Simple equilibria in dynamic bargaining games over policies

Jan Zápal *

October 25, 2011

Abstract

The paper constructs equilibria in a class of infinite horizon dynamic bargaining models in which players care about all the dimensions of a policy space. Both one-dimensional and multi-dimensional policy spaces are analysed. All the equilibria have attractive property in being simple and having intuitive structure. Equilibrium behaviour is a result of two opposing forces. One force pushes players into proposing policies as close as possible to their single period optimum. A second and strategic force pushes players in the opposite direction, in an attempt to propose policies that constrain the future proposals of all other players. The resulting dynamics of the policies is shown to converge to the most preferred policy of the median player. The paper also uncovers the multiplicity of equilibria in certain environments, which greatly complicates their computational simulation.

* Ph.D. Candidate, Department of Economics, London School of Economics and Political Science, j.zapal@lse.ac.uk.

1 Introduction

In recent years, the political science and the political economy literature has, among other things, been studying dynamic bargaining models. On the most general level, these models fully acknowledge the fact that real world policies remain in place until changed and hence the last period's policy serves as a status-quo during today's round of policy determination. Similarly, policy determined today will serve as a status-quo during the next round of policy making.

Despite considerable progress having been made, the more widespread use of dynamic bargaining models is complicated by a lack of general results that ensure the existence and uniqueness of (Markov perfect) equilibria, and by the complexity of equilibria in models where these have been derived analytically.

Faced with this complexity, a part of the literature has turned to computer simulations. What the simulations reveal is indeed the ill-behaved nature of key components of equilibria in dynamic bargaining models. The equilibrium strategies and induced preferences often lack convenient mathematical properties such as differentiability or continuity.

We hope to contribute to the growing literature on dynamic bargaining by constructing equilibria in games in which players care about all the dimensions of a policy space. All the equilibria that we derive here have very simple and arguably intuitive shape, and as such might be more useful in applied work.

We then use these equilibria to illustrate several potentially interesting features. Besides a standard comparative static exercise, we show the multiplicity of equilibria in certain environments. This multiplicity can prove challenging, especially in work where researchers use computer simulations, as the typically used value function iteration method relies on an assumption of uniqueness of the equilibrium being approximated.

Our construction can also prove beneficial in more complicated environments that are 'close' to the simple setup we use, as it readily produces value functions that can be used in the first step of the value function iteration method, in a hope to speed up convergence in the more complicated model.

The paper proceeds in the following way. We start with a brief survey of the dynamic bargaining literature in section 2 followed by a description of the model in section 3. Section 4 and 5 derives the corresponding equilibria when the policy space is assumed to be one-dimensional and multi-dimensional, respectively. Section 6 discusses our results and some possible extensions of the model in which our approach to the construction of the equilibria might still apply. Section 7 concludes. Proofs of all the propositions from the main text are included in the appendix.

2 Literature survey

We start with a description of a typical dynamic bargaining model. A set of N players interacts in an infinite horizon with discounting. In every period t, one of the players is randomly chosen to propose a policy $x_t \in \mathbb{R}^n$. This policy is then pitched against a status-quo policy q_t with a winning alternative becoming the new status-quo q_{t+1}. Players collect their utilities, given by an utility function $u_i(\cdot)$, and the bargaining moves to period $t+1$.

We distinguish the *dynamic bargaining* model just described from the closely related, but nevertheless different model with *evolving default*, in which the endogenous nature of the default policy, i.e. the alternative to the proposal, is preserved, but in which utilities are collected only once the game ends.

Both of the mentioned versions can be found in papers investigating divide-the-dollar problems, where policy space is usually $N-1$ dimensional simplex and the players only care about their share of the dollar, i.e. only about one dimension in the policy space. Kalandrakis (2004) derives the equilibrium analytically in a three player version of the dynamic bargaining model with linear utilities. Epple and Riordan (1987) on the other hand investigate a model with general utilities, but where players take fixed turns in proposing.

Diermeier and Fong (2008a,b) investigate a divide-the-dollar game with evolving default where the former model is complicated by the players having to decide about both, the size of the budget to share (with quadratic costs motivated by distortionary taxation) and about how to share it. Diermeier and Fong (2007) and Diermeier and Fong (2009) combine dynamic bargaining across periods with evolving default over individual rounds of bargaining within each period. The latter paper further adds a decision about the size of the budget with quadratic and stochastic costs. Nevertheless, all four papers just mentioned assume players derive linear utility from their share of a budget.

It is the role of concavity over players' share in a utility functions that motivates Battaglini and Palfrey (2011). They simulate equilibria in a standard divide-the-dollar three-player game with concave utilities and contrast these to the equilibria in a model with linear utilities, both theoretically and experimentally.

As opposed to divide-the-dollar games in which players only care about a single dimension of a policy space, several papers have considered models where players are interested in all the dimensions of a policy space X. We call these dynamic bargaining models over *policies*.

Among the first to investigate such a model is Baron (1996). In his paper, N players bargain over policies with $X = \mathbb{R}$ in a framework that is very similar to the one considered in (the one-dimensional part of) this paper. Fong (2005) and Baron and Herron (2003) try to expand the model by assuming $X = \mathbb{R}^2$. Both papers assume only three players with the most preferred policies on an equilateral triangle. Nevertheless, in order to derive some results they have to either put strong restrictions on X (Fong, 2005) or resort to a computer simulation of the equilibrium (Baron and Herron, 2003).

Several extensions of these models have been made. Duggan, Kalandrakis, and Manjunath (2008) make the institutional structure of their model richer by including legislature and a president with a policy veto. Alternatively, Cho (2004) includes elections in a model with three parties where voters' preferences are defined over $X \in \mathbb{R}$ and parties are interested also in the spoils of holding the office. Baron, Diermeier, and Fong (2011) have a similar model, except that $X \in \mathbb{R}^2$. In both models that include elections, it is the policy $x \in X$ that evolves endogenously, whereas spoils of the office are set to zero in case the parties do not reach an agreement.

Two papers take the dynamic bargaining to a monetary policy setting, modelling interest rate making decisions as a game with an endogenous status-quo. Riboni (2010) investigates model with N decision making players plus a public forming expectations about future monetary policy. His equilibrium then requires mutual consistency of both the decision makers' and the citizens' strategies. Riboni and Ruge-Murcia (2008) consider a model without public expectations but with the players' preferences changing stochastically from period to period. What is common to both models is that the proposal-making authority is assumed to be with a single fixed player, emulating the chairman-led nature of a typical monetary policy committee.

Faced with rather complicated equilibria that often defy attempts for analytical derivation, a series of papers have turned to computer simulations. Baron and Herron (2003) illustrate equilibrium in their model derived from simulations, along with the discontinuous property of corresponding value functions. Duggan et al. (2008) perform a similar exercise but use their simulations to investigate the welfare effects of several constitutional changes. Finally, Duggan and Kalandrakis (2011) propose a new method for the computer simulation of equilibria in dynamic bargaining games and compare its speed and robustness to several other methods. As an illustrative example they simulate equilibrium in a model with $N = 9$ and $X \in \mathbb{R}^2$.

Several other papers investigate related models that do not fit into any of the categories just mentioned. Bernheim, Rangel, and Rayo (2006) focus on a model with evolving default, where the policy space consists of a finite set of alternatives and a fixed horizon set for bargaining. They prove that if the policy space includes a Condorcet winner it will, under some conditions, be the final outcome of the bargaining independent of the original default. On the other hand, the independence of the final outcome and starting position arises without a Condorcet winner.

Battaglini and Coate (2007, 2008) analyse two related models with a tax-financed public spending used to finance public good and pork-barrel programs. The intertemporal link in their models arises due to the investment nature of the public good in the former model and possibility of debt finance in the latter.

Finally, Duggan and Kalandrakis (2010) prove the existence of a Markov perfect equilibrium in a general dynamic bargaining game. In order to smooth out the above mentioned discontinuities in the equilibrium value functions, the framework uses (possibly negligible) shocks in the utilities of the players and also a stochastic relationship between the agreed-on policy and the future status-quo. Besides the existence result, the paper proves the upper hemicontinuity of the equilibrium correspondence, suggesting a relative robustness of the equilibria to small changes in the model's parameters. As an illustrative example working paper version of the paper (Duggan and Kalandrakis, 2007) also numerically simulates equilibrium in a dynamic bargaining game with $N = 5$ and $X \in \mathbb{R}$.

3 Model

In this section we lay out our model and in the next one, we explain in detail the construction which leads to a conjectured equilibrium and derive the conditions under which the conjecture is indeed an equilibrium. Throughout, we include several examples for specific parameter values of the model.

The model of this section is simple. There is a set of N (odd) players choosing policies from $X = \mathbb{R}$ in an infinite sequence of periods $t = 0, 1, \ldots$ with discounting $\delta \in [0, 1)$. In each period t one of the players is randomly chosen to make a proposal r_t which is then pitched against the status-quo q_t policy and an alternative obtaining majority of votes is implemented and becomes a new status-quo q_{t+1}. Then players collect their utilities and bargaining moves into next period $t + 1$. Probabilities of recognition $p \in \{p_1, \ldots, p_N\}$ are fixed. The single period utility of each player i is taken to be $u_i(x) = -(x - x_i)^2$. We call the most preferred policies of players x_i *original* bliss points and order the players such that $x_1 < \ldots < x_i < x_{i+1} < \ldots < x_N$ denoting the whole vector by $x = \{x_1, \ldots, x_N\}$. We denote by $i = m$ the median player with i satisfying $|\{j|x_j < x_i\}| = |\{j|x_j > x_i\}|$ with the corresponding bliss point of $x_i = x_m$.

We focus solely on Stationary Markov Perfect equilibria (SMPE) of Maskin and Tirole (2001), in which pure strategies are measurable only with respect to the payoff relevant histories. In our model, this will imply the dependence of proposal and voting strategies on a state given by the status-quo q_t policy, not on a specific period t. For this reason we omit the time subscript from thereon.

SMPE will consist of two strategies for each player. The first one is a proposal strategy of player i when recognized in a period with a status-quo q, which we denote by $r_i(q)$. For convenience, and without the loss of generality, we assume that a proposer whose most preferred policy, out of the set of policies that would be accepted, is the current status-quo q, will indeed offer this policy, instead of offering a different policy knowing that it would be rejected.

The second strategy of each player is a voting strategy determining a player's vote when faced with status-quo q and offer r. Following Baron and Kalai (1993) we focus on *stage undominated voting* strategies, which assume that each player votes as if being pivotal and in effect for an alternative offering higher expected utility. This wipes out equilibria in which players vote against their most preferred alternative knowing their vote cannot change the resulting policy. Notice also that in any equilibrium it has to be true that the player who is indifferent between q and r votes for r.

With this structure, it is easy to see that any policy offered will always be accepted. As a result we do not have to distinguish between proposed and accepted policies and can focus solely on $r_i(q)$.

Two sets of equilibrium strategies just described give rise to a continuation value function of each player $V_i(q)$. These functions measure the expected utility from continuing the game at the beginning of each period with a status-quo q before a proposer for that given period is recognized. More formally, these can be written as

$$V_i(q) = \sum_{j=1}^{N} p_j[-(r_j(q) - x_i)^2 + \delta V_i(r_j(q))].$$

With the value functions defined, the proposal strategy of player i for a status-quo q solves

$$\max_{r \text{ accepted}|q} u_i(r) + \delta V_i(r)$$

and denoting overall expected utility by $U_i(r) = u_i(r) + \delta V_i(r)$, voting strategy of player i faced with status-quo q and alternative r dictates voting for r if and only if

$$U_i(r) \geq U_i(q).$$

4 Equilibria with $X \in \mathbf{R}$

The first result we prove greatly simplifies the derivation of decisive coalitions needed to approve any given proposal r. More specifically, we prove that, for a general set of (possibly non-Markov) equilibrium proposal and voting strategies, r is accepted, as opposed to q, if and only if the voter with a median bliss point prefers r to q. More formally we have

Proposition 1 (Dynamic median voter theorem for $X = \mathbb{R}$)**.** *For any set of proposal and voting strategies and any status-quo policy, a proposal is* *accepted if and only if it is accepted by a player with a median bliss point x_m.*

An implication of this result is that the acceptance sets for each player, when recognized to make a proposal, will be given by the shape of a median player expected utility function and will be the same for all players. A second implication is that the median player will for any status-quo offer his original bliss point. Finally, when constructing the equilibrium, we do not need to specify voting strategies of all the players relying on proposition 1 and hence focus only on the voting strategy of the median player.

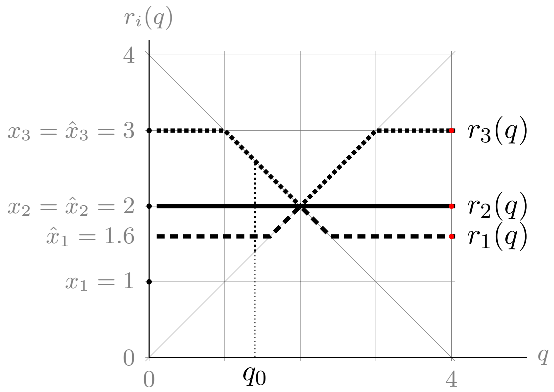
Conjectured proposal strategies will have the following shape

$$r_i(q) = \begin{cases} \max\{\min\{2x_m - q, q\}, \hat{x}_i\} & \text{for } i < m \\ \min\{\max\{2x_m - q, q\}, \hat{x}_i\} & \text{for } i > m \\ x_m & \text{for } i = m \end{cases} \tag{1}$$

where the $\hat{x}_i$s are what we call *induced* or *strategic* bliss points as those will be offered by a player i, given a large enough acceptance set, even though the original bliss point policy would be accepted as well. The vector of these can be denoted by $\hat{x} = \{\hat{x}_1, \ldots, \hat{x}_N\}$.

Example 1. *To illustrate the construction in (1) consider a simple model with $N = 3$, $x_i = i$, $p_i = \frac{1}{N}$ and $\delta = 0.9$. As will be explained below, a set of induced bliss points that qualify as an equilibrium are $\hat{x} = \{1.6, 2, 3\}$. Figure 1 illustrates the construction of (1) graphically.*

Figure 1: Conjectured equilibrium in Example 1



An intuition behind the shape of construction (1) and figure 1 follows. Given the proposal strategies, the typical acceptance set with status-quo q will be an interval between q and the policy on the other side of the median bliss point with the same distance from x_m, i.e. $2x_m - q$. One such acceptance set is indicated in the figure for status-quo q_0. This general shape arises as the overall utility of the median player decreases with a distance of a policy from x_m and hence the median player rejects any policy further from x_m compared to the status-quo.

Faced with this constraint, the decision of player i regarding her proposed policy is driven by two forces. The first force drives the proposal as close as possible to the player's original bliss point x_i increasing her current utility. The second force arises due to strategic considerations. Notice that by offering a policy that is closer to x_m player i sacrifices current utility, but potentially gains by constraining future policies to stay closer to the x_m. For a given player, this will be especially important if the probability of recognition of a player with the original bliss point on the other side of x_m is high.

The interplay of these two forces determines the shape of the construction in (1). For status-quo policies close to x_m the acceptance set is a narrow interval around x_m, the first force dominates, and both player 1 and 3 offer policies as close as possible to their bliss points. With increasing status-quo, the acceptance set widens, the second force gains force and player 1 switches to offering policy $\hat{x}_1$, for which the two forces even out, in an attempt to prevent player 3 from offering extreme policies in the future. The same logic holds for player 3 but for this player the second force is absent as player 1 does not offer extreme policies close to x_1 and hence player 3 will offer as high policies as possible given that her original bliss point x_3 is not in the acceptance set.

Finally, we need to specify a way to derive the strategic bliss points in the construction. These will be derived using an algorithm explained below, but first, we need additional piece of notation. Let us denote the general set of players in t-th step of the algorithm by $\mathbb{P}_t$, and define $p_t^+ = \sum_{i \in \mathbb{P}_t | x_i > x_m} p_i$ and $p_t^- = \sum_{i \in \mathbb{P}_t | x_i < x_m} p_i$. In words, p_t^+ is a sum of probabilities of recognition of players in set $\mathbb{P}_t$ with original bliss points above the median and analogously for p_t^-. With this notation the algorithm proceeds as follows.

Algorithm 1 (Strategic bliss points with $X = \mathbb{R}$)**.**

step 0 *Set* $\hat{x}_m = x_m$ *and* $\mathbb{P}_1 = \{1, \ldots, N\} \setminus \{m\}$

step t *For* $i \in \mathbb{P}_t$ *compute*

$$\hat{x}_{i,t} = \begin{cases} x_i + 2\delta p_t^+ (x_m - x_i) & i < m \\ x_i + 2\delta p_t^- (x_m - x_i) & i > m \end{cases}$$

and define $\mathbb{R}_t = \{i | (x_i - x_m)(\hat{x}_{i,t} - x_m) < 0\}$.

If $\mathbb{R}_t = \emptyset$ *pick a player with* $\hat{x}_{i,t}$ *closest to* x_m *out of* $\mathbb{P}_t$. *If more than one player is chosen, pick one of them in an arbitrary way. Denote the chosen player by* j. *Then* $\hat{x}_j = \hat{x}_{j,t}$ *and* $\mathbb{P}_{t+1} = \mathbb{P}_t \setminus \{j\}$. *If* $\mathbb{P}_{t+1} \neq \emptyset$, *proceed to the next step. If $\mathbb{R}_t \neq \emptyset$, proceed similarly except for picking player j out of $\mathbb{R}_t$ and setting $\hat{x}_j = x_m$.*

In words, the algorithm starts from a set of all players apart from the median, and assumes that all these players follow proposal strategies resembling the proposal strategies from figure 1 when status-quo q is close to x_m, i.e. assuming that players with $i > m$ ($i < m$) offer as high (low) policy as they can given the acceptance set of the form $[q, 2x_m - q]$.

Given these strategies, the algorithm computes the policy offering the maximum overall utility, $\hat{x}_{i,t}$, for each player and drops the player with $\hat{x}_{i,t}$ closest to x_m as this is the player first to switch into offering her strategic bliss point, i.e. the first to switch to the constant part of the equilibrium. The algorithm then proceeds similarly with a smaller set of players.

There are two possible complications. The first arises when the set $\mathbb{R}_t$, capturing the players with $\hat{x}_{i,t}$ on the other side of x_m compared to their x_i, is not empty. It is easy to confirm that this happens if and only if $2\delta p_1^+ > 1$ or $2\delta p_1^- > 1$. If this is the case, then the strategic bliss point of the chosen player is set to x_m and this player behaves in the same way as the median. Intuitively, this happens when the second force mentioned above is strong enough, which happens either when the future is important as captured by high δ, or the probability of recognition of players on the other side of x_m is high.

The second complication arises when the algorithm computes two $\hat{x}_{i,t}$s with the same distance from x_m. If this is the case the choice of which player to drop is arbitrary. This also implies that there will be two (or more) candidates for equilibria. If the algorithm at some step arrives at two players with an equal distance of $\hat{x}_{i,t}$ from x_m, eliminating one of them and proceeding, will give the first candidate equilibrium while eliminating the other will give the second candidate equilibrium.

Example 1 (continued)**.** *In the 0 step the algorithm drops the median player and sets $\hat{x}_2 = x_2 = 2$. In the first step, the algorithm computes $\hat{x}_{1,1} = 1.6$ and $\hat{x}_{3,1} = 2.4$, and by dropping the first player, finally gives $\hat{x}_3 = \hat{x}_{3,2} = 3$ as already anticipated and indeed drawn in figure 1. Notice that dropping player 3 in the first step of the algorithm would produce a symmetric around x_m but distinct set of strategic bliss points $\hat{x} = \{1, 2, 2.4\}$.*

The next example illustrates the first complication mentioned above, when either a high δ or a high probability of players on one side of the median (or both) induces a player on the other side of the median to behave as median.

Example 2 (Players behaving as median)**.** *Consider model with $N = 5$, $x_i = i$, $\delta = 0.9$ and $p = \{0.4, 0.4, 0.1, 0.05, 0.05\}$. It is easy to confirm that $\mathbb{R}_1 = \{4, 5\}$ with the algorithm dropping player 4 and $\mathbb{R}_2 = \{5\}$ with the* *algorithm dropping player 5. After two more steps, the algorithm produces* $\hat{x} = \{1, 2, 3, 3, 3\}$*. It is also easy to see that if the algorithm in the first step produces a nonempty* $\mathbb{R}$*, eliminating player, say, above the median, then all the remaining players on the same side of the median will be eliminated in the subsequent steps, and all the players on the opposite side of the median will have the strategic bliss points set to the original ones.*

Finally, we use the algorithm from above for parametrization of the model from Duggan and Kalandrakis (2007), who simulate equilibrium in a similar model. Our setup naturally lacks the utility and status-quo transition shocks their model has, but the values should be interesting for comparative purposes.

Example 3 (Duggan and Kalandrakis (2007) parametrization)**.** *Consider model with* $N = 5$*,* $x = \{1, 1.5, 2, 2.8, 3\}$*,* $\delta = 0.9$ *and* $p_i = \frac{1}{N}$*. The algorithm proceeds by eliminating players 2, 1, 4, and 5 in steps 1 through 4 respectively and produces a unique vector of strategic bliss points* $\hat{x} = \{1.72, 1.86, 2, 2.8, 3\}$*.*

We now proceed to specify the conditions under which the conjectured equilibrium constructed above is indeed an equilibrium. In order to do so we first need to construct several objects that will allow us to write the conditions in a concise way.

First, notice that the set of induced bliss points given by algorithm 1 induces a finite set of kinks in the proposal strategies. Combine all the (unique) values of q for which such kinks occur into vector $B = \{b_1, \ldots, b_k\}$ where k is the number of the kink-inducing values of q. Assume that B is ordered in such a way that $b_{j-1} < b_j$ for $j = 2, \ldots, k$.

Next, for a given status-quo q, it is helpful to split the set of players N into two subsets. Those who are on the constant part of the proposal strategies $C(q)$ and those who are not $N \setminus C(q)$. While well-defined for $X \setminus B$, $C(q)$ is not well-defined for any of the break points in B as it is not clear whether the player i for whom the $r_i(q)$ kinks at a specific q should be included in $C(q)$ or not. For this reason, let us define $C(b_j)$ for $j = 1, \ldots, k$ as a set of two sets, one including the player i (along with the rest of the non-problematic players) and one that does not (again including the non-problematic players). We regard $C(q)$ as a correspondence mapping X into sets, which is single valued for $q \in X \setminus B$ and double valued for $q \in B$.

Next, we define $p^+(q) = \sum_{i \in N \setminus C(q) | x_i > x_m} p_i$ and $p^-(q) = \sum_{i \in N \setminus C(q) | x_i < x_m} p_i$. In words, for a specific value of q, $p^+(q)$ gives the sum of probabilities of recognition of players above the median who are not on the constant part of the equilibrium. This is analogous for $p^-(q)$. Given that we view $C(q)$ as a correspondence, both $p^+(q)$ and $p^-(q)$ will be correspondences as well, mapping X into a single value for $q \in X \setminus B$ and into two values for $q \in B$. Notice also that both $p^+(q)$ and $p^-(q)$ are constant on every interval into which policy space X is divided by B, if we disregard the correspondence nature at the breaks in B. Finally, for any $b \in B$, denote by $p^+(\uparrow b)$ one of the values of $p^+(b)$, namely the one which is equal to $p^+(b-\epsilon)$ for small positive values of ϵ. Similarly, denote by $p^+(\downarrow b)$ the value of $p^+(b)$ which is equal to $p^+(b+\epsilon)$ for small positive values of ϵ. For $p^-(q)$ things are defined analogously.

With the notation in place, we are ready to state the following condition, under which the construction from (1) along with the strategic bliss points given by algorithm 1 gives an equilibrium.

Proposition 2 (Sufficient condition for equilibrium with $X = \mathbb{R}$)**.** *For $i \in N$ denote*

$$X_i^c = \begin{cases} B \cap (x_i, \hat{x}_i) & \text{for } i < m \\ B \cap (\hat{x}_i, x_i) & \text{for } i > m \end{cases}$$

with a typical element $x_{i,j}^c$. The construction from (1), with the strategic bliss points given by algorithm 1, is an equilibrium if for each i for which $X_i^c \neq \emptyset$

$$\begin{array}{ll} x_{i,j}^c - x_i + 2\delta p^+(\uparrow x_{i,j}^c)(x_i - x_m) \leq 0 & \forall j \text{ if } i < m \\ x_{i,j}^c - x_i + 2\delta p^-(\downarrow x_{i,j}^c)(x_i - x_m) \geq 0 & \forall j \text{ if } i > m \end{array}$$

The intuition behind the proposition is as follows. Take player i with $i < m$. It is easy to see she will never offer a policy on the other side of the median. Furthermore, in the proof of the proposition we show that her expected utility function is non-decreasing on $(-\infty, x_i]$ and non-increasing on $[\hat{x}_i, x_m]$. But for the construction to be an equilibrium, we need a stronger result, namely that it is non-decreasing on $(-\infty, \hat{x}_i]$. Establishing that the expected overall utility is continuous, piecewise concave and piecewise differentiable allows us to focus only on the set of points in X_i^c. This is a set of points from B between the original and the induced bliss points of player i. A condition in the proposition then makes sure that the left derivative of the expected utility is non-negative. When the condition holds, we know that on $(-\infty, x_m]$ the expected utility of player i attains a maximum at $\hat{x}_i$ and decreases on $[\hat{x}_i, x_m]$. As a result, when $\hat{x}_i$ is not in median player's acceptance set for a given q, player i will offer as low a policy as possible, and if it is in the acceptance set, she will offer $\hat{x}_i$.

Notice also that the condition in proposition 2 is stronger than needed, as it would suffice for, say, player $i < m$, for the expected utility to attain a maximum at $\hat{x}_i$ irrespective of its shape on $[x_i, \hat{x}_i]$. This indeed motivates a condition in the next proposition, which is both necessary and sufficient, but we have decided to include the condition in proposition 2 as it is extremely simple to check, as shown by the next example.

Example 1 (continued)**.** *With $\hat{x} = \{1.6, 2, 3\}$ and a set of breaks $B =$*

$\{1.6, 2.4\}$,

$$C(q) = \begin{cases} \{1,2,3\} & \textit{for } q \in (-\infty, 1] \cup [3, \infty) \\ \{1,2\} & \textit{for } q \in [1, 1.6] \cup [2.4, 3] \\ \{2\} & \textit{for } q \in [1.6, 2.4] \end{cases}$$

and the probability correspondences are

$$\begin{aligned} p^-(q) &= \begin{cases} \frac{1}{3} & \textit{for } q \in [1.6, 2.4] \\ 0 & \textit{for } q \in (-\infty, 1.6] \cup [2.4, \infty) \end{cases} \\ p^+(q) &= \begin{cases} \frac{1}{3} & \textit{for } q \in [1, 3] \\ 0 & \textit{for } q \in (-\infty, 1] \cup [3, \infty) \end{cases} \end{aligned}.$$

The condition of proposition 2 holds as $X_i^c = \emptyset$ *for* $i = 1, 2, 3$. *Indeed it is easy to show that the condition will hold for any model with* $N = 3$, $x_m - x_1 = x_3 - x_m$ *and* $p_1 = p_3$ *where players 1 and 3 denote the non-median ones.*

We next state a condition that is both sufficient and necessary for the construction above to be an equilibrium. The proposition uses set Z_i which is a set of points in $X \setminus B$ for which the overall utility of player i has a zero derivative, indicating a local maximum.

Proposition 3 (Sufficient and necessary condition for equilibrium with $X = \mathbb{R}$)**.** *For* $i \in N$ *denote*

$$X_i^c = \begin{cases} ((B \cup Z_i) \cap (x_i, \hat{x}_i)) \cup \{x_i, \hat{x}_i\} & \textit{for } i < m \\ ((B \cup Z_i) \cap (\hat{x}_i, x_i)) \cup \{x_i, \hat{x}_i\} & \textit{for } i > m \end{cases}$$

with a typical element $x_{i,j}^c$. *Arrange the elements of* $X_{i,j}^c$ *in a decreasing (increasing) order for* $i < m$ $(i > m)$ *and denote the first element by* $x_{i,0}^c$. *Finally, denote by* J_i' *the number of elements in* X_i^c.

The construction from (1) with the strategic bliss points given by algorithm 1 is then an equilibrium if and only if for each i

$$\begin{aligned} &\textstyle\sum_{j=1}^{J} \left[\frac{x^2}{2} c_1(x) + c_2(x) x \right]_{\downarrow x_{i,j}^c}^{\uparrow x_{i,j-1}^c} \geq 0 \qquad J = 1, \ldots, J_i' \textit{ if } i < m \\ &\textstyle\sum_{j=1}^{J} \left[\frac{x^2}{2} c_1(x) + c_2(x) x \right]_{\uparrow x_{i,j}^c}^{\downarrow x_{i,j-1}^c} \geq 0 \qquad J = 1, \ldots, J_i' \textit{ if } i > m \end{aligned}$$

where

$$\begin{aligned} c_1(x) &= -\frac{2}{1 - \delta(p^+(x) + p^-(x))} \\ c_2(x) &= \begin{cases} c_1(x)[-x_i + 2\delta p^+(x)(x_i - x_m)] & \textit{for } i < m \\ c_1(x)[-x_i + 2\delta p^-(x)(x_i - x_m)] & \textit{for } i > m \end{cases} \end{aligned}$$

Proposition 3 checks that $U_i(\hat{x}_i)$ is higher than $U_i(x)$ for any x in the $[x_i, \hat{x}_i]$ interval. It turns out to be enough to check a finite set of points collected in X_i^c. The proposition does not require us to construct $U_i(q)$ explicitly, as it turns out to be easier to integrate its derivative and use the fact that $U_i(x) - U_i(y) = \left[\frac{\partial U_i(x)}{\partial x}\right]_y^x$, proceeding interval by interval due to the piecewise differentiability of U_i.

We close this section by an example in which the conditions explained above might fail depending on the value of δ. It is also easy to see that both of the conditions above hold in all the preceding examples 1 through 3.

Example 4 (Possible failure of conditions for equilibrium)**.** *Consider a model with $N = 7$, $x_i = i$, $p_i = \frac{1}{N}$ and $\delta = 0.5$. Then it is relatively straightforward to check that the algorithm 1 gives eight possible arrangements of strategic bliss points, i.e. eight conjectured equilibria, and that for all those, the condition of proposition 2 and condition of proposition 3, hold.*

For the same model with $\delta = 0.9$ the set of conjectures reduces to two but both fail both conditions from above.

Finally, for $\delta = 0.95$ there are again two conjectured equilibria and for both of them the condition of proposition 2 fails while the condition of proposition 3 holds.

5 Equilibria with $X \in \mathbf{R}^n$

In this section we extend results from the previous one to the models with a multi-dimensional policy space. The setup of the model is exactly the same, except for the policy space $X = \mathbb{R}^n$. The utility of a player i is taken to be quadratic over each dimension, i.e. $u_i(x) = \sum_{j=1}^n -(x^j - x_i^j)^2$ where x^j is taken to be a policy in dimension j. Original bliss points will be vectors x_i with the preferred policy along dimension j denoted by x_i^j. Notice that $u_i(x) = ||x - x_i||^2$ where $||\cdot||$ denotes the norm.

In order to proceed we make an assumption about the arrangement of x_is in the policy space. We assume that the original bliss points are arranged in a way that ensures the existence of a core, more specifically we assume that the Plott (1967) condition is satisfied. For N odd this condition is both necessary and sufficient for the existence of a core (Austen-Smith and Banks, 2000).

This allows us to denote the player with a bliss point at the core as the median with bliss point x_m. The Plott (1967) condition then states that for each player i different from the median, there is another player j with a bliss point on the line connecting x_i with x_m but on the other side of x_m relative to x_i. For simplicity, we assume that exactly three players lie on each such line, and without loss of generality, set the bliss point of the median to be an origin of X.

Next, we prove a result similar to the dynamic median voter theorem proven in proposition 1 for a multi-dimensional policy space.

Proposition 4 (Dynamic median voter theorem for $X = \mathbb{R}^n$)**.** *For any set of proposal and voting strategies and any status-quo policy, a proposal is accepted if and only if it is accepted by a player with a median bliss point* x_m*.*

The proposition again allows us to focus on the median player who determines whether a given proposal will be accepted or not. With this result we can conjecture proposal strategies $r_i(q) : \mathbb{R}^n \to \mathbb{R}^n$ to be of the form

$$r_i(q) = \begin{cases} \frac{||q||}{||x_i||} x_i & \text{for } \frac{||q||}{||x_i||} \leq \hat{k}_i \\ \hat{k}_i x_i & \text{for } \frac{||q||}{||x_i||} \geq \hat{k}_i \end{cases}. \tag{2}$$

The proposal strategy of player i specifies, for a status-quo close to the origin, offering a policy on the line connecting origin and x_i with the same distance from the origin as the status-quo q. For a status-quo far away from origin, player i will be offering a fixed policy $\hat{k}_i x_i$, which we again term the strategic bliss point, for some $\hat{k}_i \in [0, 1]$.

The logic behind the proposal strategies is similar to the one-dimensional case. For a given status-quo q, the typical acceptance set will be a circle with the center at the origin and a radius $||q||$. There are again two forces at play. The first force pushes players into offering policies as close as possible to their original bliss points. The second strategic force pushes them into offering policies closer to the origin in an attempt to constraint future policies of the other players. For values of q close to the origin, the first force dominates and players offer a policy on a line connecting origin and x_i with the same distance from the origin as q. For values of q further away from the origin, the second force dominates and players offer fixed policies given by $\hat{k}_i x_i$.

Determining $\hat{k}_i$s is again done via a similar algorithm as in the previous section. The algorithm uses $a(i, j)$ to denote an angle between x_i and x_j. Subsequently it is easy to see that $\cos(a(i, j)) = \frac{x_i' x_j}{||x_i|| \cdot ||x_j||}$.

Algorithm 2 (Strategic bliss points with $X = \mathbb{R}^n$)**.**

step 0 Set $\hat{k}_m = 0$ *and* $\mathbb{P}_1 = \{1, \dots, N\} \setminus \{m\}$

step t For $i \in \mathbb{P}_t$ *compute*

$$\hat{k}_{i,t} = 1 - \delta \sum_{j \in \mathbb{P}_t} p_j [1 - \cos(a(i, j))]$$

and define $\mathbb{R}_t = \{i | \hat{k}_{i,t} < 0\}$.

If $\mathbb{R}_t = \emptyset$ pick a player with the smallest $\hat{k}_{i,t}||x_i||$ out of $\mathbb{P}_t$. If more than one player is chosen, pick one of them in an arbitrary way. Denote the chosen player by j. Then $\hat{k}_j = \hat{k}_{j,t}$ and $\mathbb{P}_{t+1} = \mathbb{P}_t \setminus \{j\}$. If $\mathbb{P}_{t+1} \neq \emptyset$, proceed to next step. If $\mathbb{R}_t \neq \emptyset$, proceed similarly except for picking player j out of $\mathbb{R}_t$ and setting $\hat{k}_j = 0$.

To proceed, we need to define similar objects as in the previous section, collecting all the points at which the value function induced by the construction in (2) and algorithm 2 kinks, and splitting the players into those on the constant and variable part of an equilibrium. However, we need to be concerned only about the distances from origin, not about the specific location in X.

For this purpose define B to be the collection of distances of the induced bliss points from the origin and order elements in B in an increasing order. Naturally, the first element of B is equal to 0. Next, for a given distance from origin d, let us define $C(d)$ to be a set of players on a constant part of the equilibrium proposal strategies, i.e. those players with $d > \hat{k}_i||x_i||$. For notational convenience, when we say $C(q)$ with $q \in \mathbb{R}^n$ we mean $C(||q||)$. Again $C(d)$ is well defined for $d \notin B$. For $d \in B$ we regard $C(d)$ as a correspondence giving two sets of players, one with the player for whom $d = \hat{k}_i||x_i||$ and all the players with $d > \hat{k}_i||x_i||$, and one with only the latter group of players.

Finally it will be convenient to redefine both constructions in terms of relative-to-x_i distance for player i. Hence let us define $B_i = \frac{B}{||x_i||}$ and also $C_i(k) = C(k||x_i||)$ for all $i \neq m$ and $k \in [0, \infty)$. With this notation we can state a sufficient condition for the construction just explained to be an equilibrium.

Proposition 5 (Sufficient condition for equilibrium with $X = \mathbb{R}^n$)**.** *For $i \in N \setminus \{m\}$ denote*

$$K_i^c = B_i \cap (\hat{k}_i, 1)$$

with a typical element $k_{i,j}^c$. Then the construction from (2) with the strategic bliss points given by algorithm 2 is an equilibrium if for each i for which $K_i^c \neq \emptyset$

$$1 - k_{i,j}^c - \delta \sum_{j \in N \setminus C_i(\downarrow k_{i,j}^c)} p_j [1 - cos(a(i,j))] \leq 0 \ \forall j$$

The intuition behind this result is simple. As we argue in the proof, it is enough to focus on player i offering policies on a ray starting at the origin and passing through x_i. For the construction to be an equilibrium, we want the expected utility function to first increase along this ray, until it reaches distance $\hat{k}_i||x_i||$, and then decrease. It turns out to be sufficient to focus on the interval $(\hat{k}_i||x_i||, ||x_i||)$ or in terms of the relative distance on $(\hat{k}_i, 1)$. Given the piecewise concavity of the expected utility function, the condition in the proposition ensures that at any break in B_i the expected utility is non-increasing.

Proposition 5 gives a sufficient condition which is stronger than needed but is easy to check. The next proposition states a condition that is both sufficient and necessary for equilibrium. We will again use set Z_i, which is a set of relative-to-x_i distances in $[0,\infty)\setminus B_i$ for which the expected utility function has a zero directional derivative along a ray starting at the origin and passing through x_i.

Proposition 6 (Sufficient and necessary condition for equilibrium with $X=\mathbb{R}^n$)**.** *For $i\in N\setminus\{m\}$ denote*

$$K_i^c = ((B_i\cup Z_i)\cap(\hat{k}_i,1))\cup\{\hat{k}_i,1\}$$

with a typical element $k_{i,j}^c$. Arrange the elements of $K_{i,j}^c$ in an increasing order and denote the first element by $k_{i,0}^c$. Finally denote by J_i' the number of elements in K_i^c.

Then the construction from (2) with the strategic bliss points given by algorithm 2 is an equilibrium if and only if for each $i\in N\setminus\{m\}$

$$\sum_{j=1}^{J}\left[\frac{k^2}{2}c_1(k)+c_2(k)k\right]_{\uparrow k_{i,j}^c}^{\downarrow k_{i,j-1}^c}\geq 0 \;\; J=1,\ldots,J_i'$$

where

$$\begin{aligned} c_1(k) &= -\frac{2||x_i||^2}{1-\delta\sum_{j\in N\setminus C_i(k)}} \\ c_2(k) &= c_1(k)[-1+\delta\sum_{j\in N\setminus C_i(k)}p_j[1-cos(a(i,j))]] \end{aligned}$$

We finish this section with two examples both of which assume $X=\mathbb{R}^2$.

Example 5 (Simplest example in $\mathbb{R}^2$)**.** *Consider a model with $N=5$, $p_i=\frac{1}{N}$, $\delta=0.9$ and the following bliss points*

player	*1*	*2*	*3*	*4*	*5*
x_i^1	*2*	*-2*	*0*	*0*	*0*
x_i^2	*0*	*0*	*2*	*-2*	*0*

Algorithm 2 offers four possible players to be eliminated in step 1, then 2 in steps 2 and 3. As a consequence there will be 16 possible equilibria. Eliminating players 1, 3, 2 and 4 respectively produces

player	*1*	*2*	*3*	*4*	*5*
$\hat{k}_i$	*0.28*	*0.82*	*0.46*	*1*	*0*

The set of distances at which players switch between constant and non-constant proposal strategies will be $B = \{0, 0.56, 0.92, 1.64, 2\}$ *and can be translated into relative-to-*x_i *distance* $B_i = \{0, 0.28, 0.46, 0.82, 1\}$*. With these we have*

$$C(d) = \begin{cases} \{1,2,3,4\} & \text{for } d \in [0, 0.56] \\ \{2,3,4\} & \text{for } d \in [0.56, 0.92] \\ \{2,4\} & \text{for } d \in [0.92, 1.64] \\ \{4\} & \text{for } d \in [1.64, 2] \\ \emptyset & \text{for } d \in [2, \infty] \end{cases}$$

and K_i^c *from proposition 5 will be* $K_1^c = \{0.46, 0.82\}$, $K_3^c = \{0.82\}$ *and an empty set for the remaining players. It is easy to check that both conditions from propositions 5 and 6 hold.*

Example 6 (Duggan and Kalandrakis (2011) parametrization)**.** *Consider a model with* $N = 9$, $p_i = \frac{1}{N}$, $\delta = 0.7$ *and bliss points*

player	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
x_i^1	-0.8	0.3	-0.2	0.9	0.1	-0.15	0.3	-0.9	0
x_i^2	0	0	0.2	-0.9	0.6	-0.9	0.2	-0.6	0

Algorithm 2 produces a unique set of bliss points (numbers rounded)

player	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$\hat{k}_i$	0.79	0.51	0.38	1	0.50	0.94	0.48	0.91	0

for which conditions from propositions 5 and 6 hold.

6 Discussion

In this section we discuss several topics related to the results presented so far. First, we look at the comparative static properties of the construction above. Although we do not conduct an explicit comparative static exercise, it is obvious that two variables have a strong influence on the shape of an equilibrium. Firstly, there is the discount factor δ. With the future becoming more important, all the equilibria above will be more concentrated around the bliss point of the median player. Secondly, there is a vector of recognition probabilities p. These vectors influence the equilibria in a complex way, but in general the higher the probability of recognition of a given player, the closer her opposition will be to the median.

Another observation regards the position of the most extreme player judged by the original bliss point. Notice that making player more extreme, i.e. further away from median, if this player is the last to be eliminated by the algorithms above, does not change the equilibrium behaviour of the other players. To a certain extent the same observation applies for other players. If the shape of the equilibrium remains the same, changing the bliss point of player i does not change the behaviour of other players.

Related to the shape of the equilibria explained above is the behaviour of policies over time. It is easy to see that starting from any status-quo, policies will always converge to the most preferred policy of the median player. Nevertheless, the pattern of policies during the convergence can be rather complex.

At the same time, the convergence to the median prediction is easy to avoid. Consider a similar model as above but with some probability the policy approved today will not become the next status-quo. Instead, the next status-quo would be drawn from some distribution with the cumulative distribution $F(q)$. Assuming that the distribution $F(q)$ is independent of the policy approved today, the equilibria constructed above will be the equilibria in this extended model (given some adjustment to δ). But policies in the model will not converge to the median over time.

Our construction has also uncovered the possible multiplicity of equilibria in certain environments. While not explicitly proven, it should be obvious that environments giving rise to the multiplicity are those with symmetric bliss points and equal recognition probabilities. We want to highlight this observation as the multiplicity can prove problematic for computer simulations in which a researcher chooses to simulate an often tempting symmetric environment.

At the same time it is obvious that the environments giving rise to the multiple equilibria are 'zero measure'. In example 1 we have two possible equilibria with $x = \{1, 2, 3\}$ and $p = \{1/3, 1/3, 1/3\}$. However, perturbing the environment to, say, $x = \{1, 2, 3 + \epsilon\}$ or $p = \{1/3, 1/3 + \epsilon, 1/3 - \epsilon\}$ for some small ϵ would make the equilibrium unique.

We have also made a series of simplifying assumptions. But we think the approach to the construction of simple equilibria in dynamic bargaining models taken here extends to more general environments. More specifically, we have assumed an equal δ for all players, but the construction would also be applicable to a model with player-specific δ_i. Specifying a model with a different non-quadratic utility function would also admit a similar approach, as well as an assumption in the multi-dimensional model that players attach different weights to different dimensions of a policy space, i.e. assuming utility of the form $u_i(x) = \sum_{j=1}^{n} -k_j(x^j - x_i^j)^2$ with some set of positive constants k_j. On the other hand, we do not think that the construction explained in this paper for a multi-dimensional policy space would extend to environments in which the Plott (1967) condition for core existence fails.

Finally, we admit that, for a specific parametrization of the model above, deriving strategic bliss points via algorithms 1 or 2 can be time-consuming and error prone. The same qualification applies when checking conditions for the constructions to be an equilibrium or, when deriving explicit formulas for the value functions $V_i(\cdot)$. For this reason, we have written Matlab routines for both one- and multi-dimensional models that derive the strategic bliss points, check for conditions ensuring equilibrium and derive the resulting value and expected utility functions all in a matter of seconds. Both routines are available upon request.

7 Conclusion

We provide an approach to constructing conjectured equilibria in dynamic bargaining models that produces simple and intuitive equilibrium strategies. Our results apply to models with both single and multi-dimensional policy spaces. We have also shown under which conditions the conjecture is indeed an equilibrium, some of which are straightforward to check for a given parametrization of the model.

The shape of the equilibria are in general driven by the interplay of two forces. One force pushes players into proposing policies that maximize their current utility. Another opposing and strategic force pushes players into proposing policies that constraint all the players in the future.

Our analysis shows that in dynamic bargaining models where a median is present, policies will always converge to the policy most preferred by the median. This, however, does not preclude the possibly complex behaviour of policies along the convergence path. We have also uncovered the possibility of multiple equilibria in certain symmetric environments. However none of the resulting equilibria found involves symmetric behaviour of otherwise symmetric players.

Despite the fact that our approach is not generally applicable and does not always produce equilibrium strategies, we nevertheless think it provides interesting insights into an environment with rather scarce analytical results.

References

Austen-Smith, D. and J. S. Banks (2000). *Positive Political Theory I, Collective Preferences.* Ann Arbor: The University of Michigan Press.

Baron, D. P. (1996). A dynamic theory of collective goods programs. *American Political Science Review 90*(2), 316–330.

Baron, D. P., D. Diermeier, and P. Fong (2011). A dynamic theory of parliamentary democracy. *forthcoming in Economic Theory*.

Baron, D. P. and M. C. Herron (2003). A dynamic model of multidimensional collective choice. In K. Kollman, J. H. Miller, and S. E. Page (Eds.), *Computational Models in Political Economy*, pp. 13–48. Cambridge, MA: MIT Press.

Baron, D. P. and E. Kalai (1993). The simplest equilibrium of a majority-rule division game. *Journal of Economic Theory 61*(2), 290–301.

Battaglini, M. and S. Coate (2007). Inefficiency in legislative policymaking: A dynamic analysis. *American Economic Review 97*(1), 118–149.

Battaglini, M. and S. Coate (2008). A dynamic theory of public spending, taxation, and debt. *American Economic Review 98*(1), 201–36.

Battaglini, M. and T. R. Palfrey (2011). The dynamics of distributive politics. *forthcoming in Economic Theory*.

Bernheim, D. B., A. Rangel, and L. Rayo (2006). The power of the last word in legislative policy making. *Econometrica 74*(5), 1161–1190.

Cho, S.-j. (2004). A dynamic model of parliamentary democracy. *Presented at the annual meeting of the The Midwest Political Science Association, April 15*.

Diermeier, D. and P. Fong (2007). Policy dynamics and legislative bargaining. *mimeo*.

Diermeier, D. and P. Fong (2008a). Bargaining over the budget. *mimeo*.

Diermeier, D. and P. Fong (2008b). Legislative bargaining with reconsideration. *mimeo*.

Diermeier, D. and P. Fong (2009). Policy persistence in multi-party parliamentary democracies. In E. Helpman (Ed.), *Institutions and Economic Performance*, pp. 361–406. Cambridge, MA: Harvard University Press.

Duggan, J. and A. Kalandrakis (2007). Dynamic legislative policy making. *University of Rochester Wallis Institute of Political Economy Working Paper Series No. 45*.

Duggan, J. and A. Kalandrakis (2010). Dynamic legislative policy making. *mimeo*.

Duggan, J. and A. Kalandrakis (2011). A Newton collocation method for solving dynamic bargaining games. *Social Choice and Welfare 36*(3-4), 611–650.

Duggan, J., A. Kalandrakis, and V. Manjunath (2008). Dynamics of the presidential veto: A computational analysis. *Mathematical and Computer Modelling 48*(9-10), 1570–1589.

Epple, D. and M. H. Riordan (1987). Cooperation and punishment under repeated majority voting. *Public Choice 55*(1), 41–73.

Fong, P. (2005). Dynamic legislation through bargaining. *mimeo*.

Kalandrakis, A. (2004). A three-player dynamic majoritarian bargaining game. *Journal of Economic Theory 116*(2), 294–322.

Maskin, E. and J. Tirole (2001). Markov perfect equilibrium: I. observable actions. *Journal of Economic Theory 100*(2), 191–219.

Plott, C. R. (1967). A notion of equilibrium and its possibility under majority rule. *American Economic Review 57*(4), 787–806.

Riboni, A. (2010). Committees as substitutes for commitment. *International Economic Review 51*(1), 213–236.

Riboni, A. and F. J. Ruge-Murcia (2008). The dynamic (in)efficiency of monetary policy by committee. *Journal of Money, Credit and Banking 40*(5), 1001–1032.

A1 Proofs

A1.1 Proof of proposition 1

The proof of proposition 1 builds on similar proof found in Riboni and Ruge-Murcia (2008), namely on their appendix A proof.

Proof. Note that any set of strategies for an accepted policy r_0 generates a stochastic sequence of policies $\{r_0, r_1, \ldots\}$ with implied utility for player i given as

$$U_i(r_0) = \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (r_t - x_i)^2\right].$$

Similarly accepting r'_0 generates $\{r'_0, r'_1, \ldots\}$ and gives player i

$$U_i(r'_0) = \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (r'_t - x_i)^2\right].$$

Differentiating the difference in utility that the two policies bring, with respect to bliss point of player i, gives

$$\frac{\partial[U_i(r_0) - U_i(r'_0)]}{\partial x_i} = \mathbb{E}\left[2\sum_{t=0}^{\infty} -\delta^t (r'_t - r_t)\right]$$

which is independent of x_i and hence $U_i(r_0) - U_i(r'_0)$ is linear in x_i. If, on the one hand, the median player prefers r_0 to r'_0, either all the players with a $x_i \geq x_m$ or all the players with $x_i \leq x_m$ also prefer r_0 to r'_0. As a result r_0 is accepted. If, on the other hand, the median player rejects r_0 to r'_0, either all the players with $x_i \geq x_m$ or all the players with $x_i \leq x_m$ also reject r_0 to r'_0. As a results r_0 is rejected. □

A1.2 Proof of proposition 2

We proceed is several steps, first establishing some properties of the value function induced by the construction in (1), next proving properties of the expected value function and then showing the rationale behind the condition in the proposition.

Proof. First observe that the continuation value function induced by construction (1) is continuous, piecewise concave, piecewise quadratic and symmetric around x_m. Some algebra shows that it can be written as

$$V_i(q) = \frac{h_i(q) + \sum_{j \in N \setminus C(q)} p_j u_i(r_j(q))}{1 - \delta \sum_{j \in N \setminus C(q)} p_j}$$

where $h_i(q)$ is a correspondence ensuring that $V_i(q)$ is continuous and that $V_i(x_m) = \frac{u_i(x_m)}{1-\delta}$. Its shape is similar to the shape of $p^+(q)$ in that it attains a unique value at $q \in X \setminus B$ and two values at $q \in B$. Moreover, for every $b \in B$ we require one value of $h_i(b)$ to correspond to one of the sets in $N \setminus C(b)$ and the second value of $h_i(b)$ to correspond to the other set in $N \setminus C(b)$. While this is a somewhat unusual construction, it allows us to economize on the notation later on.

With this notation it is easy to see that the derivative of the expected utility function with respect to status-quo is

$$\frac{\partial U_i(q)}{\partial q} = \begin{cases} -\frac{2}{1-\delta(p^+(q)+p^-(q))}\left[q - x_i + 2\delta p^+(q)(x_i - x_m)\right] & \text{for } q \leq x_m \\ -\frac{2}{1-\delta(p^+(q)+p^-(q))}\left[q - x_i + 2\delta p^-(q)(x_i - x_m)\right] & \text{for } q \geq x_m \end{cases}$$

where again the possibility of two values at $q = x_m$ and at all the $q \in B$ reflects the fact that the overall utility is not differentiable at those points. But it is easy to see that it possesses left and right derivatives, so we regard one value of $\frac{\partial U_i(q)}{\partial q}$ at, say, $q = x_m$ as the left derivative and the other value as the right derivative.

Now notice that the expected utility of the median player is strictly increasing on $(-\infty, x_m)$, strictly decreasing on (x_m, ∞), has unique global maximum at x_m and is symmetric around x_m. As a result, the acceptance set for the general value of q will be $[q, 2x_m - q]$ or $[2x_m - q, q]$ depending on which of the two values is larger.

Faced with the symmetry of the acceptance sets, player i never offers a policy that is on the other side of x_m compared to her x_i, due to the symmetry of the $V_i(q)$ function. As a result, we can focus only on an interval $(-\infty, x_m]$ for players $i < m$ and on an interval $[x_m, \infty)$ for players $i > m$. We will only do the former as the latter relies on a similar argument.

Take player with $i < m$. For the construction in (1) to be an equilibrium, we need the expected utility to be non-decreasing on $(-\infty, \hat{x}_i]$ and non-increasing on $[\hat{x}_i, x_m]$. The latter part is easy and follows directly from the construction of algorithm 1. Non-decreasing on $(-\infty, x_i]$ is also immediate inspecting the expression for derivative above.

This leaves $[x_i, \hat{x}_i]$ to be inspected. However, there is no need to inspect the whole interval but, given the piecewise concavity, it is enough to check the upper boundary of each interval on which the expected utility function is differentiable. We can also omit $\hat{x}_i$ as the derivative of $U_i(q)$ is zero at that point.

Proposition 2 is then the mathematical restatement of what we have just explained. Set X_i^c is a set of all points in $(x_i, \hat{x}_i)$ at which $U_i(q)$ kinks and the condition of the proposition ensures that the left derivative is non-negative at all those points. □

A1.3 Proof of proposition 3

Proof of proposition 3 uses the fact that for a differentiable continuous function $f(x)$ we have $f(x) - f(z) = [f'(a)]_z^x$. Extending this result to a continuous but only piecewise differentiable function is straightforward. If, for example, $x < y < z$ and $f(x)$ is not differentiable at y but has a left and a right derivative, we have $f(x) - f(z) = [f'(a)]_y^x + [f'(a)]_z^y$.

Proof. We show the result for $i < m$ as the argument is similar for $i > m$. We also omit repeating arguments from proof of proposition 2 and hence focus solely on the $[x_i, \hat{x}_i]$ interval.

First notice that the construction in (1) along with the bliss points derived by algorithm 1 is an equilibrium if and only if $U_i(\hat{x}_i) \geq U_i(x) \; \forall \; x \in [x_i, \hat{x}_i]$. One possible approach would be to construct the function $U_i(q)$ explicitly. Nevertheless, we propose a simpler approach as we already know the derivative of $U_i(q)$ from the proof of the previous proposition and are only interested in relative, not absolute, values.

Next, it is easy to show that the condition $U_i(\hat{x}_i) \geq U_i(x)$ fails if and only if it fails at some point of the X_i^c set, which includes all the kinks and all local maxima of the expected utility function on the $[x_i, \hat{x}_i]$ interval (the if part is obvious, the only if part follows from piecewise concavity). Hence we need to check the condition only at the points in X_i^c.

Using the derivative of $U_i(q)$ from the proof of the previous propositions, it can be written as $c_1(q)q + c_2(q)$, where $c_1(q)$ and $c_2(q)$ are given in the statement of this proposition. Integrating the derivative yields the expression $\frac{x^2}{2}c_1(x) + c_2(x)x$ and the sum then proceeds from $x_{i,0}^c = \hat{x}_i$ checking all points from X_i^c for the $U_i(\hat{x}_i) \geq U_i(x)$ condition. □

A1.4 Proof of proposition 4

Proof. The approach to the proof of proposition 4 is analogous to the proof of proposition 1. Accepting policy r_0 generates stochastic sequence of policies

$\{r_0, r_1, \ldots\}$ with the implied utility

$$U_i(r_0) = \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (r_t - x_i)'(r_t - x_i)\right].$$

Differentiating the difference in utility that r_0 and r'_0 provide, gives

$$\frac{\partial[U_i(r_0) - U_i(r'_0)]}{\partial x_i} = \mathbb{E}\left[2\sum_{t=0}^{\infty} -\delta^t (r'_t - r_t)\right]$$

which is again independent of x_i and hence $U_i(r_0) - U_i(r'_0)$ is linear in x_i. As a consequence, the derivative defines a hyperplane in $\mathbb{R}^n$ which gives all the bliss points such that any player j with a bliss point on this hyperplane will choose between r_0 and r'_0 in exactly the same way as player i. The result should now be obvious, realizing that any hyperplane going through x_m will split the remaining players into two equal size groups, at least one of which votes in the same way as the median. □

A1.5 Proof of proposition 5

The proof of this proposition is very similar to the proof of proposition 2 so we keep it brief.

Proof. First notice that the acceptance sets are circles with a center at the origin and that the continuation value function of all the players has level sets of a similar shape. It follows that player i will never offer any other policy than the policy on the line starting at the origin and going through her bliss point x_i, we call this line the i-ray.

As a consequence, we can work with the continuation value function V_i and the expected utility function U_i, mapping the i-ray into $\mathbb{R}$, instead of both functions having to map the whole policy space X into $\mathbb{R}^n$. It is then convenient to define the argument of both functions to be the distance of a policy on the i-ray from the origin, relative to $||x_i||$. With this notation, some algebra gives

$$V_i(k) = \frac{h_i(k) + \sum_{z \in N \setminus C_i(k)} p_z \sum_{j=1}^{n} - \left(\frac{x_z^j k ||x_i||}{||x_z||} - x_i^j\right)^2}{1 - \delta \sum_{z \in N \setminus C_i(k)} p_z}$$

for $k \in [0, \infty)$, with $h_i(k)$ defined similarly as in proposition 2, ensuring $V_i(k)$ is continuous and $V_i(0) = \frac{u_i(0)}{1-\delta}$. It is also immediate that $V_i(k)$ is differentiable except at points in B_i.

With this it is easy to see that the derivative of the overall utility function with respect to k is

$$\frac{\partial U_i(k)}{\partial k} = \frac{2||x_i||^2}{1 - \delta \sum_{j \in N \setminus C_i(k)} p_j}\left[1 - k - \delta \sum_{j \in N \setminus C_i(k)} p_j [1 - cos(a(i,j))]\right]$$

and that $U_i(k)$ is piecewise concave.

For the construction in (2), along with the bliss points derived via algorithm 2, to be an equilibrium, we need $U_i(k)$ to be non-decreasing on $[0, \hat{k}_i]$ and non-increasing on $[\hat{k}_i, \infty)$. Non-decreasing on $[0, \hat{k}_i]$ comes again from the construction of algorithm 2. Non-increasing on $[1, \infty)$ is also easy by inspecting the derivative of $U_i(k)$ above. This leaves $[\hat{k}_i, 1]$ to be inspected and, given piecewise concavity, it is enough to check the right derivative of $U_i(k)$ at each break B_i that falls into $(\hat{k}_i, 1)$. □

A1.6 Proof of proposition 6

The proof of proposition is very similar to the proof of proposition 3 so we only include a brief outline.

Proof. As in proposition 3 we want to make sure that $U_i(\hat{k}_i) \geq U_i(k)$ for all $k \in [\hat{k}_i, 1]$. Again, we can focus only on the points at which either $U_i(k)$ kinks or has a local maximum. This is what the set K_i^c collects. Integrating the derivative of $U_i(k)$ from the previous proposition gives $\frac{k^2}{2}c_1(k) + c_2(k)k$ and the condition of this proposition checks that $U_i(\hat{k}_i)$ is higher than $U_i(k)$ for any $k \in K_i^c$. □