Maximal lattice-free convex sets in linear subspaces

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March 30, 2009

Abstract

We consider a model that arises in integer programming, and show that all irredundant inequalities are obtained from maximal lattice-free convex sets in an affine subspace. We also show that these sets are polyhedra. The latter result extends a theorem of Lovász characterizing maximal lattice-free convex sets in $\mathbb{R}^n$.

1 Introduction

The study of maximal lattice-free convex sets dates back to Minkowski's work on the geometry of numbers. Connections between integer programming and the geometry of numbers were investigated in the 1980s starting with the work of Lenstra [21]. See Lovász [22] for a survey. Recent work in cutting plane theory [1],[2],[3],[4],[5],[8],[10],[13],[14],[15],[17],[19],[24] has generated renewed interest in the study of maximal lattice-free convex sets. In this paper we further pursue this line of research. In the first part of the paper we consider convex sets in an affine subspace of $\mathbb{R}^n$ that are maximal with the property of not containing integer points in their interior. When this affine subspace is rational, these convex sets are characterized by a result of Lovász [22]. The extension to irrational subspaces appears to be new.

The second part of the paper contains our main result. We consider a model that arises in integer programming, and show that all irredundant inequalities are obtained from maximal lattice-free convex sets in an affine subspace. The relation between lattice-free convex sets and valid inequalities in integer programming was first observed by Balas [6].

*Supported by NSF grant CMMI0653419, ONR grant N00014-03-1-0188 and ANR grant BLAN06-1-138894.

Let W be an affine subspace of $\mathbb{R}^n$. Assume that $W \cap \mathbb{Z}^n \neq \emptyset$. We say that a set $B \subset \mathbb{R}^n$ is a *maximal lattice-free convex set in* W if $B \subset W$, B is convex, has no integer point in its relative interior, and it is inclusionwise maximal with these three properties. The subspace W is said to be *rational* if it is generated by the integer points in W. So, if we denote by V the affine hull of the integer points in W, $V = W$ if and only if W is rational. If W is not rational, then the inclusion $V \subset W$ is strict. When W is not rational, we will also say that W is *irrational*.

Theorem 1. *Let $W \subset \mathbb{R}^n$ be an affine space containing an integral point and V the affine hull of $W \cap \mathbb{Z}^n$. A set $S \subset W$ is a maximal lattice-free convex set of W if and only if one of the following holds:*

(i) S is a polyhedron in W whose dimension equals $\dim(W)$*, $S \cap V$ is a maximal lattice-free convex set of V whose dimension equals* $\dim(V)$*, and for every facet F of S, $F \cap V$ is a facet of $S \cap V$;*

(ii) S is an affine hyperplane of W such that $S \cap V$ is an irrational hyperplane of V;

(iii) S is a half-space of W that contains V on its boundary.

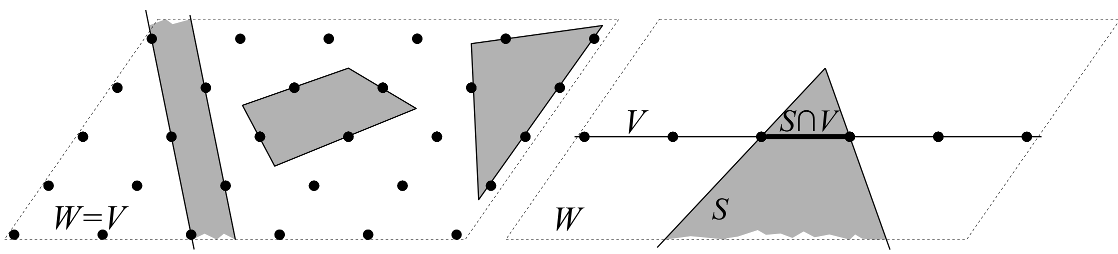

Figure 1: Maximal lattice-free convex sets in a 2-dimensional subspace (Theorem 1(i)).

A characterization of maximal lattice-free convex sets of V, needed in (i) of the previous theorem, is given by the following.

Theorem 2. (Lovász [22]) *Let V be a rational affine subspace of $\mathbb{R}^n$ containing an integral point. A set $S \subset V$ is a maximal lattice-free convex set of V if and only if one of the following holds:*

(i) S is a polyhedron of the form $S = P + L$ where P is a polytope, L is a rational linear space, $\dim(S) = \dim(P) + \dim(L) = \dim(V)$*, S does not contain any integral point in its interior and there is an integral point in the relative interior of each facet of S;*

(ii) S is an irrational affine hyperplane of V.

The polyhedron $S = P + L$ in Theorem 2(i) is called a *cylinder over the polytope P* and can be shown to have at most $2^{\dim(P)}$ facets [16].

Theorem 1 is new and it is used in the proof of our main result, Theorem 3. It is also used to prove the last theorem in [10]. Theorem 2 is due to Lovász ([22] Proposition 3.1). Lovász only gives a sketch of the proof and it is not clear how case (ii) in the above theorem arises in his sketch or in the statement of his proposition. Therefore in Section 2 we will prove both theorems for the sake of completeness.

Figure 1 shows examples of maximal lattice free convex sets in a 2-dimensional affine subspace W of $\mathbb{R}^3$. We denote by V the affine space generated by $W \cap \mathbb{Z}^3$. In the first picture W is rational, so $V = W$, while in the second one V is a subspace of W of dimension 1.

In the second part of the paper, we show a connection between maximal lattice-free convex sets in affine subspaces and mixed-integer linear programming. Suppose we consider q rows of the optimal tableau of the LP relaxation of a given MILP, relative to q basic integer variables $x_1, \dots, x_q$. Let $s_1, \dots, s_k$ be the nonbasic variables, and $f \in \mathbb{R}^q$ be the vector of components of the optimal basic feasible solution. The tableau restricted to these q rows is of the form

$$x = f + \sum_{j=1}^{k} r^j s_j, \quad x \geq 0 \text{ integral}, \ s \geq 0, \ \text{ and } s_j \in \mathbb{Z}, j \in I,$$

where $r^j \in \mathbb{R}^q$, $j = 1, \dots, k$, and I denotes the set of integer nonbasic variables. Gomory [18] proposed to consider the relaxation of the above problem obtained by dropping the nonnegativity conditions $x \geq 0$. This gives rise to the so called *corner polyhedron*. A further relaxation is obtained by also dropping the integrality conditions on the nonbasic variables, obtaining the mixed-integer set

$$x = f + \sum_{j=1}^{k} r^j s_j, \ x \in \mathbb{Z}^q, \ s \geq 0.$$

Note that, since $x \in \mathbb{R}^q$ is completely determined by $s \in \mathbb{R}^k$, the above is equivalent to

$$f + \sum_{j=1}^{k} r^j s_j \in \mathbb{Z}^q, \quad s \geq 0. \tag{1}$$

We denote by $R_f(r^1, \dots, r^k)$ the set of points s satisfying (1). The above relaxation was studied by Andersen et al. [1] in the case of two rows and Borozan and Cornuéjols [10] for the general case. In these papers they showed that the irredundant valid inequalities for $\operatorname{conv}(R_f(r^1, \dots, r^k))$ correspond to maximal lattice free convex sets in $\mathbb{R}^q$. In [1, 10] data are assumed to be rational. Here we consider the case were $f, r^1, \dots, r^k$ may have irrational entries.

Let $W = \langle r^1, \dots, r^k \rangle$ be the linear space generated by $r^1, \dots, r^k$. Note that, for every $s \in R_f(r^1, \dots, r^k)$, the point $f + \sum_{j=1}^{k} r^j s_j \in (f + W) \cap \mathbb{Z}^q$, hence we assume $f + W$ contains an integral point. Let V be the affine hull of $(f + W) \cap \mathbb{Z}^q$. Notice that $f + W$ and V coincide if and only if W is a rational space. Borozan and Cornuéjols [10] proposed to study the following *semi-infinite relaxation*. Let $R_f(W)$ be the set of points $s = (s_r)_{r \in W}$ of $\mathbb{R}^W$ satisfying

$$\begin{aligned} & f + \sum_{r \in W} r s_r \in \mathbb{Z}^q \\ & s_r \geq 0, \quad r \in W \\ & s \in \mathcal{W} \end{aligned} \tag{2}$$

where $\mathcal{W}$ is the set of all $s \in \mathbb{R}^W$ with *finite support*, i.e. the set $\{r \in W \mid s_r > 0\}$ has finite cardinality. Notice that $R_f(r^1, \ldots, r^k) = R_f(W) \cap \{s \in \mathcal{W} \mid s_r = 0 \text{ for all } r \neq r^1, \ldots, r^k\}$.

Given a function $\psi : W \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, the linear inequality

$$\sum_{r \in W} \psi(r) s_r \geq \alpha \tag{3}$$

is *valid for* $R_f(W)$ if it is satisfied by every $s \in R_f(W)$.

Note that, given a valid inequality (3) for $R_f(W)$, the inequality

$$\sum_{j=1}^{k} \psi(r^j) s_j \geq \alpha$$

is valid for $R_f(r^1, \ldots, r^k)$. Hence a characterization of valid linear inequalities for $R_f(W)$ provides a characterization of valid linear inequalities for $R_f(r^1, \ldots, r^k)$.

Next we observe how maximal lattice-free convex sets in $f+W$ give valid linear inequalities for $R_f(W)$. Let B be a maximal lattice-free convex set in $f+W$ containing f in its interior. Since, by Theorem 1, B is a polyhedron and since f is in its interior, there exist $a_1, \ldots, a_t \in \mathbb{R}^q$ such that $B = \{x \in f + W \mid a_i(x - f) \leq 1,\ i = 1 \ldots, t\}$. We define the function $\psi_B : W \to \mathbb{R}$ by

$$\psi_B(r) = \max_{i=1,\ldots,t} a_i r.$$

Note that the function ψ_B is *subadditive*, i.e. $\psi_B(r) + \psi_B(r') \geq \psi_B(r + r')$, and *positively homogeneous*, i.e. $\psi_B(\lambda r) = \lambda \psi_B(r)$ for every $\lambda \geq 0$. We claim that

$$\sum_{r \in W} \psi_B(r) s_r \geq 1$$

is valid for $R_f(W)$.

Indeed, let $s \in R_f(W)$, and $x = f + \sum_{r \in W} r s_r$. Note that $x \in \mathbb{Z}^n$, thus $x \notin \mathbf{int}(B)$. Then

$$\sum_{r \in W} \psi_B(r) s_r = \sum_{r \in W} \psi_B(r s_r) \geq \psi_B(\sum_{r \in W} r s_r) = \psi_B(x - f) \geq 1,$$

where the first equation follows from positive homogeneity, the first inequality follows from subadditivity of ψ_B and the last one follows from the fact that $x \notin \mathbf{int}(B)$.

We will show that all nontrivial irredundant valid linear inequalities for $R_f(W)$ are indeed of the type described above. Furthermore, if W is irrational, we will see that $R_f(W)$ is contained in a proper affine subspace of $\mathcal{W}$, so each inequality has infinitely many equivalent forms. Note that, by definition of ψ_B, $\psi_B(r) > 0$ if r is not in the recession cone of B, $\psi_B(r) < 0$ when r is in the interior of the recession cone of B, while $\psi_B(r) = 0$ when r is on the boundary of the recession cone of B. We will show that one can always choose a form of the inequality so that ψ_B is a nonnegative function. We make this more precise in the next theorem.

Given a point $s \in R_f(W)$, then $f + \sum_{r \in W} r s_r \in \mathbb{Z}^q \cap (f + W)$. Thus $\mathrm{conv}(R_f(W))$ is contained in the affine subspace $\mathcal{V}$ of $\mathcal{W}$ defined as

$$\mathcal{V} = \{s \in \mathcal{W} \,|\, f + \sum_{r \in W} r s_r \in V\}.$$

Observe that, given $C \in \mathbb{R}^{\ell \times q}$ and $d \in \mathbb{R}^\ell$ such that $V = \{x \in f + W \,|\, Cx = d\}$, we have

$$\mathcal{V} = \{s \in \mathcal{W} \,|\, \sum_{r \in W} (Cr) s_r = d - Cf\}. \tag{4}$$

A linear inequality $\sum_{r \in W} \psi(r) s_r \geq \alpha$ that is satisfied by every element in $\{s \in \mathcal{V} \,|\, s_r \geq 0 \text{ for every } r \in W\}$ is said to be *trivial*.

We say that inequality $\sum_{r \in W} \psi(r) s_r \geq \alpha$ *dominates* inequality $\sum_{r \in W} \psi'(r) s_r \geq \alpha$ if $\psi(r) \leq \psi'(r)$ for all $r \in W$. Note that, for any $\bar{s} \in \mathcal{W}$ such that $\bar{s}_r \geq 0$ for all $r \in W$, if $\bar{s}$ satisfies the first inequality, then $\bar{s}$ also satisfies the second. A valid inequality $\sum_{r \in W} \psi(r) s_r \geq \alpha$ for $R_f(W)$ is *minimal* if it is not dominated by any valid linear inequality $\sum_{r \in W} \psi'(r) s_r \geq \alpha$ for $R_f(W)$ such that $\psi' \neq \psi$. It is not obvious that nontrivial valid linear inequalities are dominated by minimal ones. We will show that this is the case. Note that it is not even obvious that minimal valid linear inequalities exist.

We will show that, for any maximal lattice-free convex set B of $f + W$ with f in its interior, the inequality $\sum_{r \in W} \psi_B(r) s_r \geq 1$ is a minimal valid inequality for $R_f(W)$. The main result of this paper is a converse, stated in the next theorem.

Given two valid inequalities $\sum_{r \in W} \psi(r) s_r \geq \alpha$ and $\sum_{r \in W} \psi'(r) s_r \geq \alpha'$ for R_f, we say that they are *equivalent* if there exist $\rho > 0$ and $\lambda \in \mathbb{R}^\ell$ such that $\psi(r) = \rho \psi'(r) + \lambda^T C r$ and $\alpha = \rho \alpha' + \lambda^T (d - Cf)$.

Theorem 3. *Every nontrivial valid linear inequality for $R_f(W)$ is dominated by a nontrivial minimal valid linear inequality for $R_f(W)$.*
Every nontrivial minimal valid linear inequality for $R_f(W)$ is equivalent to an inequality of the form

$$\sum_{r \in W} \psi_B(r) s_r \geq 1$$

such that $\psi_B(r) \geq 0$ for all $r \in W$ and B is a maximal lattice-free convex set in $f + W$ with f in its interior.

This theorem generalizes earlier results about the case when W is a rational space (Borozan and Cornuéjols [10]). However the proof is much more complicated. In the rational case it is immediate that all valid linear inequalities are of the form $\sum_{r \in W} \psi(r) s_r \geq 1$ with ψ nonnegative. From this, it follows easily that ψ must be equal to ψ_B for some maximal lattice-free convex set B. In the irrational case, valid linear inequalities might have negative coefficients. For minimal inequalities, however, Theorem 3 shows that there always exists an equivalent one where all coefficients are nonnegative. The function ψ_B is nonnegative if and only if the recession cone of B has empty interior. Although there are nontrivial minimal valid linear inequalities arising from maximal lattice-free convex sets whose recession cone is full dimensional, Theorem 3 states that there always exists a maximal lattice-free convex set whose recession cone is not full dimensional that gives an equivalent inequality. A crucial ingredient in showing this is a new result about sublinear functions proved in [9].

In light of Theorem 3, it is a natural question to ask what is the subset of $\mathcal{W}$ obtained by intersecting the set of nonnegative elements of $\mathcal{V}$ with all half-spaces defined by inequalities $\sum_{r\in W}\psi(r)s_r \geq 1$ as in Theorem 3. In a finite dimensional space, the intersection of all half-spaces containing a given convex set C is the closure of C. Things are more complicated in infinite dimension. First of all, while in finite dimension all norms are topologically equivalent, and thus the concept of closure does not depend on the choice of a specific norm, in infinite dimension different norms may produce different topologies. Secondly, in finite dimensional spaces linear functions are always continuous, while in infinite dimension there always exist linear functions that are not continuous. In particular, half-spaces (i.e. sets of points satisfying a linear inequality) are not always closed in infinite dimensional spaces (see Conway [12] for example).

To illustrate this, note that if $\mathcal{W}$ is endowed with the Euclidean norm, then $\mathbf{0} = (0)_{r\in W}$ belongs to the closure of $\operatorname{conv}(R_f(W))$ with respect to this norm, as shown next. Let $\bar{x}$ be an integral point in $f + W$ and let $\bar{s}$ be defined by

$$\bar{s}_r = \begin{cases} \frac{1}{k} & \text{if } r = k(\bar{x} - f), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for every choice of k, $\bar{s} \in R_f(W)$, and for k that goes to infinity the point $\bar{s}$ is arbitrarily close to $\mathbf{0}$ with respect to the Euclidean distance. Now, given a valid linear inequality $\sum_{r\in W}\psi(r)s_r \geq 1$ for $\operatorname{conv}(R_f(W))$, since $\sum_{r\in W}\psi(r)0 = 0$ the hyperplane $\mathcal{H} = \{s \in \mathcal{W} : \sum_{r\in W}\psi(r)s_r = 1\}$ separates strictly $\operatorname{conv}(R_f(W))$ from $\mathbf{0}$ even though $\mathbf{0}$ is in the closure of $\operatorname{conv}(R_f(W))$. This implies that $\mathcal{H}$ is not a closed hyperplane of $\mathcal{W}$, and in particular the function $s \mapsto \sum_{r\in W}\psi(r)s_r$ is not continuous with respect to the Euclidean norm on $\mathcal{W}$.

A nice answer to our question is given by considering a different norm on $\mathcal{W}$. We endow $\mathcal{W}$ with the norm $\|\cdot\|_H$ defined by

$$\|s\|_H = |s_0| + \sum_{r\in W\setminus\{0\}} \|r\||s_r|.$$

It is straightforward to show that $\|\cdot\|_H$ is indeed a norm. Given $A \subset \mathcal{W}$, we denote by $\bar{A}$ the closure of A with respect to the norm $\|\cdot\|_H$.

Let $\mathcal{B}_W$ be the family of all maximal lattice-free convex sets of W with f in their interior.

Theorem 4.

$$\overline{\operatorname{conv}}(R_f(W)) = \left\{ s \in \mathcal{V} \mid \begin{array}{ll} \sum_{r\in W}\psi_B(r)s_r \geq 1 & B \in \mathcal{B}_W \\ s_r \geq 0 & r \in W \end{array} \right\}.$$

Note that Theorems 3 and 4 are new even when $W = \mathbb{R}^q$. Even though data of integer programs are typically rational and studying the infinite relaxation (2) for $W = \mathbb{Q}^q$ seems natural, some of its extreme inequalities arise from maximal lattice free convex sets with irrational facets [13]. Therefore the more natural setting for (2) is in fact $W = \mathbb{R}^q$.

The paper is organized as follows. In Section 2 we will state and prove the natural extensions of Theorems 1 and 2 for general lattices. In Section 3 we prove Theorem 3, while in Section 4 we prove Theorem 4.

2 Maximal lattice-free convex sets

Given $X \subset \mathbb{R}^n$, we denote by $\langle X \rangle$ the linear space generated by the vectors in X. The underlying field is $\mathbb{R}$ in this paper. The purpose of this section is to prove Theorems 1 and 2. For this, we will need to work with general lattices.

Definition 5. *An additive group Λ of $\mathbb{R}^n$ is said to be* finitely generated *if there exist vectors $a_1, \ldots, a_m \in \mathbb{R}^n$ such that $\Lambda = \{\lambda_1 a_1 + \ldots + \lambda_m a_m \,|\, \lambda_1, \ldots, \lambda_m \in \mathbb{Z}\}$.*

If a finitely generated additive group Λ of $\mathbb{R}^n$ can be generated by linearly independent *vectors $a_1, \ldots, a_m$, then Λ is called a* lattice of the linear space $\langle a_1, \ldots, a_m \rangle$. *The set of vectors $a_1, \ldots, a_m$ is called a* basis *of the lattice Λ.*

Definition 6. *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$. Given a linear subspace L of V, we say that L is a* lattice-subspace of V *if there exists a basis of L contained in Λ.*

Given $y \in \mathbb{R}^n$ and $\varepsilon > 0$, we will denote by $B_\varepsilon(y)$ the open ball centered at y of radius ε. Given an affine space W of $\mathbb{R}^n$ and a set $S \subseteq W$, we denote by $\mathbf{int}_W(S)$ the interior of S with respect to the topology induced on W by $\mathbb{R}^n$, namely $\mathbf{int}_W(S)$ is the set of points $x \in S$ such that $B_\varepsilon(x) \cap W \subset S$ for some $\epsilon > 0$. We denote by $\mathbf{relint}(S)$ the relative interior of S, that is $\mathbf{relint}(S) = \mathbf{int}_{\mathrm{aff}(S)}(S)$.

Definition 7. *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$, and let W be a linear space of $\mathbb{R}^n$ containing V. A set $S \subset \mathbb{R}^n$ is said to be a* Λ-free convex set of W *if $S \subset W$, S is convex and $\Lambda \cap \mathbf{int}_W(S) = \emptyset$, and S is said to be a* maximal Λ-free convex set of W *if it is not properly contained in any Λ-free convex set.*

The next two theorems are restatements of Theorems 1 and 2 for general lattices.

Theorem 8. *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$, and let W be a linear space of $\mathbb{R}^n$ containing V. A set $S \subset \mathbb{R}^n$ is a maximal Λ-free convex set of W if and only if one of the following holds:*

(i) S is a polyhedron in W, $\dim(S) = \dim(W)$, $S \cap V$ is a maximal Λ-free convex set of V, and for every facet F of S, $F \cap V$ is a facet of $S \cap V$;

(ii) S is an affine hyperplane of W of the form $S = v + L$ where $v \in S$ and $L \cap V$ is a hyperplane of V that is not a lattice subspace of V;

(iii) S is a half-space of W that contains V on its boundary.

Theorem 9. *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is a maximal Λ-free convex set of V if and only if one of the following holds:*

(i) *S is a polyhedron of the form $S = P + L$ where P is a polytope, L is a lattice-subspace of V, $\dim(S) = \dim(P) + \dim(L) = \dim(V)$, S does not contain any point of Λ in its interior and there is a point of Λ in the relative interior of each facet of S;*

(ii) $\dim(S) < \dim(V)$*, S is an affine hyperplane of V of the form $S = v + L$ where $v \in S$ and L is not a lattice-subspace of V.*

2.1 Proof of Theorem 8

We assume Theorem 9 holds. Its proof will be given in the next section.

($\Rightarrow$) Let S be a maximal Λ-free convex set of W. We show that one of $(i) - (iii)$ holds. If $V = W$, then (iii) cannot occur and either (i) or (ii) follows from Theorem 9. Thus we assume $V \subset W$.

Assume first that $\dim(S) < \dim(W)$. Then there exists a hyperplane H of W containing S, and since $\mathbf{int}_W(H) = \emptyset$, then $S = H$ by maximality of S. Since S is a hyperplane of W, then either $V \subseteq S$ or $S \cap V$ is a hyperplane of V. If $V \subseteq S$, then let K be one of the two half spaces of W separated by S. Then $\mathbf{int}_W(K) \cap \Lambda = \emptyset$, contradicting the maximality of S. Hence $S \cap V$ is a hyperplane of V. We show that $P = S \cap V$ is a maximal Λ-free convex set of V. Indeed, let K be a convex set in V such that $\mathbf{int}_V(K) \cap \Lambda = \emptyset$ and $P \subseteq K$. Since $\mathrm{conv}(S \cup K) \cap V = K$, then $\mathbf{int}_W(\mathrm{conv}(S \cup K) \cap \Lambda) = \emptyset$. By maximality of S, $S = \mathrm{conv}(S \cup K)$, hence $P = K$.
Given $v \in P$, $S = v + L$ for some hyperplane L of W, and $P = v + (L \cap V)$. Applying Theorem 9 to P, we get that $L \cap V$ is not a lattice subspace of V, and case (ii) holds.

So we may assume $\dim(S) = \dim(W)$. Since S is convex, then $\mathbf{int}_W(S) \neq \emptyset$. We consider two cases.

Case 1. $\mathbf{int}_W(S) \cap V = \emptyset$.

Since $\mathbf{int}_W(S)$ and V are nonempty disjoint convex sets, there exists a hyperplane separating them, i.e. there exist $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that $\alpha x \geq \beta$ for every $x \in S$ and $\alpha x \leq \beta$ for every $x \in V$. Since V is a linear space, then $\alpha x = 0$ for every $x \in V$, hence $\beta \geq 0$. Then the half space $H = \{x \in W \mid \alpha x \geq 0\}$ contains S and V lies on the boundary of H. Hence H is a maximal Λ-free convex set of W containing S, therefore $S = H$ by the maximality assumption, so (iii) holds.

Case 2. $\mathbf{int}_W(S) \cap V \neq \emptyset$.

We claim that

$$\mathbf{int}_W(S) \cap V = \mathbf{int}_V(S \cap V). \tag{5}$$

To prove this claim, notice that the direction $\mathbf{int}_W(S) \cap V \subseteq \mathbf{int}_V(S \cap V)$ is straightforward. Conversely, let $x \in \mathbf{int}_V(S \cap V)$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap V \subseteq S$. Since $\mathbf{int}_W(S) \cap V \neq \emptyset$, there exists $y \in \mathbf{int}_W(S) \cap V$. Then there exists ε' such that $B_{\varepsilon'}(y) \cap W \subseteq S$. Since S is convex, the set $K = \mathrm{conv}((B_\varepsilon(x) \cap V) \cup (B_{\varepsilon'}(y) \cap W))$ is contained in S. Clearly $x \in \mathbf{int}_W(K)$, thus $x \in \mathbf{int}_W(S) \cap V$.

Let $P = S \cap V$. By (5) and because $\mathbf{int}_W(S) \cap \Lambda = \emptyset$, we have $\mathbf{int}_V(P) \cap \Lambda = \emptyset$. We show that P is a maximal Λ-free convex set of V. Indeed, let K be a convex set in V such that $\mathbf{int}_V(K) \cap \Lambda = \emptyset$ and $P \subseteq K$. Since $\operatorname{conv}(S \cup K) \cap V = K$, Claim (5) implies that $\mathbf{int}_W(\operatorname{conv}(S \cup K)) \cap \Lambda = \emptyset$. By maximality, $S = \operatorname{conv}(S \cup K)$, hence $P = K$.

Since $\dim(P) = \dim(V)$, by Theorem 9 applied to P, P is a polyhedron with a point of Λ in the relative interior of each of its facets. Let $F_1, \dots, F_t$ be the facets of P. For $i = 1, \dots, t$, let z_i be a point in $\mathbf{relint}(F_i) \cap \Lambda$. By (5), $z_i \notin \mathbf{int}_W(S)$. By the separation theorem, there exists a half-space H_i of W containing $\mathbf{int}_W(S)$ such that $z_i \notin \mathbf{int}_W(H_i)$. Notice that F_i is on the boundary of H_i. Then $S \subseteq \cap_{i=1}^t H_i$. By construction $\mathbf{int}_W(\cap_{i=1}^t H_i) \cap \Lambda = \emptyset$, hence by maximality of S, $S = \cap_{i=1}^t H_i$. For every $j = 1, \dots, t$, $\mathbf{int}_W(\cap_{i \neq j} H_i)$ contains z_j. Therefore H_j defines a facet of S for $j = 1, \dots, t$.

($\Leftarrow$) Let S be a set in $\mathbb{R}^n$ satisfying one of $(i), (ii), (iii)$. Clearly S is a convex set in W and $\mathbf{int}_W(S) \cap \Lambda = \emptyset$, so we only need to prove maximality. If S satisfies (iii), then this is immediate. If S satisfies (i) or (ii), suppose that there exists a closed convex set $K \subset W$ strictly containing S such that $\mathbf{int}_W(K) \cap \Lambda = \emptyset$. Let $w \in K \setminus S$. Then $\operatorname{conv}(S \cup \{w\}) \subseteq K$. We claim that the inclusion $S \cap V \subset \operatorname{conv}(S \cup \{w\}) \cap V$ is strict. This is clear when S is a hyperplane satisfying (ii). When S is a polyhedron satisfying (i), the claim follows from the fact that each facet F of S has the property that $F \cap V$ is a facet of $S \cap V$. Thus $S \cap V \subset \operatorname{conv}(S \cup \{w\}) \cap V$. By maximality of $S \cap V$, the set $\mathbf{int}_V(\operatorname{conv}(S \cup \{w\}) \cap V)$ contains a point in Λ. Now $\operatorname{conv}(S \cup \{w\}) \subseteq K$ implies that $\mathbf{int}_W(K)$ contains a point of Λ, a contradiction.

2.2 Proof of Theorem 9

Throughout this section, Λ is a lattice of a linear space V. To simplify notation, given $S \subseteq \mathbb{R}^n$, we denote $\mathbf{int}_V(S)$ simply by $\mathbf{int}(S)$.

The following standard result in lattice theory provides a useful equivalent definition of lattice (see Barvinok [7], p. 284 Theorem 1.4).

Theorem 10. *Let Λ be the additive group generated by vectors $a_1, \dots, a_m \in \mathbb{R}^n$. Then Λ is a lattice of the linear space $\langle a_1, \dots, a_m \rangle$ if and only if there exists $\varepsilon > 0$ such that $\|y\| \geq \varepsilon$ for every $y \in \Lambda \setminus \{0\}$.*

In this paper we will only need the "only if" part of the statement, which is easy to prove (see [7], p. 281 problem 5).

The following lemma proves the "only if" part of Theorem 9 when S is bounded and full-dimensional.

Lemma 11. *Let Λ be a lattice of a linear space V of $\mathbb{R}^n$. Let $S \subset V$ be a bounded maximal Λ-free convex set with $\dim(S) = \dim(V)$. Then S is a polytope with a point of Λ in the relative interior of each of its facets.*

Proof. Since S is bounded, there exist integers L, U such that S is contained in the box $B = \{x \in \mathbb{R}^d \,|\, L \leq x_i \leq U\}$. For each $y \in \Lambda \cap B$, since S is convex there exists a closed half-space H^y of V such that $S \subseteq H^y$ and $y \notin \mathbf{int}(H^y)$. By Theorem 10, $B \cap \Lambda$ is finite, therefore $\bigcap_{y \in B \cap \Lambda} H^y$ is a polyhedron. Thus $P = \bigcap_{y \in B \cap \Lambda} H^y \cap B$ is a polytope and by construction $\Lambda \cap \mathbf{int}(P) = \emptyset$. Since $S \subseteq B$ and $S \subseteq H^y$ for every $y \in B \cap \Lambda$, it follows that $S \subseteq P$. By maximality of S, $S = P$, therefore S is a polytope. We only need to show that S has a point of Λ in the relative interior of each of its facets. Let $F_1, \dots, F_t$ be the facets of S, and let $H_i = \{x \in V \mid \alpha_i x \leq \beta_i\}$ be the closed half-space defining F_i, $i = 1, \dots, t$. Then $S = \bigcap_{i=1}^t H_i$. Suppose, by contradiction, that one of the facets of S, say F_t, does not contain a point of Λ in its relative interior. □

We will also need the following famous theorem of Dirichlet.

Theorem 12 (Dirichlet)**.** *Given real numbers $\alpha_1, \dots, \alpha_n, \varepsilon$ with $0 < \varepsilon < 1$, there exist integers $p_1, \dots, p_n$ and q such that*

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{\varepsilon}{q}, \text{ for } i = 1, \dots, n, \text{ and } 1 \leq q \leq \varepsilon^{-1}. \tag{6}$$

The following is a consequence of Dirichlet's theorem.

Lemma 13. *Given $y \in \Lambda$ and $r \in V$, then for every $\varepsilon > 0$ and $\bar{\lambda} \geq 0$, there exists a point of Λ at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$.*

Proof. First we show that, if the statement holds for $\bar{\lambda} = 0$, then it holds for arbitrary $\bar{\lambda}$. Given $\varepsilon > 0$, let Z be the set of points of Λ at distance less than ε from $\{y + \lambda r \mid \lambda \geq 0\}$. Suppose, by contradiction, that no point in Z has distance less than ε from $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Then Z is contained in $B_\varepsilon(0) + \{y + \lambda r \mid 0 \leq \lambda \leq \bar{\lambda}\}$. By Theorem 10, Z is finite, thus there exists an $\bar{\varepsilon} > 0$ such that every point in Z has distance greater than $\bar{\varepsilon}$ from $\{y + \lambda r \mid \lambda \geq 0\}$, a contradiction. So we only need to show that, given $\bar{\varepsilon}$, there exists at least one point of Λ at distance at most $\bar{\varepsilon}$ from $\{y + \lambda r \mid \lambda \geq 0\}$.

Let $m = \dim(V)$ and $a_1, \dots, a_m$ be a basis of Λ. Then there exists $\alpha \in \mathbb{R}^m$ such that $r = \alpha_1 a_1 + \dots, \alpha_m a_m$. Denote by A the matrix with columns $a_1, \dots, a_m$, and define $\|A\| = \sup_{x : \|x\| \leq 1} \|Ax\|$ where, for a vector v, $\|v\|$ denotes the Euclidean norm of v. Choose $\varepsilon > 0$ such that $\varepsilon < 1$ and $\varepsilon \leq \bar{\varepsilon}/(\|A\|\sqrt{m})$. By Dirichlet's theorem, there exist $p \in \mathbb{Z}^m$ and $\lambda > 0$ such that

$$\|\alpha - \frac{p}{\lambda}\| = \sqrt{\sum_{i=1}^m \left|\alpha_i - \frac{p_i}{\lambda}\right|^2} \leq \frac{\varepsilon\sqrt{m}}{\lambda} \leq \frac{\bar{\varepsilon}}{\|A\|\lambda}.$$

Let $z = Ap + y$. Since $p \in \mathbb{Z}^m$, then $z \in \Lambda$. Furthermore

$$\|(y + \lambda r) - z\| = \|\lambda r - Ap\| = \|A(\lambda\alpha - p)\| \leq \|A\|\|\lambda\alpha - p\| \leq \bar{\varepsilon}.$$

□

Given a linear subspace of $\mathbb{R}^n$, we denote by $L^\perp$ the orthogonal complement of L. Given a set $S \subseteq \mathbb{R}^n$, the *orthogonal projection of S onto $L^\perp$* is the set

$$\text{proj}_{L^\perp}(S) = \{v \in L^\perp \mid v + w \in S \text{ for some } w \in L\}.$$

We will use the following result (see Barvinok [7], p. 284 problem 3).

Lemma 14. *Given a lattice-subspace L of V, the orthogonal projection of Λ onto $L^\perp$ is a lattice of $L^\perp \cap V$.*

Lemma 15. *If a linear subspace L of V is not a lattice-subspace of V, then for every $\varepsilon > 0$ there exists $y \in \Lambda \setminus L$ at distance less than ε from L.*

Proof. The proof is by induction on $k = \dim(L)$. Assume L is a linear subspace of V that is not a lattice-subspace, and let $\varepsilon > 0$. If $k = 1$, then, since the origin 0 is contained in Λ, by Lemma 13 there exists $y \in \Lambda$ at distance less than ε from L. If $y \in L$, then $L = \langle y \rangle$, thus L is a lattice-subspace of V, contradicting our assumption.
Hence we may assume that $k \geq 2$ and the statement holds for spaces of dimension $k - 1$. Suppose L contains a nonzero vector $r \in \Lambda$. Let

$$L' = \mathrm{proj}_{\langle r \rangle^\perp}(L), \quad \Lambda' = \mathrm{proj}_{\langle r \rangle^\perp}(\Lambda).$$

By Lemma 14, Λ' is a lattice of $\langle r \rangle^\perp \cap V$. Also, L' is not a lattice subspace of $\langle r \rangle^\perp \cap V$ with respect to Λ', because if there exists a basis $a_1, \ldots, a_{k-1}$ of L' contained in Λ', then there exist scalars $\mu_1, \ldots, \mu_{k-1}$ such that $a_1 + \mu_1 r, \ldots, a_{k-1} + \mu_{k-1} r \in \Lambda$, but then $r, a_1 + \mu_1 r, \ldots, a_{k-1} + \mu_{k-1} r$ is a basis of L contained in Λ, a contradiction. By induction, there exists a point $y' \in \Lambda' \setminus L'$ at distance less than ε from L'. Since $y' \in \Lambda'$, there exists a scalar μ such that $y = y' + \mu r \in \Lambda$, and y has distance less than ε from L.
Thus $L \cap \Lambda = \{0\}$. By Lemma 13, there exists a nonzero vector $y \in \Lambda$ at distance less than ε from L. Since L does not contain any point in Λ other than the origin, $y \notin L$. $\square$

Lemma 16. *Let L be a linear subspace of V with $\dim(L) = \dim(V) - 1$, and let $v \in V$. Then $v + L$ is a maximal Λ-free convex set if and only if L is not a lattice subspace of V.*

Proof. ($\Rightarrow$) Let $S = v + L$ and assume that S is a maximal Λ-free convex set. Suppose by contradiction that L is a lattice-subspace. Then there exists a basis $a_1, \ldots, a_m$ of Λ such that $a_1, \ldots, a_{m-1}$ is a basis of L. Thus $S = \{\sum_{i=1}^m x_i a_i \,|\, x_m = \beta\}$ for some $\beta \in \mathbb{R}$. Then, $K = \{\sum_{i=1}^m x_i a_i \,|\, \lceil \beta - 1 \rceil \leq x_m \leq \lceil \beta \rceil\}$ strictly contains S and $\mathbf{int}(K) \cap \Lambda = \emptyset$, contradicting the maximality of S.

($\Leftarrow$) Assume L is not a lattice-subspace of V. Since $S = v + L$ is an affine hyperplane of V, $\mathbf{int}(S) = \emptyset$, thus $\mathbf{int}(S) \cap \Lambda = \emptyset$, hence we only need to prove that S is maximal with such property. Suppose not, and let K be a maximal convex set in V such that $\mathbf{int}(K) \cap \Lambda = \emptyset$ and $S \subset K$. Then by maximality K is closed. Let $w \in K \setminus S$. Since K is convex and closed, then $K \supseteq \mathrm{conv}(\{v, w\}) + L$. Let ε be the distance between $v + L$ and $w + L$, and δ be the distance of $\mathrm{conv}(\{v, w\}) + L$ from the origin. By Lemma 15, since L is not a lattice-subspace of V, there exists a vector $y \in \Lambda \setminus L$ at distance $\bar{\varepsilon} < \varepsilon$ from L. Let $z = (\lfloor \frac{\delta}{\bar{\varepsilon}} \rfloor + 1) y$. By definition, z is strictly between $v + L$ and $w + L$, hence $z \in \mathbf{int}(K)$. Since z is an integer multiple of $y \in \Lambda$, then $z \in \Lambda$, a contradiction. $\square$

We are now ready to prove Lovász's Theorem.

Proof of Theorem 9. ($\Leftarrow$) If S satisfies (ii), then by Lemma 16, S is a maximal Λ-free convex set. If S satisfies (i), then, since $\mathbf{int}(S) \cap \Lambda = \emptyset$, we only need to show that S is maximal. Suppose not, and let K be a convex set in V such that $\mathbf{int}(K) \cap \Lambda = \emptyset$ and $S \subset K$. Given $y \in K \setminus S$, there exists a hyperplane H separating y from S such that $F = S \cap H$ is a facet of S. Since K is convex and $S \subset K$, then $\mathrm{conv}(S \cup \{y\}) \subseteq K$. Since $\dim(S) = \dim(V)$,

$F \subset S$ hence the $\mathbf{relint}(F) \subset \mathbf{int}(K)$. By assumption, there exists $x \in \Lambda \cap \mathbf{relint}(F)$, so $x \in \mathbf{int}(K)$, a contradiction.

($\Rightarrow$) Let S be a maximal Λ-free convex set. We show that S satisfies either (i) or (ii). Observe that, by maximality, S must be closed.

If $\dim(S) < \dim(V)$, then S is contained in some affine hyperplane H. Since $\mathbf{int}(H) = \emptyset$, we have $S = H$ by maximality of S, therefore $S = v + L$ where $v \in S$ and L is a hyperplane in V. By Lemma 16, (ii) holds.

Therefore we may assume that $\dim(S) = \dim(V)$. In particular, since S is convex, $\mathbf{int}(S) \neq \emptyset$. By Lemma 11, if S is bounded, (i) holds. Hence we may assume that S is unbounded. Let C be the recession cone of S and L the lineality space of S. By standard convex analysis, S is unbounded if and only if $C \neq \{0\}$ (see for example Proposition 2.2.3 in [20]).

Claim 1. *$L = C$.*

Let $r \in C$, $r \neq 0$. We only need to show that $S + \langle r \rangle$ is Λ-free; by maximality of S this will imply that $S = S + \langle r \rangle$. Suppose there exists $y \in \mathbf{int}(S + \langle r \rangle) \cap \Lambda$. We show that $y \in \mathbf{int}(S) + \langle r \rangle$. Suppose not. Then $(y + \langle r \rangle) \cap \mathbf{int}(S) = \emptyset$, which implies that there is a hyperplane H separating the line $y + \langle r \rangle$ and $S + \langle r \rangle$. This contradicts $y \in \mathbf{int}(S + \langle r \rangle)$. This shows $y \in \mathbf{int}(S) + \langle r \rangle$. Thus there exists $\bar{\lambda}$ such that $\bar{y} = y + \bar{\lambda} r \in \mathbf{int}(S)$, i.e. there exists $\varepsilon > 0$ such that $B_\varepsilon(\bar{y}) \cap V \subset S$. Since $y \in \Lambda$, then $y \notin \mathbf{int}(S)$, and thus, since $\bar{y} \in \mathbf{int}(S)$ and $r \in C$, we must have $\bar{\lambda} > 0$. Since $r \in C$, then $B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\} \subset S$. Since $y \in \Lambda$, by Lemma 13 there exists $z \in \Lambda$ at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Thus $z \in B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\}$, hence $z \in \mathbf{int}(S)$, a contradiction. $\diamond$

Let $P = \mathrm{proj}_{L^\perp}(S)$ and $\Lambda' = \mathrm{proj}_{L^\perp}(\Lambda)$. By Claim 1, $S = P + L$ and $P \subset L^\perp \cap V$ is a bounded set. Furthermore, $\dim(S) = \dim(P) + \dim(L) = \dim(V)$ and $\dim(P) = \dim(L^\perp \cap V)$. Notice that $\mathbf{int}(S) = \mathbf{relint}(P) + L$, hence $\mathbf{relint}(P) \cap \Lambda' = \emptyset$. Furthermore P is inclusionwise maximal among the convex sets of $L^\perp \cap V$ without points of Λ' in the relative interior: if not, given a convex set $K \subseteq L^\perp \cap V$ strictly containing P and with no point of Λ' in its relative interior, we have $S = P + L \subset K + L$, and $K + L$ does not contain any point of Λ in its interior, contradicting the maximality of S.

Claim 2. *L is a lattice-subspace of V.*

By contradiction, suppose L is not a lattice-subspace of V. Then, by Lemma 15, for every $\varepsilon > 0$ there exists $y \in \Lambda' \setminus \{0\}$ such that $\|y\| < \varepsilon$. Let V_ε be the linear subspace of $L^\perp \cap V$ generated by the points in $\{y \in \Lambda' \mid \|y\| < \varepsilon\}$. Then $\dim(V_\varepsilon) > 0$.

Notice that, given $\varepsilon' > \varepsilon'' > 0$, then $V_{\varepsilon'} \supseteq V_{\varepsilon''} \supset \{0\}$, hence there exists $\varepsilon_0 > 0$ such that $V_\varepsilon = V_{\varepsilon_0}$ for every $\varepsilon < \varepsilon_0$. Let $U = V_{\varepsilon_0}$.

By definition, Λ' is dense in U (i.e. for every $\varepsilon > 0$ and every $x \in U$ there exists $y \in \Lambda'$ such that $\|x - y\| < \varepsilon$). Thus, since $\mathbf{relint}(P) \cap \Lambda' = \emptyset$, we also have $\mathbf{relint}(P) \cap U = \emptyset$. Since $\dim(P) = \dim(L^\perp \cap V)$, it follows that $\mathbf{relint}(P) \cap (L^\perp \cap V) \neq \emptyset$, so in particular U is a proper subspace of $L^\perp \cap V$.

Let $Q = \mathrm{proj}_{(L+U)^\perp}(P)$ and $\Lambda'' = \mathrm{proj}_{(L+U)^\perp}(\Lambda')$. We show that $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$. Suppose not, and let $y \in \mathbf{relint}(Q) \cap \Lambda''$. Then, $y + w \in \Lambda'$ for some $w \in U$. Furthermore, we claim that $y + w' \in \mathbf{relint}(P)$ for some $w' \in U$. Indeed, suppose no such w' exists. Then $(y + U) \cap (\mathbf{relint}(P) + U) = \emptyset$. So there exists a hyperplane H in $L^\perp \cap V$ separating $y + U$ and $P + U$. Therefore the projection of H onto $(L + U)^\perp$ separates y and Q, contradicting $y \in \mathbf{relint}(Q)$. Thus $z = y + w' \in \mathbf{relint}(P)$ for some $w' \in U$. Since $z \in \mathbf{relint}(P)$, there exists $\bar{\varepsilon} > 0$ such that $B_{\bar{\varepsilon}}(z) \cap (L^\perp \cap V) \subset \mathbf{relint}(P)$. Since Λ' is dense in U and $y + w \in \Lambda'$, it follows that Λ' is dense in $y + U$. Hence, since $z \in y + U$, there exists $\bar{x} \in \Lambda'$ such that $\|\bar{x} - z\| < \bar{\varepsilon}$, hence $\bar{x} \in \mathbf{relint}(P)$, a contradiction. This shows $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$.

Finally, since $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$, then $\mathbf{int}(Q + L + U) \cap \Lambda = \emptyset$. Furthermore $P \subseteq Q + U$, therefore $S \subseteq Q + L + U$. By the maximality of S, $S = Q + L + U$ hence the lineality space of S contains $L + U$, contradicting the fact that L is the lineality space of S and $U \neq \{0\}$. $\diamond$

Since L is a lattice-subspace of V, Λ' is a lattice of $L^\perp \cap V$ by Lemma 14. Since P is a bounded maximal Λ'-free convex set, it follows from Lemma 11 that P is a polytope with a point of Λ' in the relative interior of each of its facets, therefore $S = P + L$ has a point of Λ in the relative interior of each of its facets, and (i) holds. $\square$

From the proof of Theorem 9 we get the following.

Corollary 17. *Every Λ-free convex set of V is contained in some maximal Λ-free convex set of V.*

Proof. Let S be a Λ-free convex set of V. If S is bounded, the proof of Lemma 11 shows that the corollary holds. If S is unbounded, Claim 1 in the proof of Theorem 9 shows that $S + \langle C \rangle$ is Λ-free, where C is the recession cone of S. Hence we may assume that the lineality space L of S is equal to the recession cone of S. The projection P of S onto $L^\perp$ is bounded. If L is a lattice-subspace, then $\Lambda' = \mathrm{proj}_{L^\perp}\Lambda$ is a lattice and P is Λ'-free, hence it is contained in a maximal Λ'-free convex set B of $L^\perp \cap V$, and $B + L$ is a maximal Λ-free convex set of V containing S. If L is not a lattice-subspace, then we may define a linear subspace U of $L^\perp \cap V$ and sets Q and Λ'' as in the proof of Claim 2. Then proof of Claim 2 shows that Q is a bounded Λ''-free convex set of $V \cap (L + U)^\perp$ and Λ'' is a lattice, thus Q is contained in a maximal Λ''-free convex set B of $V \cap (L + U)^\perp$, and $B + (L + U)$ is a maximal Λ-free convex set of V containing S. $\square$

3 Minimal Valid Inequalities

In this section we will prove Theorem 3. For ease of notation, we denote $R_f(W)$ simply by R_f in this section. A linear function $\Psi : \mathcal{W} \to \mathbb{R}$ is of the form

$$\Psi(s) = \sum_{r \in W} \psi(r) s_r, \quad s \in \mathcal{W} \tag{7}$$

for some $\psi : W \to \mathbb{R}$. Throughout the rest of the paper, capitalized Greek letters indicate linear functions from $\mathcal{W}$ to $\mathbb{R}$, while the corresponding lowercase letters indicate functions from W to $\mathbb{R}$ as defined in (7).

A function $\sigma : W \to \mathbb{R}$ is *positively homogeneous* if $\sigma(\lambda r) = \lambda\sigma(r)$ for every $r \in W$ and scalar $\lambda \geq 0$, and it is *subadditive* if $\sigma(r_1 + r_2) \leq \sigma(r_1) + \sigma(r_2)$ for every $r_1, r_2 \in W$. The function σ is *sublinear* if it is positively homogeneous and subadditive. Note that if σ is sublinear, then $\sigma(0) = 0$. One can easily show that a function is sublinear if and only if it is positively homogeneous and convex. We also recall that convex functions are continuous on their domain, so if σ is sublinear it is also continuous [20].

Lemma 18. *Let $\Psi(s) \geq \alpha$ be a valid linear inequality for R_f. Then $\Psi(s) \geq \alpha$ is dominated by a valid linear inequality $\Psi'(s) \geq \alpha$ for R_f such that ψ' is sublinear.*

Proof: We first prove the following.

Claim 1. *For every $s \in \mathcal{W}$ such that $\sum_{r\in W} rs_r = 0$ and $s_r \geq 0$, $r \in W$, we have $\sum_{r\in W} \psi(r)s_r \geq 0$.*

Suppose not. Then there exists $s \in \mathcal{W}$ such that $\sum_{r\in W} rs_r = 0$, $s_r \geq 0$ for all $r \in W$ and $\sum_{r\in W} \psi(r)s_r < 0$. Let $\bar{x}$ be an integral point in W. For any $\lambda > 0$, we define $s^\lambda \in \mathcal{W}$ by

$$s^\lambda_r = \begin{cases} 1 + \lambda s_r & \text{for } r = \bar{x} - f \\ \lambda s_r & \text{otherwise.} \end{cases}$$

Since $f + \sum_{r\in W} rs^\lambda_r = \bar{x}$, it follows that s^λ is in R_f. Furthermore $\sum_{r\in W} \psi(r)s^\lambda_r = \psi(\bar{x} - f) + \lambda(\sum_{r\in W} \psi(r)s_r)$. Therefore $\sum_{r\in W} \psi(r)s^\lambda_r$ goes to $-\infty$ as λ goes to $+\infty$. ⋄

We define, for all $\bar{r} \in W$,

$$\psi'(\bar{r}) = \inf\{\sum_{r\in W} \psi(r)s_r \,|\, \bar{r} = \sum_{r\in W} rs_r,\ s \in \mathcal{W},\ s_r \geq 0 \text{ for all } r \in W\}.$$

By Claim 1, $\sum_{r\in W} \psi(r)s_r \geq -\psi(-\bar{r})$ for all $s \in \mathcal{W}$ such that $\bar{r} = \sum_{r\in W} rs_r$ and $s_r \geq 0$ for all $r \in W$. Thus the infimum in the above equation is finite and the function ψ' is well defined. Note also that $\psi'(\bar{r}) \leq \psi(\bar{r})$ for all $\bar{r} \in W$, as follows by considering $s \in \mathcal{W}$ defined by $s_{\bar{r}} = 1$, $s_r = 0$ for all $r \in W$, $r \neq \bar{r}$.

Claim 2. *The function ψ' is sublinear*

Note first that $\psi'(0) = 0$. Indeed, Claim 1 implies $\psi'(0) \geq 0$, while choosing $s_r = 0$ for all $r \in W$ shows $\psi'(0) \leq 0$.

Next we show that ψ' is positively homogeneous. To prove this, let $\bar{r} \in W$ and $s \in \mathcal{W}$ such that $\bar{r} = \sum_{r\in W} rs_r$ and $s_r \geq 0$ for all $r \in W$. Let $\gamma = \sum_{r\in W} \psi(r)s_r$. For every $\lambda > 0$, $\lambda\bar{r} = \sum_{r\in W} r(\lambda s_r)$, $\lambda s_r \geq 0$ for all $r \in W$, and $\sum_{r\in W} \psi(r)(\lambda s_r) = \lambda\gamma$. Therefore $\psi'(\lambda\bar{r}) = \lambda\psi'(r)$.

Finally, we show that ψ' is convex. Suppose by contradiction that there exist $r', r'' \in W$ and $0 < \lambda < 1$ such that $\psi'(\lambda r' + (1-\lambda)r'') > \lambda\psi'(r') + (1-\lambda)\psi'(r'') + \epsilon$ for some positive ϵ. By definition of ψ', there exist $s', s'' \in \mathcal{W}$ such that $r' = \sum_{r\in W} rs'_r$, $r'' = \sum_{r\in W} rs''_r$, $s'_r, s''_r \geq 0$ for all $r \in W$, $\sum_{r\in W} \psi(r)s'_r < \psi'(r') + \epsilon$ and $\sum_{r\in W} \psi(r)s''_r < \psi'(r'') + \epsilon$. Since $\sum_{r\in W} r(\lambda s'_r + (1-\lambda)s''_r) = \lambda r' + (1-\lambda)r''$, it follows that $\psi'(\lambda r' + (1-\lambda)r'') \leq \sum_{r\in W} \psi(r)(\lambda s'_r + (1-\lambda)s''_r) < \lambda\psi'(r') + (1-\lambda)\psi'(r'') + \epsilon$, a contradiction. ⋄

Claim 3. *The inequality $\sum_{r\in W}\psi'(r)s_r \geq \alpha$ is valid for R_f.*

Suppose there exists $\bar{s}\in R_f$ such that $\sum_{r\in W}\psi'(r)\bar{s}_r \leq \alpha-\epsilon$ for some positive ϵ. Let $\{r^1,\dots,r^k\} = \{r\in W \mid \bar{s}_r > 0\}$. For every $i=1,\dots,k$, there exists $s^i\in W$ such that $r^i = \sum_{r\in W} r s^i_r$, $s^i_r \geq 0$, $r\in W$, and $\sum_{r\in W}\psi(r)s^i_r < \psi'(r^i)+\epsilon/(k\bar{s}_{r^i})$.

Let $\tilde{s} = \sum_{i=1}^k \bar{s}_{r^i}s^i$. Then

$$\sum_{r\in W} r\tilde{s}_r = \sum_{r\in W}\sum_{i=1}^k r\bar{s}_{r^i}s^i_r = \sum_{i=1}^k \bar{s}_{r^i}\sum_{r\in W} r s^i_r = \sum_{i=1}^k r^i\bar{s}_{r^i} = \sum_{r\in W} r\bar{s}_r,$$

hence $\tilde{s}\in R_f$. Therefore $\sum_{r\in W}\psi(r)\tilde{s}_r \geq \alpha$ since $\sum_{r\in W}\psi(r)s_r\geq\alpha$ is valid for R_f. Now

$$\begin{aligned}\sum_{r\in W}\psi(r)\tilde{s}_r &= \sum_{r\in W}\sum_{i=1}^k \psi(r)\bar{s}_{r^i}s^i_r = \sum_{i=1}^k \bar{s}_{r^i}\sum_{r\in W}\psi(r)s^i_r \\ &< \sum_{i=1}^k \bar{s}_{r^i}(\psi'(r^i)+\epsilon/(k\bar{s}_{r^i})) = \sum_{r\in W}\psi'(r^i)\bar{s}_{r^i}+\epsilon \leq \alpha,\end{aligned}$$

a contradiction. □

Lemma 19. *Let $\Psi(s)\geq\alpha$ and $\Psi'(s)\geq\alpha'$ be two equivalent valid linear inequalities for R_f.*
(i) The function ψ is sublinear if and only if ψ' is sublinear.
(ii) Inequality $\Psi(s)\geq\alpha$ is dominated by a minimal valid linear inequality if and only if $\Psi'(s)\geq\alpha'$ is dominated by a minimal valid linear inequality. In particular, $\Psi(s)\geq\alpha$ is minimal if and only if $\Psi'(s)\geq\alpha'$ is minimal.

Proof. Since $\Psi(s)\geq\alpha$ and $\Psi'(s)\geq\alpha'$ are equivalent, by definition there exist $\rho>0$ and $\lambda\in\mathbb{R}^\ell$, such that $\psi(r)=\rho\psi'(r)+\lambda^TCr$ and $\alpha=\rho\alpha'+\lambda^T(d-Cf)$. This proves (i).

Point (ii) follows from the fact that, given a function $\bar{\psi}'$ such that $\bar{\psi}'(r)\leq\psi'(r)$ for every $r\in W$, then the function $\bar{\psi}$ defined by $\bar{\psi}(r)=\rho\bar{\psi}'(r)+\lambda^TCr$, $r\in W$, satisfies $\bar{\psi}(r)\leq\psi(r)$ for every $r\in W$. Furthermore $\bar{\psi}(r)<\psi(r)$ if and only if $\bar{\psi}'(r)<\psi'(r)$. □

Given a nontrivial valid linear inequality $\Psi(s)\geq\alpha$ for R_f such that ψ is sublinear, we consider the set

$$B_\psi = \{x\in f+W \mid \psi(x-f)\leq\alpha\}.$$

Since ψ is continuous, B_ψ is closed. Since ψ is convex, B_ψ is convex. Since ψ defines a valid inequality, B_ψ is lattice-free. Indeed the interior of B_ψ is $\mathbf{int}(B_\psi)=\{x\in f+W : \psi(x-f)<\alpha\}$. Its boundary is $\mathbf{bd}(B_\psi)=\{x\in f+W : \psi(x-f)=\alpha\}$, and its recession cone is $\mathrm{rec}(B_\psi)=\{x\in f+W : \psi(x-f)\leq 0\}$. Note that f is in the interior of B_ψ if and only if $\alpha>0$ and f is on the boundary if and only if $\alpha=0$.

Remark 20. *Given a linear inequality of the form $\Psi(s)\geq 1$ such that $\psi(r)\geq 0$ for all $r\in W$,*

$$\psi(r)=\inf\{t>0 \mid f+t^{-1}r\in B_\psi\},\quad r\in W.$$

Proof. Let $r \in W$. If $\psi(r) > 0$, let t be the minimum positive number such that $f + t^{-1}r \in B_\psi$. Then $f + t^{-1}r \in \mathbf{bd}(B_\psi)$, hence $\psi(t^{-1}r) = 1$ and by positive homogeneity $\psi(r) = t$. If $\psi(r) = 0$, then $r \in \mathrm{rec}(B_\psi)$, hence $f + t^{-1}r \in B_\psi$ for every $t > 0$, thus the infimum in the above equation is 0. $\square$

This remark shows that, if ψ is nonnegative, then it is the *gauge* of the convex set $B_\psi - f$ (see [20]).

Before proving Theorem 3, we need the following general theorem about sublinear functions. Let K be a closed, convex set in W with the origin in its interior. The *polar* of K is the set $K^* = \{y \in W \,|\, ry \leq 1 \text{ for all } r \in K\}$. Clearly K^* is closed and convex, and since $0 \in \mathbf{int}(K)$, it is well known that K^* is bounded. In particular, K^* is a compact set. Also, since $0 \in K$, $K^{**} = K$ (see [20] for example). Let

$$\hat{K} = \{y \in K^* \,|\, \exists x \in K \text{ such that } xy = 1\}. \tag{8}$$

Note that $\hat{K}$ is contained in the relative boundary of K^*. Let $\rho_K : W \to \mathbb{R}$ be defined by

$$\rho_K(r) = \sup_{y \in \hat{K}} ry, \quad \text{for all } r \in W. \tag{9}$$

It is easy to show that ρ_K is sublinear.

Theorem 21 (Basu et al. [9])**.** *Let $K \subset W$ be a closed convex set containing the origin in its interior. Then $K = \{r \in W \,|\, \rho_K(r) \leq 1\}$. Furthermore, for every sublinear function σ such that $K = \{r \,|\, \sigma(r) \leq 1\}$, we have $\rho_K(r) \leq \sigma(r)$ for every $r \in W$.*

Remark 22. *Let $K \subset W$ be a polyhedron containing the origin in its interior. Let $a_1, \ldots, a_t \in W$ such that $K = \{r \in W \,|\, a_i r \leq 1,\ i = 1, \ldots, t\}$. Then $\rho_K(r) = \max_{i=1,\ldots,t} a_i r$.*

Proof. The polar of K is $K^* = \mathrm{conv}\{0, a_1, \ldots, a_t\}$ (see Theorem 9.1 in Schrijver [23]). Furthermore, $\hat{K}$ is the union of all the facets of K^* that do not contain the origin, therefore

$$\rho_K(r) = \sup_{y \in \hat{K}} yr = \max_{i=1,\ldots,t} a_i r$$

for all $r \in W$. $\square$

Remark 23. *Let B be a closed lattice-free convex set in $f + W$ with f in its interior, and let $K = B - f$. Then the inequality $\sum_{r \in W} \rho_K(r) s_r \geq 1$ is valid for R_f.*

Proof: Let $s \in R_f$. Then $x = f + \sum_{r \in W} r s_r$ is integral, therefore $x \notin \mathbf{int}(B)$ because B is lattice-free. By Theorem 21, $\rho_K(x - f) \geq 1$. Thus

$$1 \leq \rho_K(\sum_{r \in W} r s_r) \leq \sum_{r \in W} \rho_K(r s_r) \leq \sum_{r \in W} \rho_K(r) s_r,$$

where the second inequality follows from the subadditivity of ρ_K and the last from the positive homogeneity. $\square$

Lemma 24. *Given a maximal lattice-free convex set B of $f+W$ containing f in its interior, $\Psi_B(s) \geq 1$ is a minimal valid inequality for R_f.*

Proof. Let $\Psi(s) \geq 1$ be a valid linear inequality for R_f such that $\psi(r) \leq \psi_B(r)$ for all $r \in W$. Then $B_\psi \supset B$ and B_ψ is lattice-free. By maximality of B, $B = B_\psi$. By Theorem 21 and Remark 22, $\psi_B(r) \leq \psi(r)$ for all $r \in W$, proving $\psi = \psi_B$. □

Proof of Theorem 3.
Let $\Psi(s) \geq \alpha$ be a nontrivial valid linear inequality for R_f. By Lemma 18, we may assume that ψ is sublinear.

Claim 1. *If* $\mathbf{int}(B_\psi) \cap V = \emptyset$*, then* $\Psi(s) \geq \alpha$ *is trivial.*

Suppose $\mathbf{int}(B_\psi) \cap V = \emptyset$ and let $s \in \mathcal{V}$ such that $s_r \geq 0$ for every $r \in W$. Let $x = f + \sum_{r \in W} r s_r$. Since $s \in \mathcal{V}$, $x \in V$, so $x \notin \mathbf{int}(B_\psi)$. This implies

$$\alpha \leq \psi(x-f) = \psi(\sum_{r \in W} r s_r) \leq \sum_{r \in W} \psi(r) s_r = \Psi(s),$$

where the last inequality follows from the sublinearity of ψ. ⋄

Claim 2. *If $f \in V$ and $\alpha \leq 0$, then* $\mathbf{int}(B_\psi) \cap V = \emptyset$.

Suppose $f \in V$, $\alpha \leq 0$ but $\mathbf{int}(B_\psi) \cap V \neq \emptyset$. Then $\dim(\mathbf{int}(B_\psi) \cap V) = \dim(V)$, hence $\mathbf{int}(B_\psi) \cap V$ contains a set X of $\dim(V)+1$ affinely independent points. For every $x \in X$ and every $\lambda > 0$, $\psi(\lambda(x-f)) = \lambda\psi(x-f) < 0$, where the last inequality is because $x \in \mathbf{int}(B_\psi)$. Hence the set $\Gamma = f + \text{cone}\{x - f \,|\, x \in X\}$ is contained in $\mathbf{int}(B_\psi)$. Since Γ has dimension equal to $\dim(V)$ and V is the convex hull of its integral points, $\Gamma \cap \mathbb{Z}^q \neq 0$, contradicting the fact that B_ψ has no integral point in its interior. ⋄

Claim 3. *If $f \notin V$, then there exists a valid linear inequality $\Psi'(s) \geq 1$ for R_f equivalent to $\Psi(s) \geq \alpha$.*

Since $f \notin V$, $Cf \neq d$, hence there exists a row c_i of C such that $d_i - c_i f \neq 0$. Let $\lambda = (1-\alpha)(d_i - c_i f)^{-1}$, and define $\psi'(r) = \psi(r) + \lambda c_i r$ for every $r \in W$. The inequality $\Psi'(s) \geq 1$ is equivalent to $\Psi(s) \geq \alpha$. ⋄

Thus, by Claims 1, 2 and 3 there exists a valid linear inequality $\Psi'(s) \geq 1$ for R_f equivalent to $\Psi(s) \geq \alpha$. By Lemma 19, ψ' is sublinear and $\Psi(s) \geq \alpha$ is dominated by a minimal valid linear inequality if and only if $\Psi'(s) \geq \alpha'$ is dominated by a minimal valid linear inequality. Therefore we only need to consider valid linear inequalities of the form $\Psi(s) \geq 1$ where ψ is sublinear. In particular the set $B_\psi = \{x \in W \,|\, \psi(x-f) \leq 1\}$ contains f in its interior.

Let $K = \{r \in W \,|\, \psi(r) \leq 1\}$, and let $\hat{K}$ be defined as in (8).

Claim 4. *The inequality $\sum_{r \in W} \rho_K(r) s_r \geq 1$ is valid for R_f and $\psi(r) \geq \rho_K(r)$ for all $r \in W$.*

Note that $B_\psi = f + K$. Thus, by Remark 23, $\sum_{r\in W} \rho_K(r)s_r \geq 1$ is valid for R_f. Since ψ is sublinear, it follows from Theorem 21 that $\rho_K(r) \leq \psi(r)$ for every $r \in W$. $\diamond$

By Claim 4, since ρ_K is sublinear, we may assume that $\psi = \rho_K$.

Claim 5. *There exists a valid linear inequality $\Psi'(s) \geq 1$ for R_f dominating $\Psi(s) \geq 1$ such that ψ' is sublinear, $B_{\psi'}$ is a polyhedron, and* $\mathrm{rec}(B_{\psi'} \cap V) = \mathrm{lin}(B_{\psi'} \cap V)$.

Since B_ψ is a lattice-free convex set, it is contained in some maximal lattice-free convex set S by Corollary 17. The set S satisfies one of the statements (i)-(iii) of Theorem 8. By Claim 1, $\mathbf{int}(S) \cap V \neq \emptyset$, hence case (iii) does not apply. Case (ii) does not apply because $\dim(S) = \dim(B_\psi) = \dim(W)$. Therefore case (i) applies. Thus S is a polyhedron and $S \cap V$ is a maximal lattice-free convex set in V. In particular, by Theorem 9, $\mathrm{rec}(S\cap V) = \mathrm{lin}(S\cap V)$. Since S is a polyhedron containing f in its interior, there exists $A \in \mathbb{R}^{t\times q}$ and $b \in \mathbb{R}^t$ such that $b_i > 0$, $i = 1, \ldots, t$, and $S = \{x \in f + W \mid A(x - f) \leq b\}$. Without loss of generality, we may assume that $\sup_{x\in B_\psi} a_i(x - f) = 1$ where a_i denotes the ith row of A, $i = 1, \ldots, t$. By our assumption, $\sup_{r\in K} a_i r = 1$. Therefore $a_i \in K^*$, since $a_i r \leq 1$ for all $r \in K$. Furthermore $a_i \in \mathbf{cl}(\hat{K})$, since $\sup_{r\in K} a_i r = 1$.

Let $\bar{S} = \{x \in f + W \mid A(x - f) \leq e\}$, where e denotes the vector of all ones. Then $B_\psi \subseteq \bar{S} \subseteq S$. Let $Q = \{r \in W \mid Ar \leq e\}$. By Remark 22, $\rho_Q(r) = \max_{i=1,\ldots,t} a_i r$ for all $r \in W$. Since $\bar{S} \subseteq S$, $\bar{S}$ is lattice-free, by Remark 23 the inequality $\sum_{r\in W} \rho_Q(r)s_r \geq 1$ is valid for R_f. Furthermore, since $\{a_1, \ldots, a_t\} \subset \mathbf{cl}(\hat{K})$, by Claim 4 we have

$$\psi(r) = \sup_{y\in\hat{K}} yr \geq \max_{i=1,\ldots,t} a_i r = \rho_Q(r)$$

for all $r \in W$. Let $\psi' = \rho_Q$. Note that $B_{\psi'} = \bar{S}$. So, $\mathrm{rec}(B_{\psi'}) = \mathrm{rec}(\bar{S}) = \{r \in W \mid Ar \leq 0\} = \mathrm{rec}(S)$. Since $\mathrm{rec}(S \cap V) = \mathrm{lin}(S \cap V)$, then $\mathrm{rec}(B_{\psi'} \cap V) = \mathrm{lin}(B_{\psi'} \cap V)$. $\diamond$

By Claim 5, we may assume that $B_\psi = \{x \in f + W \mid A(x - f) \leq e\}$, where $A \in \mathbb{R}^{t\times q}$ and e is the vector of all ones, and that $\mathrm{rec}(B_\psi \cap V) = \mathrm{lin}(B_\psi \cap V)$. Let $a_1, \ldots, a_t$ denote the rows of A. By Claim 4 and Remark 22,

$$\psi(r) = \max_{i=1,\ldots,t} a_i r, \quad \text{for all } r \in W. \tag{10}$$

Let G be a matrix such that $W = \{r \in \mathbb{R}^q \mid Gr = 0\}$.

Claim 6. *There exists $\lambda \in \mathbb{R}^\ell$ such that $\psi(r) + \lambda^T Cr \geq 0$ for all $r \in W$.*

Given $\lambda \in \mathbb{R}^\ell$, then by (10) $\psi(r)+\lambda^T Cr \geq 0$ for every $r \in W$ if and only if $\min_{r\in W}(\max_{i=1,\ldots,t} a_i r+ \lambda^T Cr) = 0$. The latter holds if and only if

$$0 = \min\{z + \lambda^T Cr \mid ez - Ar \geq 0,\, Gr = 0\}.$$

By LP duality, this holds if and only if the following system is feasible

$$\begin{aligned} ey &= 1 \\ A^T y + C^T\lambda - G^T\mu &= 0 \\ y &\geq 0. \end{aligned}$$

Clearly the latter is equivalent to

$$\begin{aligned} A^T y + C^T\lambda - G^T\mu &= 0 \\ y \geq 0,\, y &\neq 0. \end{aligned} \tag{11}$$

Note that $\text{rec}(B_\psi \cap V) = \{r \in \mathbb{R}^q \mid Ar \leq 0,\, Cr = 0, Gr = 0\}$ and $\text{lin}(B_\psi \cap V) = \{r \in \mathbb{R}^q \mid Ar = 0,\, Cr = 0, Gr = 0\}$. Since $\text{rec}(B_\psi \cap V) = \text{lin}(B_\psi \cap V)$, the system

$$\begin{aligned} Ar &\leq 0 \\ Cr &= 0 \\ Gr &= 0 \\ e^T Ar &= -1 \end{aligned}$$

is infeasible. By Farkas Lemma, this is the case if and only if there exists $\gamma \geq 0$, λ, $\tilde{\mu}$, and τ such that

$$A^T\gamma + C^T\lambda + G^T\tilde{\mu} + A^T e\tau = 0, \quad \tau > 0.$$

If we let $y = \gamma + e\tau$ and $\mu = -\tilde{\mu}$, then (y, λ, μ) satisfies (11). By the previous argument, λ satisfies the statement of the claim. $\diamond$

Let λ as in Claim 6, and let ψ' be the function defined by $\psi'(r) = \psi(r) + \lambda^T Cr$ for all $r \in W$. So $\psi'(r) \geq 0$ for every $r \in W$. Let $\alpha' = 1 + \lambda^T(d - Cf)$. Then the inequality $\Psi'(s) \geq \alpha'$ is valid for R_f and it is equivalent to $\Psi(s) \geq \alpha$. If $\alpha' \leq 0$, then $\Psi'(s) \geq \alpha'$ is trivial. Thus $\alpha' > 0$. Let $\rho = 1/\alpha'$ and let $\psi'' = \rho\psi'$. Then $\Psi''(s) \geq 1$ is equivalent to $\Psi(s) \geq 1$. By Lemma 19(i), ψ'' is sublinear.

Let B be a maximal lattice-free convex set of $f + W$ containing $B_{\psi''}$. Such a set B exists by Corollary 17.

Claim 7. *$\psi''(r) \geq \psi_B(r)$ for all $r \in W$.*

Let $r \in \text{rec}(B_{\psi''})$. Since ψ'' is nonnegative, $\psi''(r) = 0$. Since $\text{rec}(B_{\psi''}) \subseteq \text{rec}(B)$, $\psi_B(r) \leq 0 = \psi''(r)$. Let $r \notin \text{rec}(B_{\psi''})$. Then $f + \tau r \in \mathbf{bd}(B_{\psi''})$ for some $\tau > 0$, hence $\psi''(\tau r) = 1$ and, by positive homogeneity, $\psi''(r) = \tau^{-1}$. Because $B_{\psi''} \subset B$, $f + \tau r \in B$. Since $B = \{x \in f + W \mid \psi_B(x - f) \leq 1\}$, it follows that $\psi_B(\tau r) \leq 1$, implying $\psi_B(r) \leq \tau^{-1} = \psi''(r)$. $\diamond$

Claim 7 shows that $\Psi''(s) \geq 1$ is dominated by $\Psi_B(s) \geq 1$, which is minimal by Lemma 24. By Lemma 19(ii), $\Psi(s) \geq 1$ is dominated by a minimal valid linear inequality which is equivalent to $\Psi_B(s) \geq 1$. $\square$

Example. We illustrate the end of the proof in an example. Suppose $W = \{x \in \mathbb{R}^3 \mid x_2 + \sqrt{2}x_3 = 0\}$, and let $f = (\frac{1}{2}, 0, 0)$. Note that $f + W = W$. All integral points in W are of the form $(k, 0, 0)$, $k \in \mathbb{Z}$, hence $V = \{x \in W \mid x_2 = 0\}$. Thus $\mathcal{V} = \{s \in \mathcal{W} \mid \sum_{r\in W} r_2 s_r = 0\}$.

Consider the function $\psi : W \to \mathbb{R}$ defined by $\psi(r) = \max\{-4r_1 - 4r_2, 4r_1 - 4r_2\}$. The set $B_\psi = \{x \in W \mid -4(x_1 - \frac{1}{2}) - 4x_2 \leq 1,\, 4(x_1 - \frac{1}{2}) - 4x_2 \leq 1\}$ does not contain any integer point, hence $\Psi(s) \geq 1$ is valid for R_f. Note that B_ψ is not maximal (see Figure 2).

Given $\lambda = 4$, let $\psi'(r) = \psi(r) + \lambda r_2$ for all $r \in W$. Note that $\psi'(r) = \max\{-4r_1, 4r_1\} \geq 0$ for all $r \in W$. The set $B_{\psi'} = \{x \in W \mid -4(x_1 - \frac{1}{2}) \leq 1,\, 4(x_1 - \frac{1}{2}) \leq 1\}$ is contained in the maximal lattice-free convex set $B = \{x \in W \mid -2(x_1 - \frac{1}{2}) \le 1,\ 2(x_1 - \frac{1}{2}) \le 1\}$, hence ψ' is pointwise larger than the function ψ_B defined by $\psi_B(r) = \max\{-2r_1, 2r_1\}$ and $\Psi_B(s) \ge 1$ is valid for R_f.

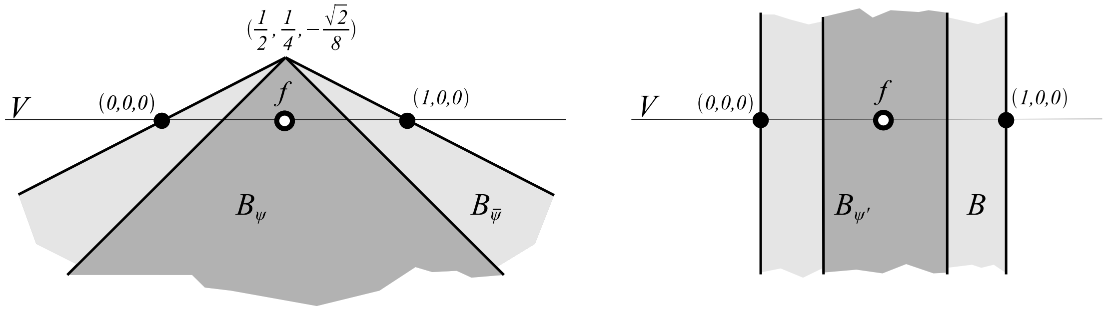

Figure 2: Lattice-free sets in the 2-dimensional space W.

By construction, the function $\bar\psi$ defined by $\psi_B(r) - \lambda r_2$ for all $r \in W$ is pointwise smaller than ψ and $\bar\Psi(s) \ge 1$ is valid for R_f. Moreover, $B_{\bar\psi} = \{x \in W \mid -2(x_1 - \frac{1}{2}) - 4x_1 \le 1,\ 2(x_1 - \frac{1}{2}) - 4x_1 \le 1\}$ is a maximal lattice-free convex set containing B_ψ. Note that the recession cones of B_ψ and $B_{\bar\psi}$ are full dimensional, hence ψ and $\bar\psi$ take negative values on elements of the recession cone. For example $\psi(0, -1, \frac{1}{\sqrt{2}}) = \bar\psi(0, -1, \frac{1}{\sqrt{2}}) = -4$. The recession cones of $B_{\psi'}$ and B coincide and are not full dimensional, thus $\psi'(0, -1, \frac{1}{\sqrt{2}}) = \psi_B(0, -1, \frac{1}{\sqrt{2}}) = 0$, since the vector $(0, -1, \frac{1}{\sqrt{2}})$ is in the recession cone of B.

4 The intersection of all minimal inequalities

In this section we prove Theorem 4. First we need the following.

Lemma 25. *Let $\psi : W \to \mathbb{R}$ be a continuous function that is positively homogeneous. Then the function $\Psi : \mathcal{W} \to \mathbb{R}$, defined by $\Psi(s) = \sum_{r \in W} \psi(r) s_r$, is continuous with respect to $(\mathcal{W}, \|\cdot\|_H)$.*

Proof: Define $\gamma = \sup\{|\psi(r)| : r \in W, \|r\| = 1\}$. Since the set $\{r \in R_f(W) : \|r\| = 1\}$ is compact and ψ is continuous, γ is well defined (that is, it is finite). Given $s, s' \in \mathcal{W}$, we will show $|\Psi(s') - \Psi(s)| \leq \gamma \|s' - s\|_H$, which implies that Ψ is continuous. Indeed

$$\begin{aligned}
|\Psi(s') - \Psi(s)| &= |\sum_{r\in W} \psi(r)(s'_r - s_r)| \\
&\leq \sum_{r\in W} |\psi(r)|\,|s'_r - s_r| \\
&= \sum_{r\in W\,:\,\|r\|=1} \sum_{\alpha>0} |\psi(\alpha r)|\,|s'_{\alpha r} - s_{\alpha r}| \\
&= \sum_{r\in W\,:\,\|r\|=1} |\psi(r)| \sum_{\alpha>0} \alpha |s'_{\alpha r} - s_{\alpha r}| \quad \text{(by positive homogeneity of } \psi\text{)} \\
&\leq \gamma \sum_{r\in W\,:\,\|r\|=1} \sum_{\alpha>0} \alpha |s'_{\alpha r} - s_{\alpha r}| \\
&= \gamma \sum_{r\in W\setminus\{0\}} \|r\| |s'_r - s_r| \quad \leq \gamma \|s' - s\|_H
\end{aligned}$$

□

Proof of Theorem 4. "$\subseteq$" By Lemma 25, Ψ_B is continuous in $(\mathcal{W}, \|\cdot\|_H)$ for every $B \in \mathcal{B}_W$, therefore $\{s \in \mathcal{W} : \Psi_B(s) \geq 1\}$ is a closed half-space of $(\mathcal{W}, \|\cdot\|_H)$. It is immediate to show that also $\{s \in \mathcal{W} : s_r \geq 0,\, r \in W\}$ is a closed set in $(\mathcal{W}, \|\cdot\|_H)$. Since $\mathcal{V} = \{s \in \mathcal{W} \,|\, \sum_{r\in W}(Cr)s_r = d - Cf\}$, and since for each row c_i of C the function $r \mapsto c_i r$ is positive homogeneous, then by Lemma 25 $\mathcal{V}$ is also closed. Thus

$$\{s \in \mathcal{V} : \Psi_B(s) \geq 1, B \in \mathcal{B}_W;\; s_r \geq 0,\, r \in W\}$$

is an intersection of closed sets, and is therefore a closed set of $(\mathcal{W}, \|\cdot\|_H)$. Thus, since it contains $\mathrm{conv}(R_f(W))$, it also contains $\overline{\mathrm{conv}}(R_f(W))$.

"$\supseteq$" We only need to show that, for every $\bar{s} \in \mathcal{V}$ such that $\bar{s} \notin \overline{\mathrm{conv}}(R_f(W))$ and $\bar{s}_r \geq 0$ for every $r \in W$, there exists $B \in \mathcal{B}_W$ such that $\sum_{r\in W} \psi_B(r)\bar{s}_r < 1$.

The theorem of Hahn-Banach implies the following.

> *Given a closed convex set A in $(\mathcal{W}, \|\cdot\|_H)$ and a point $b \notin A$, there exists a continuous linear function $\Psi : \mathcal{W} \to \mathbb{R}$ that strictly separates A and b, i.e. for some $\alpha \in \mathbb{R}$, $\Psi(a) \geq \alpha$ for every $a \in A$, and $\Psi(b) < \alpha$.*

Therefore, there exists a linear function $\Psi : \mathcal{W} \to \mathbb{R}$ such that $\Psi(\bar{s}) < \alpha$ and $\Psi(s) \geq \alpha$ for every $s \in \overline{\mathrm{conv}}(R_f)$. By the first part of Theorem 3, we may assume that $\Psi(s) \geq \alpha$ is a nontrivial minimal valid linear inequality. By the second part of Theorem 3, this inequality is equivalent to an inequality of the form $\sum_{r\in W} \psi_B(r)s_r \geq 1$ for some maximal lattice-free convex set B of W with f in its interior. □

References

[1] K. Andersen, Q. Louveaux, R. Weismantel, L. A. Wolsey, Cutting Planes from Two Rows of a Simplex Tableau, *Proceedings of IPCO XII*, Ithaca, New York (June 2007), Lecture Notes in Computer Science 4513, 1-15.

[2] K. Andersen, Q. Louveaux, R. Weismantel, An Analysis of Mixed Integer Linear Sets Based on Lattice Point Free Convex Sets, manuscript, 2007.

[3] K. Andersen, Q. Louveaux, and R. Weismantel. Integral Farkas Type Lemmas for Systems with Equations and Inequalities, to appear in *Operations Research Letters*.

[4] K. Andersen, Q. Louveaux and R. Weismantel, Mixed-Integer Sets Associated with Two Linearly Independent Rows, manuscript (July 2008).

[5] K. Andersen, C. Wagner, R. Weismantel, On an Analysis of the Strength of Mixed Integer Cutting Planes from Multiple Simplex Tableau Rows, manuscript (December 2008).

[6] E. Balas, Intersection Cuts – A New Type of Cutting Planes for Integer Programming, *Operations Research* **19** (1971) 19-39.

[7] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, Rhode Island, 2002.

[8] A. Basu, P. Bonami, G. Cornuéjols, F. Margot, On the Relative Strength of Split, Triangle and Quadrilateral Cuts, to appear in *Mathematical Programming*.

[9] A. Basu, G. Cornuéjols, G. Zambelli, Convex Sets and Minimal Sublinear Functions, manuscript (March 2009).

[10] V. Borozan, G. Cornuéjols, Minimal Valid Inequalities for Integer Constraints, technical report (July 2007), to appear in *Mathematics of Operations Research*.

[11] J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Grundlehren der mathematischen Wissenschaften, vol. 99. Berlin- Gttingen-Heidelberg, Springer, 1959.

[12] J. B. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, Springer, Berlin, 1990.

[13] G. Cornuéjols, F. Margot, On the Facets of Mixed Integer Programs with Two Integer Variables and Two Constraints, to appear in *Mathematical Programming*.

[14] S.S. Dey, L.A. Wolsey, Lifting Integer Variables in Minimal Inequalities Corresponding to Lattice-Free Triangles, *IPCO 2008*, Bertinoro, Italy (May 2008), Lecture Notes in Computer Science 5035, 463-475.

[15] S.S. Dey, L.A. Wolsey, Constrained Infinite Group Relaxations of MIPs, manuscript (March 2009).

[16] J. P. Doignon, Convexity in crystallographic lattices, *Journal of Geometry* **3** (1973) 71-85.

[17] D. Espinoza, Computing with Multi-Row Gomory Cuts, *Proceedings of IPCO XIII*, Bertinoro, Italy, (2008).

[18] R.G. Gomory, Some Polyhedra Related to Combinatorial Problems, *Linear Algebra and Applications* **2** (1969) 451-558.

[19] R.E. Gomory, Thoughts about Integer Programming, 50th Anniversary Symposium of OR, University of Montreal, January 2007, and Corner Polyhedra and Two-Equation Cutting Planes, George Nemhauser Symposium, Atlanta, July 2007.

[20] J-B. Hiriart-Urruty, C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer, Berlin, 2001.

[21] A.H. Lenstra Jr., Integer Programming with a Fixed Number of Variables, *Mathematics of Operations Research* **8** (1983), 538-548.

[22] L. Lovász, Geometry of Numbers and Integer Programming, *Mathematical Programming: Recent Developements and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989), 177-210.

[23] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York (1986).

[24] G. Zambelli, On Degenerate Multi-Row Gomory Cuts, *Operations Research Letters* **37** (2009), 21-22.