Colorings of k-balanced matrices and integer decomposition property of related polyhedra

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Abstract

We show that a class of polyhedra, arising from certain $0, 1$ matrices introduced by Truemper and Chandrasekaran, has the *integer decomposition property.* This is accomplished by proving certain coloring properties of these matrices.

1 Introduction

For any positive integer k, we say that a $0, \pm 1$ matrix A is *k-balanced* if it does not contain any square submatrix B with at most $2k$ nonzero entries in each row, such that each row and each column of B has an even number of nonzero entries, and the sum of all entries in B equals 2 modulo 4. This notion was introduced by Truemper and Chandrasekaran [11] in the $0, 1$ case and generalized by Conforti, Cornuéjols and Truemper [4]. The name "k-balanced matrices" was first adopted in [6].

A rational polyhedron P is said to have the *integer decomposition property* if, for every positive integer h and for every integral vector $y \in hP := \{hx \,|\, x \in P\}$, there exist h integral vectors $x^1, \ldots, x^h \in P$ such that $y = x^1 + \ldots + x^h$. The integer decomposition property was introduced by Baum and Trotter [1] (see [10]). The following is the main result of the paper. (For any integer k, we denote by $\mathbf{k}$ a vector with all entries equal to k.)

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Theorem 1 *Let A be an $m \times n$ k-balanced $0,1$ matrix with rows a^i, $i \in [m]$, and let S_1, S_2 be a partition of $[m]$. Then the polyhedron*

$$P = \left\{ x \in \mathbb{R}^n_+ : \begin{array}{ll} a^i x \leq b_i & i \in S_1 \\ a^i x \geq b_i & i \in S_2 \end{array} \right\} \tag{1}$$

has the integer decomposition property for every vector $b \in \mathbb{Z}^m$ such that $\mathbf{0} \leq b \leq \mathbf{k}$.

The class of k-balanced matrices is related to the theory of *totally unimodular matrices*, as we briefly explain. A real matrix is totally unimodular (t.u.) if each of its square nonsingular submatrices has determinant ± 1.

Theorem 2 (Camion [3] and Gomory [cited in [3]], Ghouila-Houri [8]) *A $0, \pm 1$ matrix A is totally unimodular if and only if A does not contain a square submatrix B with an even number of nonzero entries in each row and each column, such that the sum of all entries in B equals 2 modulo 4.*

A $0, \pm 1$ matrix which is not totally unimodular but whose proper submatrices are all totally unimodular is said *almost totally unimodular*. By Theorem 2, a $0, \pm 1$ matrix is k-balanced if and only if it has no almost totally unimodular submatrix with at most $2k$ nonzero entries in each row.

Similarly to the case of totally unimodular matrices, several polyhedra arising from k-balanced matrices have only integral vertices, as the next theorem shows. For any $m \times n$ $0, \pm 1$ matrix A, we denote by $n(A)$ the vector with m components whose i^{th} component is the number of -1's in the i^{th} row of A, and let $p(A) = n(-A)$.

Theorem 3 (Conforti, Cornuéjols and Truemper [4]) *Let A be an $m \times n$ k-balanced $0, \pm 1$ matrix with rows a^i, $i \in [m]$. Let b be a vector in $\mathbb{Z}^m$ such that $-n(A) \leq b \leq \mathbf{k} - n(A)$, and S_1, S_2 be a partition of $[m]$. Then the polytope $\{x \in \mathbb{R}^n_+ : a^i x \leq b_i$ for $i \in S_1$, $a^i x \geq b_i$ for $i \in S_2$, $x \leq \mathbf{1}\}$ is integral.*

Theorem 3 generalizes a result of Truemper and Chandrasekaran [11] valid for $0,1$-matrices. Notice that, in the latter case, Theorem 1 is a strengthening of Theorem 3. A natural question is whether the polyhedra defined in Theorem 3 have the integer decomposition property even when the k-balanced matrix A has negative entries.

See also Gijswijt [9] for results on the integer decomposition of polyhedra arising from other matrices related to total unimodularity.

The proof of Theorem 1 will follow from certain coloring properties of the class of k-balanced matrices, that we present in the next section.

2 k-equitable colorings

Given a $0, \pm 1$ matrix A with rows $a^1, \dots a^m$, let $\alpha_i := \min\{k, \lfloor \frac{p_i(A)+n_i(A)}{2} \rfloor\}$. We say that A has a *k-equitable bicoloring* if its columns can be partitioned into blue columns and red columns so that the matrix A', obtained from A by multiplying its blue columns by -1, has at least α_i positive entries and at least α_i negative entries in row i, for every $i \in [m]$. This concept is related to that of k-balancedness as follows.

Theorem 4 (Conforti, Cornuéjols, Zambelli [6]) *A $0, \pm 1$ matrix A is k-balanced if and only if every submatrix of A has a k-equitable bicoloring.*

Given a matrix A and an integer $\lambda \geq 2$, a λ-coloring of A is a partition of the columns of A into λ sets (*colors*) $I_1, \dots, I_\lambda$ (some color may be empty). Given a $0, \pm 1$ matrix A and positive integers k and λ, a *k-equitable λ-coloring* of A is a λ-coloring $I_1, \dots, I_\lambda$ of A such that I_j, I_h is a k-equitable bicoloring of the matrix $A_{I_j I_h}$ induced by the columns in $I_j \cup I_h$ for every $1 \leq j < h \leq \lambda$. We show the following.

Theorem 5 *An $m \times n$ $0,1$ matrix A is k-balanced if and only if every submatrix of A has a k-equitable λ-coloring for every integer $\lambda \geq 2$.*

Proof: If $\lambda = 2$, then the statement is equivalent to Theorem 4. We only need to show that, if A is k-balanced, then A has a k-equitable λ-coloring. Assume $\lambda \geq 3$. Let $S_1 \subseteq [m]$ be the set of indices i such that a^i has less than $k\lambda$ nonzero entries, and $S_2 = [m] \setminus S_1$. Given a partition $I_1, \dots, I_\lambda$ of the columns of A, let $n_{ij} = |\{h \in [n] \mid a^i_h = 1, h \in I_j\}|$, for every $i \in [m]$, $j \in [\lambda]$. Define

$$\mu_{ij} = \begin{cases} \max(n_{ij} - \lceil p_i(A)/\lambda \rceil, \lfloor p_i(A)/\lambda \rfloor - n_{ij}) & \text{for } i \in S_1 \\ \max(0, k - n_{ij}) & \text{for } i \in S_2 \end{cases}.$$

Choose $I_1, \dots, I_\lambda$ minimizing $\mu = \sum_{i=1}^n \sum_{j=1}^\lambda \mu_{ij}$. Observe that $I_1, \dots, I_\lambda$ is a k-equitable λ-coloring if and only if $\mu = 0$. Suppose $\mu_{st} > 0$ for some $s \in [m]$, $t \in [\lambda]$. Clearly, there exists $t' \in [\lambda]$ with the following property:

- If $s \in S_1$ and $n_{st} < \lfloor p_s(A)/\lambda \rfloor$, then $n_{st'} > \lfloor p_s(A)/\lambda \rfloor$;
- If $s \in S_1$ and $n_{st} > \lceil p_s(A)/\lambda \rceil$, then $n_{st'} < \lceil p_s(A)/\lambda \rceil$;
- If $s \in S_2$, then $n_{st'} > k$.

W.l.o.g., $t = 1$, $t' = 2$. By Theorem 4, $A_{I_1 I_2}$ admits a k-equitable bicoloring I'_1, I'_2. For $i \in [n]$ and $j = 1, 2$, let $n'_{ij} = |\{h \in [n] \,|\, a^i_h = 1, h \in I'_j\}|$, and $\mu'_{ij} = \max(n'_{ij} - \lceil p_i(A)/\lambda \rceil, \lfloor p_i(A)/\lambda \rfloor - n'_{ij})$ for $i \in S_1$, $\mu'_{ij} = \max(0, k - n'_{ij})$ for $i \in S_2$. Clearly, $n'_{i1} + n'_{i2} = n_{i1} + n_{i2}$.

Claim: $\mu'_{i1} + \mu'_{i2} \leq \mu_{i1} + \mu_{i2}$ for every $i \in [m]$, and $\mu'_{s1} + \mu'_{s2} < \mu_{s1} + \mu_{s2}$.

Case 1: $i \in S_1$.
If $2\lfloor p_i(A)/\lambda \rfloor \leq n_{i1} + n_{i2} \leq 2\lceil p_i(A)/\lambda \rceil$, then $\lfloor p_i(A)/\lambda \rfloor \leq n'_{ij} \leq \lceil p_i(A)/\lambda \rceil$ for $j = 1, 2$, so $\mu'_{i1} + \mu'_{i2} = 0 \leq \mu_{i1} + \mu_{i2}$, where the inequality is strict if $i = s$. If $n_{i1} + n_{i2} > 2\lceil p_i(A)/\lambda \rceil$, then $n'_{ij} \geq \lceil p_i(A)/\lambda \rceil$ for $j = 1, 2$ (since $\lceil p_i(A)/\lambda \rceil \leq k$), so $\mu'_{i1} + \mu'_{i2} = (n'_{i1} - \lceil p_i(A)/\lambda \rceil) + (n'_{i2} - \lceil p_i(A)/\lambda \rceil) = (n_{i1} - \lceil p_i(A)/\lambda \rceil) + (n_{i2} - \lceil p_i(A)/\lambda \rceil) \leq \mu_{i1} + \mu_{i2}$. If $i = s$, the inequality is strict since, by the choice of t and t', there exists $j \in [2]$ such that $n_{ij} - \lceil p_i(A)/\lambda \rceil < 0 \leq \mu_{sj}$. If $n_{i1} + n_{i2} < 2\lfloor p_i(A)/\lambda \rfloor$, then $n'_{ij} \leq \lfloor p_i(A)/\lambda \rfloor$ for $j = 1, 2$, thus $\mu'_{i1} + \mu'_{i2} = (\lfloor p_i(A)/\lambda \rfloor - n'_{i1}) + (\lfloor p_i(A)/\lambda \rfloor - n'_{i2}) = (\lfloor p_i(A)/\lambda \rfloor - n_{i1}) + (\lfloor p_i(A)/\lambda \rfloor - n_{i2}) \leq \mu_{i1} + \mu_{i2}$. If $i = s$, the inequality is strict since, by the choice of t and t', there exists $j \in [2]$ such that $\lfloor p_i(A)/\lambda \rfloor - n_{ij} < 0 \leq \mu_{sj}$.
Case 2: $i \in S_2$.
If $n_{i1} + n_{i2} < 2k$, then $n'_{ij} \leq k$ for $j = 1, 2$, thus $\mu'_{i1} + \mu'_{i2} = (k - n'_{i1}) + (k - n'_{i2}) = (k - n_{i1}) + (k - n_{i2}) \leq \mu_{i1} + \mu_{i2}$. If $i = s$, the inequality is strict since $k - n_{s2} < 0 = \mu_{s2}$.
If $n_{i1} + n_{i2} \geq 2k$, then $n'_{ij} \geq k$ for $j = 1, 2$, thus $\mu'_{i1} + \mu'_{i2} = 0 \leq \mu_{i1} + \mu_{i2}$, where the inequality is strict if $i = s$. This concludes the proof of the claim.

For $i \in [m]$ and $3 \leq j \leq \lambda$, let $\mu'_{ij} = \mu_{ij}$. By the previous Claim, $\mu' = \sum_{i=1}^{n} \sum_{j=1}^{\lambda} \mu'_{ij} < \sum_{i=1}^{n} \sum_{j=1}^{\lambda} \mu_{ij} = \mu$, thus $I'_1, I'_2, I_3, \ldots, I_\lambda$ contradicts the choice of $I_1, \ldots, I_\lambda$. □

We do not know if Theorem 5 holds in general for all k-balanced matrices; an answer in the affirmative was conjecture by Conforti and Zambelli for the case $k = 1$, that is for the class of *balanced matrices* (see [5]).

Theorem 5 relates previous results of Berge [2], who proved it for balanced matrices, and De Werra [7], who showed that every $m \times n$ totally unimodular matrix (hence $\lfloor \frac{n}{2} \rfloor$-balanced) has a $\lfloor \frac{n}{2} \rfloor$-equitable λ-coloring for every integer $\lambda \geq 2$ (in this case the result holds for matrices with negative entries as well).

A reduction from 1-equitable λ-colorings to 1-equitable bicolorings, similar to the one used in the proof above, was described in [5] to prove the theorem of Berge we mentioned above for $0, 1$ balanced matrices.

The proof of Theorem 5 can be turned into a strongly polynomial time algorithm to find a k-equitable λ-coloring as follows: start from an arbitrary partition $I_1, I_2, I_3, \ldots, I_\lambda$ and compute the corresponding μ. At each iteration, if $\mu = 0$ stop, else find a new partition with a smaller value of μ by computing a k-equitable bicoloring of $A_{I_t I_{t'}}$ for some appropriate choice of $t, t' \in [\lambda]$. Since computing a k-equitable bicoloring can be done in strongly polynomial time, as observed in [6], and $\mu \leq nm$ for any possible partition, then the above algorithm is strongly polynomial.

Proof of Theorem 1
Let h be a positive integer, and let y be and integral vector in hP. We need to show that y is the sum of exactly h integral vectors in P. For $h = 1$, the statement is trivial. If $h \geq 2$, consider the matrix $\bar{A}$ obtained from A by replicating y_j times the j^{th} column of A for every $j \in [n]$ (in particular, if $y_j = 0$, we remove the corresponding column). For each $j \in [n]$, let C_j be the set of indices of columns of $\bar{A}$ that are a copy of column j of A (if $y_j = 0$, let $C_j = \emptyset$). Thus $|C_j| = y_j$. Denote by $\bar{a}^i$, $i \in [m]$, the rows of $\bar{A}$. Clearly, $\bar{A}$ is still k-balanced, so by Theorem 5 there exists a k-equitable h-coloring $I_1 \ldots, I_h$ of $\bar{A}$. Let $z^1, \ldots, z^h$ be the characteristic vectors of $I_1, \ldots, I_h$, respectively. Let $x^1, \ldots, x^h$ be the vectors in $\mathbb{R}^n$ defined by $x_j^s = \sum_{t \in C_j} z_t^s$ for $s \in [h]$, $j \in [n]$. Clearly, $y = x^1 + \ldots + x^h$. We only need to show that $x^s \in P$ for every $s \in [h]$. Observe that $p(\bar{a}^i) = a^i y$ for every $i \in [m]$. Thus, if $i \in S_1$, then $a^i x^s = \bar{a}^i z^s \leq \lceil p(\bar{a}^i)/h \rceil = \lceil (a^i y)/h \rceil \leq \lceil (hb_i)/h \rceil = b_i$. If $i \in S_2$, then $a^i x^s = \bar{a}^i z^s \geq \min(\lfloor p(\bar{a}^i)/h \rfloor, k) \geq b_i$, since $k \geq b_i$ and $\lceil p(\bar{a}^i)/h \rceil = \lceil (a^i y)/h \rceil \geq b_i$. $\square$

Note that, since a k-equitable λ-coloring can be found in polynomial time, the proof of Theorem 1 gives a polynomial time algorithm that, given a positive integer h and an integral point y in hP, returns integral vectors $x^1, \ldots, x^h \in P$ such that $y = x^1 + \ldots + x^h$.

References

[1] S. Baum, L.E. Trotter, Integer Rounding and Polyhedral Decomposition of Totally Unimodular Systems, in: *Optimization and Operations Research* (Proc. Bonn 1977; R. Henn, B. Korte and W. Oettli, eds.), Lecture Notes in Economics and Math. Systems, Springer, Berlin, 1977, 15-23.

[2] C. Berge, Notes sur les Bonnes Colorations d'un Hypergraphe, *Cahiers du Centre d'études de Recherche Opérationnelle* **15** (1973), 219-223.

[3] P. Camion, Characterization of totally unimodular matrices, *Proceedings of the American Mathematical Society*, **16** (1965), 1068-1073.

[4] M. Conforti, G. Cornuéjols and K. Truemper, From Totally Unimodular to Balanced $0, \pm 1$ Matrices: A Family of Integer Polytopes, *Mathematics of Operations Research* **19** (1994), 21-23.

[5] M. Conforti, G. Cornuéjols and K. Vušković, Balanced Matrices, *Discrete Mathematics* (A. Bondy, V. Chvàtal ed.), to appear.

[6] M. Conforti, G. Cornuéjols, G. Zambelli, Bicolorings and Equitable Bicolorings of Matrices, *The Sharpest Cut, MPS/SIAM Series on Optimization* (M. Groetschel, ed.) (2004), 33-36.

[7] D. De Werra, On some Characterization of Totally Unimodular Matrices, *Mathematical Programming* **20** (1981), 14-21.

[8] A. Ghouila-Houri, Charactérisations des Matrices Totalement Unimodulaires, *Comptes Rendus de l'Académie des Sciences*, **254** (1962), 1192-1193.

[9] D. Gijswijt, Integer decomposition for polyhedra defined by nearly totally unimodular matrices, to appear in *SIAM Journal on Discrete Mathematics*.

[10] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.

[11] K. Truemper and R. Chandrasekaran, Local Unimodularity of Matrix-Vector Pairs, *Linear Algebra and its Applications*, **22** (1978), 65-78.