neighbors u and v in H and they are adjacent. Suppose x_n is not adjacent to b_2. If $h < k$ then there is a $3PC(x_nuv, b_2)$. If $h = k$ then there is a $3PC(x_nuv, x_k)$. So x_n is adjacent to b_2. $P_{x_h x_n}$ has odd length, else $(V(H) \cup V(P_{x_h x_n})) \setminus \{a, b_2\}$ induces an odd hole. Let c be the neighbor of b_2 in $H \setminus a$. Then c is adjacent to x_n. Let z be the endnode distinct from b_2 of the sector Z of (H, S) containing c, and let F be the path between c and z in Z. Since $E_S(H) \supsetneq \{ab_1, ab_2\}$, then $z \neq b_1$. Moreover F has odd length, therefore $R = a, x_1, P, x_n, c, F, z$ has odd length. Let w be the neighbor of z in $V(H) \setminus V(Z)$, then $zw \in E_S(H)$ and, by Corollary 2.28 applied to S, R and w, there is an odd number of edges of R that sees S, a contradiction.

Case 2: x_n sees an even number of edges in every sector of (H, S).

Let u be the neighbor of x_n closest to b_1 in the graph induced by $V(H) \setminus \{a, b_2\}$ and H_{ub_1} be the path between u and b_1 in the graph induced by $V(H) \setminus \{a, b_2\}$. We want to show that $P_{x_h x_n}$ has length of the same parity as the length of H_{ub_1}. If not then $u \neq b_1$ and $x_h \neq x_n$, but then $b_1, x_h, P_{x_h x_n}, x_n, u, H_{ub_1}, b_1$ is an odd hole. Let z be the endnode distinct from b_1 and b_2 of the sector Z of (H, S) containing u (the existence of such a node is guaranteed by the hypothesis $E_S(H) \supsetneq \{ab_1, ab_2\}$). Let u' be the neighbor of x_n closest to z in Z and let F be the path between u' and z in Z. Since x_n sees an even number of edges in Z, then H_{ub_1} and F have lengths of the same parity, therefore $R = a, x_1, P, x_n, u', F, z$ has odd length. Let w be the neighbor of z in $V(H) \setminus V(Z)$, then $zw \in E_S(H)$ and, by Corollary 2.28 applied to S, R and w, there is an odd number of edges of R that sees S, a contradiction.

Claim 4 *If S contains a node s with no neighbors in P, then the path $Q =$*

$a, x_1, \dots, x_n$ contains an odd number of edges that see both b_1 and b_2.

Let F be the shortest path between x_1 and s in the graph induced by $(V(H) \cup V(P) \cup \{s\}) \setminus \{a, b_1, b_2\}$. Then $H' = s, a, x_1, F, s$ is a hole. Since as sees both b_1 and b_2 and there exists a further node in P that is adjacent to both b_1 and b_2 then, by Theorem 3.6, H' contains an even number of edges that see both b_1 and b_2, but then $Q = a, x_1, P, x_n$ has an odd number of edges that see both b_1 and b_2. □

3.5 Ears on isolated edges of a hub

The objective of this section is to prove Theorem 3.2.

Lemma 3.13 *Let (H, S) be a hub of a Berge graph G such that H contains an edge uv in $E_S(H)$ that is isolated. Assume that S is maximal with such a property. Let $P = x_1, ..., x_n$ be an ear on uv. Let $R = r_1, \dots, r_\ell$ be a path in $G \setminus (V(H) \cup V(P) \cup S)$ that does not contain any node universal for S, such that r_1 has a neighbor in P, no node in $R \setminus r_1$ has a neighbor in P and r_ℓ has a neighbor in H distinct from u. Let s be the neighbor of r_ℓ closest to v in $H \setminus u$, and assume that no node in $R \setminus r_\ell$ has a neighbor in $H \setminus u$ closer to v than s, except, possibly, v itself. Then s and v have the same color w.r.t. the bicoloring induced on H by $E_S(H)$.*

Proof: By contradiction, assume s and v have distinct colors, then $s \neq v$. Let w and w' be the endnodes of the sector Z of (H, S) containing s and assume w is closer to v in $V(H) \setminus \{u\}$ than w'. Clearly $w \neq v$, and w is nonadjacent to v since uv is isolated. Let F be the shortest path between w and v in $Z \cup R \cup P \cup v$ and F' be the path between v and w in $H \setminus u$. Since $H' = v, F', w, F, v$ is a hole, then F and F' have length of the same parity. Since w and v have distinct colors in the bicoloring of H induced by $E_S(H)$, then F' has odd length, therefore F is an odd chordless path. Since $|F'| \geq 3$ then the interior of F' contains a node t that is universal for S. Corollary 2.28 applied to S, F and t implies that F contains an odd number of edges that see S, a contradiction. □

Lemma 3.14 *Let (H, S) be a hub of a Berge graph G such that H contains an edge uv in $E_S(H)$ that is isolated. Assume that S is maximal with such a property. Let $P = x_1, ..., x_n$ be an ear on uv. Let $R = r_1, \dots, r_\ell$ be a minimal* *path in $G \setminus (V(H) \cup V(P) \cup S)$ such that r_1 has a neighbor in P and r_ℓ has a neighbor in $H \setminus \{u, v\}$. Then, either R has a node universal for S, or there exist $st \in E_S(H)$ and $x \in V(P \cup R)$, such that s and t are the only neighbors of r_ℓ in H, v and s are in the same connected component of $H \setminus \{u, r\}$ and they have the same color, x is adjacent to u and v, and $H \cup P \cup R$ contains a $3PC(uvx, tsr_\ell)$.*

Proof: Let $R = r_1, \ldots, r_\ell$ be a minimal path in $G \setminus (V(H) \cup V(P) \cup S)$ such that r_1 has a neighbor in P, r_ℓ has a neighbor in $H \setminus \{u, v\}$ and no node of R is universal for S. Note that we only need to prove the statement in the case in which R does not contain any node whose only neighbors in H are u and v. In fact, if R contains such a node and r_i is the node of highest index whose only neighbors are u and v, then $P' = r_i$ is an ear on uv and $R' = r_{i+1}, R_{r_{i+1}r_\ell}, r_\ell$ is a path such that r_{i+1} has a neighbor in P', r_ℓ has a neighbor in $H \setminus \{u, v\}$, and no node of R' is adjacent to u, v and no other node of H.

Claim 1: *No node in R is adjacent to both u and v.*

Assume there exists i, $1 \leq i \leq m$, such that r_i is adjacent to u and v. Since r_i is not universal for S, then $S \cup r_i$ is anticonnected. By the maximality of S, $(H, S \cup y_i)$ is not a hub, hence uv is the only edge of H that sees $S \cup y_i$. Since uv is isolated, S does not contain any center of a twin wheel w.r.t. H, hence, by Theorem 3.6, r_i is adjacent only to u and v in H, a contradiction.

Let s be the neighbor of r_ℓ closest to v in $H \setminus u$ and t be the neighbor of r_ℓ closest to u in $H \setminus v$. To conclude, we shall prove that st is an edge of H that sees S and $st \neq uv$. Furthermore, $P = x_1$ and no node in $Q_{y_1 y_{j-1}}$ has a neighbor in H. By Lemma 3.13, s has the same color of v and t has the same color of u in the bicoloring induced on H by $E_S(H)$. By Claim 1, either $s \neq v$ or $t \neq u$. Assume, w.l.o.g., that $u \neq t$. Assume s and t are nonadjacent. Then s and t are consecutive neighbors of r_ℓ with distinct colors in H that are nonadjacent, therefore we can apply Theorem 3.10 to the path consisting of r_ℓ. Since $E_S(H)$ contains an isolated edge, then conclusion (a), (b) or (g) of Theorem 3.10 holds.

Case 1: Case (a) or (b) of Theorem 3.10 holds.

Then $E_S(H)$ consists of two nonadjacent edges uv and $u'v'$ while (H, r_ℓ) is a line wheel. Assume v and v' have the same color. By symmetry, we may assume that $u \neq t$ and v' is not adjacent to r_ℓ. Let F be the shortest path between u and r_ℓ in $P \cup R \cup u$ and let F' be the path between u and t in $H \setminus v$.

Since $u \neq t$, $H' = u, F', t, r_\ell, F, u$ is a hole, hence F' has distinct parity from F. But then, since r_ℓ sees an odd number of edges in the sector of (H, S) with endnodes u and u', the shortest path F'' from u to u' in $H \cup P \cup F \setminus \{v, v', t\}$ has odd length. By Corollary 2.28 applied to S, F'' and v', an odd number of edges of F'' see S, a contradiction.

Case 2: Case (g) of Theorem 3.10 holds.

Then $s = v$, u and t are adjacent and H contains a path v, u, t, u', v' where $u'v'$ sees S and r_ℓ is adjacent to v, t, v' but not to u or u'. Let F be the shortest path between u and r_ℓ in $P \cup R \cup u$. Since $H' = u, t, r_\ell, F, u$ is a hole, F has even parity, but then u, F, r_ℓ, v' is an odd chordless path and Corollary 2.28 applied to S, u, F, r_ℓ, v' and u', implies that an odd number of edges of F see S, a contradiction.

Therefore s and t must be adjacent and, since they have distinct colors, st sees S. To conclude, let $F = v_1, ..., v_k$ be a shortest path in $R \cup P$ such that $v_k = r_\ell$ and v_1 is adjacent to u or v . If v_1 is not adjacent to both u and v, say v_1 is not adjacent to v, then $H \cup F$ is a $3PC(sty_j, u)$, a contradiction. Therefore $P = x_1$, $v_1 = x_1$ and no node in $R \setminus r_\ell$ has a neighbor in H.

□

Lemma 3.15 *Let (H, S) be a hub of a Berge graph G such that H contains an edge uv in $E_S(H)$ that is isolated. Assume that S is maximal with such a property. Let $P = x_1, ..., x_n$ be an ear on uv. Let $Q = y_1, ..., y_m$ be a minimal path in $G \setminus (V(H) \cup V(P) \cup S)$ such that y_1 has a neighbor in P and y_m has a neighbor in the interior of some sector of (H, S). Then Q contains a node that is universal for S.*

Proof: By contradiction, assume $Q = y_1, ..., y_m$ is a minimal path in $G \setminus (V(H) \cup V(P) \cup S)$ such that y_1 has a neighbor in P, y_m has a neighbor in the interior of some sector of (H, S) and no node in Q is universal for S. We may assume that Q does not contain any node whose only neighbors in H are u and v, otherwise, if y_i is the node of highest index whose only neighbors are u and v, then $P' = y_i$ is an ear on uv and $Q' = y_{i+1}, Q_{y_{i+1}y_m}, y_m$ is a path such that y_{i+1} has a neighbor in P' and y_m has a neighbor in the interior of some sector of (H, S) but no node of Q' is adjacent to u, v and no other node of H. Also, by the same argument as in Claim 1 of the proof of Lemma 3.14, no node in Q is adjacent to both u and v.

Let y_j be the node of Q of lowest index such that y_j has a neighbor in H distinct from u and v. Let s be the neighbor of y_j closest to v in $H \setminus u$ and

t be the neighbor of y_j closest to u in $H \setminus v$. By Lemma 3.14, st is an edge of H that sees S and $st \neq uv$. Furthermore, $P = x_1$, no node in $Q_{y_1 y_{j-1}}$ has a neighbor in H and s and u have the same color. Let H_{ut} be the path in $V(H) \setminus \{v\}$ between u and t and H_{vs} be the path in $V(H) \setminus \{u\}$ between v and s. Note that H_{ut} and H_{vs} have both even length. Let y_k be the node of lowest index in Q such that $k > j$ and y_k has a neighbor in $V(H) \setminus \{s, t\}$.

Claim 1: *y_k has a neighbor both in $V(H_{ut}) \setminus \{t\}$ and in $V(H_{vs}) \setminus \{s\}$.*

Assume, w.l.o.g, that y_k has a neighbor in H_{ut} distinct from t and let p be the neighbor of y_k closest to u in H_{ut} (possibly $u = p$). By Lemma 3.13, p and u must have the same color. Let F be the shortest path between p and s in $V(Q_{y_j y_k}) \cup \{p, s\}$ and let F' be the path between u and p in H_{ut}. If y_k has no neighbors in $V(H_{vs}) \setminus \{s\}$, then $H' = u, F', p, F, s, H_{vs}, v, u$ is a hole, then $R = u, F', p, F, s$ is an odd path so, by Corollary 2.28 applied to S, R and v, R contains an odd number of edges that see S. Since u and p have the same color, then S sees an even number of edges of F', therefore S must see an odd number of edges of F, a contradiction.

Let p be the neighbor of y_k closest to u in H_{ut} and let q be the neighbor of y_k closest to v in H_{vs}. By Claim 4 and since y_k is not adjacent to both u and v, p and q are nonadjacent and, by Lemma 3.13, p has the same color of u and q has the same color of v. We can also assume, w.l.o.g., that $u \neq p$.

Then p and q are consecutive neighbors of y_k with distinct colors in H that are nonadjacent, therefore we can apply Theorem 3.10 to the path consisting of y_k. Since $E_S(H)$ contains an isolated edge, then conclusion (a), (b) or (g) of Theorem 3.10 holds.

Case 1: Case (a) or (b) of Theorem 3.10 holds.

Then $E_S(H)$ consists only of uv and st. Note that st is an isolated edge of $E_S(H)$, $P' = y_j$ is an ear of st and S is maximal with this property. Moreover $Q' = Q_{y_{j+1} y_k}$ is a path in $G \setminus (V(H) \cup V(P') \cup S)$ such that y_{i+1} has a neighbor in P' and y_k has a neighbor in the interior of a sector of (H, S). But now P' and Q' contradict Lemma 3.14.

Case 2: Case (g) of Theorem 3.10 holds.

Then $q = v$, u and p are adjacent and H contains a path v, u, p, u', v' where $u'v'$ sees S and y_j is adjacent to v, p, v' but not to u or u'.

We have two cases:

Case 2.1: $u'v' \neq st$.

Then $u'v'$ is not adjacent to st, since v' is in H_{ut} and v' and t have distinct colors. Let F be the shortest path between u and y_k in $V(P) \cup V(Q_{y_1 y_k}) \cup \{u\}$. Since $H' = u, p, y_k, F, u$ is a hole, then F is even, but then u, F, y_k, v' is an odd chordless path and Corollary 2.28 applied to S, u, F, y_k, v' and u', implies that an odd number of edges of F see S, a contradiction.

Case 2.2: $u'v' = st$.

Then $u' = t$ and $v' = s$. Let F be the shortest path between t and y_k in $V(Q_{y_j y_k}) \cup \{t\}$. Since $H' = t, p, y_k, F, t$ is a hole, then F is even, but then t, F, y_k, v is an odd chordless path and Corollary 2.28 applied to S, t, F, y_k, v and u, implies that an odd number of edges of F see S, a contradiction. □

Proof of Theorem 3.2:

Let A be a maximal set containing S such that (H, A) is a hub and uv sees A. Let B be the set containing all nodes of G that are universal for A. Let $B' = B \setminus V(H) \cup \{u, v\}$.

If $G \setminus (A \cup B')$ is not connected, then G has a skew-partition which is loose since $G \setminus (A \cup B')$ contains the endnode w of an edge in $E_A(H) \setminus \{uv\}$. We may assume, then, that $G \setminus (A \cup B')$ is connected, thus there exists a minimal path $R = r_1, \ldots, r_\ell$ in $G \setminus (V(H) \cup V(P) \cup A \cup B')$ such that r_1 has a neighbor in P and r_ℓ has a neighbor in $H \setminus \{u, v\}$. By Lemma 3.14, the only neighbors of r_ℓ in H are the endpoints s and t of an edge that sees A, and there exists $x \in V(P \cup R)$ adjacent to u and v such that $H \cup P \cup R$ contains a $3PC(uvx, tsr_\ell)$, where v and s have the same color w.r.t. the bicoloring of H induced by A. Note that the shortest path R' between x and r_ℓ with interior in R has even length. If $G \setminus (A \cup B)$ is not connected, then G contains a skew-partition. Assume that $G \setminus (A \cup B)$ is connected, then there exists a minimal path $Q = y_1, ..., y_m$ in $G \setminus (V(H) \cup V(P) \cup A \cup B)$ such that y_1 has a neighbor in P and y_m has a neighbor in the interior of some sector of (H, A), but such a path would contradict Lemma 3.15.

We shall now show that G has a balanced skew partition. Let $W = \{u, v, s, t\}$. W is a kernel, since by definition every node in G that is universal for W is either in A or it is universal for A, hence it is in B. Let C be the connected component of $G \setminus (A \cup B)$ containing the path R. Note that every pair of nonadjacent nodes $w \in \{u, v\}$ and $w' \in \{s, t\}$ is joined in C by the even path w, x, R', r_k, w'. Suppose there exist two adjacent nodes y and y' in C joined by an odd antipath F with interior in W, then F has length exactly 3 (since $\bar{G}[W]$ has diameter 2), hence $F = y, w, w', y'$ for some $w, w' \in W$ and w, y', y, w' is an odd path between w and w' so, by Lemma 3.4, G has a balanced skew partition. Therefore we may assume that every pair of adjacent nodes in C with nonneighbors in W is joined by an even antipath with interior in W, but then, by Lemma 3.5, G has a balanced skew partition. □

Corollary 3.16 *No Berge minimum imperfect graph contains a hub* (H, S) *with an ear on an isolated edge of* $E_S(H)$*.*

Proof: Follows from Theorems 2.16 and 3.2. □

3.6 Hubs in graphs containing no "large" line graphs

Throughout this section, we will assume that G is a Berge graph such that G and $\bar{G}$ contain no long prism and no $L(K_{3,3} \setminus \{e\})$. As already observed in Section 2.5, this condition implies that G cannot contain the line graph of a bipartite subdivision of a 3-connected graph.

Lemma 3.17 *Let* (H, S) *be a hub of a Berge graph* G *such that* G *and* $\bar{G}$ *contain no long prism and no* $L(K_{3,3} \setminus \{e\})$*. Let* $P = x_1, \ldots, x_n$ *be a minimal chordless path in* $G \setminus (V(H) \cup S)$ *containing no node that is universal for* S*, such that* x_1 *has a blue neighbor in* H *and* x_n *has a red neighbor (*$n = 1$ *is allowed). If there exist consecutive attachments of* P *with distinct colors that are not adjacent, then one of the following holds.*

- *(a)* $|H| = 6$*,* $n = 1$ *and there exists* $y \in S$ *such that* $V(H) \cup \{x_1, y\}$ *induces a double diamond.*
- *(b)* $n = 1$ *and there exists* $y \in S$ *nonadjacent to* x_1 *such that* (H, x_1) *and* (H, y) *are twin wheels and exactly one edge of* H *sees both* x_1 *and* y*.*
- *(c) There exists* $y \in S$ *such that* (H, y) *is a twin wheel, no node of* P *is a neighbor of* y*,* x_1 *is adjacent to the twin of* y *in* H *and no other node in* H *while* x_n *is not adjacent to both the other neighbors of* y *in* H*.*

Proof: Assume not, then P is of one of the types (a)-(c) or (f)-(k) of Theorem 3.10. If P is of type (c), then $V(H) \cup V(P) \cup \{y\}$ contains a long prism unless $n = 1$ and $|H| = 6$, so case (a) of Lemma 3.17 holds. P cannot be of type (a) by assumption. If P is of type (b), then $n = 1$, $|H| = 6$, (H, x_1) is a line wheel and $S \cup x_1$ contains an odd chordless anti-path Q of length at least 3 between x_1 and a node $y \in S$ such that (H, y) is a line wheel, no edge of H sees both x_1 and y and every intermediate node of Q is adjacent to every node in H. One can verify that $\bar{G}[V(H) \cup V(Q)]$ is the line graph of a bipartite subdivision of a 3-connected graph. If P is of type (f), then $n = 1$, H contains a subpath u, z, w, z', u' such that $E_S(H) = \{wz, wz'\}$, x_1 is adjacent to u, w and u' but not z and z', $S \cup x_1$ contains an odd chordless anti-path Q of length at least 3 between x_1 and a node $y \in S$ such that y is nonadjacent to u and u' and every intermediate node of Q is adjacent to both u and u'. One can verify that, since $|Q| \geq 3$, then $\bar{G}[V(Q) \cup \{u, z, z', u'\}]$ is a long prism $3PC(uu'y, z'zx_1)$. If P is of type (g), then $n = 1$, H contains a subpath w, z, u, z', w' such that wz and $w'z'$ are edges of $E_S(H)$, x_1 is adjacent to u, w and w' but not z and z', $S \cup x_1$ contains an even chordless anti-path Q between x_1 and a node $y \in S$ such that y is nonadjacent to u and every intermediate node of Q is adjacent to u. One can verify that, since Q has positive even length, $\bar{G}[V(Q) \cup \{w, z, u, z', w'\}]$ is a long prism $3PC(ww'u, z'zx_1)$. If P is of type (h), then $n > 1$, H contains a subpath w, z, u, z', w' such that wz and $w'z'$ are edges of $E_S(H)$, x_1 is adjacent w and w' but not u, z and z', while x_n is adjacent to u but not w, z, w' and z'. Furthermore S contains two nodes y and y' such that the only neighbors of y in $V(P) \cup \{w, z, u, z', w'\}$ are u, z, z', w, w' while the only neighbors of y' in $V(P) \cup \{w, z, u, z', w'\}$ are x_1, z, z', w, w'. One can verify that $G[V(P) \cup \{y, y', u, z, w'\}]$ is a long prism $3PC(uyz, x_1w'y'$. If P is of type (k), then $H = v, w, z, u, z', w', v$, $E_S(H) = \{wz, w'z'\}$, x_1 is adjacent only to v in H and x_n is adjacent only to u in H. Furthermore, S contains two nonadjacent nodes y and y' such that y and y' are adjacent to every node in H except v and u, respectively, and no node in P is adjacent to y or y'. One can verify that $G[V(P) \cup \{y, y', u, v, z, w'\}]$ is a long prism $3PC(uyz, vw'y')$. □

Lemma 3.18 *Let (H, S) be the hub of a Berge graph G such that G and $\bar{G}$ contain no long prism and no $L(K_{3,3} \setminus \{e\})$. Assume that S is a maximal set such that (H, S) is a hub with the further property that S does not contain any center of a twin wheel w.r.t. H. Let $P = x_1, \ldots, x_n$ be a minimal chordless path in $G \setminus (V(H) \cup S)$ containing no node universal for S such that x_1 has a red neighbor, no other node of P has a red neighbor and x_n has a blue neighbor whose neighbors in H are not red neighbors of x_1. Then one of the* *following holds:*

(1) There exist two adjacent edges ab_1, ab_2 of $E_S(H)$ such that a is the only red neighbor of x_1 in H and at least one node of P is adjacent to both b_1 and b_2. If $E_S(H) \supsetneqq \{ab_1, ab_2\}$ or if S contains a node s with no neighbors in P, then the path $Q = a, x_1, \ldots, x_n$ contains an odd number of edges that see both b_1 and b_2.

(2) $|H| = 6$, $n = 1$ and there exists $y \in S$ such that $V(H) \cup \{x_1, y\}$ induces a double diamond.

Proof: Obviously, one of the conclusions of Theorem 3.12 must occur. If conclusion (a) of Theorem 3.12 holds, then by Lemma 3.17 conclusion (2) holds (since S does not contain any center of a twin wheel) and we are done. If conclusion (b) holds, then conclusion (1) holds and we are done.

So we may assume that conclusion (c) of Theorem 3.12 holds. Then $n > 1$, $E_S(H)$ contains at least two nonadjacent edges, x_1 is adjacent to all the red endnodes of the edges of H that see S and the node x_j of lowest index adjacent to some blue node is adjacent to all the blue endnodes of the edges of H that see S. If $j > 2$, then S contains two nonadjacent nodes y and z such that y is adjacent to x_1 and to no other node of $P_{x_1x_j}$, and z is adjacent to x_j and to no other node of $P_{x_1x_j}$. If $j = 2$, then $S \cup \{x_1, x_2\}$ contains an odd chordless anti-path between x_1 and x_2.

Let uv and $u'v'$ be two nonadjacent edges of $E_S(H)$ and assume, w.l.o.g., that x_1 is adjacent to u and u' and x_j is adjacent to v and v'. If $j > 2$ then $G[V(P_{x_1x_j}) \cup \{y, z, u, v'\}]$ is a long prism $3PC(x_1yu, x_jv'z)$. If $j = 2$ then $\bar{G}[V(Q) \cup \{u, u', v, v'\}]$ is a long prism $3PC(x_1vv', x_2u'u)$. □

We say that a hub (H, S) is *good* if H has an inner blue node and an inner red node w.r.t. the bicoloring induced on H by $E_S(H)$. Equivalently, given the maximal paths $P^1, \ldots, P^k$ induced by the endnodes of the edges of $E_S(H)$, (H, S) is a good hub if and only if there exists i, $1 \leq i \leq k$, such that P^i has odd length.

Lemma 3.19 *Let (H, S) be a good hub of a Berge graph G such that G and $\bar{G}$ contain no long prism and no $L(K_{3,3} \setminus \{e\})$. Let $y \in G \setminus (V(H) \cup S)$ be a node such that $(H, S \cup y)$ is a hub. Then either $(H, S \cup y)$ is a good hub or $V(H) \cup y$ contains a hole H' such that (H', S) is a good hub with $E_S(H') \subsetneqq E_S(H)$.*

Proof: Since (H, S) is a good hub, by Lemma 3.17 every pair of consecutive neighbors of y in H with distinct colors are adjacent. Assume $(H, S \cup y)$ is not a good hub. Let $P^1, \ldots, P^k$ be the maximal paths induced by the endnodes of the edges of $E_S(H)$ and assume, w.l.o.g, that $P^1 = y_1, ..., y_m$ has odd length. If y has no neighbor in P^1, then P^1 is contained in a sector $Q = s, ..., t$ of (H, y), therefore, given $H' = y, s, Q, t, y$, (H', S) is a good hub and $E_S(H') \subsetneqq E_S(H)$. Therefore we may assume that y has a neighbor in P^1. Let r be the neighbor of y closest to y_1 in P^1 and s be the neighbor of y closest to y_m in P^1 (possibly $r = s$). Since $(H, S \cup y)$ is not a good hub, then y sees an even number of edges of P^1, therefore P^1_{rs} has even length. Since P^1 has odd length, we can assume, w.l.o.g., that $P^1_{sy_m}$ has odd length. Let $Q = s, ..., t$ be the sector of (H, y) containing $P^1_{sy_m}$, then, given $H' = y, s, Q, t, y$, (H', S) is a good hub and $E_S(H') \subsetneqq E_S(H)$ (since $(H, S \cup y)$ is a hub). □

Proof of Theorem 3.1:

Assume that, among all the good hubs contained in G, (H, S) is chosen so that $E_S(H)$ is minimal (i.e. there is no good hub (H', S') such that $E_{S'}(H') \subsetneqq E_S(H)$). Let A be a maximal set containing S such that (H, A) is a hub. Then, by Lemma 3.19 and by the minimality assumption on $E_S(H)$, (H, A) is a good hub and $E_A(H) = E_S(H)$. Let B be the set containing all the nodes that are universal for A in $G \setminus (V(H) \cup A)$ and all the blue endnodes of the edges in $E_S(H)$. If in $G \setminus (A \cup B)$ the red nodes of H are in distinct connected components than the blue nodes of H, then G has a skew partition. Otherwise there exists a chordless path $P = x_1, ..., x_n$ in $G \setminus (V(H) \cup A)$ containing no node universal for S such that x_1 is adjacent to a red node of H, no other node of P has a red node of H and x_n is adjacent to an inner blue node of H. Let x_j be the node of P with lowest index that is adjacent to a blue node b in H so that neither of the neighbors of b in H is a red neighbor of x_1. Then either conclusion (1) or (2) of Lemma 3.18 holds for $P_{x_1x_j}$. Conclusion (2) cannot hold since (H, A) is a good hub. Hence conclusion (1) holds, so there exist two adjacent edges ab_1, ab_2 of $E_A(H)$ such that a is the only red neighbor of x_1 in H and at least one node of $P_{x_1x_j}$ is adjacent to both b_1 and b_2. Since (H, A) is a good hub, $E_A(H) \supsetneqq \{ab_1, ab_2\}$ so by Lemma 3.18 the path $Q = a, x_1, \ldots, x_j$ contains an odd number of edges that see both b_1 and b_2. If $j = 1$, then $(H, A \cup x_1)$ is a hub, contradicting the maximality of A. Therefore $j > 1$ and there exists a node x_i, $i < j$, adjacent to b_1 and b_2 and to no other node in $V(H) \setminus \{a, b_1, b_2\}$.

Thus $(V(H) \cup \{x_i\}) \setminus \{a\}$ induces a hole H' and (H', A) is a good hub with $E_A(H') \subsetneqq E_S(H)$, contradicting the minimality of $E_S(H)$.

Hence G contains a skew partition A, B, C, D where C contains all the red nodes of H and D contains all the inner blue nodes of H (w.r.t. the bicoloring induced on H by $E_A(H)$). Let u be any red endpoint of some edge in $E_A(H)$, then $u \in C$ and u is universal for A, hence A, B, C, D is a loose skew partition. □

Chapter 4

Decomposition of Berge Graphs containing no proper wheels, long prisms or their complements

4.1 Introduction

In this chapter we will prove Theorem 2.26. In fact, we will prove the following, slightly stronger statement.

Theorem 4.1 *Let G be a Berge graph such that neither G nor $\bar{G}$ contains a proper wheel or a long prism. Then either G is a bipartite graph, or the line graph of a bipartite graph, or G has a loose skew partition.*

Observe that, if G does not contain a long prism, then the following strengthening of the Roussel-Rubio Lemma 2.27 holds.

Lemma 4.2 (Roussel and Rubio [69]) *Let G be a Berge graph such that G does not contain a long prism. Furthermore, $V(G)$ can be partitioned into an anticonnected set S and an odd chordless path $P = u, u', \dots, v', v$ of length at least 3 such that u and v are both universal for S. Then one of the following holds:*

(i) An odd number of edges of P see S.

(ii) $|P| = 3$ *and* $S \cup \{u', v'\}$ *contains an odd chordless antipath between* u' *and* v'*.*

Proof: If P has length at least 5 and there exist two nonadjacent nodes x, x' in S such that $(V(P) \setminus \{u, v\}) \cup \{x, x'\}$ induces a chordless path, then $V(P) \cup \{x, x'\}$ induces a long prism. □

4.2 Proof of Theorem 4.1

Let G be a Berge graph such that neither G nor $\bar{G}$ contains a proper wheel or a long prism.

Given a wheel (H, v) in G, we will say that (H, v) is *big* if $|H| \geq 6$.

4.2.1 Line graphs of $K_{3,3} \setminus \{e\}$

Lemma 4.3 *If* G *contains* $L(K_{3,3} \setminus \{e\})$ *as an induced subgraph, then either* G *is the line graph of a bipartite graph or it contains a loose skew partition.*

Proof:

Note that $L(K_{3,3} \setminus \{e\})$ is the graph consisting of nodes $\{a_1, ..., a_6, u, v\}$ where $H = a_1, ..., a_6, a_1$ is a hole, u is adjacent to a_1, a_2, a_4 and a_5 and v is adjacent to a_2, a_3, a_5 and a_6. We call this graph a *double L-wheel*, and denote it by (H, u, v), as (H, u) and (H, v) are both line wheels. Let $H' = a_1, u, a_4, a_3, v, a_6, a_1$, $Q = a_1, v, u, a_6, a_2, a_5, a_1$ and $Q' = a_2, a_5, a_3, u, v, a_4, a_2$. Then H' is a 6-hole and (H', a_2, a_5) is a double L-wheel, while Q and Q' are both 6-antiholes and (Q, a_3, a_4) (Q', a_1, a_6) are both double L-wheels in $\overline{G}$.

For $x \in V \setminus (V(H) \cup \{u, v\})$, we examine the adjacencies between x and (H, u, v). Since, as we just observed, the complement of a double line wheel is a double line wheel, then, by going to the complement, we can assume that x has at most four neighbors in (H, u, v).

Claim 1: *If* x *has at most four neighbors in* (H, u, v)*, then one of the following holds, up to the symmetries of* (H, u, v)*:*

(i) x *has no neighbor in* (H, u, v)*;*
(ii) x *is true or false twin of one of the nodes in* (H, u, v) *w.r.t.* (H, u, v)*;*
(iii) The only neighbors of x *in* (H, u, v) *are* a_1*,* a_3*,* a_4 *and* a_6*;*
(iv) The only neighbors of x *in* (H, u, v) *are* a_1 *and* a_6*;*
(v) The only neighbors of x *in* (H, u, v) *are* a_1*,* a_2 *and* u*.*

(4.1)

Proof of Claim 1: Since G does not contain any proper wheel, then (H, x) can be a universal wheel, a twin wheel, a line wheel or $V(H) \cup x$ induces a triangle-free-graph or a cap. If (H, x) is a universal wheel, then x has more than four neighbors in (H, u, v). Assume that (H, x) is a twin wheel and let $N_H(x) = \{a_{i-1}, a_i, a_{i+1}\}$. Then x is adjacent to u if and only if a_i is a neighbor of u, otherwise, if C is the hole obtained from H by replacing a_i with x, then (C, u) is a proper wheel or an odd wheel. Similarly, x is adjacent to v if and only if a_i is a neighbor of v. But then x is a twin of a_i w.r.t. (H, u, v).
Assume now that (H, x) is a line wheel. Since x has already four neighbors in (H, u, v), either x is a false twin of u or v or (iv) holds.
Assume next that $V(H) \cup x$ induces a cap. By symmetry, we can assume that x is adjacent to a_1. Assume first that x is adjacent to a_1 and a_6. If x is adjacent to both u, v, then (Q', x) is an odd wheel in $\overline{G}$. If x is adjacent to exactly one of u, v, say u, then x, u, a_2, v, a_6 induces a 5-hole. So (iv) must hold. Assume now that x is adjacent to a_1 and a_2. Then (v) must hold since, otherwise, (Q', x) is a proper wheel in $\overline{G}$.
Finally, assume that $V(H) \cup x$ induces a triangle-free graph. By symmetry we can assume that $V(H') \cup x$ also induces a triangle-free graph. If x has no neighbor in (H, u, v), (i) holds. Thus, by symmetry, we can assume that x is either adjacent to a_1 or to a_2. If x is adjacent to a_1, then x is not adjacent to a_2, a_6 and u, and (Q, x) must be a twin wheel in $\overline{G}$, hence x is a twin of a_6 w.r.t. (H, u, v). Assume then that x is adjacent to a_2 and, by symmetry and by the previous case, assume x is not adjacent to a_1, a_3, a_4 and a_6. Also, x is not adjacent to a_5, else there is a 5-hole $a_1, a_2, x, a_5, a_6, a_1$. Hence (Q, x) must be a line wheel in $\overline{G}$, so x is adjacent to v but not to u, but now (Q', x) is a proper wheel in $\overline{G}$. This completes the proof of Claim 1.

We say that a graph G' is an *extended multi line wheel* if G' can be partitioned into sets $A_1, ..., A_6$, U, V and W with the property that every node in A_i is adjacent to every node in A_{i+1} (where the indices are taken modulo 6), every node in U (resp. V) (resp. W) is adjacent to every node in A_1, A_2, A_4 and A_5 (resp. A_2, A_3, A_5, A_6) (resp. A_1, A_3, A_4, A_6) and these are the only edges with endnodes in different sets of the partition. All the sets, except at most W, are nonempty.

Since G contains a double line wheel (H, u, v), then G contains an extended multi line wheel G' such that $a_i \in A_i$, $u \in U$ and $v \in V$. Assume G'

is maximal (in terms of its node set) with this property.

Claim 2:

- *Every node of Type (4.1)(iii) w.r.t.* (H, u, v) *belongs to* W.
- *If a node* x *of Type (4.1)(ii) w.r.t.* (H, u, v) *does not belongs to* G'*, then* x *is a true twin of a node of degree 3 in* (H, u, v)*, say* a_1*, and* x *is of Type (v) w.r.t.* (H^*, u, v) *for some 6-hole* H^* *obtained from* H *by replacing* a_6 *by a node* $a_6^* \in A_6$.
- *If a node* x *is of Type (4.1)(iv) w.r.t.* (H, u, v)*, adjacent to* a_1 *and* a_6*, then* x *is universal for* $A_1 \cup A_6$ *and has no neighbor in* $G' \setminus A_1 \cup A_6$.
- *If a node* x *is of Type (4.1)(v) w.r.t.* (H, u, v)*, adjacent to* a_1, a_2 *and* u*, then* x *is universal for* $A_1 \cup A_2 \cup U$ *and has no neighbor in* $A_3 \cup A_4 \cup A_5 \cup V$.

Proof of Claim 2: By construction, every node of G' must be a twin of a node of (H, u, v) w.r.t. (H, u, v) or must be of Type (4.1)(iii). Suppose that some node x of Type (4.1)(iii) does not belong to W. Then either x is not adjacent to some node y in A_1, A_3, A_4 or A_6 or x is adjacent to some node y in A_2, A_5, U or V. Let (H^*, u^*, v^*) be the double line wheel obtained by adding y and removing the corresponding node of (H, u, v). Now x contradicts Claim 1 in (H^*, u^*, v^*) or in its complement. So the first part of Claim 2 holds. Now suppose that some node x of Type (4.1)(ii) does not belong to G'. By symmetry we can assume that x is a twin of a_1 or a_2 w.r.t. (H, u, v). If x is a twin of a_1, then either x is not adjacent to some node y in A_2, A_6 or U or x is adjacent to some node y in A_3, A_4, A_5 or V. Constructing (H^*, u^*, v^*) as above, we obtain a contradiction of Claim 1 unless y is in A_6. If x is a twin of a_2, constructing (H^*, u^*, v^*) as above, we obtain a contradiction of Claim 1 in all cases. The last two statements of Claim 2 follow similarly. This completes the proof of Claim 2.

By Claim 2, the nodes of $G \setminus G'$ partition into two sets X and Y as follows: X contains the nodes of $G \setminus G'$ that have no neighbor in $V(G')$ or are of Type (4.1)(iv) or (v) w.r.t. at least one double line wheel (H^*, u^*, v^*) where $H^* = a_1^*, \ldots, a_6^*, a_1^*$ with $a_i^* \in A_i$, $u^* \in U$, $v^* \in V$. The set Y contains the remaining nodes of $G \setminus G'$. In the complement graph $\overline{G}$, the nodes of Y have either no neighbor in $V(G')$ or are of Type (4.1)(iv) or (v) w.r.t.

at least one double line wheel (Q^*, a_3^*, a_4^*) where $Q^* = a_1^*, v^*, u^*, a_6^*, a_2^*, a_5^*, a_1^*$ with $a_i^* \in A_i$, $u^* \in U$, $v^* \in V$.

Claim 3: *Let $X_{1,2}$ be the set of nodes of X that are universal for $A_1 \cup A_2 \cup U$ and possibly adjacent to nodes of A_6 but to no other nodes of G'. Then there exists a node of A_6 that has no neighbor in $X_{1,2}$.*

Proof of Claim 3: Suppose not. Since every node of A_6 has a neighbor in $X_{1,2}$ and every node of $X_{1,2}$ has a non-neighbor in A_6, there must exist $r, s \in A_6$ and $t, z \in X_{1,2}$ such that rt and sz are edges but rz and st are not. Indeed, it is immediate to verify that the statement is true if $|A_6| \leq 2$ or $|X_{1,2}| \leq 2$. By induction on $|A_6| + |X_{1,2}|$, given $z \in X_{1,2}$, either we are done by applying the inductive hypothesis to A_6 and $X_{1,2} \setminus z$, or A_6 contains a node s with no neighbors in $X_{1,2} \setminus z$ so z and s are adjacent. Let r be a non-neighbor of z in A_6 and t be a neighbor of r in $X_{1,2}$, then rt and sz are edges but rz and st are not.

If neither rs nor zt is an edge, consider the 6-hole $L = r, t, a_2, z, s, a_5, r$. Then (L, u) is a proper wheel. If exactly one of rs and zt is an edge, there is a 5-hole r, t, a_2, z, s, r or r, t, z, s, a_5, r. If both rs and zt are edges, the nodes in $\{r, s, t, z, a_2, a_3, a_4, a_5\}$ induce a long prism. This proves Claim 3.

Claim 4: If $x_i, x_j \in X$ are universal for A_i, A_{i+1} and for A_j, A_{j+1}, respectively, where $1 \leq i < j \leq 5$, and possibly to other nodes of G', then x_i and x_j are in different connected components of $G[X]$.

Proof of Claim 4: Suppose not. Choose a pair $x_i, x_j \in X$, $i < j$, with a shortest path P connecting them in $G[X]$. By the choice of P, the internal nodes of P have no neighbor in G'. By Claim 2, there exists a double line wheel (H^*, u^*, v^*) where $H^* = a_1^*, \ldots, a_6^*, a_1^*$ with $a_i^* \in A_i$, $u^* \in U$, $v^* \in V$, such that x_i and x_j are both of Type (4.1)(iv) or (v) w.r.t. (H^*, u^*, v^*). If $j - i \geq 2$, the nodes of $H \cup P$ induce a long prism. If $j = i + 1$, it is sufficient to consider the cases $j = 2$ and $j = 3$ by symmetry. If $j = 2$, the nodes of $V(P) \cup \{a_2, a_3, a_4, a_5, u, v\}$ induce a long prism. If $j = 3$, the nodes of $V(P) \cup \{a_2, a_4, a_5, u, v\}$ induce a long prism. This proves Claim 4.

Assume that Y is nonempty. By symmetry Y contains a node y universal for $A_2 \cup A_3 \cup A_5 \cup A_6$. Furthermore, if y is of Type (4.1)(iv) in $\overline{G}$, we can assume that, in G, y is universal for $A_1 \cup A_4$ and has no neighbor in $U \cup V$. If y is of Type (4.1)(v) in $\overline{G}$, we can assume that, in G, y has no neighbor in $A_1 \cup A_4 \cup V$. Finally, if all the nodes of Y are universal for G', choose y to be any node of Y. Let A be the co-connected component of Y containing y and let B be the set of nodes $A_2 \cup A_3 \cup A_5 \cup A_6$ together with the nodes of $G \setminus G'$ that are universal for A. By Claim 4 applied to $\overline{G}$, A is universal for $A_2 \cup A_3 \cup A_5 \cup A_6$. Clearly, A is universal for $Y \setminus A$. Therefore, by Claim 4 applied to G, $A \cup B$ is a skew cutset separating V from $A_1 \cup A_4 \cup U$. By Claim 3 applied to $\overline{G}$, if y is of Type (4.1)(v) in $\overline{G}$, at least one node of U is universal for A. And if y is not of Type (4.1)(v) in $\overline{G}$, the nodes of A_1 are universal for A. So $A \cup B$ is a good skew cutset.

By the argument above applied to $\overline{G}$, if X is nonempty then G has a good skew partition. Hence we may assume that X and Y are both empty. If any of the sets $A_1, ..., A_6$, U, V, W has cardinality greater than one, then G has a star cutset. So, if G has no good skew partition, G' is a multi line wheel. □

By Lemma 4.3, we may assume that neither G nor $\bar{G}$ contains a long prism or $L(K_{3,3} \setminus \{e\})$ as an induced subgraph. By Theorem 2.24, we may assume that neither G nor $\bar{G}$ contains a good hub. In particular, neither G nor $\bar{G}$ contains a line wheel, hence, since we are assuming that there are no proper wheels in G or in $\bar{G}$, the only wheels that might be contained in G or $\bar{G}$ are universal wheels, twin wheels or triangle free wheels. Next we will show that, whenever G has a big universal wheel, G has a skew partition.

4.2.2 Big universal wheels

Lemma 4.4 *If G contains a big universal wheel, then G has a T-cutset.*

Proof: Assume G contains a universal wheel (H, x) and let A be a maximal co-connected set of $G \setminus V(H)$ such that every node in A is universal for $V(H)$. Consider a bicoloring of the nodes of H obtained by coloring the nodes of H red and blue in such a way that two nodes have the same color if and only if they have even distance in H. Let y be a node in $G \setminus (V(H) \cup A)$ that is not universal for A such that y has two nonadjacent neighbors in H. We will show that (H, y) is a triangle-free wheel and y is universal for either the red or the blue nodes of H. Let u be a node in A that is not adjacent to y. By the maximality of A, y is not universal for $V(H)$, hence y has two consecutive nonadjacent neighbors s and t in H. Let H_{st} be a path between s and t in H containing no neighbors of y. Then s and t have distance 2 in H_{st}, otherwise $H' = y, s, H_{st}, t, y$ is a big hole and (H', u) is a proper wheel (since

u is adjacent to every node but y in H'). Hence (H, y) is not a twin wheel, so $H \cup y$ is a triangle-free graph in which every pair of consecutive neighbors of y in H has distance two in H. Hence y is either universal for the red or for the blue nodes of H. So we can partition the nodes in $G \setminus (V(H) \cup A)$ that have two nonadjacent neighbors in H and that are not universal for A into sets Δ_R and Δ_B, where every node in Δ_R (resp. Δ_B) is universal for the red (resp. blue) nodes of H and has no blue (resp. red) neighbor in H. Next, we will show that either Δ_R or Δ_B is empty. Assume not and let r and b be two nodes in Δ_R and Δ_B respectively. Let st and $s't'$ be two nonadjacent edges of H where s, s' are red and t, t' are blue. If r and b are not adjacent, then $H' = r, s, t, b, t', s', r$ is a 6-hole and (H', u) is a proper wheel or a line wheel for every node u in A that is not adjacent to r or b. So r and b are adjacent and, since neither of them is universal for A, then $\overline{G}[A \cup \{r, b\}]$ contains a chordless path Q between r and b. $\overline{G}[V(Q) \cup \{s, s', t, t'\}]$ is a long prism, namely a $3PC(rtt', bs's)$, a contradiction.

Therefore we may assume, w.l.o.g., that every node in $G \setminus (V(H) \cup A)$ that has two nonadjacent neighbors in H and that is not universal for A is universal for the blue nodes of H and has no red neighbor in H. Let a be a red node of H and let b_1 and b_2 be its neighbors in H. Let B be the set of all nodes in $G \setminus (A \cup V(H)) \cup \{b_1, b_2\}$ that are universal for A. If a and $V(H) \setminus \{a, b_1, b_2\}$ lie in distinct connected components of $G \setminus (A \cup B)$, let C be the connected component of $G \setminus (A \cup B)$ containing a and $D = V(G) \setminus (A \cup B \cup C)$. Then A, B, C, D is a skew partition and, given a node t in $V(H) \setminus \{a, b_1, b_2\}$, then $t \in D$ and both a and t are universal for A, hence $A \cup B$ is a T-cutset. Hence we may assume that there exists a chordless path $P = x_1, \ldots, x_n$ in $G \setminus (A \cup B \cup V(H))$ such that x_1 is adjacent to a, x_n has a neighbor in $V(H) \setminus \{a, b_1, b_2\}$ and no intermediate node has a neighbor in $V(H) \setminus \{b_1, b_2\}$. Note that x_1 does not have two nonadjacent neighbors in H, hence $n > 1$. Also, no node in $P \setminus x_n$ has two nonadjacent neighbors in H. Note that b_1 and b_2 cannot both have neighbors in $P \setminus x_n$, otherwise let x_i and x_j be neighbors of b_1 and b_2, respectively, in $P \setminus x_n$ such that x_i and x_j are closest possible in $P \setminus x_n$. Then $H' = H \setminus a \cup V(P_{x_i x_j})$ is a hole and, for any node $u \in A$ that is not universal for $P_{x_i x_j}$, (H', u) is a proper wheel. Thus we may assume that b_1 has no neighbors in $P \setminus x_n$.

If x_n has only blue neighbors in H, then let t be the closest neighbor of x_n to b_1 and $H_{b_1 t}$ be the chordless path between b_1 and t in $H \setminus a$. $H' = a, x_1, P, x_n, t, H_{b_1 t}, b_1, a$ is a hole of even length, hence, since t and a have odd distance, P is an odd path. Let s be a neighbor of x_n distinct from b_1 and b_2.

Then $F = a, x_1, P, x_n, s$ is an odd chordless path and $F \setminus \{a, s\}$ does not have any node universal for A. By Lemma 4.2, F has length 3 and $A \cup \{x_1, x_2\}$ contains an odd chordless co-path Q between x_1 and x_2. Let w be a red node distinct from a. Then $C = a, x_1, Q, x_2, a$ is an odd anti-hole, a contradiction.

So x_n has a red neighbor in H, therefore x_n does not have two nonadjacent neighbors in H. Let t be the unique red neighbor of x_n in H. Since $|H| \geq 6$, t is not adjacent to b_1 or b_2, say b_i for $i = 1$ or 2. Let H_{b_it} be the path between b_i and t in $H \setminus a$. Since t and b_i have distinct colors, H_{b_it} has odd length, so $|H_{b_it}| \geq 3$. If x_n has no neighbors in H_{b_it}, then let u be a node of A that is not adjacent x_n. If b_i has no neighbors in P, then $H' = t, H_{b_it}, b_i, a, x_1, P, x_n, t$ is a big hole and (H', u) is a proper wheel. Otherwise let x_j be the node of highest index in P adjacent to b_i. Then $H'' = t, H_{b_it}, b_i, x_j, P_{x_jx_n}, x_n, t$ is a big hole and (H'', u) is a proper wheel. So x_n has exactly two neighbors s and t in H, s and t are adjacent and s is in H_{b_it}, so $s \neq b_1, b_2$. So s and t have no neighbor in $P \setminus x_n$. Let x_j be a node of highest index with a neighbor in $\{a, b_2\}$. We already observed that x_j cannot have two nonadjacent neighbors in H. If x_j has a unique neighbor v in $\{a, b_2\}$, then there is a $3PC(x_nst, v)$. So $j = 1$ and x_1 is adjacent to b_2 and there is a long prism $3PC(x_1ab_2, x_nts)$, a contradiction. $\square$

4.2.3 Caps

By Lemmas 4.3 and 4.4, we may assume G and $\bar{G}$ do not contain any long prism or any big wheel except twin wheels and triangle-free wheels.

A *cap* (H, v) consists of a hole H and a node v not in H that has exactly two neighbors a and b in H, and a and b are adjacent. We say that v is the *tip* of (H, v) while a and b are the *attachments* of v in H. If $|H| \geq 6$ then (H, v) is *big* else (H, v) is *small*.

Lemma 4.5 *If G contains a big cap, then G has a loose skew-partition.*

Before proving Lemma 4.5, we shall prove the following three lemmas.

Lemma 4.6 *Let Γ be a Berge graph. If Γ and $\overline{\Gamma}$ do not contain any big wheel (H, x) where x has more than $|H|/2$ neighbors in H, then Γ does not contain both a big hole and a big antihole.*

Proof: Assume, by contradiction, that Γ contains a hole H and an antihole A, where $n = |H| \geq 6$ and $m = |A| \geq 6$.

Assume first that $V(H) \cap V(A) \neq \emptyset$. It is immediate to verify that $|V(H) \cap V(A)| \leq 4$ and $V(H) \cap V(A)$ induces a chordless path or the complement of a chordless path. W.l.o.g., assume $P = \Gamma[V(H) \cap V(A)]$ is a chordless path, and let $k = |V(H) \cap V(A)|$.

By assumption, every node in $V(A) \setminus V(H)$ has at most $n/2$ neighbors in H, hence there are at most $(m-k)n/2$ edges between $V(H)$ and $V(A) \setminus V(H)$. Since P has k nodes and $k-1$ edges, and every node in A has exactly $m-3$ neighbors in A, then between $V(A) \cap V(H)$ and $V(A) \setminus V(H)$ there are exactly $k(m-3) - 2(k-1) = km - 5k + 2$ edges, hence there are at most $(m-k)n/2 - (km - 5k + 2)$ edges between $V(A) \setminus V(H)$ and $V(H) \setminus V(A)$.

Analogously, every node in $V(H) \setminus V(A)$ has at least $m/2$ neighbors in $V(A)$, hence there are at least $(n-k)m/2$ edges between $V(H) \setminus V(A)$ and $V(A)$. Also, there are exactly 2 edges between $V(A) \cap V(H)$ and $V(H) \setminus V(A)$, hence there are at least $(n-k)m/2 - 2$ edges between $V(A) \setminus V(H)$ and $V(H) \setminus V(A)$. Therefore

$$\frac{(n-k)m}{2} - 2 \leq \frac{(m-k)n}{2} - km + 5k - 2$$

implying $n + m \leq 10$, that is a contradiction since $n, m \geq 6$.

Hence we may assume that A and H are node disjoint. Every node in A has at most $n/2$ neighbors in H, hence there are at most $mn/2$ edges between $V(A)$ and $V(H)$. Every node in H has at least $m/2$ neighbors in A, hence there are at least $mn/2$ edges between $V(A)$ and $V(H)$, therefore there are exactly $mn/2$ edges between $V(A)$ and $V(H)$, every node in $V(A)$ has exactly $n/2$ neighbors in $V(H)$ and every node in $V(H)$ has exactly $m/2$ neighbors in $V(A)$. Let x be a node of A. If (H, x) is not a triangle-free wheel, then $H \cup x$ contains a hole H' of length at least 6 containing x, but H' and A have one node in common and we already showed that this is not possible. Hence, for every $x \in V(A)$, (H, x) is a triangle-free wheel. Let X and Y be the two stable sets of size $n/2$ partitioning $V(H)$, then for every $x \in V(A)$ either x is universal for X and has no neighbors in Y, or vice-versa. Since every node in H has neighbors in A and A is anticonnected, there exist two nonadjacent nodes x and y in A such that x is universal for X and has no neighbors in Y while y is universal for Y and has no neighbors in X. Let $x'y'$ and $x''y''$ be two nonadjacent edges of H with $x', x'' \in X$ and $y', y'' \in Y$, then $H' = x, x', y', y, y'', x'', x$ is a 6-hole and H' and A have two nodes in common, a contradiction. □

Note that, since G and $\bar{G}$ do not contain any big wheel except twin wheels and triangle-free wheels, then neither G nor $\bar{G}$ contains a big wheel (H, x) where x has more than $|H|/2$ neighbors in H. Hence, by Lemma 4.6, we may assume, w.l.o.g., that G contains no big antihole.

Lemma 4.7 *Assume G contains a cap (H, x) with attachments a, b. Let $P = x_1, \ldots, x_n$ be a direct connection between x and $V(H) \setminus \{a, b\}$ contained in $G \setminus H$ such that no node of P is adjacent to a. Then x_1 is adjacent to b and no other neighbor of P is adjacent to b.*

Proof: Assume first that b has no neighbors in P. If x_n has a unique neighbor t in H, then there is a $3PC(abx, t)$. If x_n has two nonadjacent neighbors in H, then there is a $3PC(abx, x_n)$. Hence x_n has exactly two neighbors t and t' in H and they are adjacent, but then either G contains a long prism $3PC(abx, tt'x_n)$, or $|H| = 4$, $n = 1$ and $V(H) \cup \{x, x_1\}$ induces an antihole of length 6. Therefore b has a neighbor in P. If $n = 1$ we are done, hence we may assume $n \geq 2$. Let t be the closest neighbor of x_n to a in H and H_{ta} be the path between a and t in $H \setminus b$, let $H' = x, x_1, P, x_n, t, H_{ta}, a, x$. Since $n \geq 2$, H' has length at least 6 and b is adjacent to a, x and some other node of P, so (H', b) is a big wheel that is not triangle-free, hence it must be a twin wheel, therefore b must be adjacent to x_1 and no other node of P. □

Lemma 4.8 *Assume G contains a connected set S, a chordless antipath $Q = y_1, ..., y_n$ disjoint from S such that y_1 and y_n have no neighbors in S and for every i, $2 \leq i \leq n-1$, y_i has a neighbor in S. Then $n \leq 4$.*

Proof: Assume, by contradiction, that $n \geq 5$. Since y_2 and y_3 have at least a neighbor in S and S is connected, there exists a chordless path $P = y_2, p_1, \ldots, p_m, y_3$ between y_2 and y_3 whose interior is contained in S. P has even length, otherwise y_n, y_2, P, y_3, y_n is an odd hole, and P has length at least 4, otherwise $P = y_2, p_1, y_3$ and, given h the smallest index such that y_h is nonadjacent to p_1 (h is well defined since y_n has no neighbors in S), then $y_1, Q_{y_1 y_h}, y_h, p_1, y_1$ is an antihole of length at least 6, a contradiction. Hence $P' = y_2, P, y_3, y_1$ is an odd path of length at least 5 and $X = V(Q) \setminus \{y_1, y_2, y_3\}$ is an anticonnected set universal for y_1 and y_2. By Lemma 4.2, the interior of P' must contain a node universal for X, a contradiction since y_n has no neighbors in the interior of P and y_3 is not adjacent to y_4. □

Proof of Lemma 4.5:

Consider a big cap (H, v) and let a, b be the attachments of x in H. Let $a_0 = a$, P^0 be the path induced by $V(H) \setminus b$, $A_0 = \{a_0\}$ and $S_0 = V(P^0) \setminus a_0$. Let $P^0, \dots, P^k$ be a sequence of chordless paths in G, where $P^i = x^i_1, \dots, x^i_{l_i}$. For every i, $1 \leq i \leq k$, let $a_i = x^i_1$, $A_i = A_{i-1} \cup a_i$ and $S_i = S_{i-1} \cup V(P^i) \setminus a_i$. Assume that the sequence $P^0, \dots, P^k$ satisfies the following properties:

1. P^i is a direct connection between x and S_{i-1} contained in $G \setminus (A_{i-1} \cup \{b\})$ such that no node in P^i is universal for A_{i-1},

2. For every i, $0 \leq i \leq k$, a_i is adjacent to b.

We will prove that, if $P = x_1, ..., x_n$ is a direct connection between x and S_k contained in $G \setminus (A_k \cup b)$ such that no node in P is universal for A_k, then x_1 is adjacent to b.

Note that this implies Lemma 4.5: obviously, P^0 is a sequence satisfying properties 1 and 2 above, hence we can consider a sequence $P^0, \dots, P^k$ ($k \geq 0$), that is the longest possible. Let $A = A_k$ and B be the set of all nodes in $V \setminus x$ that are universal for A. If $A \cup B$ is not a skew cutset that separates x from S_k, then there exists a direct connection $P = x_1, ..., x_n$ between x and S_k contained in $G \setminus (A_k \cup b)$ such that no node in P is universal for A_k. Since x_1 is adjacent to b, we can choose $P^{k+1} = P$. Now $P^0, \dots, P^{k+1}$ is a sequence satisfying properties 1 and 2, contradicting the maximality of k. Hence $A \cup B$ is a skew cutset that separates x from S_k. Let C be the connected component of $G \setminus (A \cup B)$ containing x and $D = V(G) \setminus (A \cup B \cup C)$. Then A, B, C, D is a skew partition and x is universal for A, hence it is a good skew partition.

Observe that, by construction, for every $0 \leq i \leq k$, A_i is a co-connected set and S_i is connected. Moreover, x has no neighbors in S_k and every node in A_i has a neighbor in S_i.

Claim 1: For every j, $1 \leq j \leq k$, and for every node $y \in S_j$, if y is universal for A_{j-1}, then y is the only neighbor of a_0 in $H \setminus b$.

Proof of Claim 1: By construction, for every i such that $1 \leq i \leq j$, no node in P^i is universal for A_{j-1}, hence y must be the node in P^0 adjacent to a_0.

Claim 2: For every i, $0 \leq i \leq k$, b does not see any edge of P^i.

Proof of Claim 2: The statement is trivial for $i = 0$ and it follows immediately by Lemma 4.7 for $i = 1$. Hence we may assume $i \geq 2$. Assume, by contradiction, that b sees an edge $x^i_j x^i_{j+1}$ of P^i. We will show that every node

of A_{i-1} is adjacent to x^i_j or x^i_{j+1}. Assume not, then there exists $a_h \in A_{i-1}$ such that a_h is not adjacent to x^i_j and x^i_{j+1}. Let y be the neighbor of a_h in the chordless path $x, x^i_1, P^i_{x^i_1 x^i_{j-1}}, x^i_{j-1}$ closest to x^i_j in P^i (y is well defined since every node in A_k is adjacent to x) and let F be the path from y to x^i_j in $x, x^i_1, P^i_{x^i_1 x^i_j}, x^i_j$. Let F' be a chordless path between a_h and x^i_{j+1} in the graph induced by $S_{i-1} \cup (a_h \cup V(P^i_{x^i_{j+1} x^i_{l_i}})$. By construction, no node in P^i except $x^i_{l_i}$ has a neighbor in S_{i-1}, hence $C = a_h, y, F, x^i_j, x^i_{j+1}, F', a_h$ is a big hole and b is adjacent to a_h, x^i_j and x^i_{j+1} in C, hence (C, b) is a big wheel that is neither a triangle-free nor a twin wheel, a contradiction.
Hence every node in A_{i-1} is adjacent to x^i_j or x^i_{j+1} but no node in P^i is universal for A_{i-1}, so there exists a chordless co-path $Q = y_1, \ldots, y_m$ in $G[A_{i-1}]$ such that y_1 is adjacent to x^i_j but not x^i_{j+1}, y_m is adjacent to x^i_{j+1} but not x^i_j and all the intermediate neighbors of Q are adjacent to both x^i_j and x^i_{j+1}. If $j > 1$, then x^i_j and x^i_{j+1} are not adjacent to x, hence $(x, x^i_{j+1}, y_1, Q, y_m, x^i_j, x$ is a big anti-hole. Therefore $j = 1$, and $Q' = x, x^i_2, y_1, Q, y_m, x^i_1$ is a co-path of length at least 4. Let $S = S_{i-1} \cup V(P^i)) \setminus \{x^i_1, x^i_2\}$. S is connected and neither x nor x^i_1 have neighbors in S, while, by construction, every intermediate node of Q' has a neighbor in S. Now Q' and S contradict Lemma 4.8. This completes the proof of Claim 2.

We will prove Lemma 4.5 by induction on k. If $k = 0$, then we are done by Lemma 4.7. Let us now assume, by induction, that the statement is satisfied for every big cap (H, x), and for every sequence $P_0, ..., P_j$ satisfying properties 1 and 2, whenever $j \leq k - 1$. Note that P must contain a node that is universal for A_{k-1}, otherwise $(V(P) \cup V(P^k)) \setminus a_k$ contains a direct connection P' from x to S_{k-1} and no node in P' is universal for A_{k-1} so, by induction, the first node of P', which is x_1, is adjacent to b and we are done. Let us assume, by contradiction, that x_1 is not adjacent to b.

Claim 3: No node of P is adjacent to a_k.

Proof of Claim 3: Assume, by contradiction, that a_k has a neighbor in P. Then, since by the argument above P contains a node universal for A_{k-1}, every node in A_k has a neighbor in P. For every $0 \leq i \leq k$, let $h(i)$ be the minimum index such that a_i is adjacent to $x_{h(i)}$ and let $h = \max_{0 \leq i \leq k} h(i)$. Since no node of P is universal for A_k, $h \geq 2$. If $h = 2$, then every node in A_k is adjacent to x_1 or x_2 but neither x_1 or x_2 are universal for A_k, hence A_k contains a chordless co-path $Q = y_1, \ldots, y_m$ such that y_1 is adjacent to x_1

but not to x_2, y_m is adjacent to x_2 but not x_1 and every intermediate node of Q is adjacent to both x_1 and x_2. Therefore $Q' = x, x_2, y_1, Q, y_m, x_1$ is a co-path of length at least 4. Let $S = S_k \cup (V(P) \setminus \{x_1, x_2\})$. S is a connected set and neither x_1 nor x has a neighbor in S, while every intermediate node of Q' has a neighbor in S. Therefore Q' and S contradict Lemma 4.8. Hence we can assume $h \geq 3$. Let $a_j \in A_k$ be such that $h(j) = h$. Then $C = a_j, x, x_1, P_{x_1 x_h}, x_h, a_j$ is a big hole. Since b is adjacent to both x and a_j, then (C, b) is either a cap or a twin wheel. If (C, b) is a cap, let F be a shortest path between x_h and b in $S_k \cup V(P_{x_h x_n}) \cup b$, then $C' = x, x_1, P_{x_1 x_h}, x_h, F, b, x$ is a hole and a_j is adjacent to x, b and x_h in C', therefore (C', a_h) is a big wheel that is neither a triangle-free wheel nor a twin wheel. Hence (C, b) must be a twin wheel so b is adjacent either to x_1 or to x_h. In the former case we are done. Now assume that b is adjacent to x_h. $C' = x, x_1, P_{x_1 x_h}, x_h, b, x$ is a big hole. Since every node in A_k has a neighbor in $P_{x_1 x_h}$, then (C', a_i) must be a twin wheel for every $a_i \in A_k$, hence every node in A_k is adjacent to x_1 or x_h. Since no node in P is universal for A_k and A_k is co-connected, there exists two nonadjacent nodes a_s and a_t in A_k such that a_s is adjacent to x_1 and not to x_h, and a_t is adjacent to x_h and not to x_1. $C'' = x, x_1, P_{x_1 x_h}, x_h, a_t, x$ is a big hole and (C'', a_s) is a cap where x, x_1 are the attachments of a_s in C''. By construction S_k contains a direct connection F form a_s to $C'' \setminus \{x_1, x\}$, but no node in S_k is adjacent to x or to x_1, hence F contradicts Lemma 4.7. This completes the proof of Claim 3.

Claim 4: b does not have any neighbor in P.

Proof of Claim 4: Assume by contradiction that x_j, for some $1 \leq j \leq n$, is adjacent to b. Let F be a chordless path between a_k and x_1 in $S_k \cup V(P)$. Since a_k has no neighbor in P, then P is a subpath of F and $C = x, x_1, F, a_k, x$ is a hole, b is adjacent to x, a_k and x_j in P but x_j is not adjacent to x (otherwise $j = 1$ and b is adjacent to x_1) and x_j is not adjacent to a_k (because, by Claim 3, a_k has no neighbors in P), hence (C, b) is a big wheel that is neither a twin wheel nor a triangle-free wheel, a contradiction.

Claim 5: $S_k \cup V(P)$ contains a chordless path $F = y_1, \ldots, y_{m+1}$ between x_1 and b such that a_k is adjacent to y_m, and no other node in F and y_1 is universal for A_{k-1}.

Proof of Claim 5: Let $F = y_1, \ldots, y_{m+1}$ be a chordless path between x_1 and b in $S_k \cup V(P)$, where $y_1 = x_1$ and $y_{m+1} = b$. Note that, since b is not adjacent to x_1, then $C = x, x_1, F, b, x$ is a hole. Since b has no neighbor in P, then P

is a subpath of F and y_m is in S_k. By Claim 1, y_m is not universal for A_{k-1} since y_m is adjacent to b.

Since P contains a node x_j universal for A_{k-1} and (C, a_i) is a wheel that is not triangle-free for each $a_i \in A_{k-1}$, every node in A_{k-1} must be adjacent to x_1. If y_m is adjacent to a_k we are done. Otherwise (C, a_k) is a cap where x, b are the attachments of a_k in C. Let $Z = z_1, \ldots, z_l$ be a direct connection form a_k to $V(C) \setminus \{x, b\}$, contained in $S_k \setminus V(F)$. Since no node in Z is adjacent to x, then by Lemma 4.7 z_1 is adjacent to b and no node in $Z \setminus z_1$ is adjacent to b. Hence, given y the closest neighbor of z_l to x_1 in F, $F' = x_1, F_{x_1 y}, z_l, Z, z_1, b$ is a chordless path between x_1 and b in $S_k \cup V(P)$. Note that $F' = y'_1, \ldots, y'_{m'+1}$, where $y'_1 = x_1$ and $y'_{m'+1} = b$ and a_k is adjacent to $y'_{m'}$ and no other node in F'. Thus F' satisfies the statement of Claim 5.

Let j, $0 \leq j \leq k$, be the index such that $y_m \in V(P^j)$. Note that, since y_m and b are adjacent to a_k, then, by Claim 2, $y_m \neq x_2^k$, hence $j < k$. This implies that P^k consists of only one node, namely a_k.

Claim 6: a_k is universal for A_{k-2} and a_k is not adjacent to a_{k-1}.

Proof of Claim 6: If a_k is universal for A_{k-2}, then by construction a_k is not adjacent to a_{k-1}. Assume, by contradiction, that a_k is not universal for A_{k-2}. Then $(V(P^k) \cup V(P^{k-1})) \setminus \{a_{k-1}\}$ contains a direct connection $\tilde{P}^{k-1} = \tilde{x}_1^{k-1}, \ldots, \tilde{x}_{l'_{k-1}}^{k-1}$ from x to S_{k-2} such that no node in $\tilde{P}^{k-1}$ is universal for A_{k-2} (obviously, $\tilde{P}^{k-1}$ contains $P^k = a_k$ and $\tilde{x}_1^{k-1} = a_k$). Let $\tilde{a}_{k-1} = \tilde{x}_1^{k-1}$, $\tilde{A}_{k-1} = A_{k-2} \cup \{\tilde{a}_{k-1}\}$ and $\tilde{S}_{k-1} = S_{k-2} \cup V(\tilde{P}^{k-1})$. Let $\tilde{P} = \tilde{x}_1, \ldots, \tilde{x}_{n'}$ be a direct connection contained in $(V(P) \cup V(P^{k-1})) \setminus (\tilde{S}_{k-1} \cup \{a_{k-1}\})$ from x to $\tilde{S}_{k-1}$. By Claim 3 and by construction of P^{k-1}, $\tilde{P}$ does not contain any node universal for $\tilde{A}_{k-1}$. But $\tilde{x}_1 = x_1$, x_1 is not adjacent to b, contradicting the inductive hypothesis. This proves Claim 6.

Let h be the lowest index such that $2 \leq h \leq l_j$ such that x_h^j is adjacent to b (one such index exists since $y_m \in V(P^j) \setminus a_j$).

Claim 7: $h \geq 5$ and every node in A_{j-1} has a neighbor in $P^j_{x_2^j x_h^j}$.

Proof of Claim 7: By Claim 2, $h \geq 3$, hence $\tilde{H} = b, x_1^j, P^j_{x_1^j x_h^j}, x_h^j, b$ is a hole. We first show that every node in A_{j-1} has a neighbor in $P^j_{x_2^j x_h^j}$. Assume not, then there exists q, $0 \leq q \leq j-1$, such that a_q has no neighbor in $P^j_{x_2^j x_h^j}$. Let Z be a shortest path between a_q and x_h^j in $S_{j-1} \cup V(P^j_{x_h^j x_{l_j}^j})$. Then by construction no node in $P^j_{x^j_2 x^j_{h-1}}$ has a neighbor in Z and x^j_1 has no neighbor in $Z \setminus a_q$. If a_q is not adjacent to a_j, then $C = x, a_j, P^j_{x^j_1 x^j_h}, x^j_h, Z, a_q, x$ is a big hole, otherwise $C' = a_j, P^j_{x^j_1 x^j_h}, x^j_h, Z, a_q, a_j$ is a big hole. In both cases, either (C, b) or (C', b) is a big wheel that is neither a twin wheel nor a triangle-free wheel, a contradiction. To conclude the proof of Claim 6, we have only to show that $h \geq 5$. Note that h must be odd, otherwise $\tilde{H}$ is an odd hole. Assume then, by contradiction, that $h = 3$. Then, since every node in A_{j-1} is adjacent to x^j_2 or x^j_3 but no node in P^j is universal for A_{j-1}, A_{j-1} contains a chordless co-path $Q = q_1, \ldots, q_s$ such that q_1 is adjacent to x^j_2 but not x^j_3, q_s is adjacent to x^j_3 but not x^j_2, and every intermediate node of Q is adjacent to both x^j_2 and x^j_3. But then $x, x^j_3, q_1, Q, q_s, x^j_2, x$ is a big anti-hole, a contradiction. This completes the proof of Claim 7.

Let $\tilde{H} = b, x^j_1, P^j_{x^j_1 x^j_h}, x^j_h, b$. By Claim 7, $\tilde{H}$ is a big hole.

Claim 8: $j < k - 1$.

Proof of Claim 8: We already observed that $j < k$. Assume, by contradiction, that $j = k - 1$. Let $\tilde{P}^0 = \tilde{H} \setminus b$ and $\tilde{a}_0 = x^j_1$. Let $\tilde{P}^1 = \tilde{x}^1_1, \ldots, \tilde{x}^1_l$ be a direct connection between x and $V(\tilde{P}^0) \setminus \{\tilde{a}_0\}$ contained in $\{a_k\} \cup V(P^j_{x^j_h y_m})$. By construction, $\tilde{a}_0$ has no neighbors in $\tilde{P}^1$. Let $\tilde{a}_1 = \tilde{x}^1_1 = a_k$. Therefore the sequence $\tilde{P}^0$, $\tilde{P}^1$ satisfies properties 1 and 2 at the beginning of the proof. Let $\tilde{P} = \tilde{x}_1, \ldots, \tilde{x}_{n'}$ be a direct connection between x and $(V(\tilde{P}^0) \cup V(\tilde{P}^1)) \setminus \{\tilde{a}_0, \tilde{a}_1\}$ contained in $V(F) \cup V(P^j_{x^j_h y_m})$, where F is the path found in Claim 5. Obviously, $\tilde{x}_1 = x_1$, no node in $\tilde{P}$ is universal for $\{\tilde{a}_0, \tilde{a}_1\}$ and $\tilde{x}_1$ is not adjacent to b. If $k > 1$, then $\tilde{x}_1$ not adjacent to b contradicts the inductive hypothesis on k. So $k = 1$ and $\tilde{a}_0 = a_0$, $\tilde{H} = H$, $\tilde{P}^0 = P^0$, $\tilde{a}_1 = a_1$, $\tilde{P}^1 = P^1 = a_1$ and $\tilde{P} = P$. Then, by Claims 3 and 4, a_1 and b have no neighbors in P, by Claim 5 $S_2 \cup V(P)$ contains a chordless path $F = y_1, \ldots, y_{m+1}$ between x_1 and b such that a_1 is adjacent to y_m and no other node in F, $y_1 = x_1$ is adjacent to a_0 and no node other node in $F \setminus y_1$. Hence y_m must be the neighbor of b in $H \setminus a_0$, so a_1 is adjacent in H to b and y_m but not to a_0. If a_1 has no further neighbors in H, then x, a_0, P^0, y_m, a_1, x is an odd hole, therefore (H, a_1) must be a twin wheel and a_1 is adjacent to the neighbor c of y_m in $H \setminus b$. Since y_m is the only neighbor of a_1 in F, then c is not a node of F, hence x_n is adjacent to y_m. $H' = x, x_1, P, x_n, y_m, a_1, x$ is a hole and (H', a_0) is a cap where x, x_1 are the attachments of a_0 in H'. $H \setminus \{a, b\}$ contains a direct connection P' from x to $V(H') \setminus \{x, x_1\}$ whose first node, that is the neighbor of a in $H \setminus b$, is not adjacent to x. By Lemma 4.7 the first node of P' must be adjacent to x_1, hence $n = 1$ and x_1 is adjacent in H to a, y_m and the neighbor of a in $H \setminus b$. Therefore (H, x_1) is a big wheel that is neither a triangle-free wheel nor a twin wheel, a contradiction. This completes the proof of Claim 8.

Claim 9: $j > 0$.

Proof of Claim 9: Assume $j = 0$, then y_m is the neighbor of b in $H \setminus a$. By Claim 8, $j < k - 1$, so by Claim 6 a_k is adjacent to a_0. Hence, in H, a_k is adjacent to a_0, b and y_m, so (H, a_k) is a twin wheel. Let $b' = a_k$, $H' = H \cup b' \setminus b$ is a big hole. (H', x) is a cap where the attachments of x in H' are a and b'. Note that $P^0 = H' \setminus b'$ and, by Claim 6, for every i, $0 \leq i \leq k-2$, a_i is adjacent to b'. Now P^{k-1} is a direct connection from x to S_{k-2} in $G \setminus (A_{k-2} \cup \{b'\})$ such that no node in P^{k-1} is universal for A_{k-2}, but $a_{k-1} = x_1^{k-1}$ is not adjacent to b', contradicting the inductive hypothesis. This completes the proof of Claim 9.

Assume that (H, x), $P^0, \ldots, P^k$, P and F are chosen so that j is largest possible, where the sequence $P^0, \ldots, P^k$ satisfies properties 1 and 2, P is a direct connection between x and S_k contained in $G \setminus (A_k \cup b)$ such that no node in P is universal for A_k and x_1 is not adjacent to b, and F satisfies Claim 5.

By Claim 7, the hole $\tilde{H}$ has length at least 6 and every node in A_{j-1} has a neighbor in $\tilde{H} \setminus \{a_j, b\}$. Let $\tilde{a}_0 = a_j$, $\tilde{P}^0 = \tilde{H} \setminus b$, $\tilde{S}_0 = V(\tilde{P}^0) \setminus b$ and $\tilde{A}_0 = \{a_0\}$. Since A_{j-1} is co-connected, there exists a bijection σ between $\{1, \ldots, j\}$ and $\{0, \ldots, j-1\}$ such that, if we define $\tilde{a}_i = a_{\sigma(i)}$ for every i, $1 \leq i \leq j$, and, for every $1 \leq q \leq j$, $\tilde{A}_q = \{\tilde{a}_i \,|\, 0 \leq i \leq q\}$, then for every q, $1 \leq q \leq j$, $\tilde{a}_q$ is not universal for $\tilde{A}_{q-1}$. Note that $\tilde{A}_j = A_j$ and every node in $\tilde{A}_j$ has a neighbor in $\tilde{S}_0$. For every i such that $1 \leq i \leq j$, we define $\tilde{S}_i = \tilde{S}_0$ and $\tilde{P}^i = \tilde{a}^i$.

For every i such that $j < i \leq k$, let $\tilde{a}_i = a_i$, $\tilde{A}_i = A_i$ and define recursively, for $i = j+1$ to k, the path $\tilde{P}^i$ and the set $\tilde{S}_i$ has follows: $\tilde{P}^i = \tilde{x}_1^i, \ldots, \tilde{x}_{l'_i}^i$ is a direct connection between x and $\tilde{S}_{i-1}$ contained in $V(P^i) \cup S_{i-1}$, while $\tilde{S}_i = \tilde{S}_{i-1} \cup V(\tilde{P}^i)) \setminus \{\tilde{x}_1^i\}$. By construction, $\tilde{S}_i \subseteq S_i$, P^i is a subpath of $\tilde{P}^i$ and $\tilde{x}_1^i = \tilde{a}_i$ is adjacent to b. Moreover, since $\tilde{P}^i$ is contained in $V(P^i) \cup S_{i-1}$, no node in $\tilde{P}^i$ is universal for $\tilde{A}_{i-1}$. Let $\tilde{P} = \tilde{x}_1, \ldots, \tilde{x}_{n'}$ be a direct connection from x to $\tilde{S}_k$ contained in $V(P) \cup S_k$. Since $\tilde{S}_k \subseteq S_k$, P is a subpath of $\tilde{P}$. Therefore $\tilde{x}_1 = x_1$ is not adjacent to b. Finally, since $\tilde{P}$ is contained in

$V(P) \cup S_k$, no node in $\tilde{P}$ is universal for $\tilde{A}_k$. By Claims 3 and 4, $\tilde{a}_k$ and b have no neighbors in $\tilde{P}$ and by Claim 5 $\tilde{S}_k \cup V(\tilde{P})$ contains a chordless path $\tilde{F} = \tilde{y}_1, \dots, \tilde{y}_{m'+1}$ between $\tilde{x}_1$ and b such that $\tilde{a}_k$ is adjacent to $\tilde{y}_{m'}$ and no other node in $\tilde{F}$ and $\tilde{y}_1$ is universal for A_{k-1}. Let j', $0 \leq j' \leq k$, be the index such that $\tilde{y}_{m'} \in V(\tilde{P}^{j'})$. By Claims 6-9, $1 \leq j' \leq k-2$. On the other hand, since $\tilde{S}_j = \tilde{S}_0$, $j' > j$ contradicting our choice of (H, x), $P^0, \dots, P^k$, P and F so that j is largest possible. □

By Lemmas 4.3-4.5, we can assume that G does not contain any big cap, any big antihole or any big wheel except twin wheels and triangle-free wheels.

Lemma 4.9 *If G contains a small cap, then G has a T-cutset.*

Proof:
Claim 1: Let (H, x) be a small cap where a, b denote the attachments of x in H, and let $P = x_1, \dots, x_n$ be a direct connection from x to $V(H) \setminus \{a, b\}$ in $G \setminus (V(H) \cup \{x\})$. If a has no neighbors in P, then $n = 1$ and x_1 is adjacent to both neighbors of a in H.

Proof of Claim 1: By Lemma 4.7 x_1 is adjacent to b and no other node in P is adjacent to b. Let a' and b' be, respectively, the neighbors of a in $H \setminus b$ and the neighbor of b in $H \setminus a$. If x_n is not adjacent to a', then $H' = x, x_1, P, x_n, b', a', a, x$ is a big hole and (H', b) is a proper wheel. So a' is adjacent to x_n. If $n = 1$ we are done, hence we may assume $n > 1$. If x_n is adjacent to b', then $H'' = x, x_1, P, x_n, a', a, x$ is a big hole and (H'', b') is a big cap. So x_n is not adjacent to b', $C = b, x_1, P, x_n, a', b', b$ is a big hole and (C, x) is a big cap, a contradiction. This proves Claim 1.

Let $Q = y_1, \dots, y_m$ be the longest chordless path in $\overline{G}$. Note that the complement of a small cap is a chordless path on 5 nodes, so, if G contains a small cap, then Q has at least 5 nodes (i.e. $m \geq 5$). Let (H, y_3) be the cap induced by $\{y_i \mid 1 \leq i \leq 5\}$, where $H = y_1, y_5, y_2, y_4$ and y_1, y_5 are the attachments of y_3 in H. Define A to be a maximal co-connected set contained in $G \setminus \{y_i \mid 2 \leq i \leq 5\}$ such that $y_1 \in A$ with the property that every node in A is adjacent to y_3, y_4, y_5 but not y_2. Note that, for every $y \in A$, $Q \setminus y_1 \cup y$ is a chordless co-path. Otherwise, there exists j, $6 \leq j \leq m$, such that y_j is not adjacent to y. Assume j is the lowest such index. Then $C = y, y_2, Q_{x_2 x_j}, y_j, y$ is a big anti hole, a contradiction. Let B be the set of all nodes in $V(G) \setminus \{y_3, y_4\}$ that are universal for A. If $A \cup B$ is a cutset separating y_3 and $\{y_2, y_4\}$, then let C be the connected component of

$G \setminus (A \cup B)$ containing y_3 and let $D = V(G) \setminus (A \cup B \cup C)$. Then A, B, C, D is a skew-partition, $y_3 \in C$ is universal for A and $y_4 \in D$ is universal for A, hence $A \cup B$ is a T-cutset.

Next we will show that $A \cup B$ is a cutset separating y_3 and $\{y_2, y_4\}$. Assume not. Then there exists a direct connection $P = x_1, \dots, x_n$ in $G \setminus (A \cup B)$ between y_3 and $\{y_2, y_4\}$. If there exists a node $y \in A$ with no neighbors in P, then consider $H' = H \cup y \setminus y_1$. H' is a hole of length 4 and (H', y_3) is a small cap. By Claim 1, $n = 1$ and x_1 is adjacent to y_4 and y_5. If x_1 is adjacent to y_2, then $x_1, y, y_2, Q_{y_2 y_5}, y_5$ is a path in $\overline{G}$. Since Q is the longest path in $\overline{G}$, then $Q \setminus y_1 \cup \{x_1, y\}$ is not a chordless path. Therefore x_1 has a neighbor (in $\overline{G}$) in $Q \setminus y_1$. Let j be the lowest index such that x_1 is adjacent to y_j in $\overline{G}$. Then $6 \leq j$ and $C = x_1, y, y_2, Q_{y_2 y_j}, y_j, x_1$ is a big anti-hole in G, a contradiction. Hence x_1 is not adjacent to y_2, so $A \cup x_1$ is a co-connected set, x_1 is adjacent to y_3, y_4 and y_5 but not y_2, contradicting the maximality of A.
So every node in A must have a neighbor in P. For every $y \in A$ let $h(y)$ be the minimum index such that y is adjacent to $x_{h(y)}$, and let $h = \max_{y \in A} h(y)$. If $h > 2$, then let $x \in A$ be such that $h = h(x)$ and let $H' = x, y_3, x_1, P_{x_1 x_h}, x_h, x$. H' is a big hole and y_5 is adjacent to x and y_3 in H'. Since (H', y_5) is not a big cap, then (H', y_5) must be a twin wheel, hence y_5 is adjacent to either x_1 or x_h. If y_5 is adjacent to x_1, then let F be a shortest path from y_5 to x_h in $V(P_{x_h x_n}) \cup \{y_2, y_4, y_5\}$, then $H'' = y_5, x_1, P_{x_1 x_h}, x_h, F, y_5$ is a big hole and (H'', y_3) is a big cap. If y_5 is adjacent to x_h, then let $H'' = H' \cup y_5 \setminus x$. Since, by definition of h, every node of A has a neighbor in $P_{x_1 x_h}$ and every node in A is adjacent to y_3 and y_5, then (H'', y) is a twin wheel for every $y \in A$. Since no node in P is universal for A and A is co-connected, then there exists two nonadjacent nodes u and v in A such that u is adjacent to x_1 and not to x_h, and v is adjacent to x_h and not to x_1. Therefore $V(H) \cup \{u, v\} \setminus \{y_5\}$ induces a big cap, a contradiction. Therefore $h \leq 2$ and, since no node in P is universal for A, $h = 2$ and every node in A is adjacent to x_1 or x_2. Since x_1 and x_2 are not universal for A and A is co-connected, there exists a chordless co-path $Z = z_1, \dots, z_k$ contained in A such that z_1 is adjacent to x_1 but not x_2, z_k is adjacent to x_2 but not x_1 and all the intermediate nodes of Z are adjacent to both x_1 and x_2. If x_2 is not adjacent to y_4, then $y_4, x_2, z_1, Z, z_k, x_1, y_4$ is a big anti-hole. Then x_2 is adjacent to y_4, so $y_4, y_3, x_2, z_1, Z, z_k, x_1, y_4$ is a big anti-hole, a contradiction.
$\square$

4.2.4 Meyniel graphs

Lemmas 4.3, 4.4, 4.5 and 4.9 imply that, if G and $\bar{G}$ do not contain a proper wheel or a long prism, then, if G contains a cap, G has a loose skew-partition. Next we have to address the case in which G does not contain any cap. Note that the class of Berge graphs containing no caps coincide with the class of Meyniel graphs (see Section 2.5.2), that is the class of graphs in which every odd cycle has at least 2 chords. As we already mentioned, Burlet and Fonlupt [7] showed that Meyniel graphs are either bipartite or can be decomposed by amalgams and clique cutsets, and both this decompositions imply the presence of a star cutset. Hoàng [57], gave a short proof of a weaker result, namely:

Theorem 4.10 *If G is a Meyniel graph, then either G is bipartite or $\bar{G}$ contains a star-cutset or a U-cutset.*

For the sake of completeness, we give a proof of Theorem 4.10, essentially following [57].

Proof: If G is not bipartite, then, since G is Berge, G contains three pairwise adjacent nodes u, v and w. Let U and V be, respectively, the set of neighbors of u and v in $\bar{G}$, and let S be the connected component of $\bar{G} \setminus (U \cup V)$ containing w. Let U', V' and X be, respectively, the set of nodes in $U \setminus V$, $V \setminus U$ and $U \cap V$ that are adjacent to some node of S in $\bar{G}$. Note that, if $U' = \emptyset$ or $V' = \emptyset$, then $\{v\} \cup V' \cup X$ or $\{u\} \cup U' \cup X$, is a star cutset of $\bar{G}$ centered, respectively, at v or u. Hence we may assume $U' \neq \emptyset$ and $V' \neq \emptyset$. Next we show that, in $\bar{G}$, every node in U' is adjacent to every node in V'. Assume not and let $u' \in U'$ and $v' \in V'$ be nonadjacent in $\bar{G}$ and let x_u and x_v be, respectively, neighbors (in $\bar{G}$) of u' and v' in S at minimum distance in $\bar{G}[S]$. Let Q be a shortest path between x_u and x_v in $\bar{G}[S]$. Then $u, u', x_u, Q, x_v, v', v$ is a chordless path containing at least 5 nodes, hence $G[V(Q) \cup \{u, u', v, v'\}]$ contains a small cap, a contradiction. If no connected component of $G[U' \cup V' \cup X]$ intersects both U' and V', then let A be the union of all connected components of $G[U' \cup V' \cup X]$ intersecting U' and let $B = (U' \cup V' \cup X) \setminus A$. Then $A \cup B$ is a cutset separating S and $\{u, v\}$ in $\bar{G}$, u is universal for A while v is universal for B, so $A \cup B$ is a U-cutset. Hence we can assume that there are nodes $u' \in U'$ and $v' \in V'$ such that there exists a chordless path P between u' and v' in $G[X \cup \{u', v'\}]$. Since, in G, u' is not adjacent to v', P has length at least two, so $H = v, u', P, v', u, v$

is a big hole. S is an anticonnected set (in G) and, by definition of U', V' and X, no node in P is universal for S (in G). But then S sees exactly one edge in H, namely uv. Since G contains no caps, then every node in S has a neighbor in $H \setminus \{u, v\}$. By Theorem 3.6, there exist two nonadjacent nodes x and y in S such that (H, x) and (H, y) are twin wheels and the only edge of H that sees both x and y is uv. As one can readily verify, $H \cup \{x, y\}$ contains a big cap, a contradiction. □

Chapter 5

Recognizing Balanced and Balanceable Matrices

5.1 Introduction

A $0, \pm 1$ matrix is *balanced* if it does not contain a square submatrix with exactly two nonzero elements in each row and in each column such that the sum of all entries equals 2 modulo 4. This notion was introduced by Berge [3] for $0, 1$ matrices, and generalized to $0, \pm 1$ matrices by Truemper [75]. Berge [4] proved that balanced $0, 1$ matrices are perfect: indeed, as we shall see, a stronger property holds, as several polyhedra are integral whenever the constraint matrix is balanced. The question if there exists a polynomial-time algorithm to decide whether or not a given $0, \pm 1$ matrix is balanced, has been settled by Conforti, Cornuéjols and Rao [26] for the $0, 1$ case, and by Conforti, Cornuéjols, Kapoor and Vušković [23] for the general case. The proof of this fact is based on a decomposition theorem for the class of balanced matrices, very much in the same spirit as the decomposition theorem for Berge graphs discussed in Chapter 2. Unfortunately, the proof of this theorem is long and difficult. In this chapter we will provide a simpler, self contained polynomial-time algorithm to recognize balanced matrices, which does not rely on the decomposition theorem for balanced matrices [80]. The algorithm uses ideas from Conforti, Cornuéjols and Rao [26], Conforti, Cornuéjols, Kapoor and Vušković [23] and Chudnovsky and Seymour [15]. Interestingly, Kapoor [58] showed that deciding if a matrix contains a minimally unbalanced submatrix intersecting a prescribed column is NP-complete.

A $0,1$ matrix A is said to be *balanceable* if its nonzero entries can be signed $+1$ or -1 so that the resulting $0,\pm1$ matrix A' is balanced. The problem of deciding if a given $0,1$ matrix is balanceable can be reduced to that of deciding if a $0,\pm1$ matrix A is balanced using a signing algorithm due to Camion [9], that will be described in section 5.1.2. In fact, Camion's algorithm assigns values ±1 to the nonzero entries of A in such a way that, if A is balanceable, the resulting matrix A' is balanced. Therefore, one can just apply Camion's algorithm to A, and then test if A' is balanced. This was the only approach known so far to solve the recognition problem for balanceable matrices. On the other hand, Truemper [76] gave a co-NP characterization for this class of matrices by showing that, if a matrix is not balanceable, then it must contain some special submatrix, whose structure is well described in terms of its bipartite representation. This indicates that one could try to recognize balanceable matrices by looking for these forbidden submatrices. An algorithm that does precisely was given by Conforti and Zambelli [38], and it will be described in this chapter.

5.1.1 Notations and definitions

Often, it will be convenient to work with the bipartite representation of a matrix. Given a $0,1$ matrix A, the *bipartite representation of* A is the bipartite graph G where the two sides of the bipartition are the sets R and C of rows and columns of A, respectively, and there is an edge between $i \in R$ and $j \in C$ if and only if $a_{ij} = 1$. Clearly, A is balanced if and only if its bipartite representation does not contain a *hole* of length 2 modulo 4 as an induced subgraph (a hole is a chordless cycle). A bipartite graph is *balanced* if it does not contain any hole of length 2 modulo 4. For general $0,\pm1$, matrices, the most convenient setting to work with, is their signed bipartite representation. A *signed bipartite graph* is a pair (G,σ) where G is a bipartite graph and σ is a *signing* of the edges, that is a function from $E(G)$ to $\{1,-1\}$. Given a $0,\pm1$ matrix A, the *signed bipartite representation of* A is the signed bipartite graph (G,σ) where G is the bipartite representation of the support matrix of A and σ is defined, for each edge ij, by $\sigma(ij) = a_{ij}$. For any subgraph F of G, we define

$$\sigma(F) = \sum_{e \in E(F)} \sigma(e).$$

It is immediate to verify that a 0 ± 1 matrix A is balanced if and only if its signed bipartite representation does not contain a hole H such that $\sigma(H) \equiv 2 \bmod 4$ as an induced subgraph. We will say that such a hole is *unbalanced*, and a signed bipartite graph is *balanced* if it contains no unbalanced hole. Observe that, given a cut $(S, \bar{S})$ of G (where S is a subset of the nodes of G), if we define a signing σ' by

$$\sigma'(ij) = \begin{cases} \sigma(ij) & \text{if } ij \notin (S, \bar{S}) \\ -\sigma(ij) & \text{if } ij \in (S, \bar{S}) \end{cases},$$

it is easy to see that (G, σ) is balanced if and only if (G, σ') is balanced (since, for any hole H, $\sigma(H) \equiv \sigma'(H) \bmod 4$). We call this operation *scaling* along the cut $(S, \bar{S})$.

In the remainder of the chapter, G will always be a bipartite graph. Given a graph F and two nodes x and y of F, $d_F(x, y)$ denotes the length of the shortest path between x and y contained in F.

The following two graphs will play an important role in the remainder of the paper. Given two nonadjacent nodes a and b in distinct sides of the bipartition, a *3-path configuration* between a and b is a graph consisting of three chordless paths P^1, P^2, P^3 between a and b such that, for every $1 \leq i < j \leq 3$, no node in the interior of P^i belongs to or has a neighbor in the interior of P^j. We say that P^1, P^2, P^3 form a 3-path configuration.
A *wheel* consists of a hole H and a node v outside H with at least 3 distinct neighbors in H, and is denoted by (H, v). The node v is called the *center* of the wheel. A wheel (H, v) for which v has k neighbors in H is said a k-wheel. (H, v) is an *odd wheel* if it is a wheel and v has an odd number of neighbors in H.

It is easy to see that if G contains a 3-path configuration or an odd wheel, then G is not balanceable. In fact, if F is a 3-path configuration or an odd wheel contained in G, then F contains an odd number of edges, and each edge is contained in exactly 2 holes. Denote by $\mathcal{H}$ the family of all holes in F. For any signing σ of F, $\sum_{H \in \mathcal{H}} \sigma(H) = 2\sigma(F) \equiv 2 \bmod 4$, therefore there exists a hole H such that $\sigma(H) \equiv 2 \bmod 4$.

Truemper showed that the converse is also true.

Theorem 5.1 (Truemper [76]) *A bipartite graph G is balanceable if and only if it does not contain a 3-path configuration or an odd wheel*

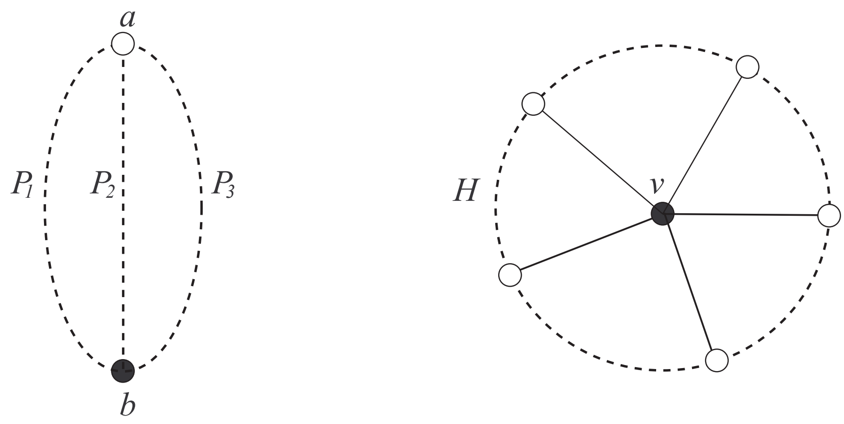

Figure 5.1: A 3-path configuration and a wheel.

Thus, deciding if a graph G is balanceable is equivalent to determining if G contains a 3-path configuration or an odd wheel. A nice proof of Theorem 5.1 can be found in [34].

5.1.2 Camion's Signing Algorithm

As we already mentioned, Camion [9] gave a polynomial time algorithm to sign the nonzero entries of a $0,1$ matrix A so that the resulting matrix A' is balanced if A is balanceable. Here we describe such an algorithm. Observe that, given a set S of rows an columns of A', then multiplying the rows and columns of A' in S by -1 corresponds to scaling along $(S, \bar{S})$ in the bipartite representation G of A'. Given a maximal forest F of G and an edge e of F, there exists a cut $(S, \bar{S})$ of G such that $E(F) \cap (S, \bar{S}) = \{e\}$. Thus, one can multiply some of the rows and columns of A' so that the entries in A'' corresponding to the edges in F have an arbitrarily fixed sign. This simple observation is central to the algorithm of Camion.

Claim 5.2 *There exists a polynomial time algorithm with the following specifications:*

- **Input** *A balanceable* $0,1$ *matrix* $A = (a_{ij})$*, a maximal forest* F *of its bipartite representation* G*, and a signing* σ *of* F*.*

- **Output** *The unique balanced matrix* $A' = (a'_{ij})$ *such that* $|a'_{ij}| = a_{ij}$, *and* $a'_{ij} = \sigma(ij)$ *for every* $ij \in E(F)$.

Camion's Algorithm: Let $G_0 = F$, $n = |E(G)|$.
For $i = 0, \ldots, n - |E(F)| - 1$, do the following:

1. Choose an edge $e_i \in E(G) \setminus E(G_i)$ and a path P_i in G_i between its two endnodes so that $|P_i|$ is minimum over all possible choices of e_i and P_i;

2. Define $\sigma(e_i) \equiv -\sigma(P_i)$ mod 4, and $G_{i+1} = (V(G), E(G_i) \cup \{e_i\})$.

Define $A'_{ij} = \sigma(ij)$ for every $ij \in E(G)$, 0 otherwise.

Correctness: At each iteration, the edge e_i and the path P_i form a hole H_i of G_{i+1} which, by the choice of e_i and P_i, is also a hole in G. The only way to extend the signing constructed so far so that $\sigma(H_i) \equiv 0$ mod 4 is to assign $\sigma(e_i) \equiv -\sigma(P_i)$ mod 4. Since we know that there exists a balanced signing of G which extends the signing of F, then the signing produced by the algorithm is the only possible. □

We already observed that testing if a matrix is balanceable can be reduced, via Camion's algorithm, to testing if a matrix is balanceable. The converse is also true. Suppose we have a polynomial time algorithm to test if a matrix is balanceable, and we wish to test if a given $0, \pm 1$ matrix A is balanced. Let B be the support matrix of A. Test if B is balanceable. If it is not, then output that A is not balanced. Else, let F be a maximal forest in the bipartite representation of B and let $\sigma(ij) = a_{ij}$ for every $ij \in E(F)$. Apply Camion's algorithm to B, F, and σ, to obtain a balanced matrix B'. Since B' is unique, then A is balanced if and only if $A = B'$.

5.1.3 Overview

In section 5.2 we will discuss some polyhedral properties of balanced matrices related to the classical models of set packing and set covering. In the reminder of the chapter we will provide two algorithms: one to recognize balanced signed bipartite graphs, and one to test if a given bipartite graph is balanceable. In Section 5.3 we will provide an algorithm to recognize whether a bipartite graph has a 3-path configuration, while in Section 5.4 an algorithm is presented, to recognize if a bipartite graph not containing any

3-path configuration, contains a detectable 3-wheel (which is a special type of odd wheel). By Theorem 5.1, if G contains any of these graphs, then G is not balanceable, so (G, σ) is not balanced for any signing σ.

In Section 5.5 we study some properties of the odd wheels of minimum cardinality in a bipartite graph. Understanding what are the possible adjacencies between smallest odd wheels and the remaining nodes of the graph will be fundamental for the remainder development of the algorithm. Finally, in sections 5.6 and 5.7 we give the algorithms for recognizing balanced and balanceable matrices, respectively. Both algorithms have running time $O(|V(G)|^9)$.

5.2 Properties of balanced matrices

The following theorem has been proved by Berge [4] for $0,1$ matrices, and by Conforti and Cornuéjols [20] for the general case. If A is a $0,\pm 1$ matrix, we denote by $n_i(A)$ the number of -1's in the ith row of A.

Theorem 5.3 *Let A be an $m \times n$ balanced $0,\pm 1$ matrix with rows a^i, $i \in [m]$, and let S_1, S_2, S_3 be a partition of $[m]$. Then*

$$\begin{aligned} R(A) = \{x \in \mathbb{R}^n \quad : \quad & a^i x \leq 1 - n_i(A) \text{ for } i \in S_1 \\ & a^i x = 1 - n_i(A) \text{ for } i \in S_2 \\ & a^i x \geq 1 - n_i(A) \text{ for } i \in S_3 \\ & \mathbf{0} \leq x \leq \mathbf{1}\} \end{aligned}$$

is an integral polytope.

A proof of a more general result will be given in Chapter 6. Note that, if A has only nonnegative entries and $S_2, S_3 = \emptyset$, then Theorem 5.3 implies that $0,1$ balanced matrices are perfect. This can be easily seen also as a consequence of Theorem 2.6 and the Strong Perfect Graph Theorem 2.10, as if a $0,1$ matrix A does not contain a 3×3 unbalanced matrix, then the undominated rows of A form the node-clique incidence matrix of some graph G, and if G contains an odd hole of length k then A contains a $k \times k$ unbalanced submatrix, while if G contains an odd antihole, then A contains a 5×5 unbalanced submatrix; hence, if A is balanced, then G is perfect, thus A is perfect. Indeed, balanced matrices have an even stronger property.

A linear system $Ax \leq b$ is *totally dual integral* (TDI) if the linear program $\max\{cx \mid Ax \leq b\}$ has an optimal dual solution y for every integral vector c for which the linear program has a finite optimum. Edmonds and Giles [47] showed that, if $Ax \leq b$ is TDI and b is integral, then $P = \{x \mid Ax \leq b\}$ is a integral polyhedron.

Theorem 5.4 *Let A be an $m \times n$ balanced $0, \pm 1$ matrix with rows a^i, $i \in [m]$, and let S_1, S_2, S_3 be a partition of $[m]$. Then the system*

$$\begin{cases} a^i x \leq 1 - n(a^i) & \text{for } i \in S_1 \\ a^i x = 1 - n(a^i) & \text{for } i \in S_2 \\ a^i x \geq 1 - n(a^i) & \text{for } i \in S_3 \\ \mathbf{0} \leq x \leq \mathbf{1}\} & \end{cases}$$

is totally dual integral.

Theorem 5.4 was proven by Fulkerson, Hoffman and Oppenheim [50] for the $0, 1$ case, and by Conforti and Cornuéjols [20] in the general case.
We refer the reader to [30] for a survey on balanced matrices.

5.3 Detecting a 3-path configuration

We say that a 3-path configuration is *smallest* in G if it contains the minimum number of nodes among all 3-path configurations in G.

Claim 5.5 *Let Π be a smallest 3-path configuration in $G = (R, C; E)$. Assume Π is formed by the paths $P^i = a, a_i, \ldots, b_i, b$, $i \in [3]$, where $a \in R$, $b \in C$. For every $i \in [3]$, let m_i be a node of P^i such that $|d_{P^i}(a_i, m_i) - d_{P^i}(b_i, m_i)| \leq 1$. Let X be the set of nodes of G with no neighbors in $\{a, b, a_2, a_3, b_2, b_3\}$, and P be a shortest path between a_1 and m_1 in $G[X \cup \{a_1, m_1\}]$. Then $Q^1 = a, a_1, P, m_1, P^1_{m_1 b_1}, b_1, b$ is a chordless path and Q^1, P^2, P^3 form a smallest 3-path configuration.*

Symmetrically, analogous statements hold for every P^i, $i \in [3]$, and all possible pairs a_i, m_i and m_i, b_i

Proof: Let $P = p_1, \ldots, p_k$ where $a_1 = p_1$ and $m_1 = p_k$. If $a_1 = m_1$ or a_1 is adjacent to m_1, then the statement holds trivially, hence we may assume $|P^1| \geq 5$ and $m_1 \neq b_1$, therefore m_1 has no neighbors in P^2 or P^3.

If no node in the interior of P belongs to or has a neighbor in P^2 or P^3 then, given Q^1 the shortest path between a and b with interior in $V(P \cup P^1_{m_1b_1})$, Q^1, P^2, P^3 form a 3-path configuration between a and b which, by the minimality of Π and the choice of P, must have the same cardinality as Π, hence $Q^1 = a, a_1, P, m_1, P^1_{m_1b_1}, b_1, b$ and we are done.

Assume, then, that there exists h, $2 \leq h \leq k-1$, such that p_h belongs to or has a neighbor in P^2 or P^3, and let h be maximum with this property. Note that, by definition, p_h does not belong to P^2 or P^3.
Suppose p_h has at least two distinct neighbors in $P^2 \cup P^3$. If $p_h \in R$, let Q^1 be the shortest path between p_h and b in $P_{p_hp_k} \cup P^1_{m_1b}$, let Q^2 be the (unique) shortest path between p_h and b in $(p_h \cup P^2 \cup P^3) \setminus b_3$ and Q^3 be the (unique) shortest path between p_h and b in $(p_h \cup P^2 \cup P^3) \setminus b_2$. Then Q^1, Q^2, Q^3 form a 3-path configuration between p_h and b which is strictly shorter than Π since $|Q^1| < |P^1|$ and $|Q^2| + |Q^3| \leq |P^2| + |P^3|$. Similarly, if $p_h \in C$, let Q^1 be the shortest path between a and p_h in $P_{p_hp_k} \cup P^1_{am_1}$, let Q^2 be the (unique) shortest path between a and p_h in $(p_h \cup P^2 \cup P^3) \setminus a_3$ and Q^3 be the (unique) shortest path between a and p_h in $(p_h \cup P^2 \cup P^3) \setminus a_2$. Then Q^1, Q^2, Q^3 form a 3-path configuration Π' between a and p_h. Since $|P^1_{a_1m_1}| \leq |P^1_{b_1m_1}| + 1$ and $h \geq 2$, then

$$\begin{aligned} |Q^1| &\leq |P| - 1 + |P^1_{am_1}| \leq |P^1_{am_1}| + |P^1_{a_1m_1}| - 1 \\ &\leq |P^1_{am_1}| + |P^1_{m_1b_1}| < |P^1|. \end{aligned} \tag{5.1}$$

Furthermore, $|Q^2| + |Q^3| \leq |P^2| + |P^3|$, hence Π' has cardinality strictly smaller than Π, a contradiction.
Therefore, we may assume that p_h has a unique neighbor x in $P^2 \cup P^3$, say $x \in V(P^2)$. If $x \in R$, then let Q^1 be the shortest path between x and b in $x \cup P_{p_hm_1} \cup P^1_{m_1b}$, let $Q^2 = x, P^2_{xb}, b$ and $Q^3 = x, P^2_{xa}, a, P^3, b$. Then Q^1, Q^2, Q^3 form a 3-path configuration between x and b which has cardinality strictly smaller than Π since $|Q^2| + |Q^3| = |P^2| + |P^3|$ and

$$|Q^1| \leq |P| - 1 + |P^1_{m_1b}| + 1 \leq |P^1_{a_1m_1}| + |P^1_{m_1b}| < |P^1|.$$

If $x \in C$, then let Q^1 be the shortest path between x and a in $x \cup P_{p_hm_1} \cup P^1_{am_1}$, let $Q^2 = a, P^2_{ax}, x$ and $Q^3 = a, P^3, b, P^2_{bx}, x$. Then Q^1, Q^2, Q^3 form a 3-path configuration Π' between x and a. If $h = 2$, then $|Q^1| = 3 < |P^1|$, otherwise $h \geq 3$ and $|Q^1| \leq |P| + |P^1_{am_1}| - 1 < |P^1|$. Since $|Q^2| + |Q^3| = |P^2| + |P^3|$, then Π' has cardinality strictly smaller than Π, a contradiction. $\square$

Claim 5.6 *There exists a $O(|V(G)|^9)$ algorithm with the following specifications:*

- **Input** *A bipartite graph G.*
- **Output** *Either:*
  1. *a 3-path configuration Π, or*
  2. *it determines that G does not contain any 3-path configurations.*

Algorithm:
For every 6 tuple $a_1, a_2, a_3, b_1, b_2, b_3$ such that:

- $a_i \in R$, $b_i \in C$ for every $i \in [3]$,
- a_i is nonadjacent to b_j for every $i \neq j$,
- there exist nonadjacent nodes x and y such that x is adjacent to a_1, a_2, a_3 and y is adjacent to b_1, b_2, b_3;

do the following:

1. For $i = 1, 2, 3$, compute the set $X(i)$ of nodes that are not adjacent to any of x, y, a_j or b_j for $j \neq i$.
2. For $i = 1, 2, 3$, for every node $m \in X(i)$, compute the paths $Q^i(m)$ and $R^i(m)$ (if they exist), where $Q^i(m)$ is the shortest path between a_i and m in $G[X(i) \cup a_i]$ and $R^i(m)$ is the shortest path between b_i and m in $G[X(i) \cup b_i]$.
3. For $i = 1, 2, 3$, for every node $m \in X(i) \cup a_i$, define $P^i(m)$ as follows: if a_i is adjacent to b_i, then $P^i(a_i) = a_i, b_i$ and $P^i(m)$ is undefined for every $m \in X(i)$; else $P^i(a_i)$ is undefined and for every $m \in X(i)$ satisfying the following

   (i) $Q^i(m)$ and $R^i(m)$ both exist

   (ii) No node in $Q^i(m)$, except m, belongs to or has a neighbor in $R^i(m)$

   let $P^i(m) = x, a_i, Q^i(m), m, R^i(m), b_i, y$, else, if $Q^i(m)$ and $R^i(m)$ do not satisfy (i) and (ii), $P^i(m)$ is undefined.

4. For every $m \in X(i) \cup a_i$ such that $P^i(m)$ is defined, compute the set $Y_i(m)$ of nodes that do not belong or have a neighbor in the interior of $P^i(m)$.

5. For every $1 \leq i < j \leq 3$, and for every $m_i \in X(i) \cup a_i$ and every $m_j \in X(j) \cup a_j$, verify that the interior of $P^j(m_j)$ is contained in $Y_i(m_i)$. If this is the case, say that the pair m_i, m_j is (i,j)-good.

6. Verify if there exists a triple m_1, m_2, m_3 such that $m_i \in X(i) \cup a_i$ for $i \in [3]$ and such that m_i, m_j is (i,j)-good for every $1 \leq i < j \leq 3$. If such a triple exist, output the graph Π induced by $P^1(m_1), P^2(m_2), P^3(m_3)$ and stop.

Otherwise output the fact that G contains no 3-path configuration.

Correctness: It takes time $O(|V(G)|)^8$ to compute all possible 6-tuples a_1, a_2, a_3, b_1, b_2, b_3 as above, and there are $O(|V(G)|)^6$ of them. For each 6-tuple, each step from 1 through 6 takes time $O(|V(G)|)^3$, therefore the total running time is $O(|V(G)|)^9$.
If for some 6-tuple, in step 6 the algorithm outputs a graph Π induced by $P^1(m_1)$, $P^2(m_2)$, $P^3(m_3)$, then Π is a 3-path configuration between x and y, since step 3 ensures that $P^i(m_i)$ is a chordless path between x and y for every $i \in 3$, while steps 5 and 6 guarantee that no node in the interior of $P^i(m_i)$ belongs to or has a neighbor in the interior of $P^j(m_j)$ for every $1 \leq i < j \leq 3$.
We only need to verify that, if G contains some 3-path configuration, then the algorithm will detect one. Let $\tilde{\Pi}$ be a smallest 3-path configuration in G. Let $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ be the 3-paths inducing $\tilde{\Pi}$, where $\tilde{P}_i = a, a_i, \ldots, b_i, b$. Then there exist nonadjacent nodes x and y such that x is adjacent to a_i and y is adjacent to b_i for every $i \in [3]$ (since $x = a$ and $y = b$ would satisfy such condition). For $i = 1, 2, 3$, let P^i be the shortest path between x and y with interior contained in the interior of $\tilde{P}_i$. Then P^1, P^2, P^3 form a 3-path configuration Π with at most as many nodes as $\tilde{\Pi}$, hence Π and $\tilde{\Pi}$ must have the same cardinality and $P^i = x, a_i, \ldots, b_i, y$. For every $i \in [3]$, let m_i be a node of P^i such that $|d_{P^i}(a_i, m_i) - d_{P^i}(b_i, m_i)| \leq 1$, in particular we may assume that, when a_i and b_i are adjacent, $m_i = a_i$. Then, by 5.5, given $Q^1 = x, a_1, Q^1(m_1), m_1, P^1_{m_1 b_1}, b_1, x$, where $Q^1(m_1)$ is the path computed in step 2 of the algorithm, Q^1, P^2, P^3 forms a 3-path configuration between x and y. By repeating the argument, we conclude that the paths $P^1(m_1), P^2(m_2), P^3(m_3)$ computed by the algorithm form a 3-path configuration between x and y, hence the algorithm would have output the correct answer. □

5.4 Detectable 3-wheels

A 3-wheel (H, v) is *detectable* if two of the neighbors of v in H have distance two in H. If (H, v) has the minimum number of nodes among all possible detectable 3-wheels, we say that (H, v) is a *smallest* detectable 3-wheel.

Claim 5.7 *Let $G = (R, C; E)$ be a bipartite graph containing no 3-path configurations. Let (H, v) be a smallest detectable 3-wheel in G. Let u, v_1 and v_2 be the neighbors of v in H, where v_1 and v_2 are both adjacent to a node w in H. Let u_1 and u_2 be the two neighbors of u in H such that the two maximal paths P^1 and P^2 in $H \setminus \{u, w\}$ have endpoints u_1, v_1 and u_2, v_2, respectively. Let s be the neighbor of u_1 in P^1. Let X be the set of nodes with no neighbors in $\{u, v, w, u_2, v_2\}$. Let P be a shortest path between v_1 and s in $G[X \cup \{v_1, s\}]$. Then $H' = v_1, P, s, u_1, u, u_2, P^2, v_2, w, v_1$ is a hole and (H', v) is a smallest detectable 3-wheel.*

Proof: Let $P = p_1, \ldots, p_k$, where $p_1 = v_1$ and $p_k = s$. W.l.o.g., $v \in R$ and $u \in C$. If no node in the interior of P belongs to or has a neighbor in P^2, then $H' = v_1, P, s, u_1, u, u_2, P^2, v_2, w, v_1$ is a hole, hence by construction (H', v) is a detectable 3-wheel which is smallest since $|P| \leq |P^1| - 1$. We may therefore assume that there exists h, $2 \leq h \leq k - 1$, such that p_h belongs to or has a neighbor in P^2. Assume h is the highest such index. Then p_h does not belong to P^2. Suppose p_h has exactly one neighbor in P^2, say x. If $x \in R$, then let Q^1 be the shortest path between x and u in $P_{p_h p_k} \cup x, u_1, u$, let $Q^2 = x, P^2_{x v_2}, v_2, v, u$ and $Q^3 = x, P^2_{x u_2}, u_2, u$. Then $|Q^i| \geq 3$ and Q^1, Q^2, Q^3 form a 3-path configuration between x and u, a contradiction. If $x \in C$, then let Q^1 be the shortest path between x and v in $P_{p_h p_k} \cup P^1_{s v_1} \cup \{v, x\}$, $Q^2 = x, P^2_{x v_2}, v_2, v$ and $Q^3 = x, P^2_{x u_2}, u_2, u, v$. Q^1, Q^2, Q^3 form a 3-path configuration between x and v. Hence we may assume that p_h has at least 2 neighbors in P^2. Let x and y be the neighbors of p_h in P^2 that are closest, respectively, to v_2 and u_2. If $p_h \in R$, let Q^1 be the shortest path between p_h and u in $P_{p_h p_k} \cup u_1, u$, let $Q^2 = p_h, x, P^2 x v_2, v_2, v, u$ and $Q^3 = p_h, y, P^2_{y u_2}, u_2, u$. Then Q^1, Q^2, Q^3 form a 3-path configuration between p_h and u. If $p_h \in C$, then let Q^1 be the shortest path between p_h and v in $P_{p_h p_k} \cup P^1_{s v_1} \cup v$, $Q^2 = p_h, x, P^2_{x v_2}, v_2, v$ and $Q^3 = p_h, y, P^2_{y u_2}, u_2, u, v$. Q^1, Q^2, Q^3 form a 3-path configuration between x and v, a contradiction. $\square$

Claim 5.8 *There exists a $O(|V(G)|^9)$ algorithm with the following specifications:*

- **Input** *A bipartite graph G containing no 3-path configuration.*
- **Output** *Either:*
  1. *a detectable 3-wheel, or*
  2. *it determines that G does not contain any detectable 3-wheel.*

Algorithm:
For every 7 tuple $u_1, u_2, v, v_1, v_2, w, s$ such that:

- v and w are both adjacent to v_1 and v_2
- there exists a node x such that x is adjacent to v, u_1, u_2 but not to w
- s is adjacent to u_1
- either $s = v_1$, or s has no neighbors in $\{u_2, v, v_2, x, w\}$.

do the following:

1. Compute the set X of nodes that do not belong to or have a neighbor in $\{u_2, v, v_2, x, w\}$.
2. Compute the shortest path P, if any, between v_1 and s in $G[X \cup \{v_1\}]$.
3. Verify that the only neighbor of u_1 in P is s, if this is the case let $P^1 = v_1, P, s, u_1$, otherwise P^1 is undefined.
4. If P^1 is not undefined, compute the set Y of all nodes that do not belong to or have a neighbor in $P^1 \cup \{w, x\}$.
5. Compute, if one exists, a chordless path P^2 between u_2 and v_2 with interior contained in Y. If P^2 exists, then let

$$H = w, v_1, P^1, u_1, x, u_2, P^2, v_2, w;$$

   output (H, v) and stop.

Otherwise output the fact that G does not contain any detectable 3-wheel.

Correctness: It takes time $O(|V(G)|)^8$ to compute all possible 7-tuples u_1, u_2, v, v_1, v_2, w, s as above, and there are $O(|V(G)|)^7$ of them. For every

7-tuple, step 4 takes time $O(|V(G)|^2)$, while all other steps take linear time, thus the overall running time is $O(|V(G)|)^9$.
Obviously, when the algorithm outputs a graph (H, v), such graph is a detectable 3-wheel.
Suppose that G contains some detectable 3-wheel. We want to show that the algorithm will output one. Let $(\tilde{H}, v)$ be a smallest detectable 3-wheel in G. Let u, v_1 and v_2 be the neighbors of v in $\tilde{H}$, where v_1 and v_2 are both adjacent to a node w in $\tilde{H}$. Let u_1 and u_2 be the two neighbors of u in $\tilde{H}$ such that the two maximal paths $\tilde{P}_1$ and $\tilde{P}_2$ in $\tilde{H} \setminus \{u, w\}$ have endpoints u_1, v_1 and u_2, v_2, respectively. Let s be the neighbor of u_1 in $\tilde{P}_1$. Then the 7-tuple $u_1, u_2, v, v_1, v_2, w, s$ satisfies the properties described in the algorithm, hence at some stage the algorithm will examine it. Let x be a node adjacent to v, u_1, u_2 but not to w or s (such a node exists since $x = u$ satisfies such condition). Let u_1' and u_2' be neighbors of x in $\tilde{H}$, such that u_i' is closest possible to v_i in $\tilde{P}_i$, $i = 1, 2$, and let Q^i be the path between v_i and u_i' in $\tilde{P}_i$. Then $H' = w, v_1, Q^1, u_1', x, u_2', Q^2, v_2, w$ is a hole and (H', v) is a detectable 3-wheel with at most as many nodes as $(\tilde{H}, v)$, therefore $Q^i = \tilde{P}_i$, for $i = 1, 2$. Let P be the shortest path between v_1 and s in $G[X \cup v_1]$ computed by the algorithm in step 2. Then, by 5.7, $P^1 = v_1, P, s, u_1$ is a path and the algorithm will verify this in step 3. Finally, there exists a chordless path P^2 between u_2 an v_2 with interior in the set Y computed at step 5 of the algorithm, since $\tilde{P}_2$ is such a path, therefore $H = w, v_1, P^1, u_1, x, u_2, P^2, v_2, w$ is a hole and (H, v) is detectable 3-wheel. □

5.5 Major nodes on a smallest odd wheel

We say that (H, x) is a *smallest odd wheel* in G, if (H, x) is an odd wheel in G with the minimum number of nodes. Given a hole H, we say that a vertex $v \in V(G) \setminus V(H)$ is major for H if $N_H(v)$ is not contained in a subpath of H of length 2, and denote by $M(H)$ the set of major nodes for H.

In this section, we are interested in deriving some properties of the nodes that are major for a given hole H, such that (H, x) is a smallest odd wheel for some node x. Given a chordless path or a hole Q and a set $X \subseteq V(G)$ with at least two distinct elements in Q, an *X-sector of Q* is a maximal subpath of Q whose interior does not contain an element in X.

Claim 5.9 *Let (H, x) be a smallest odd wheel in G, and let y be a major node for H nonadjacent to x. Let $Q = q_1, ..., q_k$ be a proper subpath of H,* *and let $u, v \in \{x, y\}$, $u \neq v$, such that $q_1, q_k \in N(u)$, and v has an odd number of neighbors in Q. Then one of the following holds:*

(i) G contains a 3-path configuration Π such that the three holes in Π have length smaller then H.

(ii) Q contains an odd number of $N(u)$-sectors in which v has exactly one neighbor, while all other $N(u)$-sectors of Q contain an even number of neighbors of v. Furthermore, if x and y are in the same side of the bipartition, and $S = s_1, \ldots, s_h$ is an $N(u)$-sector of Q where v has exactly one neighbor, then v is adjacent to s_1 or s_h.

Proof: Since v has an odd number of neighbors in H, there is an odd number of $N(u)$-sectors of Q containing an odd number of neighbors of v. Let $S = s_1, \ldots, s_h$ be such a sector. If v has at least 3 neighbors in S, then $V(S) \cup \{u, v\}$ induces an odd wheel (H', v), and (H', v) has less nodes than (H, x) since x and y are both major nodes. Therefore v has exactly one neighbor s_i, $1 \leq i \leq h$, in S. Assume that x and y are in the same side of the bipartition. If $i = 1$ or $i = h$ we are done, hence we may assume $3 \leq i \leq h - 2$. Suppose u and v both have neighbors in $V(H) \setminus V(S)$. Then there exists a path P between u and v with interior in $V(H) \setminus V(S)$. Then $P^1 = s_i, v, P, u$, $P^2 = s_i, S_{s_i s_h}, s_h, u$, $P^3 = s_i, S_{s_1 s_i}, s_1, u$ induce a 3-path configuration between s_i and x, and $|P_i| + |P_j| < |H|$, for $1 \leq i < j \leq 3$. Since v is major, v has at least one neighbor in $V(H) \setminus V(S)$, therefore u has exactly two neighbors in H, so $u = y$, $v = x$ and $S = Q$. Let x', x'' be the neighbors of x closest to q_1 and q_k, respectively, in the path Q' induced by $V(H) \setminus \{q_2, \ldots, q_{k-1}\}$. Let P' and P'' be the unique paths in Q' between q_1 and x', and q_k and x'', respectively. Then $P^1 = x, x', P', q_1$, $P^2 = x, s_i, Q_{s_i q_1}, q_1$ and $P^3 = x, x'', P'', q_k, y, q_1$ induce a 3-path configuration between x and q_1, and $|P_i| + |P_j| < |H|$, for $1 \leq i < j \leq 3$. □

Claim 5.10 *Let (H, x) be a smallest odd wheel in G and y be a major node for H. One of the following holds:*

(i) G contains a 3-path configuration Π such that the three holes in Π have length smaller then H.

(ii) y has an odd number of neighbors in H.

Proof: Suppose, by contradiction, that (i) and (ii) do not hold. In particular, y has an even number of neighbors in H. Let $X = N_H(x)$ and $Y = N_H(y)$.

Case 1: x and y are in distinct sides of the bipartition.

Assume x and y are adjacent. One can easily verify that there exists $u, v \in \{x, y\}$, $u \neq v$, such that u has a positive even number of neighbors in some $N_H(v)$-sector $S = s_1, \ldots, s_k$ of H. Thus, given $H' = v, s_1, S, s_k, v$, (H', u) is an odd wheel, and $|H'| < |H|$ since x and y are major, a contradiction.

Henceforth we may assume that x and y are nonadjacent. Since x has an odd number of neighbors in H, then there exists a Y-sector $S = s_1, ..., s_k$ of H containing an odd number of neighbors of x. By 5.9, x has exactly one neighbor, say s_i, in S. Let $z', z'' \in V(H) \setminus V(S)$ be the nodes in $X \cup Y$ that are closer to s_1 and s_k, respectively, in the path Q induced by $V(H) \setminus \{s_2, \ldots, s_{k-1}\}$. Let P' and P'' be the unique paths in Q between z' and s_1, and z'' and s_k, respectively.

(5.10.1) *At least one of z' and z'' is adjacent to y.*

Suppose z' and z'' are adjacent to x. If $i \geq 3$ or $i \leq k-2$, say $i \leq k-2$, then there is a 3-path configuration induced by the paths $P^1 = s_i, S_{s_i s_k}, s_k$, $P^2 = s_i, x, z'', P'', s_k$, $P^3 = s_i, S_{s_i s_1}, s_1, y, s_k$, and $|P_i| + |P_j| < |H|$, for $1 \leq i < j \leq 3$. So $S = s_1, s_2, s_3$ and $i = 2$. Let $H' = x, z', P', s_1, y, s_3, P'', z'', x$; (H', s_i) is an odd wheel with at most as many nodes as (H, a). Thus $|H'| = |H|$, since (H, a) is a smallest odd wheel, and z', z'' have a common neighbor in Q. Since y has an even number of neighbors in H, then s_1, s_3 are the only neighbors of y in H, a contradiction since y is major for H. This concludes the proof of (5.10.1).

Thus we may assume, w.l.o.g., that z' is adjacent to y. Let S' be the X-sector containing s_1 and z', and let x' be the endnode of S' distinct from s_i. Since y has at least two neighbors in S', then by 5.9 y must have an even number of neighbors in S'. Therefore, since x has an odd number of neighbors in H and y as an even number of neighbors in H, both x and y have neighbors in $V(H) \setminus (V(S') \cup V(S))$, so there exists a path P between x and y with interior in $V(H) \setminus (V(S) \cup V(S'))$. Let y' be the neighbor of y closest to x' in S'. Consider the paths $P^1 = x, P, y$, $P^2 = x, s_i, S_{s_i s_1}, s_1, y$, $P^3 = x, x', S'_{x'y'}, y', y$. Then $|P_i| + |P_j| < |H|$, for $1 \leq i < j \leq 3$, and P^1, P^2, P^3 induce a 3-path configuration unless the neighbor y'' of y in P is adjacent to x'. Therefore y'' is the only neighbor of y in $V(H) \setminus (V(S) \cup V(S'))$, so s_k and y'' are the endnodes of a Y-sector S'' of H containing an odd number of neighbors of x. Thus S'' contains exactly one neighbor of x. Now s_i and x' are the nodes of $(V(H) \setminus V(S'')) \cap (X \cup Y)$ closest to s_k and y'', respectively, in the subpath induced by $V(H)$ minus the interior of S''; but s'_i and x' are both adjacent to x, contradicting (5.10.1).

Case 2: x and y are in the same side of the bipartition.

Since x has an odd number of neighbors in H, there exists an odd number of Y-sectors where x has an odd number of neighbors. By 5.9, each of these sectors contains exactly one neighbor of x, and such neighbor is an endpoint of the sector. Thus $X \cap Y \neq \emptyset$. Suppose $|X \cap Y| = 1$, and let $z \in X \cap Y$. Then there exists a unique Y-sector S where x has an odd number of neighbors, and $z \in S$. Let S' be the Y-sector, distinct from S, containing z. Then x has an even number of neighbors in S'. Let z', z'' be the endnodes of S and S', respectively, distinct from z, and Q be the subpath of H between z' and z'' that does not contain z. Then x has an odd number of neighbors in Q, so there exists a Y-sector of Q, which is also a Y sector of H distinct from S, where x has an odd number of neighbors, a contradiction.

Therefore $|X \cap Y| \geq 2$. Notice that $|(X \cup Y) \setminus (X \cap Y)|$ is odd, thus there exists an $X \cap Y$-sector $Q = q_1, ..., q_k$ of H containing an odd number of elements of $|(X \cup Y) \setminus (X \cap Y)|$. So there exists $u, v \in \{x, y\}$ such that u has an even number of neighbors in Q, v has an odd number of neighbors in Q, and both endnodes of Q are adjacent to u. By 5.9, Q contains an odd number of $N(u)$-sectors where the only neighbor of v is one of the endnodes, and all other $N(u)$-sectors contain an even number of neighbors of v. Since the only common neighbors of u and v in Q are q_1 and q_k, then there is exactly one $N(u)$-sector of Q containing an odd number of neighbors of v, and it has q_1 or q_k as an endnode, say q_1. Let $2 \leq i \leq j \leq k-1$ be the minimum and maximum index, respectively, such that $q_i, q_j \in X \cup Y$. Then q_i is adjacent to u and q_j is adjacent to v, so the path $Q' = Q_{q_1 q_j}$ has both endnodes adjacent to v, and u has an odd number of neighbors in Q'. By 5.9, there exists an $N(v)$-sector of Q' where the only neighbor of u is one of the endnodes, which is impossible since q_1 is the only common neighbor of u and v in Q' and q_i is adjacent to u. □

Claim 5.11 *Let (H, x) be a smallest odd wheel in G. One of the following holds:*

(i) G contains a 3-path configuration Π such that the three holes in Π have length smaller then H.

(ii) There exist $a \in V(H) \cap R$ and $b \in V(H) \cap C$ such that $N(a) \supset M(H) \cap C$ and $N(b) \supset M(H) \cap R$.

Proof: Assume that (i) does not hold. The statement is obvious if $|H| \leq 6$, hence we may assume $|H| \geq 8$. By symmetry, we only need to prove the statement for $M(H) \cap C$. We will procede by induction on $|M(H) \cap C|$.

(5.11.1) *5.11 holds if $|M(H) \cap C| \leq 2$.*

The statement is trivial if $|M(H) \cap C| \leq 1$. Let $\{x, y\} = M(H) \cap C$. By 5.10, x has an odd number number of elements in H, thus there exists an $N_H(y)$ sector of H where x has an odd number of neighbors, so by 5.9 this sector contains a common neighbor of x and y. This concludes the proof of (5.11.1).

Assume $|M(H) \cap C| = 3$ and let $\{x, y, z\} = M(H) \cap C$. Let $X = N_H(x)$, $Y = N_H(Y)$ and $Z = N_H(z)$. By contradiction, suppose that there is no node in $X \cap Y \cap Z$.

(5.11.2) *Let $Q = q_1, ..., q_n$ be a subpath of H such that $q_1, q_k \in X \cup Y$, and z has an odd number of neighbors in Q. Then there exists an odd number of $X \cup Y$-sectors of Q containing an odd number of neighbors of z. Furthermore, if z has an odd number of neighbors in an $X \cup Y$-sector $S = s_1, ...s_h$ of Q, then s_1 and s_h are either both in X or both in Y, and $N_S(z) \subset \{s_1, s_h\}$.*

Clearly, there exists an odd number of $X \cup Y$-sectors of Q containing an odd number of neighbors of z. Let $S = s_1, ...s_h$ be such a sector. If $s_1, s_h \in X$ or $s_1, s_h \in Y$, then the claim follows from 5.9. Thus we may assume $s_1 \in X \setminus Y$ and $s_h \in Y \setminus X$. By (5.11.1), there exists a node $t \in V(H)$ adjacent to both x and y, therefore $t \notin V(S)$ and $H' = t, x, s_1, S, s_h, y, t$ is a hole of length smaller than H. Since z is not adjacent to t, then z has an odd number of neighbors in H', so z must have exactly one neighbor in H', say s_i, $1 \leq i \leq h$. We may assume, w.l.o.g., that $i > 1$. By (5.11.1), there exists a node $r \in V(H)$ adjacent to both x and z, therefore $r \notin V(S)$ and $r \neq t$. The paths $P^1 = x, s_1, S s_1 s_i, s_i$, $P^2 = x, r, z, s_i$, $P^3 = x, t, y, s_k, S_{s_k s_i}, s_i$ induce a 3-path configuration, and $|P_i| + |P_j| < |H|$, for $1 \leq i < j \leq 3$, a contradiction. This concludes the proof of (5.11.2).

By (5.11.1), the sets $X \cap Y$, $X \cap Z$ and $Y \cap Z$ are all nonempty. Let $W = (X \cap Y) \cup (X \cap Z) \cup (Y \cap Z)$. Notice that $|X| + |Y| + |Z| = |(X \cup Y \cup Z) \setminus W| + 2|W|$, and $|X|$, $|Y|$, $|Z|$ are all odd, thus $(X \cup Y \cup Z) \setminus W$ has odd cardinality. So there exists a W-sector $Q = q_1, ..., q_k$ of H that contains an odd number of elements of $(X \cup Y \cup Z) \setminus W$. It is easy to see that at an odd number of nodes in $\{x, y, z\}$ has an odd number of neighbors in Q. We may assume, w.l.o.g., that z has an odd number of neighbors in Q, while $|X \cap V(Q)|$ and $|Y \cap V(Q)|$ have the same parity. Clearly $q_1, q_k \in X \cup Y$. Since the only nodes in Q that are adjacent to both z and either one of x or y are q_1 and q_k, then, by (5.11.2), there is exactly one $X \cup Y$-sector S of Q such that z has an odd number of neighbors in S, and such a sector must contain either q_1 or q_k. We may assume, w.l.o.g., that $S = S_{q_1 q_i}$, $i < k$, $q_1 \in X \cap Z$ and $q_i \in X$. Let $j < k$ be the maximum index such that $q_j \in X \cup Y \cup Z$.

(5.11.3) $q_k \notin Z$.

Suppose $q_k \in Z$. If $q_j \in X \cup Y$, then $Q_{q_j q_k}$ is an $X \cup Y$-sector of Q distinct from S containing an odd number of neighbors of z, a contradiction. Thus $q_j \in Z$. Since the number of neighbors of x and y in Q have the same parity, and they are not both adjacent to q_k, then either x or y has an odd number of neighbors in $Q' = Q_{q_1 q_j}$. Let $u, v \in \{x, y\}$, $u \neq v$ such that u has an odd number of neighbors in Q'. Since $q_1, q_j \in Z \cup N(v)$, then by (5.11.2), there exists a $Z \cup N(v)$-sector S' of Q' such that u has a unique neighbor in S', and such neighbor is an endnode of S'. Thus S' must contain q_1, so $u = x$, but $q_i \in V(S')$ is adjacent to x, a contradiction. This concludes the proof of (5.11.3).

By (5.11.3), $q_k \in X \cap Y$. Also, by (5.11.3) and by symmetry, we may assume that x has an even number of neighbors in Q, thus y has an even number of neighbors in Q as well. Note that $q_1, q_k \in X \cup Z$, and there are an even number of $X \cup Z$-sectors of Q where y has an odd number of neighbors. Since the only node in Q that is adjacent to y and to either x or z is q_k, then, by (5.11.2), there are no $X \cup Z$-sectors of Q where y has an odd number of neighbors. Thus $q_j \in Y$, x has an odd number of neighbors in $Q' = q_1, \ldots q_j$, and $q_1, q_j \in Y \cup Z$. By (5.11.2), there exists a $Y \cup Z$-sector S' of Q' such that x has a unique neighbor in S', and such neighbor is an endnode of S'. Thus S' must contain q_1, but $q_i \in V(S')$ is adjacent to x, a contradiction.

Henceforth we may assume $|M(H) \cap C| \geq 4$. Let $x_1, x_2, x_3, x_4 \in M(H) \cap C$. By induction, there exist nodes $s_1, s_2, s_3 \in V(H)$ such that s_i is adjacent to every node in $M(H) \cap C$ except x_i, $i = 1, 2, 3$. Thus $H' = x_1, s_2, x_3, s_1, x_2, s_3, x_1$ is a hole of length 6 and (H', x_4) is an odd wheel (since x_4 is adjacent to s_1, s_2, s_3), a contradiction. □

5.6 Recognizing balanced graphs

In this section we show how to decide if a signed bipartite graph (G, σ) contains an unbalanced hole. In 5.6.1 we show how to generate a polynomial-size family of induced subgraphs of G with the property that, if G is not balanced, then one member of this family contains an unbalanced hole of minimum size that has no major nodes. In 5.6.2 we show that a smallest unbalanced hole with this property can be detected easily. Finally, in 5.6.3, we give the complete description of the algorithm.

5.6.1 Cleaning a smallest unbalanced hole

Given a signed graph (G, σ), an unbalanced hole H is *smallest* if it has minimum length among all unbalanced holes. We say that H is *clean* if there is no major node for H in G. A *cleaner* for H is a subset X of $V(G) \setminus V(H)$ that contains all major nodes for H (i.e. H is clean in $G \setminus X$). We say that (G, σ) is *clean* if either G is balanced, or G contains a smallest unbalanced hole that is clean.

We will provide an algorithm running in time $O(n^7)$ which will constructs a family $\mathcal{C}$ of subsets of $V(G)$ containing $O(n^6)$ elements such that if H is a smallest unbalanced hole in (G, σ), then $\mathcal{C}$ contains a cleaner for H. The following is due to Conforti and Rao [35].

Claim 5.12 *Let H be a smallest unbalanced hole in (G, σ), and $x \in V(G) \setminus V(H)$ be a major node for H. Then (H, x) is a smallest odd wheel.*

Proof: If not then the graph F induced by $V(H) \cup \{x\}$ has an even number of edges, and each edge is contained in 2 holes of F. Let $\mathcal{H}$ be the family of holes in F. Then $H \in \mathcal{H}$ and $|C| < |H|$ for every $C \in \mathcal{H}$ (since x is major). Thus $\sum_{C \in \mathcal{H}} \sigma(C) = 2\sigma(F) \equiv 0 \bmod 4$, and, since $\sigma(H) \equiv 2 \bmod 4$, there exists $C \in \mathcal{H}$ such that $\sigma(C) \equiv 2 \bmod 4$, a contradiction. □

The following was proven by Conforti and Rao [35], but only in the unsigned case (or, equivalently, the case in which $\sigma(e) = 1$ for every $e \in E(G)$).

Claim 5.13 *Let H be a smallest unbalanced hole in (G, σ). There exist $a \in V(H) \cap R$ and $b \in V(H) \cap C$ such that $N(a) \supset M(H) \cap C$ and $N(b) \supset M(H) \cap R$.*

Proof: If $M(H)$ is empty then the statement holds trivially. Otherwise, by 5.12, (H,x) is a smallest odd wheel for every $x \in M(H)$. Since H is the smallest unbalanced hole, than G does not contain a 3-path configuration in which the three holes have each length smaller then $|H|$. Hence case (ii) of 5.11 holds. □

This provides the following cleaning algorithm.

Claim 5.14 *There exists a $O(|V(G)|^7)$ algorithm with the following specifications:*

- **Input** *A signed bipartite graph (G,σ).*
- **Output** *A family $\mathcal{C}$ of $O(|V(G)|^6)$ subsets of $V(G)$ such that, if H is a smallest unbalanced hole in G, then there exists an element of $\mathcal{C}$ that is a cleaner for H.*

Algorithm:
For every 6-tuple of nodes $u_1,\ldots,u_6$, such that u_1,u_2,u_3 and u_4,u_5,u_6 induce paths, and $u_2 \in C$, $u_5 \in R$, compute

$$X(u_1,\ldots,u_6) = N(u_2) \cup N(u_5) \setminus \{u_1,u_2,u_3,u_4,u_5,u_6\}.$$

Let $\mathcal{C}$ be the family containing $X(u_1,\ldots,u_6)$ for every possible choice of $u_1,\ldots,u_6$.

Correctness: The running time of the algorithm is obviously $O(|V(G)|^7)$ and $\mathcal{C}$ has $O(|V(G)|^6)$ elements. We only need to show that, if (G,σ) contains a smallest unbalanced hole H, then $\mathcal{C}$ contains a cleaner for H. By 5.13, there exist two nodes $u_2 \in V(H) \cap C$, $u_5 \in V(H) \cap R$, such that every node in $M(H) \cap R$ is adjacent to u_2 and every node in $M(H) \cap C$ is adjacent to u_5. Let u_1, u_3 be the neighbors of u_2 in H and u_4, u_6 be the neighbors of u_5 in H. Then the algorithm will examine the 6-uple $u_1,\ldots,u_6$, and clearly $X(u_1,\ldots,u_6)$ is a cleaner for H. □

5.6.2 Detecting a clean smallest unbalanced hole

Claim 5.15 *Let (G,σ) be a signed bipartite graph containing no 3-path configuration and no detectable 3-wheel. Let H be a clean smallest unbalanced hole of (G,σ), u and v be two nonadjacent nodes of H and P^1, P^2 be the two internally node-disjoint subpaths of H between u and v, where $|P^1| \leq |P^2|$. Let P be a shortest path between u and v in G. Then the following hold:*

(*i*) $|P| = |P^1|$

(*ii*) *Either* $H' = u, P, v, P^2, u$ *is a clean smallest unbalanced hole, or* $|P^1| = |P^2|$ *and* $H'' = u, P, v, P^1, u$ *is a clean smallest unbalanced hole.*

Proof: The statement is obvious if H is an unbalanced hole of length 4, hence we may assume $|H| \geq 6$. Let $H = h_1, \ldots, h_{2s}, (h_{2s+1} = h_1)$ where $h_1 = u$, $s \geq 3$. Let $\overrightarrow{H}$ be the directed cycle obtained by orienting the edges of H from h_i to h_{i+1} for every $1 \leq i \leq h_{2s}$. For any two distinct nodes x and y in H, let H_{xy} be the underlaying graph of the directed path from x to y in $\overrightarrow{H}$. W.l.o.g., $P^1 = H_{uv}$ and $P^2 = H_{vu}$, and $v = h_m$ for some $3 \leq m \leq s+1$. Let $P = p_0, \ldots, p_{k+1}$, where $p_0 = u$ and $p_{k+1} = v$. We will prove 5.15 by induction on k.

If $k = 1$, then p_1 has exactly two neighbors in H, namely u and v (since H is clean), and they are contained in a subpath of H of length 2, say u, w, v. Hence, by 5.10, $H' = u, p_1, v, P^2, u$ is a an unbalanced hole of the same length as H. We only need to prove that H' is clean. Assume not and let x be a major node for H'. Since x is not major for H, then x is adjacent to p_1 but not to w, x has exactly 2 neighbors in P^2 and they are contained in a path of length 2. But then (H', x) is a detectable 3-wheel, a contradiction. Hence we may assume $k \geq 2$.

(5.15.1) *Either:*

(*i*) *no node of* $P_{p_1p_k}$ *belongs to or has a neighbor in* $H_{h_{m+1}h_{2s}}$, *or*

(*ii*) $|P| = |H_{uv}| = |H_{vu}| = s$, $\sigma(P) = \sigma(H_{vu})$ *and no node of* $P_{p_1p_k}$ *belongs to or has a neighbor in* $H_{h_2h_{m-1}}$.

Assume that there is a node of $P_{p_1p_k}$ that belongs to or has a neighbor in $H_{h_{m+1}h_{2s}}$, then there exists a j, $m+1 \leq j \leq 2s$, such that there are chordless paths Q^1 and Q^2 between h_j and u and h_j and v, respectively, with interior contained in the interior of P. Therefore

$$\begin{aligned} |Q^1| + |Q^2| &\leq k+3 \leq m+1 \leq 2s+3-m = (2s+2-j) + (j-m+1) \\ &\leq (|H_{h_ju}|+1) + (|H_{vh_j}|+1). \end{aligned}$$

Since $|Q^1|$ has the same parity as $|H_{h_ju}|$ and $|Q^2|$ has the same parity as $|H_{vh_j}|$, then, by symmetry, we may assume $|Q^1| \leq |H_{h_ju}|$. We can also argue that either $|Q^1| < |P|$ and $j < 2s$, or $|Q^2| \leq |H_{vh_j}|$, $|Q^2| < |P|$ and $j > m + 1$. In fact, if $|Q^1| = |P|$, then $Q^1 = u, p_1, \ldots, p_k, h_j$ and $Q^2 = h_j, p_k, v$, hence $|Q^2| \leq |H_{vh_j}|$ and $|Q^2| < |P|$. Furthermore, if $j = 2s$, then $|Q^2| < |P| \leq |H_{vh_j}| + 1$ (since u cannot be adjacent to both h_1 and p_1) and $j > m+1$. Thus, by symmetry, we may assume $|Q^1| \leq |H_{h_j u}|$, $|Q^1| < |P|$ and $j < 2s$. By inductive hypothesis,

$$d_G(h_j, u) = d_H(h_j, u) = \min(2s + 1 - j, j - 1)$$

and $d_G(h_j, u) \leq |Q^1| < |P| \leq m - 1 < j - 1$, hence $d_G(h_j, u) = 2s + 1 - j = |Q^1| < s$. By induction, $\sigma(Q^1) = \sigma(H_{h_j u})$ and $H' = u, H_{uh_j}, h_j, Q^1, u$ is a clean smallest unbalanced hole. We obtain a directed cycle $\overrightarrow{H'}$ by orienting the edges of H' to agree with the orientation of the edges in H_{uh_j}, and define H'_{xy} for every x, y in H as before.
Let u' be the neighbor of h_j in Q^1. Then there exists a subpath P' of P between u' and v of length $k + 2 - |Q^1| = k + j + 1 - 2s < k + 1$. By induction, $|H'_{u'v}| \leq k + j + 1 - 2s$ or $|H'_{vu'}| \leq k + j + 1 - 2s$. But $|H'_{u'v}| > |H_{uv}| \geq k + 1 > |P'|$, hence $|H'_{vu'}| \leq k + j + 1 - 2s$. This implies $j - m + 1 \leq k + j + 1 - 2s$, so $2s \leq k + m$, but $m \leq s + 1$ and $k \leq s - 1$, hence $m = s + 1$, $k = s - 1$, $d_G(u'v) = d_{H'}(u', v) = |P'|$ and $\sigma(P') = \sigma(H'_{vu'})$. By induction, $H'' = u', H'_{u'v}, v, P', u'$ is a clean smallest unbalanced hole. Since H_{uv} is contained in $H'_{u'v}$, then no node in the interior of P' belongs to or has a neighbor in $H_{h_2 h_{m-1}}$. Since every node in $P_{p_1 p_k}$ is either a node of P^1 or a node of P', then no node of $P_{p_1 p_k}$ belongs to or has a neighbor in $H_{h_2 h_{m-1}}$. Finally

$$\begin{aligned}
\sigma(P) &= \sigma(P_{uu'}) + \sigma(P_{u'v}) = \sigma(Q^1) - \sigma(u'h_j) + \sigma(P') \\
&= \sigma(H_{h_j u}) - \sigma(u'h_j) + \sigma(H'_{vu'}) \\
&= \sigma(H_{h_j u}) - \sigma(u'h_j) + \sigma(H_{vh_j}) + \sigma(u'h_j) \\
&= \sigma(H_{vu}) \qquad (5.2)
\end{aligned}$$

This concludes the proof of 5.15.1.

By 5.15.1 and symmetry, we may assume that no node of $P_{p_1 p_k}$ belongs to or has a neighbor in $H_{h_{m+1} h_{2s}}$.

(5.15.2) *Either*

(*i*) $|H_{uv}| = |P|$ *and* $\sigma(H_{uv}) = \sigma(P)$, *or* *(ii)* $|P| = |H_{uv}| = |H_{vu}| = s$, $\sigma(P) = \sigma(H_{vu})$ *and no node of* $P_{p_1p_k}$ *belongs to or has a neighbor in* $H_{h_2h_{m-1}}$.

Clearly, if $\sigma(H_{uv}) = \sigma(P)$, then $H' = u, P, v, H_{vu}, u$ is an unbalanced hole of length at most $|H|$, hence $|P| = |H_{uv}|$ and 5.15.2 holds. Suppose then that $\sigma(H_{uv}) \neq \sigma(P)$. If no node in $P_{p_1p_k}$ belongs to or has a neighbor in $H_{h_2h_{m-1}}$, then u and v must be on the same side of the bipartition, else H_{uv}, H_{vu}, P would induce a 3-path configuration between u and v. Thus P has even length and $H'' = u, H_{uv}, v, P, u$ is an unbalanced hole strictly smaller than H unless $|P| = |H_{uv}| = |H_{vu}| = s$, thus case (ii) holds. Therefore there exists j, $2 \leq j \leq m-1$, such that there are chordless paths Q^1 and Q^2 between h_j and u and h_j and v, respectively, with interior contained in the interior of P. By 5.15.1 and symmetry, we may assume $m \leq s$. We have

$$|Q^1| + |Q^2| \leq k + 3 \leq m + 1 = j + (m + 1 - j) = (|H_{uh_j}| + 1) + (|H_{h_jv}| + 1)$$

and, by the same argument as in 5.15.1, we may assume $|Q^1| \leq j - 1$, $|Q^1| < |P|$ and $j > 2$. By induction, $|Q^1| = d_G(u, h_j) = d_H(u, h_j) = j - 1$, $\sigma(Q^1) = \sigma(H_{uh_j}$ and $H' = u, Q^1, h_j, H_{h_ju}, u$ is a clean smallest unbalanced hole.
Let u' be the neighbor of h_j in Q^1 and let P' be the path between u' and v in P. Then

$$|P'| = k + 2 - |Q^1| = k - j + 3$$

thus, by induction, $\sigma(P') = \sigma(H'_{u'v})$. Finally, with a calculation very similar to the one in (5.2),

$$\begin{aligned}
\sigma(P) &= \sigma(P_{uu'}) + \sigma(P_{u'v}) \\
&= \sigma(H_{uh_j}) - \sigma(u'h_j) + \sigma(H_{h_jv}) + \sigma(u'h_j) \\
&= \sigma(H_{uv}).
\end{aligned}$$

This completes the proof of 5.15.2.

By 5.15.1, 5.15.2 and by symmetry, we may assume that $H' = u, P, v, H_{vu}, u$ is a smallest unbalanced hole. To conclude the proof of 5.15 we only need to show that H' is clean. Suppose, by contradiction, that H' is not clean, and let x be a major vertex for H'. If x has at least two neighbors in H_{vu}, then such neighbors are contained in a subpath of H of length 2, thus x is adjacent to h_i and h_{i+2} for some $m \leq i \leq 4s + 1$ and has no other neighbors in H. Thus $H'' = h_i, x, h_{i+2}, H_{h_{i+2}h_i}, h_i$ is a clean smallest unbalanced hole and the interior of P contains a neighbor of x, whence, by 5.15.1 applied to H'' and P, it must be the case that $|P| = s$, $\sigma(P) = \sigma(H'_{vu}) = \sigma(H_{vu})$ and no node of $P_{p_1 p_k}$ belongs to or has a neighbor in $H'_{h_2 h_{m-1}} = H_{h_2 h_{m-1}}$. Since $\sigma(H) \equiv 2 \bmod 4$, and $\sigma(H_{uv}) = \sigma(P) = \sigma(H_{vu})$, then u and v are in distinct sides of the bipartition and H_{uv}, H_{vu}, P induce a 3-path configuration between u and v. Thus x has at most one neighbor in H_{vu} and at least 2 neighbors in the interior of P. Let p_i and p_j be the neighbors of x in P of lowest and highest index, respectively. Then $j = i + 2$, else $u, P_{up_i}, p_i, x, p_j, P_{p_j v}, v$ is a path between u and v strictly shorter than P. But then x has exactly 3 neighbors in H', and two of these neighbors have distance 2 in H', so (H', x) is a detectable 3 wheel, a contradiction. □

Claim 5.16 *There exists a $O(|V(G)|^4)$ algorithm with the following specifications:*

- **Input** *A clean signed bipartite graph (G, σ) containing no 3-path configuration and no detectable 3-wheel.*
- **Output** *Either*
  - *(i) An unbalanced hole H,*
  - *(ii) Determines that (G, σ) is balanced.*

Algorithm:

For every possible pair of nodes u_1, u_2, do the following:

1. compute the shortest path P between u_1 and u_2.
2. compute the set X of nodes that do not belong to or have a neighbor in the interior of P.
3. for every node u_3 in X at distance 2 from u_1 in $G[X \cup \{u_1\}]$, compute the shortest paths $P^1(u_3)$ and $P^2(u_3)$ between u_1 and u_3 in $G[X \cup \{u_1\}]$ and between u_2 and u_3 in $G[X \cup \{u_2\}]$ (if one exists), respectively.
4. for every such $u_3 \in X$, verify that no node in $P^1(u_3) \setminus u_3$ belongs to or has a neighbor in $P^2(u_3) \setminus u_3$. If this is the case, define

$$H(u_1, u_2, u_3) = u_1, P, u_2, P^2(u_3), u_3, P^1(u_3), u_1;$$

otherwise let $H(u_1, u_2, u_3)$ be undefined.

5. If $\sigma(H(u_1, u_2, u_3)) \equiv 2 \bmod 4$, then output the unbalanced hole $H = H(u_1, u_2, u_3)$ and stop.

Otherwise output that G is balanced.

Correctness: checking if (G, σ) has an unbalanced hole of length 4 takes time $O(|V(G)|^4$. For every possible pair u_1 and u_2, the running time of step 1 is linear while step 2 through 4 take time $O(|V(G)|^2)$ (in fact, step 4 takes linear time for every choice of u_3 since $P^1(u_3)$, by definition, has constant length 2). Hence the overall running time is $O(|V(G)|^4)$. Obviously, when the algorithm outputs an unbalanced hole it is correct. We need to verify that the algorithm is always correct when it outputs that G is balanced. Assume G is not balanced. Since G is clean, there exists a clean smallest unbalanced hole H, with $|H| = 2s$, and $s \geq 3$ since $|H| \geq 6$. Let u_1, u_2, u_3 be three nodes in H such that $d_H(u_1, u_3) = 2$, while $d_H(u_1, u_2) = d_H(u_2, u_3) = s - 1$. Let P, $P^1(u_3)$, $P^2(u_3)$ be the paths computed by the algorithm for the triple u_1, u_2, u_3. Let Q^1 and Q^2 be the subpaths of H between u_1 and u_2 such that Q^1 does not contain u_3 and Q^2 contains u_3. Then, by our choice of u_1, u_2, u_3, $|Q^1| < |Q^2|$, hence, by 5.15, $H' = u_1, P, u_2, Q^2, u_1$ is a clean smallest unbalanced hole. By repeating the same argument for $P^1(u_3)$ and $P^2(u_3)$, we argue that $H(u_1, u_2, u_3) = u_1, P, u_2, P^2(u_3), u_3, P^1(u_3), u_1$ is a clean smallest unbalanced hole, hence the algorithm would have output it correctly. □

5.6.3 The recognition algorithm

At this point we are ready to give the complete description of the $O(|V(G)|^9$ algorithm to decide if a signed bipartite graph is balanced. For the sake of clarity, let us first describe a slightly simpler algorithm, with running time $O(|V(G)|^{10})$: run first algorithm 5.6 and then algorithm 5.8 to determine if a graph has a 3-path configuration or a detectable 3-wheel. If so, stop and output that (G, σ) is not balanced. Otherwise, apply algorithm 5.14 to obtain a family $\mathcal{C}$ of subsets of $V(G)$, and finally apply algorithm 5.16 to $G \setminus X$ for every $X \in \mathcal{C}$. If for some X the algorithm detects an unbalanced hole, then output the fact that (G, σ) is not balanced. Otherwise, output the fact the (G, σ) is balanced. Notice that, in this case, the output is correct, otherwise G would contain a smallest unbalanced hole H, but $\mathcal{C}$ would contain cleaner X for H and the algorithm would output an unbalanced hole in $G \setminus X$.

In order to achieve the running time $O(|V(G)|^9$, we need to perform cleaning and look for a smallest unbalanced hole at the same time.

Theorem 5.17 *There exists a $O(|V(G)|^9)$ algorithm with the following specifications:*

- **Input** *A signed bipartite graph (G, σ).*
- **Output** *Determines whether (G, σ) is balanced or not.*

Algorithm:

1. Apply the algorithm 5.6. If G contains a 3-path configuration, then output the fact that (G, σ) is not balanced and stop.
2. Apply the algorithm 5.8. If G contains a detectable 3-wheel, then output the fact that (G, σ) is not balanced and stop.
3. For every 7-tuple of nodes $u_1, \ldots, u_7$, such that u_1, u_2, u_3 and u_4, u_5, u_6 induce a path, $u_2 \in C$, $u_5 \in R$, u_7 is nonadjacent to u_2, do the following
   (a) Compute $X(u_1, \ldots, u_6) = N(u_2) \cup N(u_5) \setminus \{u_1, u_2, u_3, u_4, u_5, u_6\}$.
   (b) Compute the shortest paths $P_1(u_1, \ldots, u_7)$ and $P_2(u_1, \ldots, u_7)$ between u_1 and u_7 and u_3 and u_7 in $G \setminus X(u_1, \ldots, u_6)$, respectively.
   (c) If no node in the interior of $P_1(u_1, \ldots, u_7)$ belongs to or has a neighbor in $P_2(u_1, \ldots, u_7)$, define:
   $$H(u_1, \ldots, u_7) = u_1, P_1(u_1, \ldots, u_7), u_7, P_2(u_1, \ldots, u_7), u_3, u_2, u_1;$$
   (d) Compute $\sigma(H(u_1, \ldots, u_7))$. If $\sigma(H(u_1, \ldots, u_7)) \equiv 2 \bmod 4$, output that G is not balanced and stop.

   Otherwise output that G is balanced.

Correctness: Both step 1 and step 2 take time $O(|V(G)|^9)$. Step 3 performs computations (a)-(d) at most $|V(G)|^7$ times. Steps (a), (b), and (d) can be performed in time $O(|V(G)|)$, while step (c) can be performed in time $O(|V(G)|^2)$, thus the running time is $O(|V(G)|^9)$ as claimed.

We need to show that the algorithm is correct. If G contains a 3-path configuration or a detectable 3-wheel, then by 5.6 and 5.8 the algorithm will output correctly that (G, σ) is not balanced. We only need to prove that, if G does not contain a 3-path configuration or a detectable 3-wheel, but

G contains an unbalanced hole, then step 3 will output that (G,σ) is not balanced. Let H be a smallest unbalanced hole in G. Then by 5.13 there exist two subpaths u_1, u_2, u_3 and u_4, u_5, u_6 of H such that every major node for H is adjacent to u_2 or u_5. The set $X(u_1, ..., u_6)$ computed in step (a) is a cleaner for H, as shown in the proof of 5.14. Let u_7 be the node at distance $|H|/2$ from u_2 in H. Clearly, the paths Q^1 and Q^2 between u_1 and u_7 and between u_3 and u_7 in H, respectively, have length strictly less then $|H|/2$, thus, by an argument similar to the one in the proof of 5.16, $H(u_1, \ldots, u_7) = u_1, P_1(u_1, \ldots, u_7), u_7, P_2(u_1, \ldots, u_7), u_3, u_2, u_1$ is a smallest unbalanced hole, where $P_1(u_1, \ldots, u_7)$ and $P_2(u_1, \ldots, u_7)$ are the paths computed in step (b). Thus step (d) will output that (G,σ) is not balanced. □

5.6.4 Refinements

For unsigned graphs, the problem of detecting an unbalanced hole when 3-path configurations and detectable 3-wheels are not present can be solved faster, in time $O(|V(G)|^7)$. Notice that this does not improve the total running time in the general case, since algorithms 5.6 and 5.8 run in time $O(|V(G)|^9)$. The speedup is allowed by the following.

Claim 5.18 *Let G be a bipartite graph and H be a smallest unbalanced hole in G. Then one of the following holds:*

(i) Every node in $M(H) \cap R$ is adjacent to every node in $M(H) \cap C$.

(ii) There exist two adjacent nodes a and b in H such that every major node for H is adjacent to a or b.

Proof: If $|H| = 6$, case (ii) must always occur, hence we may assume $|H| \geq 10$. Assume that there exist $a \in M(H) \cap R$ and $b \in M(H) \cap C$ such that a and b are not adjacent. Fix an orientation on H and let $a_1, \ldots, a_k$ be the neighbors of a in H in the order they appear according to such orientation starting from a_1. For $1 \leq i \leq k$ let A_k be the subpath of H between a_i and a_{i+1} (where $a_{k+1} = a_1$) containing no neighbors of a in its interior.

(5.18.1) *Up to symmetry, b is adjacent to the neighbor of a_1 in A_1, say b_1, all neighbors of b distinct from b_1 are contained in A_k and the neighbor of b closest to a_k in A_k has distance 3 modulo 4 from a_k.*

Let $\mathcal{I}$ be the family of all maximal subpaths of H with no neighbor of a or b in the interior. By definition the endnodes of every element I in $\mathcal{I}$ are neighbors of a and b. Since every major node for H has an odd number of neighbors in H, then $\mathcal{I}$ has an even number of elements. Every path in $\mathcal{I}$ of even length has both endnodes adjacent either to a or b and it must have length 2 modulo 4 (else either $I \cup a$ or $I \cup b$ induce an unbalanced hole strictly smaller than H). Since H has length 2 modulo 4, then there must be an even number of paths in $\mathcal{I}$ with odd length, and the sum of all lengths of such paths must be 2 modulo 4. Obviously, every odd path must have one endnode adjacent to a and the other adjacent to b. Suppose there are exactly 2 odd paths in $\mathcal{I}$, say I' and I'' with endnodes $a' \in N(a)$, $b' \in N(b)$ and $a' \in N(a)$, $b' \in N(b)$ respectively. Then a', b', a'', b'' are all distinct otherwise, w.l.o.g., $a' = a''$ and the subpath of H between b' and b'' not containing a' contains at least one neighbor of a, therefore there exists another path in $\mathcal{I}$ of odd length. Also, a' and b'' are not adjacent, otherwise a', b'' would be an odd path in $\mathcal{I}$. Analogously a'' and b' are not adjacent, hence $H' = a, a', I', b', b, b'', I'', a'', a$ is an unbalanced hole smaller than H. Thus there are at least 4 paths of odd length in $\mathcal{I}$. Furthermore there exist 3 paths I_1, I_2, I_3 in $\mathcal{I}$ each of length q modulo 4 for some $q \in \{1, 3\}$ (since either $\mathcal{I}$ contains at least 6 odd paths, or $\mathcal{I}$ contains exactly 4 odd paths whose total length must be 2 modulo 4). Since $\mathcal{I}$ contains at least 6 elements, we may assume that I_2 and I_3 have no node in common. If no node in I_2 is adjacent to a node in I_3, then $I_2 \cup I_3 \cup \{a, b\}$ induces an unbalanced hole strictly smaller than H. Thus an endnode of I_2, say a_1 w.l.o.g., is adjacent to an endnode of I_3, say b_1. This implies that I_2 and I_3 have length 1 modulo 4, $I_1 = a_1, b_1$ and there are no other paths in $\mathcal{I}$ of length 1. This argument also show that there exists a unique path I_4 in $\mathcal{I}$ of length 3 modulo 4, therefore we may assume that b_1 is in A_1, all neighbors of b distinct from b_1 are in A_k, and I_4 is the shortest path in A_k between a_k and a neighbor of b. This proves 5.18.1.

If every major node for H is adjacent to a_1 or b_1, then we are done. Hence we may assume that there exists $x \in M(H) \cap C$ nonadjacent to b_1.

(5.18.2) *x and b are both adjacent to the neighbor of a_1 in A_k, say b_2, and $|A_1| > 2$.*

By 5.11 there exists a node b' in H adjacent to both b and x. Assume first that x is adjacent to a, then $b' = b_2$, else a, x, b', b, b_1, a_1, a is a 6-hole. If x has no neighbors in A_1, then $H' = a, a_2, A_1(a_2, b_1), b_1, b, b_2, x, a$ is an unbalanced hole strictly smaller than H, as one can readily verify. Hence x' has a neighbor in A_1, distinct from b_1 by assumption, therefore $|A_1| > 2$. Hence we may assume that x is not adjacent to a. Since $b' \neq b_1$ and all neighbors of b in

H distinct from b_1 are contained in A_k, then $b' \in A_k$. Since a and x are not adjacent, then by 5.18.1 we have two cases.
Case (1): every neighbor of x in H except one, say b'', is contained in A_k, and b'' is either the neighbor of a_1 in A_1 or the neighbor of a_k in A_{k-1}. Since $b'' \neq b_1$ by assumption, then b'' is the neighbor of a_k in A_{k-1}, but then, given I the path between b_1 and b'' in $H \setminus A_k$, b, b_1, I, b'', x, b', x is an unbalanced hole strictly smaller than H, a contradiction.
Case (2): b' is the only neighbor of x in A_{k-1}. In this case, either $b' = b_2$ and all neighbors of x in H distinct from b_2 are contained in A_1, hence $|A_1| > 2$ and we are done, or b' is the neighbor of a_k in A_{k-1}, contradicting 5.18.1, since b' is also adjacent to b and the neighbor of b closest to a_k in A_k has distance 3 modulo 4 from a_k itself.
This conclude the proof of 5.18.2.

By 5.18.2, every node in $M(H) \cap C$ is adjacent to b_1 or b_2. If there exists $y \in M(H) \cap C$ such that y is not adjacent to b_2, then by 5.11, x and y have a common neighbor b' in H and $x, b', y, b_1, a_1, b_2, x$ is a 6 hole, a contradiction. Thus b_2 is adjacent to every node of $M(H) \cap C$. If a_1 is adjacent to every node of $M(H) \cap R$ then we are done, therefore, by 5.18.2 and by symmetry, every node in $M(H) \cap R$ is adjacent to the neighbor of b_1 in H distinct from a_1, say a'. In particular $a' = a_2$ and $|A_1| = 2$, contradicting 5.18.2. □

For signed graphs, the previous statement does not hold.

Claim 5.19 *There exists a $O(|V(G)|^5)$ algorithm with the following specifications:*

- **Input** *A bipartite graph G.*
- **Output** *A family $\mathcal{C}$ of $O(|V(G)|^4)$ subsets of $V(G)$ such that, if H is a smallest unbalanced hole in G, then there exists an element of $\mathcal{C}$ that is a cleaner for H.*

Algorithm:

1. For every chordless path P of length 3, $P = u_1, u_2, u_3, u_4$ define
   $X(P) = (N(u_2) \cup N(u_3)) \setminus V(P)$ and
   $Y(P) = (N(u_1) \cap N(u_3)) \cup (N(u_2) \cap N(u_4))$.
2. Let $\mathcal{C}$ be the family containing $X(P)$ and $Y(P)$ for every chordless path P of length 3.

Correctness: The running time of the algorithm is obviously $O(|V(G)|^5)$ and $\mathcal{C}$ has $O(|V(G)|^4)$ elements. We only need to show that, if G contains a smallest unbalanced hole H, then $\mathcal{C}$ contains a cleaner for H. If H contains two adjacent nodes u_2 and u_3 such that every major node for H is adjacent to u_2 or u_3, then let u_1 be the neighbor of u_2 in H distinct from u_3, and u_4 be the neighbor of u_3 in H distinct from u_2. $X(u_1, u_2, u_3, u_4)$ is obviously a cleaner for H.
Otherwise, by 5.18, every node in $M(H) \cap R$ is adjacent to every node in $M(H) \cap C$. By 5.11, there exist nodes u_1 and u_4 in H such that u_1 is adjacent to every node in $M(H) \cap R$ and u_4 is adjacent to every node in $M(H) \cap C$. Let a', a'' be the neighbors of u_1 in H and b', b'' the neighbors of u_4 in H. Then there exists $x', x'' \in M(H) \cap R$ such that x' is not adjacent to a' and x'' is not adjacent to a''. If $x' \neq x''$, then $u_4, x', a'', u_1, a', x'', u_4$ is a 6-hole, a contradiction. Let $u_2 = x' = x''$. Analogously, there exists a node $u_3 \in M(H) \cap C$ that is nonadjacent to both b' and b''. It is immediate to verify that $Y(u_1, u_2, u_3, u_4)$ is a cleaner for H. □

Notice that, if G does not contain 3-path configuration or a detectable 3-wheel, then we can run algorithm 5.19 to obtain a family $\mathcal{C}$ with $O(|V(G)|^4)$ subsets of $V(G)$, and the run algorithm 5.16 on $G \setminus X$ for every $X \in \mathcal{C}$. The total running time is $O(|V(G)|^8)$. The running time of $O(|V(G)|^7)$ can be achieved by performing 5.19 and 5.16 together. Namely, when we examine a 4-tuple $u_1, \ldots, u_4$ to generate $X(P)$ and $Y(P)$, we can now examine a 5th node u_5. Eventually, if there is a smallest unbalanced holes, either $X = X(P)$ or $X = Y(P)$ is a cleaner for H, and u_5 has distance less then $|H|/2$ from both u_1 and u_4. If we compute the shortest paths in $G \setminus X$ between every pair of nodes among u_1, u_4, and u_5, then, by an argument similar to the one in 5.16, these three paths induce an unbalanced hole. So, for each 5-tuple, we need to compute 3-shortest paths, in time $O(|V(G)|)$, and to check that we obtained an unbalanced hole, in time $O(|V(G)|^2)$. thus the total running time is $O(|V(G)|^7)$.

5.7 Recognizing balanceable graphs

In this section we will describe an algorithm that, given a bipartite graph G, decides whether or not G contains a 3-path configuration or an odd wheel. By 5.1, this is equivalent to decide if G is balanceable.

First, we will prove a technical lemma that is the analogue of 5.15 for odd wheels. Given a smallest odd wheel (H, x), a *cleaner* for H is a subset of $V(G) \setminus V(H)$ that contains all major nodes for H. Given two smallest odd wheels (H, x) and (H', y), we say that (H, x) *dominates* (H', y) (or (H', y) *is dominated by* (H, x)) if $M(H') \subseteq M(H)$. In particular, if X is a cleaner for (H, x) disjoint from H', then it is also a cleaner for H'.

Claim 5.20 *Let G be a bipartite graph containing no 3-path configuration and no detectable 3-wheel. Let (H, x) be a smallest odd wheel of G, u and v be two nonadjacent nodes of H and P^1, P^2 be the two internally node-disjoint subpaths of H between u and v, where $|P^1| \leq |P^2|$. Let P be a shortest path between u and v in $G' = G \setminus M(H)$. Then the following hold:*

- *(i)* $|P| = |P^1|$
- *(ii) Either $H' = u, P, v, P^2, u$ is a hole, and (H', x) is a smallest odd wheel dominated by (H, x); or $|P^1| = |P^2|$, $H'' = u, P, v, P^1, u$ is a hole, and (H'', x) is a smallest odd wheel dominated by (H, x).*

Proof: Let $H = h_1, \ldots, h_{2s}, (h_{2s+1} = h_1)$ where $h_1 = u$, $s \geq 3$. Let $\overrightarrow{H}$ be the directed cycle obtained by orienting the edges of H from h_i to h_{i+1} for every $1 \leq i \leq 2s$. For any two distinct nodes a and b in H, let $H(a, b)$ be the underlaying graph of the directed path from a to b in $\overrightarrow{H}$. W.l.o.g., $P^1 = H(u, v)$ and $P^2 = H(v, u)$, and $v = h_m$ for some $3 \leq m \leq s + 1$. Let $P = p_0, \ldots, p_{k+1}$, where $p_0 = u$ and $p_{k+1} = v$. We will prove 5.15 by induction on k.

If $k = 1$, then p_1 has exactly two neighbors in H, namely u and v (since $p_1 \notin M(H)$), and they are contained in a subpath of H of length 2, say u, w, v. Hence $H' = u, p_1, v, P^2, u$ is a hole of the same length as H. Suppose (H', x) is not an odd wheel, then x is adjacent to exactly one node $y \in \{p_1, w\}$. Let $z \in \{p_1, w\}$, $z \neq y$, and let u', v' be the neighbors of x distinct from u, v, closest to u and v in P^2, respectively. Then $C = u, z, v, H(v, v'), v', x, u', H(u', u), u$ is a hole and (C, y) is a detectable 3-wheel, a contradiction. We only need to prove that (H', x) is dominated by (H, x). Assume not and let $y \neq x$ be a major node for H' that is not major for H. Since y is not major for H, then y is adjacent to p_1 but not to w, y has exactly 2 neighbors in P^2 and they are contained in a path of length 2. But then (H', y) is a detectable 3-wheel, a contradiction. Hence we may assume $k \geq 2$.

(5.15.1) *Either:*

(i) no node of $P_{p_1p_k}$ belongs to or has a neighbor in $H(h_{m+1}, h_{2s})$, or

(ii) $H'' = u, P, v, P^1, u$ is a hole, and (H'', x) is a smallest odd wheel.

Assume that there is a node of $P_{p_1p_k}$ that belongs to or has a neighbor in $H(h_{m+1}, h_{2s})$, then there exists a j, $m+1 \leq j \leq 2s$, such that there are chordless paths Q^1 and Q^2 between h_j and u and h_j and v, respectively, with interior contained in the interior of P. Therefore

$$\begin{aligned}|Q^1| + |Q^2| &\leq k+3 \leq m+1 \leq 2s+3-m = (2s+2-j)+(j-m+1)\\ &\leq (|H(h_j,u)|+1)+(|H(v,h_j)|+1).\end{aligned}$$

Since $|Q^1|$ has the same parity as $|H(h_j, u)|$ and $|Q^2|$ has the same parity as $|H(v, h_j)|$, then, by symmetry, we may assume $|Q^1| \leq |H(h_j, u)|$. We can also argue that either $|Q^1| < |P|$ and $j < 2s$, or $|Q^2| \leq |H(v, h_j)|$, $|Q^2| < |P|$ and $j > m+1$. In fact, if $|Q^1| = |P|$, then $Q^1 = u, p_1, \ldots, p_k, h_j$ and $Q^2 = h_j, p_k, v$, hence $|Q^2| \leq |H(v, h_j)|$ and $|Q^2| < |P|$. Furthermore, if $|Q^1| < |P|$ and $j = 2s$, then $|Q^2| < |P| \leq |H(v, h_j)| + 1$ (since h_j cannot be adjacent to both h_1 and p_1) and $j > m+1$. Thus, by symmetry, we may assume $|Q^1| \leq |H(h_j, u)|$, $|Q^1| < |P|$ and $j < 2s$. By inductive hypothesis,

$$d_{G'}(h_j, u) = d_H(h_j, u) = \min(2s+1-j, j-1)$$

and $d_{G'}(h_j, u) \leq |Q^1| < |P| \leq m-1 < j-1$, hence $d_{G'}(h_j, u) = 2s+1-j = |Q^1| < s$. Since $V(Q^1) \subseteq V(G) \setminus M(H)$, then by induction $C = u, H(u, h_j), h_j, Q^1, u$ is a hole and (C, x) is a smallest odd wheel dominated by (H, x). We obtain a directed cycle $\overrightarrow{C}$ by orienting the edges of C to agree with the orientation of the edges in $H(u, h_j)$, and define $C(a, b)$ for every a, b in C as before.

Let u' be the neighbor of h_j in Q^1. Then $P' = P_{u'v}$ is a path between u' and v of length $k+2-|Q^1| = k+j+1-2s < k+1$, and $V(P') \subseteq V(G) \setminus M(H) \subseteq V(G) \setminus M(C)$. By induction, $|C(u', v)| \leq k+j+1-2s$ or $|C(v, u')| \leq k+j+1-2s$. But $|C(u', v)| > |H(u, v)| \geq k+1 > |P'|$, hence $|C(v, u')| \leq k+j+1-2s$. This implies $j-m+1 \leq k+j+1-2s$, so $2s \leq k+m$, but $m \leq s+1$ and $k \leq s-1$, hence $m = s+1$, $k = s-1$, $d_{G'}(u'v) = d_C(u', v) = |P'|$. By induction, $C' = u', C(u', v), v, P', u'$ is a hole, and (C, x) is a smallest odd wheel. Clearly, $C' = H''$, where $H'' = u, P, v, P^1, u$. This concludes the proof of 5.15.1.

By 5.15.1 and symmetry, we may assume that no node of $P_{p_1p_k}$ belongs to or has a neighbor in $H(h_{m+1}, h_{2s})$.

(5.15.2) *Either*

(*i*) $H' = u, P, v, P^2, u$ *is a hole, and* (H', x) *is a smallest odd wheel; or*

(*ii*) $|P^1| = |P^2|$, $H'' = u, P, v, P^1, u$ *is a hole, and* (H'', x) *is a smallest odd wheel.*

Assume that no node in $P_{p_1p_k}$ belongs to or has a neighbor in $H(h_2, h_{m-1})$, then u and v must be on the same side of the bipartition, else $H(u, v)$, $H(v, u)$, and P would induce a 3-path configuration between u and v. If (H'', x) is an odd wheel, then it is a smallest one, since $|H''| \leq |H|$, and case (*ii*) occurs. Thus $V(P_i) \cup V(P)$, $i = 1, 2$, contains either exactly one neighbor of x, or an even number of neighbors of x. We may assume that x is nonadjacent to both u and v, otherwise the number of neighbors of x in $H(u, v)$ and in $H(v, u)$ have distinct parities, so either (H', x) or (H'', x) is an odd wheel. Thus we may assume x and v are nonadjacent. If x is adjacent to u, we may assume that either $H(u, v) \setminus u$ and $H(v, u) \setminus u$ both contain neighbors of x, or $P \setminus u$ contains a neighbor of x, otherwise all neighbors of x in H are contained in $H(u, v)$ or $H(v, u)$, and no neighbor of x is contained in $P \setminus u$, but then either (H', x) or (H'', x) is a smallest odd wheel. Thus, if u_1, u_2, u_3 are the neighbors of x closest to v in $H(u, v)$, $H(v, u)$ and P, respectively, then u_1, u_2, u_3 are pairwise distinct, thus $Q^1 = x, u_1, H(u_1, v), v$, $Q^2 = x, u_2 H(v, u_2), v$ and $Q^3 = x, u_3, P u_3 v, v$ induce a 3-path configuration (since x and v are in distinct sides of the bipartition, because x and u are adjacent). Thus we may assume that both u and v are nonadjacent to x. This implies that the number of neighbors of x in $H(u, v)$ and in $H(v, u)$ have distinct parities, so x has an odd number of neighbors on the hole C, where $C = H'$ or $C = H''$. This implies that x has exactly one neighbor, say x' in C, while x has at least 2 neighbors in the chordless path P' contained in H whose interior is disjoint from C. Let u' and v' be the neighbors of x in P' closest to u and v, respectively. Clearly u, v, u', v' are pairwise distinct. Let Q and Q' be the two distinct subpaths of C between x' and u such that v is in Q'. If u and x are in distinct sides of the bipartition, then the paths $Q^1 = u, Q, x', x$, $Q^2 = u, P_{uu'}, u', x$ and $Q^3 = u, Q'(u, v), v, P_{vv'}, v', x$ form a 3-path configuration, a contradiction. Thus x' and u are in distinct sides of the bipartition. Since $|H(u, v)|, |H(v, u)|, |P| \geq 3$, then x' is not adjacent to both u and v, say, w.l.o.g., x' is nonadjacent to u. The paths Q, Q', $Q'' = u, P_{uu'}, x, x'$ induce a 3-path configuration.

Therefore we may assume there exists j, $2 \leq j \leq m-1$, such that there are chordless paths Q^1 and Q^2 between h_j and u and h_j and v, respectively, with interior contained in the interior of P. By 5.15.1 and symmetry, we may assume $m \leq s$. We have

$$|Q^1|+|Q^2| \leq k+3 \leq m+1 = j+(m+1-j) = (|H(u,h_j)|+1)+(|H(h_j,v)|+1)$$

and, by the same argument as in 5.15.1, we may assume $|Q^1| \leq j-1$, $|Q^1| < |P|$ and $j > 2$. By induction, $|Q^1| = d_{G'}(u,h_j) = d_H(u,h_j) = j-1$, $C = u, Q^1, h_j, H(h_j,u), u$ is a hole and (C,x) is a smallest odd wheel dominated by (H,x).
Let u' be the neighbor of h_j in Q^1 and let P' be the path between u' and v in P. Then

$$|P'| = k+2-|Q^1| = k-j+3 \leq |C(u',v)|$$

thus, by induction, $C' = u', P', v, C(v,u'), u'$ is a hole and (C',x) is a smallest odd wheel. Clearly, $C' = H'$. This concludes the proof of 5.15.2.

By 5.15.2 and by symmetry, we may assume that (H',x) is a smallest odd wheel. To conclude the proof of 5.15 we only need to show that (H',x) is dominated by (H,x). Suppose there exists a major node y for H' that is not major for H. Then the neighbors of y in H are contained in a subpath of H of length 2. Also, the neighbors of y in P are contained in a subpath of P of length 2, otherwise let i,j, $0 \leq i < j \leq k+1$ be the minimum and maximum index, respectively, such that p_i and p_j are adjacent to y; then $P' = u, P_{up_i}, p_i, y, p_j, P_{p_jv}, v$ is a path in G' strictly shorter then P, a contradiction. Therefore y has at most 3 neighbors in H', and two of them are contained in a subpath of H' of length 2. Thus (H',y) is a detectable 3-wheel, a contradiction. □

Claim 5.21 *There exists a $O(|V(G)|^9)$ algorithm with the following specifications:*

- **Input** *A bipartite graph G.*
- **Output** *Determines whether G is balanceable or not.*

Algorithm:

1. Apply the algorithm in 5.6. If G contains a 3-path configuration, then output the fact that G is not balanceable and stop.

2. Apply the algorithm in 5.8. If G contains a detectable 3-wheel, then output the fact that G is not balanceable and stop.

3. For every 7-tuple of nodes $u_1, \ldots, u_7$, such that u_1, u_2, u_3 and u_4, u_5, u_6 induce a path, $u_2 \in C$, $u_5 \in R$, u_7 is nonadjacent to u_2, do the following

   (a) Compute $X(u_1, \ldots, u_6) = N(u_2) \cup N(u_5) \setminus \{u_1, u_2, u_3, u_4, u_5, u_6\}$.

   (b) Compute the shortest paths $P_1(u_1, \ldots, u_7)$ and $P_2(u_1, \ldots, u_7)$ between u_1 and u_7 and u_3 and u_7 in $G \setminus X(u_1, \ldots, u_6)$, respectively.

   (c) If no node in the interior of $P_1(u_1, \ldots, u_7)$ belongs to or has a neighbor in $P_2(u_1, \ldots, u_7)$, define:

   $$H(u_1, \ldots, u_7) = u_1, P_1(u_1, \ldots, u_7), u_7, P_2(u_1, \ldots, u_7), u_3, u_2, u_1;$$

   (d) For each $x \in X(u_1, \ldots, u_6)$ check if $(H(u_1, \ldots, u_7), x)$ is an odd wheel. If it is, output that G is not balanceable and stop.

   Otherwise output that G is balanceable.

Correctness: Both step 1 and step 2 take time $O(|V(G)|^9)$. Step 3 performs computations (a)-(d) at most $|V(G)|^7$ times. Steps (a) and (b) can be performed in time $O(|V(G)|)$, while steps (c) and (d) can be performed in time $O(|V(G)|^2)$, thus the running time is $O(|V(G)|^9)$ as claimed.

We need to show that the algorithm is correct. If G contains a 3-path configuration or a detectable 3-wheel, then by 5.6 and 5.8 the algorithm will output correctly that G is not balanceable. We only need to prove that, if G does not contain a 3-path configuration or a detectable 3-wheel, but G contains an odd wheel, then step 3 will output that G is not balanceable. Let (H, x) be a smallest odd wheel in G. Then by 5.11 there exist two subpaths u_1, u_2, u_3 and u_4, u_5, u_6 of H such that every major node for H is adjacent to u_2 or u_5. The set $X(u_1, ..., u_6)$ computed in step (a) is a cleaner for H, by an argument similar to the one in 5.19. Let u_7 be the node at distance $|H|/2$ from u_2 in H. Clearly, the paths Q_1 and Q_2 between u_1 and u_7 and between u_3 and u_7 in H, respectively, have length strictly less then $|H|/2$, thus, by an argument similar to the one in the proof of 5.16, $H(u_1, \ldots, u_7) =$

$u_1, P_1(u_1, \ldots, u_7), u_7, P_2(u_1, \ldots, u_7), u_3, u_2, u_1$ is a hole and $(H(u_1, \ldots, u_7), x)$ is a smallest odd wheel, where $P_1(u_1, \ldots, u_7)$ and $P_2(u_1, \ldots, u_7)$ are the paths computed in step (b). Since $x \in X(u_1, \ldots, u_6)$, then step (d) will output that G is not balanceable. $\square$

Chapter 6

Bicolorings and equitable bicolorings of matrices

6.1 Introduction

A real matrix is *totally unimodular* (t.u.) if every nonsingular square submatrix has determinant ± 1 (note that every t.u. matrix must be a $0, \pm 1$ matrix). A $0, \pm 1$ matrix which is not totally unimodular but whose submatrices are all totally unimodular is said *almost totally unimodular*. Camion [9] proved the following:

Theorem 6.1 (Camion [9] and Gomory [cited in [9]]) *Let A be an almost totally unimodular $0, \pm 1$ matrix. Then A is square,* $\det A = \pm 2$*, and A^{-1} has only $\pm\frac{1}{2}$ entries. Furthermore, each row and each column of A has an even number of nonzero entries and the sum of all entries in A equals 2 modulo 4.*

For any positive integer k, we say that a $0, \pm 1$ matrix A is *k-balanced* if it does not contain any almost totally unimodular submatrix with at most $2k$ nonzero entries in each row. Obviously, an $m \times n$ $0, \pm 1$ matrix A is balanced if and only if it is 1-balanced, while A is totally unimodular if and only if A is k-balanced for some $k \geq \lfloor n/2 \rfloor$. The class of k-balanced matrices was introduced by Truemper and Chandrasekaran [77] in the $0, 1$ case and generalized by Conforti, Cornuéjols and Truemper in [27], who also showed that several polyhedral properties of balanced and totally unimodular matrices extend to the class of k-balanced matrices. In Section 6.3 we characterize the class of

k-balanced matrices in terms of k-*equitable bicolorings*, that are special partitions of the columns of a $0, \pm 1$ matrix. This characterization generalizes results of Ghouila-Houri [52] for totally unimodular matrices, and Berge [4] and Conforti and Cornuéjols [20] for balanced matrices. In section 6.4, we further discuss some coloring properties of the class of k-balanced matrices. Finally, we will use such properties to prove that certain polyhedra arising from k-balanced $0, 1$ matrices have the *integer decomposition property.*

6.2 k-balanced matrices and integral polyhedra

The following is a classical result of Hoffman and Kruskal.

Theorem 6.2 (Hoffman and Kruskal [56]) *Let A be an $m \times n$ totally unimodular matrix. Then $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ is an integral polyhedron for every $b \in \mathbb{Z}^m$.*

The next Theorem, proved by Conforti, Cornuéjols and Truemper, provides a generalization of Theorem 6.2, as we shall explain later. Given an $m \times n$ $0, \pm 1$ matrix, denote by $n(A)$ the vector with m components whose ith component $n_i(A)$ is the number of -1's in the ith row of A, and let $p(A) = n(-A)$.

Theorem 6.3 (Conforti, Cornuéjols and Truemper [27]) *Let A be an $m \times n$ k-balanced $0, \pm 1$ matrix with rows a^i, $i \in [m]$, b be a vector with entries b_i, $i \in [m]$, and let $\mathcal{S} = S_1, S_2, S_3$ be a partition of $[m]$. Then*

$$
\begin{aligned}
P(A, b, \mathcal{S}) = \{x \in \mathbb{R}^n \quad : \quad & a^i x \leq b_i \text{ for } i \in S_1 \\
& a^i x = b_i \text{ for } i \in S_2 \\
& a^i x \geq b_i \text{ for } i \in S_3 \\
& \mathbf{0} \leq x \leq \mathbf{1}\} \qquad (6.1)
\end{aligned}
$$

is an integral polytope for all integral vectors b such that $-n(A) \leq b \leq \mathbf{k} - n(A)$.

Proof: Suppose not and let A be a counterexample with the minimum number of rows and columns, and $\bar{x}$ be a fractional vertex of $P(A, b, \mathcal{S})$. We may assume that, for some $1 \leq k \leq n$, $\bar{x}_1, \ldots, \bar{x}_k$ are all fractional and $\bar{x}_{k+1}, \ldots, \bar{x}_n$ are all $0, 1$. Since $\bar{x}$ is a vertex, then there exists a $k \times n$ row submatrix A' of A, such that $A'x = b'$, where b' is the corresponding subvector of b, and such that the submatrix $C = (c_{ij})$ of A' formed by the columns of A' with index at most k is nonsingular. Let $A' = (C, D)$, where $D = (d_{ij})$ is a $k \times (n-k)$ matrix, and assume, w.l.o.g., that the rows of A' are indexed by $[k]$. Define a vector $\bar{b} \in \mathbb{Z}^k$ by $\bar{b}_i = b_i - \sum_{j=k+1}^n d_{ij}\bar{x}_j$, for $i \in [k]$. Then $\bar{b}_i \geq -n_i(C)$, since $b_i = \sum_{j=1}^k c_{ij}\bar{x}_i + \sum_{j=k+1}^n d_{ij}\bar{x}_j \geq -n_i(C) + \sum_{j=k+1}^n d_{ij}\bar{x}_j$, and $\bar{b}_i \leq k - n_i(C)$, since $b_i \leq k - n_i(A) = k - n_i(C) - n_i(D) \leq k - n_i(C) + \sum_{j=k+1}^n d_{ij}\bar{x}_j$. Since $(\bar{x}_1, \ldots, \bar{x}_k)$ is a fractional vertex of $\{x \in \mathbb{R}^k \mid Cx = \bar{b}, \mathbf{0} \leq x \leq \mathbf{1}\}$, then, by the minimality of A, $A = C$.

Thus $\bar{x}$ has only fractional entries, and A is square and nonsingular. Let $i \in [n]$, and let $\bar{A}$ be the submatrix of A obtained by deleting row i, and $\bar{b}$ be the corespondent subvector of b. Then, by the minimality of A, $\bar{P} = \{x \in \mathbb{R}^n \mid \bar{A}x = \bar{b}, \mathbf{0} \leq x \leq \mathbf{1}\}$ is integral, and it is nonempty since $\bar{x} \in \bar{P}$. $\bar{P}$ has dimension 1, hence it has only two vertices z_1 and z_2, so $\bar{x} = \lambda z_1 + (1-\lambda)z_2$ for some $0 < \lambda < 1$. Since $0 < x_i < 1$ for every $i \in [n]$, then $z_1 + z_2 = \mathbf{1}$, thus

$$p(\bar{A}) - n(\bar{A}) = \bar{A}\mathbf{1} = \bar{A}(z_1 + z_2) = 2\bar{b} \leq 2\mathbf{k} - 2n(\bar{A}).$$

Thus the total number of nonzero elements in row j of $\bar{A}$ is $p_j(\bar{A}) + n_j(\bar{A}) \leq 2k$. By repeating the previous argument for a different choice of $i \in [n]$, we conclude that every row of A has at most $2k$ nonzero elements per row and per column. Also, by Theorem 6.2, A is not totally unimodular, hence it contains an almost t.u. submatrix, contradicting the fact that A is k-balanced. $\square$

Notice that a totally unimodular matrix is k-balanced for every positive integer k. In fact, Theorem 6.3 generalizes Hoffman and Kruskal's characterization of total unimodular matrices. Let b be an integral vector and $\bar{x}$ be a vertex of $Q(A, b) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. Let A' be the matrix obtained by replicating the ith column of A $\lceil |\bar{x}_i| \rceil$ times, and multiplying such column by $\bar{x}_i/|\bar{x}_i|$. Let C_i be the set of indices of columns of A' that correspond to the ith column of A. Consider the vector $\bar{y}$, with as many entries as the number of columns of A', defined as follows: for every $i \in [n]$, fix to 1 $\lfloor |\bar{x}_i| \rfloor$ of the entries of $\bar{y}$ with index in C_i, and fix to $|x_i| - \lfloor |\bar{x}_i| \rfloor$ the remaining entry (if any). Let $\mathcal{S} = S_1, S_2, S_3$, where $S_1 = \{i \in [m] \mid b^i \geq 0\}$, $S_2 = \emptyset$, $S_3 = [m] \setminus S_1$. It is immediate to verify that $\bar{y}$ is a vertex of $P(A', b, \mathcal{S})$, and $\bar{y}$ is integral if and only if $\bar{x}$ is integral. Since A is totally unimodular, then A' is totally unimodular as well, thus, by Theorem 2.16, $P(A', b, \mathcal{S})$ is integral, so $Q(A, b)$ is integral.

Also, when $k = 1$, Theorem 6.3 is equivalent to Theorem 5.3. It is not known whether the system defined by the constraints in (6.1) is totally dual integral. Clearly, by Theorem 5.4 such system is TDI if $-n(A) \leq b \leq \mathbf{1} - n(A)$. Next we observe that the system is TDI also for $k = 2$ if the matrix A has only nonnegative entries.

Proposition 6.4 *Let A be an $m \times n$ 2-balanced $0, 1$ matrix with rows a^i, $i \in [m]$, b be a vector with entries b_i, $i \in [m]$, and let $\mathcal{S} = S_1, S_2, S_3$ be a partition of $[m]$. Then the system*

$$\begin{array}{ll} a^i x \leq b_i & \textit{for } i \in S_1 \\ a^i x = b_i & \textit{for } i \in S_2 \\ a^i x \geq b_i & \textit{for } i \in S_3 \\ x \geq \mathbf{0} & \end{array} \tag{6.2}$$

is totally dual integral for every integral vector b such that $0 \leq b \leq \mathbf{2}$.

Proof: Let $c \in \mathbb{Z}^n$ such that the linear program $\min\{cx \,|\, x \text{ satisfies } (6.2)\}$ has a finite optimum. The dual is

$$\begin{array}{lll} \max & ub & \\ & uA \leq c & \\ & u_i \leq 0 & i \in S_1 \\ & u_i \geq 0 & i \in S_3 \end{array} \tag{6.3}$$

We need to show that (6.3) has an optimal solution with integer components. The proof is by induction on the number of rows of A. We may assume that $b_\ell \leq 1$ for some $\ell \in [m]$, otherwise $b = \mathbf{2}$ and we may consider the problem $\max\{u\mathbf{1} \,|\, uA \leq c,\ u_i \leq 0 \text{ for } i \in S_1,\ u_i \geq 0 \text{ for } i \in S_3\}$ that has the same optimal solutions as (6.3). Let $\bar{u}$ be an optimal solution for (6.3). Let A^ℓ be the matrix obtained from A by removing row a^ℓ, and u^ℓ be the vector obtained from u by removing the entry u_ℓ. By the inductive hypothesis, the system

$$\begin{array}{lll} \max & \sum_{i=1, i\neq \ell}^{m} b_i u_i & \\ & u^\ell A^\ell \leq c - a^\ell \lfloor \bar{u}_\ell \rfloor & \\ & u_i \leq 0 & i \in S_1 \setminus \{\ell\} \\ & u_i \geq 0 & i \in S_3 \setminus \{\ell\} \end{array} \tag{6.4}$$

has an integral optimal solution $\tilde{u}$ (since $\bar{u}^\ell$ is a feasible solution). By Theorem 6.3, $\sum_{i=1}^m b_i\bar{u}_i$ is integer, thus

$$\sum_{i=1, i\neq\ell}^{m} b_i\tilde{u}_i \geq \left\lceil \sum_{i=1, i\neq\ell}^{m} b_i\bar{u}_i \right\rceil = \sum_{i=1}^{m} b_i\bar{u}_i - \lfloor b_\ell\bar{u}_\ell \rfloor = \sum_{i=1}^{m} b_i\bar{u}_i - b_\ell\lfloor\bar{u}_\ell\rfloor$$

since $b_\ell \leq 1$. Thus $(\tilde{u}_1, \ldots, \tilde{u}_{\ell-1}, \lfloor\bar{u}_\ell\rfloor, \tilde{u}_{\ell+1}, \ldots, \tilde{u}_m)$ is an integral optimal solution for (6.3). □

6.3 k-equitable bicolorings

A $0, \pm 1$ matrix A has an *equitable bicoloring* if its columns can be partitioned into red and blue columns so that, for every row of A, the sum of the entries in the red columns differs by at most one from the sum of the entries in the blue columns. Ghouila-Houri [52] gave the following characterization of the class of totally unimodular matrices.

Theorem 6.5 (Ghouila-Houri [52]) *A $0, \pm 1$ matrix A is totally unimodular if and only if every submatrix of A has an equitable bicoloring.*

An analogous of Theorem 6.5 holds for balanced matrices. A $0, \pm 1$ matrix is *bicolorable* if its columns can be partitioned into blue columns and red columns so that every row with at least two nonzero entries contains either two nonzero entries of opposite sign in columns of the same color or two nonzero entries of the same sign in columns of different colors. A partition with this property is a *bicoloring* of A. Berge [4] showed that a $0, 1$ matrix A is balanced if and only if every submatrix of A is bicolorable. Conforti and Cornuéjols [20] extended this result to $0, \pm 1$ matrices.

Theorem 6.6 *A $0, \pm 1$ matrix A is balanced if and only if every submatrix of A has a bicoloring.*

Given a $0, \pm 1$ matrix A with rows a^i, $i \in [m]$, let $\alpha_i := \min\left(\lfloor\frac{p_i(A)+n_i(A)}{2}\rfloor, k\right)$. We say that A has a *k-equitable bicoloring* if its columns can be partitioned into blue columns and red columns so that the matrix A', obtained from A by multiplying its blue columns by -1, has at least α_i positive entries and at least α_i negative entries in row a^i, for every $i \in [m]$.

One can readily verify that an $m \times n$ $0, \pm 1$ matrix A is bicolorable if and only if A has a 1-equitable bicoloring, while A has an equitable bicoloring if and only if A has a k-equitable bicoloring for $k \geq \lfloor n/2 \rfloor$. The following theorem provides a characterization of the class of k-balanced matrices, which is a generalization of both Theorems 6.5 and 6.6.

Theorem 6.7 (Conforti, Cornuéjols, Zambelli [32]) *A $0, \pm 1$ matrix A is k-balanced if and only if every submatrix of A has a k-equitable bicoloring.*

Proof: Assume first that A is k-balanced and let B be any submatrix of A. Assume, up to row permutation, that

$$B = \begin{pmatrix} B' \\ B'' \end{pmatrix}$$

where B' is the row submatrix of B determined by the rows of B with $2k$ or fewer nonzero entries. Consider the system

$$\begin{aligned} B'x &\geq \left\lfloor \frac{B'\mathbf{1}}{2} \right\rfloor \\ -B'x &\geq -\left\lceil \frac{B'\mathbf{1}}{2} \right\rceil \\ B''x &\geq \mathbf{k} - n(B'') \\ -B''x &\geq \mathbf{k} - n(-B'') \\ \mathbf{0} \leq x &\leq \mathbf{1} \end{aligned} \tag{6.5}$$

Since B is k-balanced, also $\begin{pmatrix} B \\ -B \end{pmatrix}$ is k-balanced. Therefore the constraint matrix of system (6.5) above is k-balanced. One can readily verify that $-n(B') \leq \left\lfloor \frac{B'\mathbf{1}}{2} \right\rfloor \leq \mathbf{k} - n(B')$ and $-n(-B') \leq -\left\lceil \frac{B'\mathbf{1}}{2} \right\rceil \leq \mathbf{k} - n(-B')$. Therefore, by Theorem 6.3 applied with $S_1 = S_2 = \emptyset$, system (6.5) defines an integral polytope. Since the vector $(\frac{1}{2}, ..., \frac{1}{2})$ is a solution for (6.5), the polytope is nonempty and contains a $0, 1$ point $\bar{x}$. Color a column i of B blue if $\bar{x}_i = 1$, red otherwise. It can be easily verified that such a bicoloring is, in fact, k-equitable.

Conversely, assume that A is not k-balanced. Then A contains an almost totally unimodular matrix B with at most $2k$ nonzero elements per row. Suppose that B has a k-equitable bicoloring, then such a bicoloring must be equitable since each row has, at most, $2k$ nonzero elements. By Theorem 6.1, B has an even number of nonzero elements in each row. Therefore the sum of the columns colored blue equals the sum of the columns colored red, therefore B is a singular matrix, a contradiction. □

Given a $0/\pm 1$ matrix A and positive integer k, one can find in polynomial time a k-equitable bicoloring of A or a certificate that A is not k-balanced as follows:

Find a basic feasible solution of (6.5). If the solution is not integral, A is not k-balanced by Theorem 6.3. If the solution is a 0,1 vector, it yields a k-equitable bicoloring as in the proof of Theorem 6.7.

A basic feasible solution can to (6.5) can be found in polynomial time using an algorithm of Megiddo [65] that, given optimal solutions for both the primal and the dual, determines basic optimum solutions both for the primal and for the dual. In fact, given a feasible solution $\bar{x}$ for some system of constraints $A \leq x$, $x \geq 0$, $\bar{x}$ is obviously optimal for the problem $\max\{0 \mid A \leq x, x \geq 0\}$ and $\bar{y} = 0$ is optimal for the dual $\min\{by \mid y^{\mathsf{T}} A \geq 0, y \geq 0\}$, hence an optimal basic feasible solution can be computed using Megiddo's algorithm. Since the vector $(\frac{1}{2}, ..., \frac{1}{2})$ is a feasible solution of (6.5), a basic feasible solution of (6.5) can be derived in strongly polynomial time.

6.4 λ-colorings

Given a matrix A and an integer $\lambda \geq 2$, a λ-coloring of A is a partition of the columns of A into λ sets (*colors*) $I_1, \ldots, I_\lambda$ (some of the colors may be empty).

Let A be a $0,1$ matrix and a^i, $i \in [m]$, be the rows of A. We say that *color h occurs in row a^i* if there exists $j \in I_h$ such that $a^i_j = 1$. We say that a $0,1$ matrix A is *λ-colorable* if there exists a λ-coloring of A such that, for every $i \in [m]$, the number of colors occurring in a^i is $\min(p_i(A), \lambda)$. Obviously, this is the maximum number of colors that can occur in a^i. We call such a λ-coloring a *good λ-coloring*. It is immediate to verify that the definition of good 2-coloring is equivalent to the definition of bicoloring we gave in the previous section when restricted to $0,1$ matrices. The following was proven by Berge [5]:

Theorem 6.8 (Berge) *A $0,1$ matrix A is balanced if and only if every submatrix of A is λ-colorable for every integer $\lambda \geq 2$.*

Observe that $I_1, \ldots, I_\lambda$ is a λ-coloring of A if and only if I_j, I_h is a bicoloring of the matrix $A_{I_j I_h}$ induced by the columns in $I_j \cup I_h$ for every $1 \le j < h \le \lambda$. Thus one may extend the definition of λ-coloring to $0, \pm 1$ matrices by saying that a $0, \pm 1$ matrix A is λ-*colorable* if and only if there exists a λ-coloring $I_1, \ldots, I_\lambda$ of A such that I_j, I_h is a bicoloring of $A_{I_j I_h}$ for every $1 \le j < h \le \lambda$. It was conjecture by Conforti and Zambelli that a statement analogous to 6.8 should hold for 0 ± 1 matrices:

Conjecture 6.9 *A $0, \pm 1$ matrix A is balanced if and only if every submatrix of A is λ-colorable for every integer $\lambda \ge 2$.*

A weaker definition of λ-colorability was given by De Werra [45]. Such definition has the same requirement as the definition above in the rows in which all entries are either all nonnegative or all non-positive, while in the other rows it only demands that a $+1$ and a -1 receive the same color. De Werra showed that every $0, \pm 1$ balanced matrix satisfies this condition.

A result in the spirit of Conjecture 6.9 holds for totally unimodular matrices. We say that a 0 ± 1 matrix A has an *equitable λ-coloring* if there exists a λ-coloring $I_1, \ldots, I_\lambda$ so that, for each row a^i and for every color I_h, $\lfloor p_i(A)/\lambda \rfloor \le \sum_{j \in I_s} a^i_j \le \lceil p_i(A)/\lambda \rceil$. Equivalently, $I_1, \ldots, I_\lambda$ is an equitable λ-coloring if I_j, I_h is an equitable bicoloring of $A_{I_j I_h}$ for every $1 \le j < h \le \lambda$. De Werra [43] showed that totally unimodular matrices can be characterized in terms of equitable λ-colorings.

Theorem 6.10 (De Werra) *A matrix A is totally unimodular if and only if every submatrix of A has an equitable λ-coloring for every integer $\lambda \ge 2$.*

Proof: Necessity follows immediately from Theorem 6.7 (in fact, we only need to consider 2-colorings). For the other direction, consider the system

$$\begin{array}{c} \lfloor A\mathbf{1}/\lambda \rfloor \le Ax \le \lceil A\mathbf{1}/\lambda \rceil \\ \mathbf{0} \le x \le \mathbf{1} \end{array} \tag{6.6}$$

Since A is totally unimodular, (6.6) defines an integral polytope, which is nonempty since $\lambda^{-1}\mathbf{1}$ is a solution. Thus (6.6) has an integral solution $\bar{x}$. Let $\bar{A}$ be the matrix induced by the columns corresponding to the zero entries of $\bar{x}$. By induction on λ, $\bar{A}$ has an equitable $(\lambda - 1)$-coloring $I_1, \ldots, I_{\lambda-1}$. Let I_λ be the set of columns corresponding to the positive entries of $\bar{x}$. Then $I_1, \ldots, I_\lambda$ is an equitable λ-coloring. □

We give a common generalization of good and equitable λ-colorings as follows. Given a $0, \pm 1$ matrix A and integers k and λ, a *k-equitable λ-coloring* of A is a λ-coloring $I_1, \ldots, I_\lambda$ of A such that I_j, I_h is a k-equitable bicoloring of $A_{I_j I_h}$ for every $1 \leq j < h \leq \lambda$. Clearly, a good 1-equitable λ-coloring is a good λ-coloring, while an $\lfloor n/\lambda \rfloor$-equitable λ-coloring is an equitable λ-coloring (where n is the number of columns).

Next, we show that k-balanced $0,1$ matrices have a k-equitable λ-coloring for every λ. Notice that, in the $0,1$ case, a k-equitable λ-coloring is a λ coloring such that, in the ith row, the number of ones of each color is at least k if $p_i(A) \geq k\lambda$, and it is either $\lfloor p_i(A)/\lambda \rfloor$ or $\lceil p_i(A)/\lambda \rceil$ otherwise.

Theorem 6.11 *A $0,1$ matrix A is k-balanced if and only if every submatrix of A has a k-equitable λ-coloring for every integer $\lambda \geq 2$.*

Proof: If $\lambda = 2$, then the statement is equivalent to Theorem 6.7. We only need to show that, given an $m \times n$ $0,1$ matrix A, A has a k-equitable λ-coloring. Assume $\lambda \geq 3$. Let $S_1 \subseteq [m]$ be the set of indices such that a^i has less than $k\lambda$ nonzero entries for each $i \in S_1$, and $S_2 = [m] \setminus S_1$. Given a partition $I_1, \ldots, I_\lambda$ of the columns of A, let, for every $i \in [m]$, $j \in [\lambda]$, $n_{ij} = |\{h \in [n] \,|\, a^i_h = 1, h \in I_j\}|$. Define

$$\mu_{ij} = \begin{cases} \max(n_{ij} - \lceil p_i(A)/\lambda \rceil, \lfloor p_i(A)/\lambda \rfloor - n_{ij}) & \text{for } i \in S_1 \\ \max(0, k - n_{ij}) & \text{for } i \in S_2 \end{cases}.$$

Choose $I_1, \ldots I_\lambda$ minimizing $\mu = \sum_{i=1}^{n} \sum_{j=1}^{\lambda} \mu_{ij}$. Observe that $I_1, \ldots, I_\lambda$ is a k-equitable λ-coloring if and only if $\mu = 0$. Suppose $\mu_{st} > 0$ for some $s \in [m]$, $t \in [\lambda]$. Clearly, there exists $t' \in [\lambda]$ with the following property:

- If $s \in S_1$ and $n_{st} < \lfloor p_i(A)/\lambda \rfloor$, than $n_{st'} > \lfloor p_s(A)/\lambda \rfloor$;
- If $s \in S_1$ and $n_{st} > \lceil p_s(A)/\lambda \rceil$, than $n_{st'} < \lceil p_s(A)/\lambda \rceil$;
- If $s \in S_2$, than $n_{st'} > k$.

W.l.o.g., $t = 1$, $t' = 2$. By Theorem 6.7, $A_{I_1 I_2}$ admits a k-equitable bicoloring I'_1, I'_2. For $i \in [n]$ and $j = 1, 2$, let $n'_{ij} = |\{h \in [n] \,|\, a^i_h = 1, h \in I'_j\}|$, and $\mu'_{ij} = \max(n'_{ij} - \lceil p_i(A)/\lambda \rceil, \lfloor p_i(A)/\lambda \rfloor - n'_{ij})$ for $i \in S_1$, $\mu'_{ij} = \max(0, k - n'_{ij})$ for $i \in S_2$. Clearly, $n'_{i1} + n'_{i2} = n_{i1} + n_{i2}$.

Claim: $\mu'_{i1} + \mu'_{i2} \leq \mu_{i1} + \mu_{i2}$ for every $i \in [m]$, and $\mu'_{s1} + \mu'_{s2} < \mu_{s1} + \mu_{s2}$.

Let $i \in S_1$. If $2\lfloor p_i(A)/\lambda \rfloor \le n_{i1}+n_{i2} \le 2\lceil p_i(A)/\lambda \rceil$, then $\lfloor p_i(A)/\lambda \rfloor \le n_{ij} \le \lceil p_i(A)/\lambda \rceil$ for $j = 1,2$, so $\mu'_{i1} + \mu'_{i2} = 0 \le \mu_{i1} + \mu_{i2}$ (where the inequality is strict if $i = s$). If $n_{i1} + n_{i2} > 2\lceil p_i(A)/\lambda \rceil$, then $n'_{ij} \ge \lceil p_i(A)/\lambda \rceil$ for $j = 1,2$ (since $\lceil p_i(A)/\lambda \rceil \le k$), so $\mu'_{i1} + \mu'_{i2} = (n'_{i1} - \lceil p_i(A)/\lambda \rceil) + (n'_{i2} - \lceil p_i(A)/\lambda \rceil) = (n_{i1} - \lceil p_i(A)/\lambda \rceil) + (n_{i2} - \lceil p_i(A)/\lambda \rceil) \le \mu_{i1} + \mu_{i2}$. If $i = s$, the inequality is strict since, by the choice of t and t', there exists $j \in [2]$ such that $n_{ij} - \lceil p_i(A)/\lambda \rceil < 0 \le \mu_{sj}$. If $n_{i1}+n_{i2} < 2\lfloor p_i(A)/\lambda \rfloor$, then $n'_{ij} \le \lfloor p_i(A)/\lambda \rfloor$ for $j = 1,2$, thus $\mu'_{i1}+\mu'_{i2} = (\lfloor p_i(A)/\lambda \rfloor - n'_{i1}) + (\lfloor p_i(A)/\lambda \rfloor - n'_{i2}) = (\lfloor p_i(A)/\lambda \rfloor - n_{i1}) + (\lfloor p_i(A)/\lambda \rfloor - n_{i2}) \le \mu_{i1} + \mu_{i2}$. If $i = s$, the inequality is strict since, by the choice of t and t', there exists $j \in [2]$ such that $\lfloor p_i(A)/\lambda \rfloor - n_{ij} < 0 \le \mu_{sj}$. Let $i \in S_2$. If $n_{i1} + n_{i2} < 2k$, then $n'_{ij} \le k$ for $j = 1,2$, thus $\mu'_{i1} + \mu'_{i2} = (k - n'_{i1}) + (k - n'_{i1}) = (k - n_{i1}) + (k - n_{i1}) \le \mu_{i1} + \mu_{i2}$. If $i = s$, the inequality is strict since $k - n_{s2} < 0 = \mu_{s2}$. If $n_{i1} + n_{i2} \ge 2k$, then $n'_{ij} \ge k$ for $j = 1,2$, thus $\mu'_{i1} + \mu'_{i2} = 0 \le \mu_{i1} + \mu_{i2}$ (where the inequality is strict if $i = s$). This concludes the proof of the claim.

For $i \in [m]$ and $3 \le j \le \lambda$, let $\mu'_{ij} = \mu_{ij}$. By the previous Claim, $\mu' = \sum_{i=1}^{n} \sum_{j=1}^{\lambda} \mu'_{ij} < \sum_{i=1}^{n} \sum_{j=1}^{\lambda} \mu_{ij} = \mu$, thus $I'_1, I'_2, I_3, \ldots, I_\lambda$ contradicts the choice of $I_1, \ldots, I_\lambda$. □

Clearly, Theorem 6.11 implies Theorem 6.8, and it also implies Theorem 6.10 for 0, 1 matrices. A natural question is whether or not every k-balanced $0, \pm 1$ matrix has a k-equitable λ-coloring for every $\lambda \ge 2$. If true, this would obviously imply also Conjecture 6.9.

Observe that the proof of Theorem 6.11 can be turned into a polynomial time algorithm to find such a k-equitable λ-coloring as follows: start from an arbitrary partition $I_1, I_2, I_3, \ldots, I_\lambda$ and compute the corresponding μ. At each iteration, if $\mu = 0$ then stop, else find a new partition with a smaller value of μ by computing a k-equitable bicoloring of $A_{I_t I_{t'}}$ for some appropriate choice of $t, t' \in [\lambda]$. Since computing a k-equitable bicoloring can be done in polynomial time, as we observed in section 6.3, and $\mu \le nm$ for any possible partition, then the above algorithm is polynomial.

Theorem 6.11 has polyhedral consequences. A rational polyhedron P is said to have the *integer decomposition property* if and only if, for every positive integer h and for every integral vector $y \in hP := \{hx \mid x \in P\}$, there exist h integral vectors $x^1, \ldots, x^h \in P$ such that $y = x^1 + \ldots + x^h$. This notion was introduced by Baum and Trotter [1]. We show the following [81].

Theorem 6.12 *Let A be an $m \times n$ k-balanced $0,1$ matrix with rows a^i, $i \in [m]$, and let $\mathcal{S} = S_1, S_2$ be a partition of $[m]$. Then the polyhedron*

$$P = \{x \in \mathbb{R}^n \mid \begin{array}{ll} a^i x \leq b_i & \text{for } i \in S_1 \\ a^i x \geq b_i & \text{for } i \in S_2 \\ x \geq \mathbf{0}\} & \end{array} \tag{6.7}$$

has the integer decomposition property for every integral vector b such that $\mathbf{0} \leq b \leq \mathbf{k}$.

Proof: Let $h \geq 2$ be an integer, and let y be and integral vector in hP. Consider the matrix A' obtained from A by replicating y_j times the jth column of A for every $j \in [n]$ (in particular, if $y_j = 0$, we remove the corresponding column). For each $j \in [n]$, let C_j be the set of indices of columns of A' that are a copy of column j of A (thus $|C_j| = y_j$). Clearly, A' is also k-balanced, so by Theorem 6.11 there exists a k-equitable h-coloring $I_1 \ldots, I_h$ of A'. Let $z^1, \ldots, z^h$ be the characteristic vectors of $I_1, \ldots, I_h$, respectively. Let $x^1, \ldots, x^h$ be the vectors in $\mathbb{Z}^n$ defined by $x^s_j = \sum_{t \in C_j} z^s_t$ for $s \in [h]$. Clearly, $y = x^1 + \ldots + x^h$. We only need to show that $x^s \in P$ for every $s \in [h]$. Observe that $p_i(A') = a^i y$ for every $i \in [m]$. Thus, if $i \in S_1$, then $a^i x^s = a'_i z^s \leq \lceil p_i(A')/h \rceil \leq b_i$. If $i \in S_2$, then $a^i x^s = a'_i z^s \geq \min(\lfloor p_i(A')/h \rfloor, k) \geq b_i$. □

It is known that the class of polyhedra defined in (6.1) has the integer decomposition property whenever $k = 1$ (De Werra [44]) or $k \geq \lfloor n/2 \rfloor$ (Baum and Trotter [1]). Thus Theorem 6.12 generalizes both results whenever the matrix A is a $0,1$ matrix. Does the class of polytopes defined by (6.1) have the integer decomposition property?

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