Note on the Bellman Functional Equation, Existence with Upper Hemicontinuous Feasibility Correspondence

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Abstract

This short note establishes existence and uniqueness of the solution to the Bellman functional equation, arising in dynamic programming. The result differs from the one presented in Stokey and Lucas (1989) in that it does not require the feasibility correspondence to be continuous and shows that upper hemicontinuity suffices. The note also relaxes assumption of continuity of the objective function and requires it to be only upper semicontinuous. Finally, we provide a sketch of possible applications of the result.

Theorem 4.6 in Stokey and Lucas (1989) shows existence and uniqueness of the solution to the equation

$$v(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta v(y)].$$

Defining the usual T operator and writing

$$(Tf)(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta f(y)] \tag{1}$$

the theorem assumes X is a convex subset of $\mathbb{R}^n$, the feasibility correspondence $\Gamma : X \twoheadrightarrow X$ is non-empty, compact-valued and continuous and the objective function $F : A \to \mathbb{R}$ with $A = \{(x, y) \in X \times X |\ y \in \Gamma(x)\}$ is bounded and continuous with $\beta < 1$.

The argument is as follows. Assuming f is continuous and bounded, the maximum exists and, by the Theorem of the Maximum, Tf is continuous. Hence, $T : C(X) \to C(X)$, where $C(X)$ is a space of continuous and bounded functions defined on X, and since $C(X)$ is a complete metric space

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with uniform metric and T is a contraction by the Blackwell's sufficient conditions for contraction, it follows, by the Contraction Mapping Theorem, that the solution exists and is unique.

We show that this result can be extended in two respects. First, the assumption of the continuity of Γ, which guarantees continuity of Tf, can be replaced with the upper hemicontinuity (u.h.c.). Furthermore, the assumption of F being continuous can be replaced with a weaker notion of upper semicontinuity (u.s.c.). Under these conditions, T can still be shown to possess a unique fixed point. The argument is essentially the same, making sure the maximum in (1) exists, replacing Theorem of the Maximum with its weaker version and ensuring the space of bounded u.s.c. functions, defined on X, is a complete metric space.

Since the notion of upper semicontinuity is not well known in the economic literature, we provide its definition.

Definition 1 (upper semicontinuous function)**.** *A function $f : X \to \bar{\mathbb{R}}$ on a topological space X is upper semicontinuous at $x \in X$ if, for each $\epsilon > 0$, there exists a neighbourhood U of x such that $f(y) \leq f(x) + \epsilon$ for all y in U. It is upper semicontinuous if it is upper semicontinuous $\forall x \in X$.*

An alternative definition, sometimes used, takes a sequence $\{x_n\}$ and defines u.s.c. as a function that satisfies $x_n \to x \Rightarrow \limsup_n f(x_n) \leq f(x)$ which is, indeed, the same requirement (Bourbaki, 2007, Chapter IV.6, Proposition 4). Yet, another definition requires the set $\{x \in X | f(x) < c\}$ to be open for any $c \in \mathbb{R}$, which is equal to the previous definition (Aliprantis and Border, 2006, Lemma 2.42).

Intuitively, u.s.c. functions are allowed to jump but, when they do so, the value of the function at the jump is 'the higher of the two'. The advantage of the u.s.c. functions is that they possess maxima on compact intervals. Having said that, we are ready to state the main result.

Theorem 1. *Let X be a convex subset of $\mathbb{R}^n$, $\Gamma : X \twoheadrightarrow X$ nonempty, compact valued and upper hemicontinuous correspondence, $F : A \to \mathbb{R}$ on $A = \{(x, y) \in X \times X |\ y \in \Gamma(x)\}$ bounded and upper semicontinuous function, $SC(X)$ space of bounded upper semicontinuous functions $f : X \to R$ with the sup norm $\|f\| = \sup_{x \in X} |f(x)|$ and $\beta < 1$. Then, the T operator, defined in (1), maps $SC(X)$ into itself and has a unique fixed point $v = Tv$.*

The strategy of the proof is in the following. First, we make sure that a maximum in (1) exists, next we show that Tf is u.s.c. and, hence, T maps $SC(X)$ into itself. Next, we observe that T is a contraction and, hence, has a unique fixed point, provided that $SC(X)$ is complete. As is customary, we view the normed vector space $(X, \|\cdot\|)$ as a metric space on X with the uniform metric $d(f, g) = \|f - g\|$.

Proof. First observe that, for any $x \in X$, the function $F(x, \cdot) + \beta f(\cdot)$ is u.s.c. and is maximized on a compact, non-empty set $\Gamma(x)$, hence, the maximum exists (Aliprantis and Border, 2006, Theorem 2.43).

Furthermore, as Γ is u.h.c., T is u.s.c. (Aliprantis and Border, 2006, Lemma 17.30) and it is clearly bounded. Hence, $T : SC(X) \to SC(X)$.

Next, we need to make sure that T satisfies conditions under which Blackwell's Theorem (Aliprantis and Border, 2006, Theorem 3.53) holds. Denoting by $B(X)$ the space of bounded functions defined on X, we need T to map a closed linear subspace of $B(X)$ that includes constant functions into itself. Furthermore, we need T to satisfy *monotonicity* and *discounting*.

That $SC(X)$ is a linear subspace of $B(X)$ that includes constant functions follows trivially. To establish that $SC(X)$ is closed, we observe that $B(X)$ is complete and that any complete subset of a complete metric space is closed (Berberian, 1999, Chapter III.4, Theorem 1). Hence, if we can establish that $SC(X)$ is complete, then closedness follows.

To establish that $SC(X)$, with the uniform metric, is a complete metric space, we adopt the approach of the proof of theorem 3.1 in Stokey and Lucas (1989), with appropriate modifications. We find a function f to which a Cauchy sequence of functions $\{f_n\}$ converges, we show the sequence converges in the uniform metric and, finally, that $f \in SC(X)$.

First, fix $x \in X$ and take a sequence $\{f_n(x)\}$, which satisfies

$$|f_n(x) - f_m(x)| \leq \sup_{y \in X} |f_n(y) - f_m(y)| = \|f_n - f_m\|$$

and which satisfies the Cauchy criterion and, hence, converges to a limit $f(x)$.

Second, we need to show that $\{f_n\}$ converges in the uniform metric. Pick $\epsilon > 0$ and $N := N(\epsilon)$, such that $n, m \geq M \Rightarrow \|f_n - f_m\| \leq \epsilon/2$ (which can be done). For any $x \in X$ and all $n, m \geq N$

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(m)| + |f_m(x) - f(x)| \\ &\leq \|f_n - f_m\| + |f_m(x) - f(x)| \\ &\leq \epsilon/2 + |f_m(x) - f(x)|. \end{aligned}$$

As $f_m(x) \to f(x)$, choose $m(x)$ for each $x \in X$ such that $|f_m(x) - f(x)| \leq \epsilon/2$. As x was arbitrary, it follows that $\|f_n - f\| \leq \epsilon$ for $\forall n \geq N$ and, as ϵ was arbitrary, we have convergence in the uniform metric.

Third, we need to show that f is bounded and u.s.c., the first of which follows readily. To show the u.s.c. part, pick $\epsilon > 0$ and k such that $\|f_k - f\| \leq \epsilon/3$. As $f_n \to f$, this can be done. Then, choose δ such that $\|x - y\|_E < \delta \Rightarrow f_k(y) < f_k(x) + \epsilon/3$ where $\|\cdot\|_E$ is a usual Euclidean distance and it can be done by u.s.c. of f_k. Finally,

$$\begin{aligned} f(y) - f(x) &= f(y) - f_k(y) + f_k(y) - f_k(x) + f_k(x) - f(x) \\ &\leq |f(y) - f_k(y)| + f_k(y) - f_k(x) + |f_k(x) - f(x)| \\ &\leq 2\|f - f_k\| + f_k(y) - f_k(x) \\ &\leq \epsilon. \end{aligned}$$

Furthermore, it is easy to confirm that $g \leq f$ implies $Tg \leq Tf$ (monotonicity) and that there exists $\beta \in (0,1)$, such that $T(f+c) \leq Tf + \beta c$ for any constant function c (discounting). Hence, by Blackwell's Theorem, T is a contraction and it has a unique fixed point, which concludes the proof. □

To sketch an application of the above result, consider a bargaining model with an endogenous status-quo, i.e. a model where policy enacted today constitutes the status-quo for the next period of bargaining. To maintain simplicity, assume that there are only two players in the model, indexed by i and j, and focus on a stationary Markov perfect equilibrium, in which strategies in every period depend only on the current status-quo policy.

Denoting the current status-quo by x, the policy to be chosen by p, continuous utility functions of the players by $F_i(x,p)$, $F_j(x,p)$ and an agent whose turn it is to make a proposal by i, the equilibrium condition is a Bellman equation

$$V_i(x) = \max_{p \in A_j(x)} \{F_i(x,p) + \delta V_i(p)\} \tag{2}$$

where V_i is a continuation value function induced by the equilibrium. The acceptance set of the player j is

$$A_j(x) = \{p | F_j(x,p) + \delta V_j(p) \geq F_j(x,x) + \delta V_j(x)\}.$$

Applying the theorem and ensuring that the technical conditions hold, if we can prove upper hemicontinuity of the acceptance correspondence A_j, then, we know that the solution to (2) exists and is unique. But, proving the upper hemicontinuity of A_j is easy, once we prove the continuity of the V_j function. To see the continuity of V_j implies an upper hemicontinuity of A_j, take two sequences $\{x_\alpha\} \to x$ and $\{p_\alpha\} \to p$, such that $p_\alpha \in A_j(x_\alpha)\ \forall\alpha$. Then, we need to show that $p \in A_j(x)$. Assume that $p \notin A_j(x)$, then we have

$$\begin{gathered} F_j(x_\alpha, p_\alpha) + \delta V_j(p_\alpha) \geq F_j(x_\alpha, x_\alpha) + \delta V_j(x_\alpha)\ \forall\alpha \\ F_j(x,p) + \delta V_j(p) < F_j(x,x) + \delta V_j(x) \end{gathered}$$

which, after summing, gives

$$\begin{aligned} &[F_j(x_\alpha, p_\alpha) + \delta V_j(p_\alpha)] - [F_j(x,p) + \delta V_j(p)] > \\ &\qquad [F_j(x_\alpha, x_\alpha) + \delta V_j(x_\alpha)] - [F_j(x,x) + \delta V_j(x)]\ \ \forall\alpha \end{aligned}$$

and, upon taking the limit as $\alpha \to \infty$ on both sides, violates the continuity of $F_j + \delta V_j$. Notice also, that the continuity of V_j is necessary for the upper hemicontinuity of A_j. Indeed, it is easy to construct examples with upper semicontinuous or lower semicontinuous V_j inducing A_j that is not upper hemicontinuous.

The theorem can be used in two ways. First, upon deriving an equilibrium that gives rise to continuous V functions, the theorem implies that the equilibrium is unique in the class of equilibria that give rise to the continuous V functions.

Alternatively, without deriving the equilibrium explicitly, it suffices to show that any equilibrium gives rise to continuous V's and the theorem can be used to prove uniqueness and, more importantly, the existence of such an equilibrium.

More broadly, the theorem can be used in settings where the proposer makes 'take it or leave' it policy proposals to the other players. In order for the proposal to be accepted, it has to provide a higher utility than the status-quo. Provided that the acceptance sets in an equilibrium are cut-out by continuous functions, the acceptance correspondences will be upper hemicontinuous and, hence, the proposer's optimization problem will possess a unique solution.

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