

# Wealth inequality and collective action <sup>☆</sup>

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## Abstract

We study the effect of inequality in the distribution of endowments of private inputs (*e.g.*, land, wealth) that are complementary in production with collective inputs (*e.g.*, contribution to public goods such as irrigation and extraction from common-property resources) on efficiency in a class of collective action problems. We focus on characterizing the joint surplus maximizing level of inequality, making due distinction between contributors and non-contributors, in a framework that allows us to consider a wide variety of collective action problems ranging from pure public goods to impure public goods to commons. We show that while efficiency increases with greater equality *within* the groups of contributors and non-contributors, so long the externalities (positive or negative) are significant, there is an optimal degree of inequality *between* these groups.

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## 1. Introduction

Does inequality affect adversely the resolution of collective action problems? Evidence ranging from the micro (Wade, 1994; Alesina and La Ferrara, 2000; Bardhan, 2000; Dayton-

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Johnson, 2000; Khwaja, 2004) to the macro level (Knack and Keefer, 1997; Banerjee, 2004; Banerjee et al., 2005) strongly suggests that the propensity of individuals to join groups, to participate in social activities, to cooperate in various collective action problems, or to contribute to public goods and services is negatively related to inequality.

The relationship between collective action problems and inequality arises in a variety of contexts: maintenance of infrastructure in rural areas,<sup>1</sup> measures of social capital,<sup>2</sup> cooperation in water allocation and field channel maintenance,<sup>3</sup> extraction of common property resources (henceforth, CPR) such as forests,<sup>4</sup> and community level forest conservation efforts.<sup>5</sup> Some of these collective action problems involve positive externalities (pure or impure public goods) while some of them include negative externalities (commons). What is missing is a common theoretical framework that links inequality to efficiency across the whole range of collective action problems as those listed. The goal of this paper is to provide such a framework.

While our framework is relevant to the above examples, for the sake of a concrete benchmark we think of a collective action problem in agriculture. Suppose there is inequality in the distribution of endowments of a private input (say, land). Producers use this private input and a collective good (say, irrigation water) to produce a private good (say, rice). The private and collective inputs are complements in the production function. Water use by one farmer creates a negative externality by making less available for others (as in a CPR) but effort toward maintenance generates positive externalities (as in a public good).

The standard assumption of decreasing returns implies that one would expect that a more equal distribution of the private input across production units will improve efficiency. If the market for this input operated well, it would make sure that the input is allocated efficiently. There is considerable evidence, however, suggesting that the market for inputs such as land or capital does not operate frictionlessly and the wealth of a person determines how much of the input is used in her production unit.<sup>6</sup>

In the presence of collective action problems inequality of private endowments such as land or wealth may generate an effect that goes in the opposite direction. Indeed, in his pioneering work on collective action, Olson (1965) makes a strong case in favor of inequality since the greater the interest in the collective good of any single member, the greater the likelihood that this member will get such a significant proportion of the total benefit from the collective good that he will gain from its provision, even if he has to pay all of the cost himself. This role of “elites” has been

<sup>1</sup> Khwaja (2004) finds a negative effect of inequality in his study of 132 community maintained infrastructure projects in rural areas of Northern Pakistan.

<sup>2</sup> Using survey data on group membership and data on U. S. localities, Alesina and La Ferrara (2000) find that, after controlling for many individual characteristics, participation in social activities is significantly lower in more unequal localities. See also La Ferrara (2003).

<sup>3</sup> Using data from India and Mexico respectively, Bardhan (2000) and Dayton-Johnson (2000) find that the Gini coefficient for landholdings among irrigators has a significant negative effect on cooperation in water allocation and field channel maintenance.

<sup>4</sup> Baland et al. (in press) study the link between firewood collection and inequality using data from Nepal.

<sup>5</sup> Somanathan et al. (2006) study community management of forests in the Indian Central Himalayas and find that greater equality in land ownership increases conservation of pine forests.

<sup>6</sup> Evans and Jovanovic (1989) analyzed panel data from the National Longitudinal Survey of Young Men (NLSY), which surveyed a sample of 4000 men in the US between the ages of 14–24 in 1966 almost every year between 1966–81, and found that entrepreneurs are limited to a capital stock of no more than one and one-half times their wealth when starting a new venture. There is also a large literature showing that small farms are more efficient than large farms in developing countries (see Berry and Cline, 1979).

confirmed in field studies, such as Wade's (1994) authoritative study on irrigation systems in South Indian villages.<sup>7</sup>

In our setting the marginal benefit from contributing is increasing in the private input. Thus, redistributions increasing the wealth of the richer players at the expense of non-contributing poorer players would achieve a greater amount of the public good, and *other things being constant*, this should increase efficiency. This is how Olson's original argument shows up in our model. However, the gain from increasing the collective input has to be measured against the cost arising from worsening the allocation of the private input in the presence of decreasing returns, so the overall effect is not obvious.

Olson considered pure public goods only, which have the property that only the largest player contributes (see Cornes and Sandler, 1996). Our paper goes further, answering the following questions. Does Olson's argument hold for a more general class of collective goods which also includes impure public goods and CPRs? If we look at a welfare measure like joint surplus instead of the level of provision of the collective good, is it possible that some degree of inequality may be preferable to perfect equality? Is it possible for the allocation under some degree of inequality to Pareto-dominate the allocation under perfect equality? If more than one player contributes, how does inequality within the group of contributors *versus* inequality between contributors and non-contributors affect efficiency?

We show that the total provision (or, in the case of common property resources, total extraction) of the collective input is a concave function of the private input endowment for well-known production functions (e.g., Cobb–Douglas and CES). This means that asset inequality lowers the total provision of (pure and impure) public goods, and lowers the total extraction from the CPR. In addition, we provide a precise characterization of the optimal distribution of wealth that maximizes joint surplus in the general case of imperfect convertibility between the private and collective inputs. We show that the joint surplus maximizing wealth distribution features equality of wealth *within* each of the groups of contributing and non-contributing players. The contrast with the conclusions of Olson is quite sharp. The key assumptions leading to this result are: market imperfections that prevent the efficient allocation of the private input across production units, decreasing returns to scale, and some properties of the production function shared by widely used functional forms such as Cobb–Douglas and CES. With constant returns to scale, we show that joint surplus is independent of the distribution of wealth as in the well-known “distribution-neutrality” theorem.

Next, we consider the question of inequality *between* the groups of contributing and non-contributing players. In particular, does perfect equality among *all* players maximize joint surplus? We provide a limited answer to this question. It turns out that perfect equality is not always optimal. If wealth were equally distributed among all, the average wealth of contributors is low and this could reduce the level of the collective good. In contrast, concentrating all wealth in the hand of one person maximizes the average wealth of contributors, but involves significant efficiency losses due to the decreasing returns. The optimal surplus-maximizing distribution of wealth achieves a compromise between these two opposing forces by featuring equality *within* each of the groups of contributors and non-contributors but inequality *between* those groups.

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<sup>7</sup> See also Baland and Platteau (1997) who provide some very interesting examples where richer agents tend to play a leading role in collective action in a decentralized setting. For example, in rural Mexico the richer members of the population take the initiative in mobilizing labor to manage common lands and undertake conservation measures such as erosion control.

Under constant returns to scale, we obtain a complete characterization of the joint surplus maximizing level of inequality in all the cases ranging from pure public goods to commons. We show that, for public goods (pure and impure), inequality between contributors and non-contributors is still optimal. However, for commons where everyone contributes, distribution neutrality obtains as in Warr (1983). For the case of public goods, we also show that an allocation with some degree of inequality can *Pareto-dominate* the allocation under perfect equality.

*Related literature.* Following Olson's pioneering work, there is a large literature in public economics that addresses the question of inequality in collective action problems. An example is the "distribution neutrality" result mentioned above (Warr, 1983; Bergstrom et al., 1986) where in a Nash equilibrium the wealth distribution within the set of contributors does not matter for the amount of public goods provision.

Subsequent work has shown that this result depends on several crucial assumptions.<sup>8</sup> Namely, the neutrality breaks down if the set of contributors changes as a result of the redistribution or if individual contributions are not perfect substitutes in the public good production. For example, Cornes (1993) and Cornes and Sandler (1996) show that if individual contributions are complements, then greater inequality reduces the equilibrium level of the public good.<sup>9</sup> Another crucial assumption is the "purity" of the public good — Andreoni (1989) and Cornes and Sandler (1994) demonstrate that distribution neutrality can fail for impure public goods. Subsequent research has also examined, in the context of pure public goods, the effect of inequality on *joint surplus* and not just the equilibrium level of the public good. Itaya et al. (1997) show that redistribution from non-contributors to contributors can increase joint surplus. Cornes and Sandler (2000) find that Pareto improvements are also possible under some conditions.<sup>10</sup>

The literature on how inequality affects inequality in the case of commons is much smaller. An exception is Sandler and Arce (2003) who show that pure public goods and commons display a benefit-cost duality. As a result, the standard Olsonian result that the poor free ride on the rich in the case of pure public goods, is reversed in the case of commons. While suggesting that maximum profits may result in the case of commons if a single firm is active whereas, in the case of public goods, a more widespread participation by benefit recipients is efficient, the paper does not provide a formal characterization of the joint-surplus maximizing level of inequality. Baland and Platteau (1997) also discuss the effect of inequality on efficiency in the case of commons, but their exercise involves studying inequality in terms of the constraints on the contributions of individual players (which they interpret as credit constraints). Since under commons people contribute more than they should, if a tightening of these constraints prevents them from doing so, efficiency might go up.<sup>11</sup>

While the existing literature provides a fairly complete characterization of the effects of inequality in the case of pure public goods, it lacks a general framework encompassing the whole spectrum of externalities from pure public goods to commons. This is where the main contribution of the paper lies. While our model is admittedly not completely general, it is tractable

<sup>8</sup> Cornes and Sandler (1996) provide an excellent review of this literature.

<sup>9</sup> Baland et al. (in press) extend this line of research and show, using a CES production function, that there is a threshold value of the elasticity of substitution such that whether perfect equality is joint-surplus maximizing or not depends on whether the elasticity of substitution is below or above this threshold value.

<sup>10</sup> See also Boadway and Hayashi (1999).

<sup>11</sup> Dayton-Johnson and Bardhan (2002) also look at this question in a dynamic framework.

and the fact that it nests the distribution neutrality result for pure public goods as a special case lends the comparison with other types of collective action problems a certain amount of continuity.

Other than in its focus and scope, our analysis differs the distribution neutrality literature in another important dimension. In the standard distribution neutrality framework, the contributions towards the public good and the private good are fully convertible. This is not the case in our paper. Our motivation for assuming imperfect convertibility arises from the fact that in many real-world collective action problems, especially in developing countries, the contribution towards the collective input often takes the form of labor. If we interpret the private input as capital then our assumption reflects an important issue of economic inequality: labor is not freely convertible into capital. Typically, labor and capital are not perfect substitutes in the production technology, and because of credit market imperfections<sup>12</sup> capital does not flow freely from the rich to the poor to equate marginal returns.

The rest of the paper proceeds as follows. In Section 2 we describe our theoretical model and main assumptions. Section 3 characterizes the equilibrium in our setting and contains the main results about the effects of inequality on total collective good provision and joint surplus. In Section 4 we discuss several extensions of the basic framework that deal with convertibility, substitutability and complementarity between the private and collective inputs. Section 5 concludes and provides directions for future research. All proofs are in the Appendix.

## 2. The model

There are  $n > 1$  players. Each player uses two inputs,  $k_i$  and  $z_i$ , to produce a final good. The input  $k_i$  is a purely *private* good, such as land, capital, or managerial inputs. We assume that there is no market for this input and so a player is restricted to choose  $k_i \leq w_i$ <sup>13</sup> where  $w_i$  is the exogenously given endowment of this input of player  $i$ . While we will focus on this interpretation, there is an alternative one which views  $w_i$  as capturing some characteristic of a player, such as a skill or a taste parameter.<sup>14</sup> In contrast,  $z_i$  is a *collective* good in the sense that it involves some externalities, positive or negative. We assume that each player chooses some action  $x_i$  which can be thought of as her effort that goes into using a common property resource or contributing towards the collective good. Let  $X \equiv \sum_{i=1}^n x_i$  be the sum total of the actions chosen by the players. The individual actions aggregate into the collective input in the following simple way  $z_i = bx_i + cX$ . The profit (surplus) function of player  $i$  is<sup>15</sup>:

$$\pi_i = f(w_i, z_i) - x_i$$

Note that the input  $x_i$  appears twice in the profit function, once on its own as a private input, and once in combination with the quantities used or supplied by other agents. This implies that

<sup>12</sup> In developing countries the markets for land and credit are often highly imperfect due to badly defined property rights and inefficient legal systems.

<sup>13</sup> Given that  $k_i$  is non-tradeable,  $k_i = w_i$  will be chosen to maximize profits.

<sup>14</sup> The assumption that the market for the private input does not exist at all, while stark, is not crucial for our results. All that is needed is that the amount a person can borrow or the amount of land she can lease in depends positively on how wealthy she is. Various models of market imperfections, such as adverse selection, moral hazard, costly state verification or imperfect enforcement of contracts will lead to this property.

<sup>15</sup> We will also refer to  $\Pi = \sum_{i=1}^n \pi_i$  as joint surplus or joint profits later in the paper.

the private return to a player, *i.e.*,  $\frac{\partial z_i}{\partial x_i}$ , always exceeds the social return as long as  $b > 0$ . The input  $X$  can be a good (*e.g.*, R&D, education) or a bad (*e.g.*, any case of congestion or pollution). This formulation allows each player to receive a different amount of benefit from the collective input which depends on the action level they choose. In contrast, for pure public goods every player receives the same benefits irrespective of their level of contribution. This case, as well as many others (involving both positive and negative externalities) appear as special cases of our formulation as we will see shortly. Following the distribution neutrality literature we assume that the cost of supplying  $x_i$  units of the collective input, is simply  $x_i$  and that the production function,  $f$  exhibits non-increasing returns with respect to the private and collective inputs  $w_i$  and  $z_i$ .

Let  $\mathbf{w} = (w_1, w_2, \dots, w_n)$  denote the vector of wealth levels of the players. Assume that the  $w_i$  are arranged in descending order of magnitude, *i.e.*,  $w_1 > w_2 > \dots > w_n > 0$  and let  $W \equiv \sum_{i=1}^n w_i$  denote the total amount of wealth of the  $n$  players. We make the following assumption about the production function:

**Assumption 1.**  $f(w, z)$  is a strictly increasing and strictly concave function that is twice continuously differentiable on  $R_+^2$  with respect to both arguments,  $f_{12} \geq 0$ ,  $\lim_{w \rightarrow 0} f_2(w, z) = 0$  and  $f$  satisfies the Inada endpoint conditions.

We also make an assumption about the parameters  $b$  and  $c$ :

**Assumption 2.**

$$b \geq 0 \text{ and } b + cn > 0 \tag{A2}$$

Notice that we allow  $c$  to be positive, negative or zero. Only when  $c < 0$  does the second inequality in Eq. (A2) become relevant ensuring that  $|c|$  is not too large. The inequality  $b + cn > 0$  implies that if a social planner chooses the level of the collective input, she would choose a positive level of  $x_i$  for at least one player which is the natural case to be interested in. In addition, the same inequality also turns out to be a condition on the reaction functions of the players which ensures the stability of the equilibrium (see the Appendix). Finally, the Inada conditions of  $f$  together with the condition  $b + cn > 0$  ensure that  $z_i$  is always positive in equilibrium (see Section 3.1 below). When  $c = 0$  we have the case of a pure private good — there are no externalities. For  $b = 0$  and  $c > 0$  we have the case of pure public goods, *i.e.* the one on which most of the existing literature has focused. For  $b > 0$  and  $c > 0$  we have the case of impure public goods as defined by Cornes and Sandler (1996). For  $b > 0$  and  $c < 0$  we have a version of the commons problem: by increasing her action relative to those of the others, an individual gains.

We begin our analysis by the following result which shows how the choice of  $x_i$  by player  $i$  depends on how much wealth she has:

**Lemma 1.**  $\gamma(w) \equiv \arg \max_{z \geq 0} \{f(w, z) - z\}$  is strictly positive for  $w > 0$  and is an increasing function.

Notice that the property of  $\gamma$  in the Lemma is not satisfied if  $w$  and  $z_i$  are substitutes in the production function. We will discuss this case in Section 4. Several widely used in economics functional forms that display complementarity among the inputs satisfy the conditions stipulated in this lemma. These include the following:

- (a) the Cobb–Douglas production function,  $f(w, z) = w^\alpha z^\beta$  with  $\alpha + \beta \leq 1$ . In this case  $\gamma(w) = \beta^{\frac{1}{1-\beta}} w^{\frac{\alpha}{1-\beta}}$  which is clearly an increasing function;



- (b) the generalized CES production function, namely,  $f(w, z) = (\delta w^\rho + (1 - \delta)z^\rho)^{\frac{k}{\rho}}$  under parameter restrictions that ensure non-increasing returns to scale ( $k \leq 1$ ), and complementarity between  $w$  and  $z$  ( $\rho < k$ ). We work out the details of this case in the Appendix. For  $k=1$  (constant returns to scale) it is possible to solve for  $\gamma(w)$  explicitly, which turns out to be  $\left(\frac{\delta}{\delta - (1-\delta)^{\frac{1}{1-\rho}}}\right)^{\frac{1}{\rho}} (1 - \delta)^{\frac{1}{1-\rho}} w$ .

An important question from the economic point of view is that while clearly both a rich and a poor individual would increase their contribution to the collective input if their wealth increases by the same amount, who would want to expand their contribution to the collective input more? The answer to this question is crucial for determining the effect of the distribution of wealth on the total amount of the collective good,  $X$  and joint profits. This effect depends on the curvature of  $\gamma$ , which in turn depends on the third-order properties of the production function. Intuitively, the question is whether diminishing returns with respect to  $z$  would kick in at a faster or a slower rate at a higher wealth level. This would determine whether for a richer person a relatively small or large increase in  $z$  would restore her individual optimum compared to a poorer person. It turns out that the widely used functional forms in economics where inputs are complements have the property that diminishing returns kick in at the same or faster rate the higher is the wealth level. This implies that  $\gamma$  is linear or strictly concave. We make the following technical assumption about the production function which we show is equivalent to  $\gamma$  having this property.

**Assumption 3.**

$$h(w, z) \equiv \frac{\partial f(w, z)}{\partial z} \text{ is quasi-concave and } C^2.$$

The following lemma proves that  $h$  being (weakly) quasi-concave is equivalent to  $\gamma$  being (weakly) concave:

**Lemma 2.** *Suppose Assumption 1 holds.  $\gamma(w)$  is concave if and only if  $h(w, z) \equiv \frac{\partial f(w, z)}{\partial z}$  is quasi-concave.*

For the Cobb–Douglas and CES production functions,  $\gamma(w)$  is strictly concave under decreasing returns to scale ( $\alpha + \beta < 1$  for Cobb–Douglas and  $k < 1$  for CES) and linear under constant returns to scale ( $\alpha + \beta = 1$  or  $k = 1$ ). The following lemma provides additional characterization of the class of production functions for which  $\gamma$  is linear:

**Lemma 3.** *Suppose Assumption 1 holds. If  $f(w, z)$  is homogeneous of degree 1 then  $\gamma(w) = Aw$  where  $A$  is a positive constant.*

The Cobb–Douglas production  $w^\alpha z^\beta$  with  $\alpha + \beta = 1$  and the CES production function  $(\delta w^\rho + (1 - \delta)z^\rho)^{\frac{k}{\rho}}$  with  $k=1$  are examples of production functions that are homogeneous of degree 1 and we have already verified that  $\gamma(w)$  is linear in these cases.

**3. The decentralized equilibrium**

We characterize the decentralized equilibrium in the following two steps. First, for a given distribution of the private input  $\mathbf{w} = \{w_i\}_{i=1..n}$ , we solve for the optimal contributions of each

agent,  $\hat{x}_i$ , the total contribution  $X$ , and the joint surplus,  $\Pi$ . Second, we look for the distributions of  $w_i$ , which maximize the total contribution and joint profits to be able to analyze the effects of inequality on these two variables.

### 3.1. Characterization of the Nash equilibrium for a given distribution of the private input

Let us consider the decentralized Nash equilibrium allocation. Player  $i$  takes the contribution of other players,  $X_{-i}$ , as given and solves:

$$\max_{x_i \geq 0} \pi_i = f(w_i, bx_i + cX) - x_i$$

The first-order conditions are

$$f_2(w_i, bx_i + cX)(b + c) - 1 \leq 0$$

$$x_i \geq 0$$

together with the standard Kuhn–Tucker complementary slackness condition. Let the function  $g(w_i) > 0$  denote the solution to<sup>16</sup>

$$f_2(w_i, g(w_i))(b + c) = 1$$

Notice that under Assumptions 1–3  $g(w)$  is increasing and concave in  $w$ .

Suppose  $m \leq n$  players contribute in equilibrium. By the assumed complementarity between  $w_i$  and  $z_i$ , for a given value of  $z_i$ ,  $f_2(w_i, z_i)$  is increasing in  $w_i$ , thus  $g(w_i)$  is increasing. In addition, by Assumption 2,  $\frac{\partial z_i}{\partial x_i} = b + c > 0$ . Therefore, regardless of the value of  $c$  (in particular, even if  $c < 0$ ), the richest player has the greatest marginal benefit from contributing, followed by the second richest, etc. In the case of pure public goods ( $b = 0$ ) the amount of the collective input enjoyed by each player is the same. Therefore, only the richest player contributes in equilibrium, i.e.,  $m = 1$ . In contrast, for impure public goods ( $b > 0$ ), the amount of the collective input,  $z_i$  consumed by each player is different. Thus, even though the second richest player’s marginal return from contributing is lower than that of the richest player, she can contribute less, enjoy a lower level of  $z$ , and still attain an interior equilibrium. As a result the set of contributors will consist of the  $m$  richest agents.

Let us denote by  $\hat{x}_i$  the optimal contribution of player  $i$  and let  $X = \sum_{i=1}^m \hat{x}_i$ . We assume that the wealth of the richest player exceeds some threshold level so that  $x_1 > 0$ . Given  $z_1 = g(w_1) > 0$  and Assumption 2, we will have  $x_1 = \frac{g(w_1)}{b+c} > 0$ .

The latter inequality implies that  $m \geq 1$ . Then, the equilibrium conditions for the optimal individual contributions are as follows:

$$\hat{x}_i = \begin{cases} \frac{g(w_i) - cX}{b}, & i = 1, \dots, m \\ 0, & i = m + 1, \dots, n. \end{cases} \tag{1}$$

$$X(m) = \frac{\sum_{i=1}^m g(w_i)}{b + mc} \tag{2}$$

$$g(w_m) \geq \frac{c \sum_{i=1}^m g(w_i)}{b + mc} > g(w_{m+1}) \tag{3}$$

<sup>16</sup> This function would be identical to  $\gamma(\cdot)$  defined in the previous section if  $b+c=1$  and it obviously inherits the properties of  $\gamma$  discussed above.



Condition (1) states that, for all contributing agents, the first order condition must hold as equality. Condition (2) is equivalent to  $X(m) = \sum_{i=1}^m \hat{x}_i$ . It states that the total contribution must have a value that is consistent with the individual maximization problems of all  $m$  contributors, and is obtained by adding up the  $m$  first-order conditions from Eq. (1). The third condition is the most interesting one, as it determines the size of the contributing group,  $m$ . To see why it should hold, note that the  $m+1$ -th agent would not contribute if  $f_2(w_{m+1}, cX(m))(b+c) < 1$ , since any further contribution has a marginal benefit that is lower than marginal cost. This condition is equivalent to  $cX(m) > g(w_{m+1})$ , which is exactly what the second inequality in Eq. (3) states. By the same logic, the  $m$ -th agent would not be contributing if  $g(w_m) < cX(m-1) = \frac{c \sum_{i=1}^{m-1} g(w_i)}{b+(m-1)c}$ , which can be rearranged as  $g(w_m) < \frac{c \sum_{i=1}^m g(w_i)}{b+mc}$ . Thus, if she is to contribute, the condition  $g(w_m) \geq \frac{c \sum_{i=1}^k g(w_i)}{b+mc}$  must be satisfied. The number of contributing agents,  $m$  is the smallest integer for which Eq. (3) is satisfied. If the left inequality in Eq. (3) holds for  $m=1, 2, \dots, n$ , then all agents contribute.

The following lemma, together with the fact that  $g$  is increasing, ensures the existence of a unique value of  $m$  that solves Eq. (3):

**Lemma 4.** *If  $k+1 \leq m$ , then the function  $s(k) = \frac{c \sum_{i=1}^k g(w_i)}{b+kc}$  is increasing in  $k$ . If  $k > m$  the function  $s(k)$  is decreasing in  $k$ .*

Several useful observations follow directly from Eqs. (1)–(3):

**Observation 1.** For the case of a pure public good ( $b=0, c \geq 0$ ) Eq. (3) cannot hold for  $m > 1$  as that would imply

$$g(w_m) \geq \frac{\sum_{i=1}^m g(w_i)}{m}$$

which is impossible given that  $w_1 > w_2 > \dots > w_n$  by assumption and Lemma 1 showing that  $g(\cdot)$  is increasing. Thus, for pure public goods only the richest player contributes. This has the implication that even when the difference in the wealth between the richest player and second richest player is arbitrarily small, the former provides the entire amount of the public good.<sup>17</sup>

**Observation 2.** For the case of pure private goods ( $c=0$ ), there is no interdependence across players and all of them will contribute.

**Observation 3.** For the case of commons ( $c < 0$ ), the expression  $\frac{c \sum_{i=1}^m g(w_i)}{b+mc}$  is negative for all  $m \leq n$  while  $g(w_i)$  are positive and hence all agents will always contribute.

**Observation 4.** For those player who contribute in equilibrium, Eq. (1) can be rewritten in the form of a reaction function:

$$\hat{x}_i = \frac{1}{b+c} \{g(w_i) - cX_{-i}\}, i = 1, \dots, m$$

where  $X_{-i} \equiv \sum_{j=1, j \neq i}^m \hat{x}_j$ . We show in the Appendix that the condition for stability of the Nash equilibrium is  $b+cm > 0$  which is ensured by Assumption 2. The individual contributions are

<sup>17</sup> Given this, it is possible that the richest agent is worse off than the second richest. See the discussion at the end of Section 3.2.

strategic complements for  $c < 0$  and strategic substitutes for  $c > 0$ . Formally, this follows from the fact that  $\frac{\partial^2 \pi}{\partial x_i \partial X_{-i}} = cf_{22}(w_i, bx_i + cX)(b + c)$ . Intuitively, the reason is that the contributions of various players are perfect substitutes in the payoff function, and in the case of public goods (commons) an increase in the contribution of others is similar to an increase (decrease) in the player’s own contribution, which reduces (increases) the marginal return of her contribution due to diminishing returns in the collective input  $z$ .

3.2. *Effect of wealth inequality on total contributions and joint profits*

From Eq. (2),

$$X = \frac{\sum_{i=1}^m g(w_i)}{b + mc} \equiv \tilde{g}(\mathbf{w})$$

Under Assumptions 1–3  $X = \tilde{g}(\mathbf{w})$  is the sum of  $m$  concave functions and as such is a concave function itself. Moreover, as these functions are identical and receive the same weight, if we hold the number of contributors constant the total contribution is maximized when all contributing agents have equal amounts of the private input. Therefore we have:

**Proposition 1.** *Suppose Assumptions 1–3 are satisfied. If  $g(w_i)$  is strictly concave in  $w_i$  then  $X$  is strictly concave in  $\mathbf{w}$  and is maximized when all contributing agents have equal amounts of the private input. If  $g(w_i)$  is linear in  $w_i$  then  $X$  is linear in  $\mathbf{w}$ .*

Recall that Assumption 3 implies that diminishing returns with respect to the collective input used by the  $i$ -th individual set in at a faster rate at a higher wealth level, and so the optimal level of the collective input is a concave function of the wealth level (Lemma 2). Proposition 1 follows from this assumption, and the fact that the collective input used by the  $i$ -th individual is a linear function of the individual’s own contribution and the contribution of other players.

To see this more clearly, consider the two-player version of the game where player 1 has wealth  $w + \varepsilon$  and player 2 has wealth  $w - \varepsilon$  where  $\varepsilon > 0$ . From the first order conditions of the two players:

$$\begin{aligned} g(w + \varepsilon) &= (b + c)x_1 + cx_2 \\ g(w - \varepsilon) &= cx_1 + (b + c)x_2 \end{aligned}$$

Therefore, the reaction functions are:

$$\begin{aligned} x_1 &= \frac{1}{b + c} \{g(w + \varepsilon) - cx_2\} \\ x_2 &= \frac{1}{b + c} \{g(w - \varepsilon) - cx_1\} \end{aligned}$$

Consider the effect of an increase in  $\varepsilon$ . The direct effect is to increase  $x_1$  and reduce  $x_2$ . For the case of positive externalities ( $c > 0$ ) the indirect effects move in the same direction, while for the case of negative externalities, the indirect effects move in the opposite direction. For example, in the former case, a reduction in  $x_2$  stimulates a further increase in  $x_1$  and an increase in  $x_1$  leads to a further decrease in  $x_2$ . The stability condition ensures that indirect effects in the successive rounds shrink in terms of size. Linearity of the reaction functions implies that the total effects of a change in  $\varepsilon$  on  $x_1$  and  $x_2$  are linear combinations of the direct effects on the two players. The fact that the reaction functions of both players have the same slope in our setup implies additionally that the

direct effects of redistribution on the contribution of the two players receive the same weight in  $\frac{dX}{de}$  and so it follows directly from the concavity of  $g(\cdot)$  that  $X$  is decreasing in  $\varepsilon$ .

The effect of wealth inequality on  $X$  has implications which are quite different from those available so far in the public economics literature. Our analysis shows that greater equality among those who contribute towards the collective good will increase the value of  $X$ . Therefore a more equal wealth distribution among contributors will increase the equilibrium level of the collective input.<sup>18</sup> In addition, any redistribution of wealth from non-contributors to contributors that does not affect the set of contributors will also increase  $X$ .<sup>19</sup> In terms of the two-player example, this implies that as long as both players contribute, any inequality in the distribution of wealth reduces  $X$ . But with sufficient inequality if one player stops contributing then any further increases in inequality will increase  $X$ .

Let us now turn to the normative implications of changes in the distribution of wealth. Under the first-best, which can be thought of a centralized equilibrium where players choose their contributions to maximize joint surplus, the first-order condition for player  $i$  is:

$$f_2(w_i, bx_i + cX)(b + nc) \leq 1.$$

The difference with the decentralized equilibrium is that now individuals look at the social marginal product of their contribution to the collective input, *i.e.*,  $f_2(w_i, bx_i + cX)(b + nc)$  as opposed to the private marginal product, *i.e.*,  $f_2(w_i, bx_i + cX)(b + c)$ . Then it follows directly that those who will contribute will contribute more (less) than in the decentralized equilibrium if  $c > 0$  ( $c < 0$ ). Also, the number of contributors will be higher (lower) than in the decentralized equilibrium if  $c > 0$  ( $c < 0$ ).

Therefore, for the case of positive externalities, the total contribution in a decentralized equilibrium is less than the efficient (*i.e.*, joint surplus maximizing) level. Conversely, for the case  $c < 0$ , total contributions exceed the socially efficient level. From this one might want to conclude that greater inequality among contributors increases efficiency in the presence of negative externalities and reduces efficiency if there are positive externalities.<sup>20</sup> Indeed, the literature on the effect of wealth (or income) distribution on collective action problems have typically focussed on the size of total contributions. However, that is inappropriate as the correct welfare measure is joint surplus.

In the presence of decreasing returns to scale the distribution of the private input across firms will have a direct effect on joint profits irrespective of its effect on the size of the collective input. In particular, greater inequality will reduce efficiency by increasing the discrepancy between the marginal returns to the private input across different production units. In the case of negative externalities, these two effects of changes in the distribution of the private input work in different directions, while in the case of positive externalities, they work in the same direction. We proceed to formally analyze this issue.

Using Conditions (1)–(3) agent  $i$ 's surplus is:

$$\begin{aligned} \pi_i(w_i, x_i, X) &= f(w_i, g(w_i)) - \frac{g(w_i) - cX}{b}, & i = 1..m \text{ (contributors)} \\ \pi_i(w_i, x_i, X) &= f(w_i, cX), & i = m + 1..n \text{ (non-contributors)} \end{aligned}$$

<sup>18</sup> See also Vicary (2006) who shows that equality is beneficial for public good provision in a different setting.

<sup>19</sup> In the above formula for  $X$ , holding  $m$  constant a redistribution from non-contributors to contributors will increase  $w_i$  ( $i=1, 2, \dots, m$ ) with the increase being strict for some  $i$ .

<sup>20</sup> Note however that a sufficiently large degree of inequality among contributors may reduce  $X$  below the first-best level in the  $c < 0$  case.

Joint surplus is given by:

$$\Pi = \sum_{i=m+1}^n f(w_i, cX) + \sum_{i=1}^m f(w_i, g(w_i)) - \frac{\sum_{i=1}^m g(w_i)}{b + mc}.$$

Let us denote the joint surplus of contributing players by  $\Pi^c \equiv \sum_{i=1}^m f(w_i, g(w_i)) - \frac{\sum_{i=1}^m g(w_i)}{b+mc}$  and that of non-contributors by  $\Pi^n \equiv \sum_{i=m+1}^n f(w_i, cX)$ .

First, consider the effect of distribution of wealth among non-contributors. This is trivial, since  $f(w_i, cX)$  is concave, and hence  $\sum_{i=m+1}^n f(w_i, cX)$  is concave as well. Therefore perfect equality of wealth among non-contributors will maximize their joint profits. Note that even if  $f$  is homogeneous of degree 1, this is still true. Next, let us consider the effect of distribution of wealth among contributors. Let

$$\tilde{\pi}(w) \equiv f(w, g(w)) - \frac{g(w)}{b + mc}.$$

Notice that  $\Pi^c \equiv \sum_{i=1}^m \tilde{\pi}(w_i)$ . The following lemma helps characterize the effect of wealth inequality on the contributors' total surplus,  $\Pi^c$ :

**Lemma 5.** *Suppose Assumptions 1–3 hold. Then:*

- (a) *If  $g(w)$  is concave and  $c \geq 0$ , or  $c < 0$  but  $|c|$  small, then  $\tilde{\pi}(w) \equiv f(w, g(w)) - \frac{g(w)}{b + mc}$  is concave.*
- (b) *If  $f$  is homogeneous of degree one then  $\tilde{\pi}(w)$  is linear in  $w$ .*

The intuition behind this result is the following. In the absence of externalities (*i.e.*,  $c=0$ ) if we want to find the effect of a change in  $w$  on the profit of a player, we can focus only on the direct effect and ignore the indirect effect *via* the envelope theorem. As a result, the second derivative of the profit function also depends only on the direct effect through  $w$ . In contrast, in the presence of externalities, we must take into account the indirect effect of  $w$  on  $X$  that affects other players. This residual term, which is a fraction of  $X$  (namely,  $\frac{1}{b+c} - \frac{1}{b+mc} = \frac{(m-1)c}{(b+c)(b+mc)}$ ) increases joint profits for  $c > 0$  and reduces them for  $c < 0$  compared to the case where  $c=0$ . Since we already know that  $X$  is concave, in the former case this reinforces the concavity of the joint profit function but in the latter case the effect goes the other way. As a result, for  $c < 0$  a sufficient condition to ensure concavity of  $\tilde{\pi}(w)$  is  $|c|$  to be small.

Lemma 5 implies immediately that for  $c \geq 0$  and for  $c < 0$  but  $|c|$  small,  $\Pi^c = \sum_{i=1}^m \tilde{\pi}(w_i)$  is concave in the wealth of contributors so that greater equality will increase joint profits. As a result, perfect equality of wealth among contributors maximizes their joint surplus. For the special case where  $f(w, z)$  is homogeneous of degree one  $\Pi^c$  is linear in the wealth of the contributors. In this case a redistribution of wealth among contributors will not affect joint surplus. However, equalizing wealth among non-contributors will still maximize  $\Pi^n$ . For  $c < 0$  but  $|c|$  large (while continuing to satisfy Assumption 1) we cannot determine the curvature of  $\Pi^c$  in general.

It turns out that, for the Cobb–Douglas case with decreasing returns,  $\tilde{\pi}(w)$  is strictly concave even if  $c < 0$  and  $|c|$  not necessarily very small. In this case,  $\tilde{\pi}(w) = w^{\frac{\alpha}{1-\beta}} [(b+c)\beta]^{\frac{1}{1-\beta}} \left( \frac{1}{\beta(b+c)} - \frac{1}{b+mc} \right)$ . In order for total contribution to be positive, we need  $\tilde{\pi}(w_1) > 0$ . This inequality holds if and only if  $b(1-\beta) > (\beta-m)c$ , which holds for  $c \geq 0$  or  $c < 0$  and  $|c| < \frac{b(1-\beta)}{m-\beta}$ . We assume this to be true — otherwise no player ever contributes. Hence  $\tilde{\pi}(w)$  is concave in  $w$  for  $\alpha + \beta < 1$ . In the case of constant returns, *i.e.*,  $\alpha + \beta = 1$ ,  $\tilde{\pi}(w)$  is linear.

Given the initial wealth distribution  $\mathbf{w}$ , there is some  $m \geq 1$  such that players with wealth  $w \geq w_m$  contribute and those with  $w \leq w_{m+1}$  do not. For this value of  $m$ , it is clear by the concavity of  $\Pi$ , that the wealth distribution maximizing joint surplus should have  $w_i = \hat{w}$  for all  $i = 1, \dots, m$  and  $w_j = \tilde{w} < \hat{w}$  for all  $j = m+1, \dots, n$ . subject to the following two conditions:

$$(n - m)\tilde{w} + m\hat{w} = W$$

$$g(\hat{w}) \geq \frac{cmg(\hat{w})}{b + mc} > g(\tilde{w}).$$

The first of the above conditions can be rewritten as:

$$\tilde{w} = \frac{W - m\hat{w}}{n - m}$$

Using this, the expression for joint profits becomes:

$$\Pi = (n - m)f\left(\frac{W - m\hat{w}}{n - m}, \frac{cmg(\hat{w})}{b + mc}\right) + m\left[f(\hat{w}, g(\hat{w})) - \frac{g(\hat{w})}{b + mc}\right]. \tag{4}$$

Also, the total contribution is  $X = m\frac{g(\hat{w})}{b+mc}$ . The following result characterizes the joint surplus maximizing wealth distribution for a given  $m$ .

**Proposition 2.** *Suppose Assumptions 1–3 are satisfied,  $c \geq 0$  and if  $c < 0$ ,  $|c|$  is small. For a given  $m$  the joint profit maximizing wealth distribution under private provision of the public good involves equalizing the wealths of all non-contributing players to some  $\tilde{w} > 0$  and also those of all contributing players to some  $\hat{w} > \tilde{w}$ .*

This result shows that maximum joint surplus is achieved for both contributors and non-contributors, if there is no intra-group inequality. This is a direct consequence of joint profit of each group being concave in the wealth levels of the group members. The contrast with the conclusions of both Olson and the distribution neutrality literature is quite sharp. The key assumptions leading to the result are, market imperfections that prevent the efficient allocation of the private input across production units, and some technical properties of the production function that are shared by widely used functional forms such as Cobb–Douglas and CES under decreasing returns to scale. With constant returns to scale, the joint profits within the group of contributors are independent of the distribution of wealth as in the distribution neutrality theorem.

In the above result we did not talk about inter-group inequality. Formally, we took  $m$  as given while considering alternative wealth distributions. An obvious question to ask is, what is the joint profit maximizing distribution of wealth when we can also choose the number of contributors,  $m$ . For example, does perfect equality among all players maximize joint surplus? This turns out to be a difficult question. Below we provide a partial answer to this question for the cases of both positive and negative externalities. Let us first look at the case of positive externalities ( $c > 0$ ). Suppose all players are contributing when wealth is equally distributed. Then from Proposition 2 we know that limited redistribution that does not change the number of contributors cannot improve efficiency. This immediately suggests the following result:

**Corollary to Proposition 2.** *Suppose all players contribute under perfect equality. Then, if after a redistribution all players continue to contribute, joint profits cannot increase.*

But suppose we redistribute wealth from one player to the other  $n - 1$  players up to the point where this player stops contributing. Recall that when the group size is  $m < n$ , then  $X = m\frac{g(\hat{w})}{b+mc}$ .

It is obvious that an increase in the average wealth of contributing players keeping the number of contributors fixed will increase  $X$ . It turns out that an increase in  $m$  holding the average wealth of contributors constant will always increase  $X$ .<sup>21</sup> However, if we simultaneously decrease  $m$  from  $n$  to  $n - 1$  and increase the average wealth of contributors, it is not clear whether  $X$  will go up or not. If  $X$  goes down then we can unambiguously say that joint profits are lower due to this redistribution (for  $c > 0$ ) since the effect of this policy on the efficiency of allocation of the private input across production units is definitely negative. However, if  $X$  goes up then there is a trade off: the increase in  $X$  benefits all players (since  $c > 0$ ), including the player who is too poor to contribute now, but this has to be balanced against the greater inefficiency in the allocation of the private input.

To analyze the effect of wealth distribution on joint profits when some players do not contribute we restrict attention to the comparison between joint profits under perfect equality (*i.e.*, when all players have wealth  $\bar{w} \equiv \frac{W}{n}$ ) and the wealth distribution that is obtained by a redistribution that leads to  $k$  contributing and  $n - k$  non-contributing players. From the discussion above, we know that under our assumptions all players contribute under perfect equality. We focus on studying only the efficient wealth distributions, *i.e.* ones which achieve maximum joint surplus. Since any intra-group inequality among the contributors and non-contributors reduces joint surplus we assume that all  $k$  contributors have wealth  $\bar{w} + \frac{\varepsilon}{k}$  and all  $n - k$  non-contributors have wealth  $\bar{w} - \frac{\varepsilon}{n-k}$  after the redistribution. Let us denote by  $\Pi^E$  the joint surplus under perfect equality and with  $\Pi^I(\varepsilon)$  the one under the unequal wealth distribution of the above type. Let also total wealth be normalized to  $n\bar{w}$ . A player stops contributing if

$$g\left(\bar{w} - \frac{\varepsilon}{n - k}\right) < cX = \frac{kcg\left(\bar{w} + \frac{\varepsilon}{k}\right)}{b + kc} \tag{5}$$

Let  $\varepsilon^*$  be defined as the indifference point between contributing and not contributing, *i.e.* the solution to

$$g\left(\bar{w} - \frac{\varepsilon^*}{n - k}\right) = \frac{kcg\left(\bar{w} + \frac{\varepsilon^*}{k}\right)}{b + kc} \tag{6}$$

Let  $\tilde{\varepsilon}$  denote the degree of inequality maximizing  $\Pi^I(\varepsilon)$  subject to  $\varepsilon \geq \varepsilon^*$ , *i.e.* when there are non-contributors in equilibrium.<sup>22</sup> The level of joint surplus when there are  $k$  contributors each with wealth  $\bar{w} + \frac{\tilde{\varepsilon}}{k}$  and  $(n - k)$  non-contributors each with wealth  $\bar{w} - \frac{\tilde{\varepsilon}}{n-k}$  is:

$$\begin{aligned} \Pi^I(\varepsilon) = & kf\left(\bar{w} + \frac{\varepsilon}{k}, g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) + (n - k)f\left(\bar{w} - \frac{\varepsilon}{n - k}, \frac{kc}{b + kc}g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) \\ & - \frac{kcg\left(\bar{w} + \frac{\varepsilon}{k}\right)}{b + kc}. \end{aligned}$$

<sup>21</sup> Formally, this is because  $\frac{m}{b+mc}$  is increasing in  $m$ . The intuition is, the new entrant to the group of contributors will contribute a positive amount, which would reduce the incentive of existing contributors to contribute due to diminishing returns. However, in the new equilibrium  $X$  must go up, as otherwise the original situation could not have been an equilibrium.

<sup>22</sup> Clearly, if  $\varepsilon < \varepsilon^*$  Eq. (5) cannot hold (since  $g$  is an increasing function and all players contribute).



Differentiating with respect to  $\varepsilon$  we get:

$$\begin{aligned} \frac{d\Pi^j(\varepsilon)}{d\varepsilon} = & \left[ f_1\left(\bar{w} + \frac{\varepsilon}{k}, g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) - f_1\left(\bar{w} - \frac{\varepsilon}{n-k}, \frac{kc}{b+kc}g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) \right] \\ & + f_2\left(\bar{w} - \frac{\varepsilon}{n-k}, \frac{kc}{b+kc}g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) \frac{(n-k)c}{b+kc}g'\left(\bar{w} + \frac{\varepsilon}{k}\right)f \\ & + g'\left(\bar{w} + \frac{\varepsilon}{k}\right) \left[ \frac{1}{b+c} - \frac{1}{b+kc} \right] \end{aligned} \tag{7}$$

where we used the fact that  $f_2\left(\bar{w} + \frac{\varepsilon}{k}, g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) = \frac{1}{b+c}$  by the definition of  $g$ . The following lemma helps us characterize the optimal degree of inequality:

**Lemma 6.**  $\frac{\partial \Pi^j(\varepsilon^*)}{\partial \varepsilon} < 0$  implies that  $\frac{\partial \Pi^j(\varepsilon)}{\partial \varepsilon} < 0$  for all  $\varepsilon \geq \varepsilon^*$  and so  $\tilde{\varepsilon} = \varepsilon^*$ . Conversely, if  $\frac{\partial \Pi^j(\varepsilon^*)}{\partial \varepsilon} > 0$  then  $\tilde{\varepsilon} > \varepsilon^*$ .

The above lemma implies that, if  $\frac{\partial \Pi^j(\varepsilon^*)}{\partial \varepsilon} < 0$ , then we have a corner solution, i.e.  $\Pi^j(\varepsilon)$  is maximized at  $\tilde{\varepsilon} = \varepsilon^*$ . Now we are ready to prove:

**Proposition 3.**

- (a) For pure public goods ( $b=0$  and  $c > 0$ ) perfect equality among the agents is never joint profit maximizing.
- (b) For pure private goods ( $b > 0$  and  $c=0$ ) perfect equality is always joint profit maximizing.

We noted a special property of pure public goods in the previous section (Observation 1), namely, even if the difference in the wealth between the richest player and the second richest player is arbitrarily small, the former provides the entire amount of the public good with everyone else free riding on her. This property is the key to explain why perfect equality is not joint profit maximizing in this case. Start with a situation where all players except for one have the same wealth level, and this one player has a wealth level which is higher than that of others by an arbitrarily small amount. As a result this player is the single contributor to the public good. A small redistribution of wealth from other players to this player, keeping the average wealth of the other players constant, will have three effects on joint profits: the effect due to the worsening of the allocation of the private input, the effect of the increase in  $X$  on the payoff of the non-contributing players, and the effect of the increase in  $X$  on the payoff of the single contributing player. The result in the proposition follows from the fact that the first effect is negligible since by assumption the extent of wealth inequality is very small, the second effect is positive, and the third effect can be ignored by the envelope theorem. It should also be noted that this result goes through for both constant and decreasing returns to scale. The second part of Proposition 3 follows from the fact that when  $c=0$  a player will always choose  $x_i > 0$  however small her wealth level. Then all players are contributors so long as they have non-zero wealth and it follows directly from Proposition 2 that perfect equality will maximize joint profits.

For the case of impure public goods ( $b, c > 0$  under decreasing returns to scale we can provide only a local characterization:

**Proposition 4.** Consider the case of impure public good subject to decreasing returns to scale, i.e.  $c > 0$  and suppose that Assumptions 1–3 hold. Then:

- (a) Given  $c, n$  there exist some  $b_1, b_2 > 0$  such that for all  $b \in [\bar{b}_2, \infty)$  perfect equality is always joint profit maximizing, whereas for  $b \in [0, \bar{b}_1]$  perfect equality is never joint profit maximizing.

(b) Given  $b, n$  there exists some  $\bar{c} > 0$  such that for all  $c \in (0, \bar{c}]$  perfect equality is always joint profit maximizing.

Two opposing forces are at work in this case — the “decreasing returns to scale” effect calling for equalizing the wealth of agents and the “dominant player” effect due to the positive externality calling for re-distribution towards the richest players as there is a positive effect on the payoffs of the non-contributing players. Each of the two effects can dominate the other depending on the parameter values. The direct effect of an increase in the richer player contribution on her own payoff can once again be ignored by the envelope theorem.

While we cannot provide a full characterization of the case of decreasing returns, due to the existence of two opposing forces, we can provide some illustrative examples using the Cobb–Douglas production function  $f(w, z) = w^\alpha z^\beta$  for a two player game. In Figs. 1 and 2 we plot how the difference between joint profits under perfect equality and under inequality (where the degree of inequality is chosen to maximize joint profits given that only one player contributes) vary with  $b$  and  $c$  for several alternative sets of values of  $\alpha$  and  $\beta$ . As we can see from the figures: (a) there is a unique  $\bar{b}$  such that  $\Pi^E \geq \Pi^I(\bar{\varepsilon})$  for  $b \geq \bar{b}$  and  $\Pi^E < \Pi^I(\bar{\varepsilon})$  for  $b < \bar{b}$ ; and (b) there is a unique  $\bar{c}$  such that  $\Pi^E \geq \Pi^I(\bar{\varepsilon})$  for  $c \leq \bar{c}$  and  $\Pi^E < \Pi^I(\bar{\varepsilon})$  for  $c > \bar{c}$ .

We now turn to the case of negative externalities ( $b > 0, c < 0$ ) under decreasing returns to scale. We show that:

**Proposition 5.** Consider the case of commons, ( $c < 0$ ) under decreasing returns to scale and suppose that Assumptions 1–3 hold. Then there exist two critical values of  $c, c_1 < c_0 < 0$  such that:

- (a) For  $c \in (c_0, 0)$  perfect equality is a local maximum of the joint profit function.
- (b) For  $c \in (-\frac{b}{n}, c_1)$  perfect equality is never joint profit maximizing.

In the commons case, Assumption 1 implies that all agents contribute. Notice that then we can write the joint profits function as:

$$\Pi = k\hat{\pi}(w_1) + (n - k)\hat{\pi}(w_2) + \frac{(n - 1)c}{(b + c)}X$$

where  $\hat{\pi}(w) \equiv f(w, g(w)) - \frac{g(w)}{b+c}$ ,  $w_1$  is the wealth of rich players and  $w_2$  is the wealth level of poor players. Notice also that  $\hat{\pi}'(w) = f_1(w, g(w)) + (f_2(w, g(w)) - \frac{1}{b+c})g'(w) = f_1(w, g(w))$  from the definition of  $g(w)$ . Therefore, given the definition of  $g(w)$  and the concavity of  $f$ ,  $\hat{\pi}$  is strictly concave. On the other hand, the term  $\frac{(n-1)c}{(b+c)}X$  is convex given that  $c < 0$  and Proposition 1. Intuitively, joint profits are the sum of individual profits ignoring the externality of a player’s action on others, and the sum total of the externality terms. The former is concave in the wealth distribution but in the case of negative externalities, the latter is convex. For  $c$  small enough (in absolute value) the decreasing returns to scale effect dominates, i.e. joint profits are maximized at perfect equality but for  $c$  large (in absolute value) the “cost of negative externality” term, which is convex, dominates and so greater inequality leads to higher joint profits.

Finally, we turn to the case of constant returns to scale. In this case we are able to provide a complete characterization of how joint profits depend on the wealth distribution for the whole range of collective action problems under our formulation:

**Proposition 6.** If the production function displays constant returns to scale then:

- (a) Perfect equality is never joint profit maximizing for impure public goods (i.e.,  $c > 0$ ). Moreover, it is possible to have inequality Pareto-dominate perfect equality.

(b) In the pure private good case ( $c=0$ ) and in the commons case ( $c<0$ ) joint profits are independent of the wealth distribution.

This result is driven by two forces: the linear “demand” for the collective input (coming from CRS) and the fact that all agents contribute under for  $c \leq 0$ . Under CRS the desired (equilibrium) levels of the collective good are linear in the wealths of contributors and so total provision is distribution neutral. Since the production function is homogeneous of degree one in the private input and the collective input, this means output and hence surplus is also linear in the wealth levels of contributors. In the case of pure private goods and the case of commons all agents are contributors and that is why distribution neutrality obtains overall. In the case of pure and impure public goods ( $c > 0$ ) there are some non-contributors in equilibrium and the non-contributors’ welfare is non-linear in their wealth. In particular, a redistribution from non-contributors to contributors increases the level of the collective input and therefore, overall surplus.

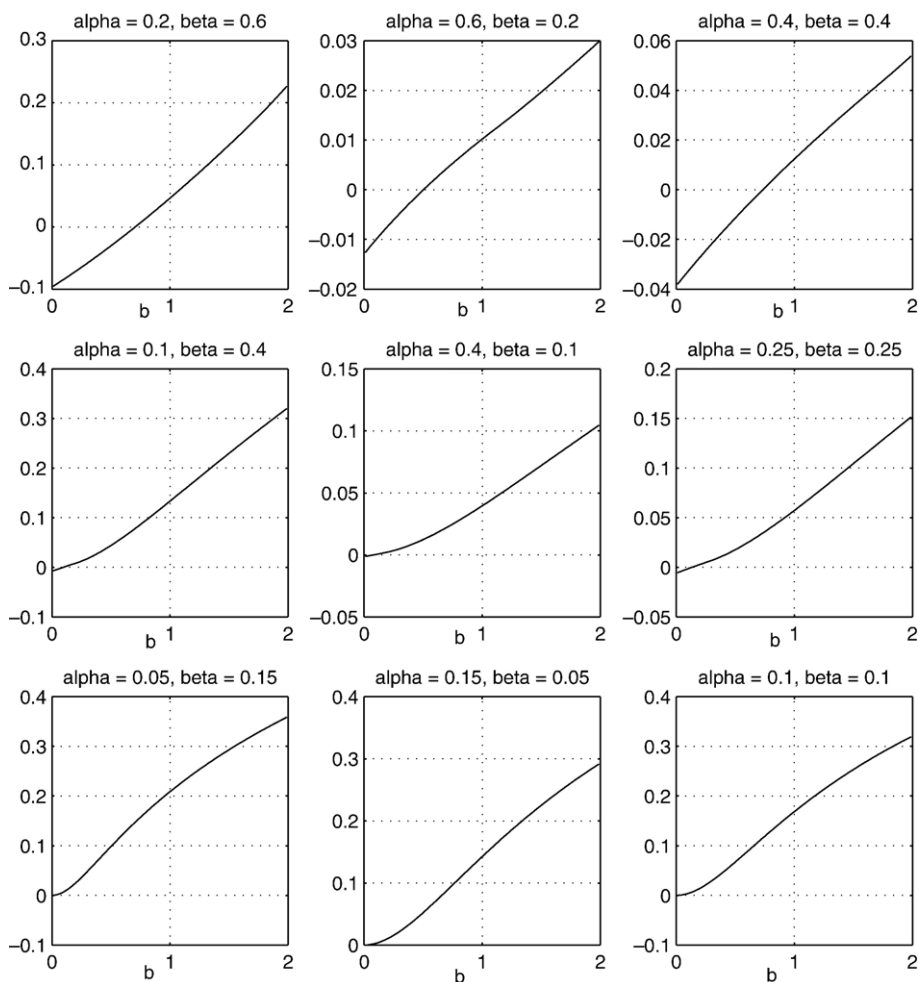


Fig. 1. Difference in joint surplus under perfect equality and optimal inequality.

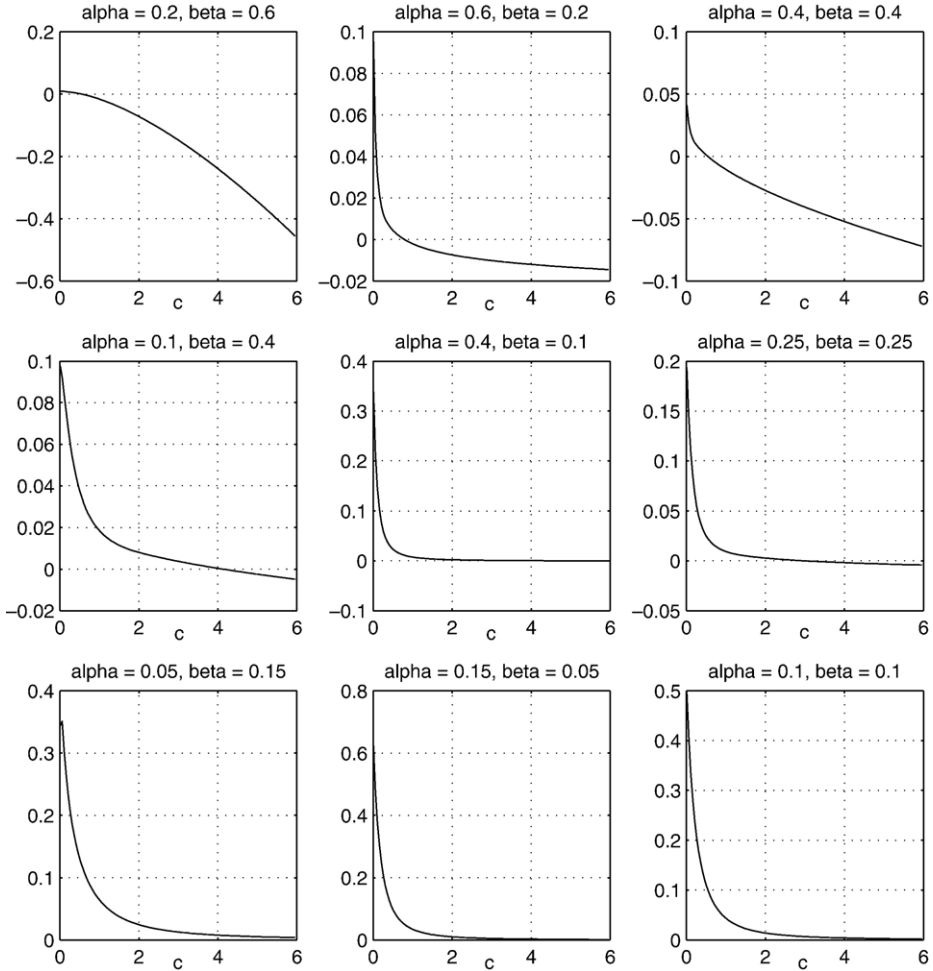


Fig. 2. Difference in joint surplus under perfect equality and optimal inequality.

The logic for part (a) is similar to that of the pure public goods case. In that case the richest person is the only contributor even when the wealth difference between him and the second richest player is very small and therefore a small amount of inequality does not result in large losses due to the inefficient allocation of the private input. With constant returns to scale and for impure public goods, the difference between the wealth level of the contributors and non-contributors need not be very small. However, joint profits of contributors depend only on their total wealth and not how it is distributed. As a result, creating some inequality from the point where only player is exactly indifferent between contributing and not, to the point where she strictly prefers not to contribute, involves a small loss due the inefficient allocation of the private input. Unlike the pure public goods case, this could involve a significant amount of inequality relative to the perfectly equal wealth distribution.

In part (a) we showed that, under some circumstances, it is possible to have some degree of inequality among agents Pareto-dominate the allocation under perfect equality. If we think of a two player set up, starting with perfect equality if we redistribute wealth from one player to the other, the poorer player is initially strictly better off than the rich player because she is free-riding

on the rich player who contributes most of the good and bears a large share of the costs. This is the starkest possible demonstration of what Olson called the “exploitation of the great by the small”.<sup>23</sup> However, if we continue increasing inequality, eventually the loss of the private input offsets the gain from free riding on the provision of the public good for the poorer player. This makes it possible that the two players get the same level of surplus at some positive level of inequality and that this surplus is higher than the level they get at perfect equality.

#### 4. Extensions

##### 4.1. Convertibility between the private input and contribution to the collective input

It is important for our result that  $x_i$  and  $w_i$  are different types of goods and one cannot be freely converted into the other. Suppose the individual can freely allocate a fixed amount of wealth between two uses, namely, as a private input and as her contribution to the collective input. This is the formulation chosen by the literature on distribution-neutrality (e.g., Warr, 1983; Bergstrom et al., 1986; Cornes and Sandler, 1996; Itaya et al., 1997). This literature focuses on pure public goods, i.e., where  $z_i = cX$ . For ease of comparability, let us consider this case first. Let  $k_i$  denote the amount of the private input chosen by player  $i$ . Then player  $i$ 's decision problem is to maximize  $f(k_i, cX)$  with respect to  $k_i$  and  $x_i$  subject to the budget constraint  $k_i + x_i \leq w_i$ . The first-order condition of an individual who contributes a positive amount in equilibrium is

$$f_1(k_i, cX) = cf_2(k_i, cX), \quad i = 1, 2, \dots, m.$$

As  $k_i = w_i - x_i$  from the budget constraint of the individual, and  $x_i + X_{-i} = X$  for all  $i = 1, 2, \dots, m$ , this condition implicitly defines the following function:

$$w_i - x_i = h(X).$$

Summing across all players who contribute in equilibrium, we get  $X + mh(X) = W$ . This equation can be solved for  $X$  which therefore depends only on total wealth,  $W$  and not on its distribution. Joint profits will also be independent of the distribution of wealth.

The above formulation is similar to that of a consumer allocating a fixed amount of money to alternative goods in order to maximize utility. An alternative formulation to capture free convertibility between  $k_i$  and  $x_i$  is to pose the problem as that of a firm maximizing profits by choosing inputs which can be sold or purchased from the market at a given price. One could think of  $k_i$  as capital which has a given price  $r$  such that a firm that has an excess of it (relative to its endowment  $w_i$ ) can sell it to other firms, and a firm that has a shortage of it can buy it at the same price, say  $r$ . Similarly, one can think of  $x_i$  as labor that can be used to contribute towards the collective input, or sold in the labor market at price  $w$ .<sup>24</sup> Now the first order condition of a contributing player,  $i$ , is

$$f_1(k_i, cX) = \frac{r}{w} cf_2(k_i, cX), \quad i = 1, 2, \dots, m.$$

This condition is the same as in the previous formulation, except for the multiplicative constant  $\frac{r}{w}$  and so the distribution neutrality result goes through.

<sup>23</sup> See also Cornes and Sandler (2000) who derive a similar result for the pure public goods case in the standard distribution-neutrality setting. In their framework, among other conditions, the number of players needs to be at least three for the result to go through, whereas in our framework there is no such restriction on the number of players.

<sup>24</sup> In our framework labor is not directly used in production. We can think of another sector which uses labor. Alternatively, we can extend the basic model by adding labor as a third input. The distribution neutrality result will go through.

Turning now to impure public goods, *i.e.*, where  $b > 0$ , the first order condition for player  $i$  according to the first formulation is:

$$f_1(k_i, bx_i + cX) = (b + c)f_2(k_i, bx_i + cX), \quad i = 1, 2, \dots, m.$$

It is clear that in general the distribution neutrality result will not go through now. It will go through only for some special cases, such as when  $f(w, z)$  is homothetic. In this case, the values of  $k_i$  and  $z_i$  at a point of individual optimum satisfy the condition

$$\frac{k_i}{bx_i + cX} = A$$

where  $A$  is a positive constant. It is readily verified that the distribution neutrality result holds in this case. In general, our analysis shows that, relaxing the assumption of perfect convertibility of the private input and the contribution to the collective input implies that the distribution neutrality result no longer holds. Specifically, greater equality among contributors always improves efficiency for impure public goods (*i.e.*,  $c > 0$ ) while for collective inputs subject to negative externalities, the effect of inequality on efficiency is ambiguous. In the latter case, we characterize the conditions under which we can sign the effect of inequality on efficiency. Our results do not depend on the production functions being homothetic, but in the general case even with free convertibility, distribution neutrality can break down if the collective input is not a pure public good, as is well recognized in the literature (see for example, Bergstrom et al., 1986; Cornes and Sandler, 1996).

#### 4.2. Substitutability between the private and the collective input

Above, we assumed that the private input and the public good are complements in the production function. In this section we examine the implications of these being substitutes. For simplicity, we examine the case where  $w$  and  $z$  are perfect substitutes:  $\pi(w_i, x_i, X) = f(w + bx_i + cX) - x_i$ , where  $f$  is increasing and strictly concave and  $b$  and  $c$  satisfy Assumption 2. The first order conditions for the agent's problem is:

$$(b + c)f'(w_i + bx_i + cX) \leq 1$$

with strict equality when  $x_i > 0$ . Let us denote by  $w^*$  the solution to  $f'(w) = \frac{1}{b+c}$ , which exists and is unique given our assumptions. In contrast to the complements case, it is now the poorest player who has the highest marginal product of contributing. In the pure public good case ( $b=0$ ) the poorest player will be the only contributor if  $w_n < w^*$  and if  $w_n \geq w^*$  the public good will not be provided at all.

As before, joint surplus goes up if wealth is equally distributed among non-contributors. Also, we cannot say for sure whether the optimal distribution of wealth involves perfect equality, or some inequality among the contributor (the poorest agent) and the rest. This is clearly seen for the case of the pure public good ( $b=0$ ). For simplicity, suppose there are two players with wealth levels  $w_1 = \bar{w} + \varepsilon$  and  $w_2 = \bar{w} - \varepsilon$  and, in addition assume that  $c=1$ . Now joint profits are:  $\Pi(\varepsilon) = f(w^*) - \{w^* - (\bar{w} - \varepsilon)\} + f(\bar{w} + \varepsilon + w^* - (\bar{w} - \varepsilon)) = f(w^*) - w^* + \bar{w} + f(2\varepsilon + w^*) - \varepsilon$  and so  $\Pi'(\varepsilon) = 2f'(2\varepsilon + w^*) - 1$ . We know that  $f'(2\varepsilon + w^*) - 1 \leq 0$  since by definition  $f'(w^*) - 1 = 0$  but whether  $2f'(2\varepsilon + w^*) - 1 \leq 0$  or  $> 0$  cannot be determined *a priori*.

For the intuition behind this result, notice that, those who choose  $x_i > 0$ , *i.e.*, the poorest players, use the efficient amount of the input. Other players have more than the efficient level of the input in their production units. Any redistribution from the poor to the rich players does not



affect the profit of the former as they exactly compensate for this by increasing their contribution. Since rich players have more than the efficient level of the input in their firms, normally a transfer of an additional unit of wealth would reduce joint profits since the marginal gain to the rich player is less than the marginal cost to the poor player. But every extra unit of wealth received by the rich player increases the input received by her firm by *twice* the amount because of the increase in the effort by the poor player and as a result it is not clear whether joint profits increase or decrease.

#### 4.3. Complementarity between the individual relative contribution and the total contribution

Above we studied the case where the player's own contribution and the total contribution of all players are perfect substitutes in determining the benefit from the collective input enjoyed by a player,  $z_i$ . In this section we consider an alternative formulation where they can be complements:

$$z_i = \left(\frac{x_i}{X}\right)^\theta X^\gamma$$

where  $0 \leq \theta \leq 1$  and  $0 \leq \gamma \leq 1$ . According to this specification, each player not only gains from the total contribution, but her gains are greater, the greater her contribution is relative to the total. This induces people to choose a higher level of  $x_i$  which benefits others through the term  $X^\gamma$ . But it also reduces how much others can enjoy the collective good by a congestion effect captured by the term  $\left(\frac{x_i}{X}\right)^\theta$ . If the latter effect is unimportant compared to the former, then we have a public good and indeed for  $\theta=0$  we have the textbook case of a pure public good. But if it is the other way around then the congestion effect dominates the beneficial externality effect and in the limit, for  $\theta=1$  we have the textbook case of the commons. When these two effects exactly balance each other out ( $\theta=\gamma$ ), we have the case of the a pure private good.

Analytically, this case turns out to be quite hard to characterize even when we assume a specific form of the production function, namely Cobb–Douglas, and consider a two player game. We have shown that<sup>25</sup> if we compare the allocations under perfect equality (both players have the same level of wealth) and perfect inequality (one player has all the wealth and the other player has nothing) joint surplus is always higher under perfect equality for non-negative externalities (*i.e.*,  $\theta \geq \gamma$ ). However, if there are substantial negative externalities then under some parameter values joint surplus will be higher under perfect inequality. The intuition for this result lies in the fact that, when the negative externality problem is very severe, then under perfect equality the players choose their actions related to the collective input at too high a level relative to the joint surplus maximizing solution. Perfect inequality converts the model to a one player game and hence eliminates this problem. On the other hand, due to joint diminishing returns to the private input and the collective input, joint surplus is lower under perfect inequality compared to perfect equality if there were no externalities. What this result tells us is that perfect inequality is desirable only when the negative externality problem is severe and when the extent of diminishing returns is not too high.

If, instead of comparing the allocations under perfect equality and perfect inequality, we consider the effects of a continuous change in inequality on total contributions and joint profits, the results are not clear-cut. We prove that, in the case of commons, its total use ( $X$ ) decreases with increasing wealth inequality and joint profits per unit of total contributions (*i.e.*,  $\Pi/X$ ), or what one may call the average rate of return on the collective input, increases with inequality. But

<sup>25</sup> The proofs of all the results in this section are in the working paper version available from the authors upon request.

the absolute level of joint profits may increase or decrease with inequality. Numerical simulations suggest that joint profits in general decrease with inequality, except for the case of substantial negative externalities. In the case of public goods (pure and impure), we prove that the average rate of return on the public good input decreases with inequality. But as the extent of positive externalities become large (approaching the pure public goods case) the total amount of public good provision (and the absolute amount of the joint profits) may increase with inequality. However there exists a range of moderate presence of positive externalities such that total contributions as well as joint profits decrease with inequality.

## 5. Concluding remarks

In this paper we analyzed the effect of inequality in the distribution of endowment of private inputs that are complementary in production with collective inputs (*e.g.*, contributions to public goods such as irrigation and extraction from common-property resources) on efficiency in a class of collective action problems. In an environment where transaction costs prevent the efficient allocation of private inputs across individuals, and the collective inputs are provided in a decentralized manner, we have characterized the optimal second-best distribution of the private input. We show that while efficiency increases with greater equality *within* the group of contributors and non-contributors, in some situations there is an optimal degree of inequality *between* the groups.

The link between inequality and collective action identified here suggests an alternative mechanism<sup>26</sup> that could help explain the relationship between initial wealth inequality and the pace and pattern of economic development.<sup>27</sup> Indeed, empirical studies that show that reducing inequality (*e.g.*, through land reform) can have large positive productivity effects have suggested that this is likely to be due partly to better resolution of collective action problems (Banerjee et al., 2002).

Some limitations of our model suggest several directions of potentially fruitful research. First, our model is static. It is important to extend it to the case where both the wealth distribution and the efficiency of collective action are endogenous. For example, it is possible to have multiple stationary states with high (low) wealth inequality leading to low (high) incomes to the poor due to low (high) level of provision of public goods, which *via* low (high) mobility can sustain an unequal (equal) distribution of wealth. Also, in the dynamic case it could be interesting to analyze the effects of inequality on the sustainability of cooperation in a situation of repeated games. Second, technological non-convexities and differential availability of exit options seriously affect collective action in the real world, while our model ignores them.<sup>28</sup> For example, the public good may not be generated if the total amount of contribution is below a certain threshold (*e.g.* Cornes, 1993). This is the case for renewable resources like forests or fishery where a minimum stock is necessary for regeneration, or in the case of fencing a common pasture. Third, the empirical literature suggests that even when the link between inequality and collective action is consistent with our results, the mechanisms involved may be quite different in some cases. For example, transaction costs in conflict management and costs of negotiation may be higher in situations of higher inequality. Fourth, following the public economics

<sup>26</sup> Other alternatives include borrowing constraints (*e.g.*, Galor and Zeira, 1993; Banerjee and Newman, 1993) or political economy (Alesina and Rodrik, 1994).

<sup>27</sup> See Benabou (1996) and Banerjee and Duflo (2003) for empirical evidence on this.

<sup>28</sup> The model of Dayton-Johnson and Bardhan (2002) examines the effect of inequality on resource conservation with two periods and differential exit options for the rich and the poor in the case when technology is linear. Baland and Platteau (1997) discuss the effect of non-convex technology in a static model.

literature, in this paper we focus mainly on the free-rider problem arising in a collective action setting. Hence, the main issue is the sharing of the costs of collective action. But there is another problem, often called the “bargaining problem”, whereby collective action breaks down because the parties involved cannot agree on the sharing of the benefits.<sup>29</sup> Inequality matters in this problem as well. For example, bargaining can break down when one party feels that the other party is being unfair in sharing the benefits (there is ample evidence for this in the experimental literature on ultimatum games).

More generally, social norms of cooperation and group identification may be difficult to achieve in highly unequal environments. Putnam (1993) in his well-known study of regional disparities of social capital in Italy points out that “horizontal” social networks (*i.e.*, those involving people of similar status and power) are more effective in generating trust and norms of reciprocity than “vertical” ones. Knack and Keefer (1997) also find that the level of social cohesion (which is an outcome of collective action) is strongly and negatively associated with economic inequality. Finally, we focus only on the voluntary provision of public goods and do not consider the possibility that the players might elect a decision maker who can tax them and choose the level of provision of the collective good. The role of inequality in such a framework is an important topic for future research.<sup>30</sup>

### Appendix A

**Proof of Lemma 1.** Consider the first order condition,  $f_2(w, z) - 1 = 0$ . By Assumption 1,  $f_2(w, z) > 0$  for all  $w > 0$  and  $\lim_{w \rightarrow 0} f_2(w, z) = 0$ . Therefore,  $\gamma(w) > 0$  for all  $w > 0$ . By concavity, a global maximum exists and  $f_{22}(w, z) < 0$ . By definition,  $f_2(w, \gamma(w)) - 1 = 0$ . Notice that under our assumptions  $\gamma(w)$  is differentiable, and hence continuous. In particular,  $\frac{d\gamma(w)}{dw} = -\frac{f_{12}}{f_{22}} > 0$ .  $\square$

**Proof of Lemma 2.** By the definition of  $h(w, z)$ ,  $h(w, \gamma(w)) = 1$ . Totally differentiating with respect to  $w$  we get  $h_1 + h_2 \frac{d\gamma(w)}{dw} = 0$ , or,  $\frac{d\gamma(w)}{dw} = -\frac{h_1}{h_2}$ . Notice that  $h_1 = \frac{\partial^2 f(w, z)}{\partial z \partial w} \geq 0$  (as  $w$  and  $z$  are complements) and  $h_2 = \frac{\partial^2 f(w, z)}{\partial z^2} < 0$  (by strict concavity). Differentiating again with respect to  $w$  we get:

$$\frac{d^2 \gamma(w)}{dw^2} = -\frac{h_1^2 h_{22} + h_2^2 h_{11} - 2h_1 h_2 h_{12}}{h_2^3}$$

The condition  $\frac{d^2 \gamma(w)}{dw^2} \leq 0$  is equivalent to the determinant  $\begin{vmatrix} 0 & h_1 & h_2 \\ h_1 & h_{11} & h_{12} \\ h_2 & h_{12} & h_{22} \end{vmatrix}$  being  $\leq 0$  which in turn is equivalent to  $h(w, z)$  being quasi-concave (see Theorem 21.20 of Simon and Blume, 1994).  $\square$

**Proof of Lemma 3.** Since  $f(w, z)$  is homogeneous of degree 1,  $f_2(w, z)$  is homogeneous of degree 0. If  $\lambda > 0$ ,  $f_2(\lambda w, \lambda \gamma(w)) = f_2(w, \gamma(w))$ . Since by definition  $f_2(w, \gamma(w)) = 1$ , so  $f_2(\lambda w, \lambda \gamma(w)) = f_2(w, \gamma(w)) = 1$ . Then it must be true that  $\gamma(\lambda w) = \lambda \gamma(w)$  which means  $\gamma(w) = Aw$  where  $A > 0$  is a constant.  $\square$

**Proof of Lemma 4.** Since agent  $k+1$  contributes a positive amount by assumption,  $g(w_{k+1}) > \frac{c \sum_{i=1}^k g(w_i)}{b+kc}$ . Straightforward algebra shows that this is equivalent to the inequality  $\frac{c \sum_{i=1}^{k+1} g(w_i)}{b+(k+1)c} > \frac{c \sum_{i=1}^k g(w_i)}{b+kc}$ . The second part of the lemma is proved in the same way.  $\square$

<sup>29</sup> See for example, Elster (1989).

<sup>30</sup> Olszewski and Rosenthal (2004) address this question for pure public goods within the framework of the distribution neutrality literature.

**Proof of Lemma 5.**

(a) Totally differentiating with respect to  $w$  we get:

$$\frac{\partial \tilde{\pi}(w)}{\partial w} \equiv f_1(w, g(w)) + \left( f_2(w, g(w)) - \frac{1}{b + mc} \right) g'(w).$$

From the definition of  $g(w)$  and the first-order condition of a contributing player,  $f_2(w, g(w)) = \frac{1}{b+c}$ . Substituting in, we get

$$\frac{\partial \tilde{\pi}(w)}{\partial w} \equiv f_1(w, g(w)) + \frac{(m - 1)c}{(b + c)(b + mc)} g'(w).$$

Totally differentiating once again with respect to  $w$ :

$$\frac{\partial^2 \tilde{\pi}(w)}{\partial w^2} \equiv f_{11}(w, g(w)) + f_{12}(w, g(w))g'(w) + \frac{(m - 1)c}{(b + c)(b + mc)} g''(w).$$

From the Proof of Lemma 1,  $g'(w) = -\frac{f_{12}}{f_{22}}$ . Therefore,  $f_{11} f_{11} + f_{12}g'(w) = \frac{f_{11}f_{22} - f_{12}^2}{f_{22}^2} < 0$  since  $f(w, z)$  is concave. Therefore,  $\frac{\partial^2 \tilde{\pi}(w)}{\partial w^2}$  is negative if one of the following holds: (i)  $g(w)$  is concave and  $c > 0$ ; (ii)  $c = 0$  or (iii)  $c < 0$  and  $|c|$  small.

(b) The second part of the lemma follows from the fact that if  $f(w, z)$  is homogeneous of degree one, then  $g(\cdot)$  is linear and  $\tilde{\pi} = f(\lambda w, \lambda g(w)) - \frac{g(\lambda w)}{b+mc} = \lambda \left[ f(w, g(w)) - \frac{g(w)}{b+mc} \right]$  is linear as well.  $\square$

**Proof of Proposition 2.** For a given value of  $m$  it follows from the concavity of the profit functions of both contributors and non-contributors that there should not be any intra-group heterogeneity. Also,  $\hat{w} > \tilde{w}$  given that contributors must be richer than non-contributors (see Eqs. (1)–(3)). It is never optimal to set  $\tilde{w}$  at a very low level given the Inada endpoint conditions, namely,  $\lim_{\tilde{w} \rightarrow 0} f_1(\tilde{w}; cX) = \infty$ . Since  $\hat{w} > \tilde{w}$  it would never be optimal to make  $\hat{w}$  arbitrarily small, since that would mean  $\tilde{w}$  would be even smaller and almost all of  $W$  would be left unused.  $\square$

**Proof of Lemma 6.** From Lemma 5, we know that the joint profit of contributing players is concave in  $\varepsilon$ . Also, it can be directly verified that the joint profit of non-contributors is concave in  $\varepsilon$ . Differentiating the terms in Eq. (7) that relate to non-contributing players and using the superscript  $n$  to denote these players we get

$$\begin{aligned} & \frac{1}{n - k} f_{11}^n - \frac{c}{b + kc} g' \left( \bar{w} + \frac{\varepsilon}{k} \right) f_{12}^n \\ & + f_2^n \left( \bar{w} - \frac{\varepsilon}{n - k}, \frac{kc}{b + kc} g \left( \bar{w} + \frac{\varepsilon}{k} \right) \right) \frac{1}{k} \frac{(n - k)c}{b + kc} g'' \left( \bar{w} + \frac{\varepsilon}{k} \right) \\ & + \frac{(n - k)c}{b + kc} g' \left( \bar{w} + \frac{\varepsilon}{k} \right) \left\{ -\frac{1}{n - k} f_{21}^n + f_{22}^n \frac{c}{b + kc} g' \left( \bar{w} + \frac{\varepsilon}{k} \right) \right\}. \end{aligned}$$

This expression is negative since all the terms are negative. Therefore  $\Pi^I(\varepsilon)$  is concave in  $\varepsilon$  and so

$$\frac{\partial \Pi^I(\varepsilon)}{\partial \varepsilon} \leq 0 \text{ as } \varepsilon \geq \tilde{\varepsilon}.$$

The claim in the lemma follows directly from the above.  $\square$

**Proof of Proposition 3.**

- (a) If  $b=0$  Eq. (6) implies that  $\varepsilon^*=0$  i.e. any degree of inequality can be sustained in an equilibrium with non-contributors. Consider the derivative in Eq. (7) evaluated at  $\varepsilon=0$  (i.e. around the point of perfect equality). We have that:

$$\frac{d\Pi^I(0)}{d\varepsilon} = \frac{d\Pi^I(\varepsilon^*)}{d\varepsilon} = f_2\left(\bar{w}, \frac{kc}{b+kc}g(\bar{w})\right) \frac{(n-k)c}{b+kc}g'(\bar{w}) + g'(\bar{w}) \left[ \frac{c(k-1)}{(b+c)(b+kc)} \right] > 0$$

as all the terms are positive. Then by Lemma 6,  $\tilde{\varepsilon} > \varepsilon^*=0$ , i.e., perfect equality is never joint profit maximizing in the case of pure public goods.

- (b) In the case of pure private goods ( $c=0$ ), Eq. (3) is clearly satisfied for any redistribution of wealth among the agents, i.e. all of them always contribute. But then it follows directly from Proposition 2 that greater inequality reduces joint profits.  $\square$

**Proof of Proposition 4.**

- (a) Differentiating both sides of Eq. (6) with respect to  $b$  we get:

$$g'\left(\bar{w} - \frac{\varepsilon^*}{n-k}\right) \left(-\frac{1}{n-k}\right) \frac{\partial \varepsilon^*}{\partial b} = \frac{kcg'\left(\bar{w} + \frac{\varepsilon^*}{k}\right) \frac{1}{k} \frac{\partial \varepsilon^*}{\partial b} (b+kc) - kcg\left(\bar{w} + \frac{\varepsilon^*}{k}\right)}{(b+kc)^2}$$

i.e.

$$\frac{\partial \varepsilon^*}{\partial b} = \frac{kcg\left(\bar{w} + \frac{\varepsilon^*}{k}\right)}{cg'\left(\bar{w} + \frac{\varepsilon^*}{k}\right)(b+kc) + g'\left(\bar{w} - \frac{\varepsilon^*}{n-k}\right) \frac{1}{n-k} (b+kc)^2} > 0$$

Therefore,  $w_1 \equiv \bar{w} + \frac{\varepsilon^*}{k}$  is increasing in  $b$  and  $w_2 \equiv \bar{w} - \frac{\varepsilon^*}{n-k}$  is decreasing in  $b$ . Given the definition of  $\varepsilon^*$ , and the fact that  $f_2(z, g(z)) = \frac{1}{b+c}$  we get

$$f_2\left(\bar{w} - \frac{\varepsilon^*}{n-k}, \frac{kc}{b+kc}g\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\right) = f_2\left(\bar{w} - \frac{\varepsilon^*}{n-k}, g\left(\bar{w} - \frac{\varepsilon^*}{n-k}\right)\right) = \frac{1}{b+c}$$

Therefore Eq. (7) evaluated at  $\varepsilon^*$  can be written as:

$$\frac{d\Pi^I(\varepsilon^*)}{d\varepsilon} = \frac{(n-1)c}{(b+c)(b+kc)}g'(w_1) + f_1(w_1, g(w_1)) - f_1(w_2, g(w_2)). \tag{8}$$

Note that when  $b \rightarrow \infty$ ,  $\varepsilon^* \rightarrow (n-k)\bar{w}$ . Therefore,  $w_1 \rightarrow \frac{n}{k}\bar{w}$  and  $w_2 \rightarrow 0$ . From the Inada conditions  $f_1(w_2, g(w_2)) \rightarrow \infty$  and thus  $\frac{d\Pi^I(\varepsilon^*)}{d\varepsilon} = -\infty < 0$  while the other two terms in the above expression are finite and non-negative. Since the function  $\Pi^I(\varepsilon^*)$  is continuous, this proves that there exists some  $\bar{b}_2 > 0$  (which would, in general, depend on  $n, k$  and  $c$ ) such

that for  $b \in [\bar{b}_2, \infty)$ ,  $\frac{d\Pi^I(\varepsilon^*)}{d\varepsilon} < 0$ , implying that  $\Pi^I(\varepsilon)$  is maximized at  $\varepsilon^*$  (the minimum degree of inequality needed to have non-contributing agents in equilibrium), *i.e.* the problem has a corner solution. Let

$$\begin{aligned} \Pi^C\left(\bar{w} + \frac{\varepsilon}{k}, \bar{w} - \frac{\varepsilon}{n-k}\right) &\equiv k \left[ f\left(\bar{w} + \frac{\varepsilon}{k}, g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) - \frac{g\left(\bar{w} + \frac{\varepsilon}{k}\right)}{b + nc} \right] \\ &+ (n-k) \left[ f\left(\bar{w} - \frac{\varepsilon}{n-k}, g\left(\bar{w} - \frac{\varepsilon}{n-k}\right)\right) - \frac{g\left(\bar{w} - \frac{\varepsilon}{n-k}\right)}{b + nc} \right] \end{aligned}$$

denote joint profits when there are  $k$  agents with wealths  $\bar{w} + \frac{\varepsilon}{k}$  and  $n-k$  agents with wealths  $\bar{w} - \frac{\varepsilon}{n-k}$ , all of whom contribute. From the definition of  $\varepsilon^*$ ,  $\Pi^C\left(\bar{w} + \frac{\varepsilon^*}{k}, \bar{w} - \frac{\varepsilon^*}{n-k}\right) = \Pi^I(\varepsilon^*)$ . Also,  $\Pi^E > \Pi^C\left(\bar{w} + \frac{\varepsilon^*}{k}, \bar{w} - \frac{\varepsilon^*}{n-k}\right)$  from Proposition 2. Thus perfect equality maximizes joint surplus for large enough values of  $b$ . From Proposition 3, part (a) and the continuity of the joint profit function, we know that there exists  $\bar{b}_1 > 0$  such that if  $b \in [0, \bar{b}_1]$  such that perfect equality is never joint profit maximizing.

- (b) The proof is very similar to that of part (a). Differentiating both sides of Eq. (6) with respect to  $c$  we get:

$$\begin{aligned} &g'\left(\bar{w} - \frac{\varepsilon^*}{n-k}\right) \left(-\frac{1}{n-k}\right) \frac{\partial \varepsilon^*}{\partial c} \\ &= \frac{\left[ kcg'\left(\bar{w} + \frac{\varepsilon^*}{k}\right) \frac{1}{k} \frac{\partial \varepsilon^*}{\partial c} + kg\left(\bar{w} + \frac{\varepsilon^*}{k}\right) \right] (b + kc) - k^2cg\left(\bar{w} + \frac{\varepsilon^*}{k}\right)}{(b + kc)^2} \end{aligned}$$

*i.e.*

$$\frac{\partial \varepsilon^*}{\partial c} = \frac{-bkg\left(\bar{w} + \frac{\varepsilon^*}{k}\right)}{kcg'\left(\bar{w} + \frac{\varepsilon^*}{k}\right)(b + kc) + g'\left(\bar{w} - \frac{\varepsilon^*}{n-k}\right) \frac{1}{n-k}(b + kc)^2} < 0.$$

The above implies that  $w_1 \equiv \bar{w} + \frac{\varepsilon^*}{k}$  is decreasing in  $c$  and  $w_2 \equiv \bar{w} - \frac{\varepsilon^*}{n-k}$  is increasing in  $c$ . At  $c=0$  we have (assuming  $g(0)=0$ ),  $\varepsilon^* = (n-k)\bar{w}$ ,  $w_1 = \frac{n}{k}\bar{w}$  and  $w_2=0$ . Now let us look at Eq. (8) once again. From the Inada conditions  $f_1(w_2, g(w_2))=\infty$  and thus  $\frac{d\Pi^I(\varepsilon^*)}{d\varepsilon} = -\infty < 0$  as the other two terms in Eq. (8) are finite and non-negative (the first term is actually 0). Since  $f$  is concave by assumption,  $f_1(z, g(z))$  is decreasing in  $z$  and as  $w_1$  is decreasing in  $c$  the second term above is increasing in  $c$ . Similarly, as  $-f_1(z, g(z))$  is increasing in  $z$  and  $w_2$  is increasing in  $c$ , the third term is increasing in  $c$  as well. The latter imply that  $f_1(w_1, g(w_1)) - f_1(w_2, g(w_2)) < 0$  and it increases towards zero as  $c \rightarrow \infty$  (Fact A). By a continuity argument, the above shows that there exists some  $\bar{c} > 0$  (depending on  $n, k, b$ ) such that for  $c \in [0, \bar{c}]$ ,  $\frac{d\Pi^I(\varepsilon^*)}{d\varepsilon} < 0$ , *i.e.*  $\Pi^I(\varepsilon)$  is maximized at  $\varepsilon^*$  (the minimum degree of inequality needed to have non-contributing agents) as. But  $\Pi^E > \Pi^C\left(\bar{w} + \frac{\varepsilon^*}{k}, \bar{w} - \frac{\varepsilon^*}{n-k}\right) = \Pi^I(\varepsilon^*)$  and so perfect equality maximizes joint surplus for small  $c$ . Unfortunately, we cannot obtain in general a limiting result for  $c \rightarrow \infty$  as the term in Eq. (8) containing  $g'$  is positive but decreasing in  $c$  towards zero which combined with Fact A does not necessarily imply the existence of some  $c_2 > 0$  such that  $\frac{d\Pi^I(\varepsilon^*)}{d\varepsilon} > 0$  for  $c > c_2$ .  $\square$

**Proof of Proposition 5.** From Assumption 1 we know that  $f_2(w, z)=\infty$  as  $z$  approaches 0 from above. Therefore all agents contribute in equilibrium, *i.e.*  $m=n$ . Also we know that  $X$  is



maximized when wealths are equalized as it is equal to  $\frac{\sum g(w_i)}{b+nc}$ . The individual contributions then equal:

$$x_i = \frac{g(w_i) - cX}{b} = \frac{(b + (n - 1)c)}{b(b + nc)}g(w_i) - \frac{c}{b(b + nc)} \sum_{j \neq i} g(w_j) > 0$$

as  $c < 0$  and  $b + (n - 1)c > b + nc > 0$  by Assumption 2. Since all agents contribute joint surplus equals:

$$\Pi = \sum f(w_i, g(w_i)) - \frac{\sum g(w_i)}{b + nc}.$$

Let us start at perfect equality, i.e.  $w_i = w$  and consider a redistribution giving  $k$  of the agents  $w + \frac{\varepsilon}{k}$  and the rest  $w - \frac{\varepsilon}{n-k}$ ,  $\varepsilon > 0$ . We then have:

$$\begin{aligned} \Pi &= k \left[ f\left(\bar{w} + \frac{\varepsilon}{k}, g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) - \frac{g\left(\bar{w} + \frac{\varepsilon}{k}\right)}{b + nc} \right] \\ &+ (n - k) \left[ f\left(\bar{w} - \frac{\varepsilon}{n - k}, g\left(\bar{w} - \frac{\varepsilon}{n - k}\right)\right) - \frac{g\left(\bar{w} - \frac{\varepsilon}{n - k}\right)}{b + nc} \right]. \end{aligned}$$

Let us see how a change in  $\varepsilon$  affects joint profits:

$$\begin{aligned} \frac{\partial \Pi}{\partial \varepsilon} &= f_1\left(\bar{w} + \frac{\varepsilon}{k}, g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) + g'\left(\bar{w} + \frac{\varepsilon}{k}\right) \left[ f_2\left(\bar{w} + \frac{\varepsilon}{k}, g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) - \frac{1}{b + nc} \right] \\ &- f_1\left(\bar{w} - \frac{\varepsilon}{n - k}, g\left(\bar{w} - \frac{\varepsilon}{n - k}\right)\right) \\ &- g'\left(\bar{w} - \frac{\varepsilon}{n - k}\right) \left[ f_2\left(\bar{w} - \frac{\varepsilon}{n - k}, g\left(\bar{w} - \frac{\varepsilon}{n - k}\right)\right) - \frac{1}{b + nc} \right] \end{aligned}$$

We have  $f_2(z, g(z)) = \frac{1}{b+c}$  from the first-order conditions. So:

$$\begin{aligned} \frac{\partial \Pi}{\partial \varepsilon} &= \left[ f_1\left(\bar{w} + \frac{\varepsilon}{k}, g\left(\bar{w} + \frac{\varepsilon}{k}\right)\right) - f_1\left(\bar{w} - \frac{\varepsilon}{n - k}, g\left(\bar{w} - \frac{\varepsilon}{n - k}\right)\right) \right] \\ &+ \frac{(n - 1)c}{(b + c)(b + nc)} \left[ g'\left(\bar{w} + \frac{\varepsilon}{k}\right) - g'\left(\bar{w} - \frac{\varepsilon}{n - k}\right) \right] \end{aligned} \tag{9}$$

Evaluating the above at  $\varepsilon = 0$ , we have that

$$\frac{\partial \Pi}{\partial \varepsilon} \Big|_{\varepsilon=0} = 0$$

i.e.  $\varepsilon = 0$  is a critical point for the joint surplus function. Denote  $w_1 = \bar{w} + \frac{\varepsilon}{k}$  and  $w_2 = \bar{w} - \frac{\varepsilon}{n-k}$ . The second derivative of  $\Pi$  is:

$$\begin{aligned} \frac{\partial^2 \Pi}{\partial \varepsilon^2} &= \frac{1}{k} [f_{11}(w_1, g(w_1)) + f_{12}(w_1, g(w_1))g'(w_1)] + \frac{1}{n - k} [f_{11}(w_2, g(w_2)) \\ &+ f_{12}(w_2, g(w_2))g'(w_2)] + \frac{(n - 1)c}{(b + c)(b + nc)} \left[ \frac{1}{k} g''(w_1) + \frac{1}{n - k} g''(w_2) \right]. \end{aligned}$$

At  $\varepsilon=0$  the above equals:

$$\frac{n}{k(n-k)} \left[ f_{11}(\bar{w}) + f_{12}(\bar{w})g'(\bar{w}) + \frac{(n-1)c}{(b+c)(b+nc)}g''(\bar{w}) \right].$$

The first term within the brackets is negative (recall from the Proof of Lemma 5 that  $f_{11}f_{11} + f_{12}g'(w) = \frac{f_1 f_{22} - f_{12}^2}{f_{22}} < 0$  as  $f(z, w)$  is concave) but the second term is positive. Therefore we cannot sign the derivative in general. For  $c \rightarrow 0$ , however, we know it is going to be negative by the concavity of  $f$ , i.e.  $\varepsilon=0$  is a local maximum. Recall that by Assumption 2,  $b + nc > 0$ , or  $c > -\frac{b}{n}$ . Suppose  $c$  is large enough in absolute value such that  $b + nc$  is close enough to 0. Then the last term within the square brackets becomes arbitrarily large and so  $\frac{\partial^2 \Pi}{\partial \varepsilon^2} > 0$  i.e.  $\varepsilon=0$  is a local minimum. Therefore by a continuity argument, if  $c$  is close to zero, i.e. for all  $c$  in some interval  $[c_0, 0)$  perfect equality is locally joint profit maximizing. If however  $c$  is large in absolute value, i.e.  $c \in [-\frac{b}{n}, c_1)$  and so  $b + cn$  close to 0, the second term above is arbitrarily large and therefore joint profits are maximized at some positive degree of inequality.  $\square$

**Proof of Proposition 6.**

- (a) Since we assume constant returns to scale, it follows from Lemma 3 that  $g(w)=Aw$ , where  $A$  is a positive constant. A player stops contributing if

$$A\left(\bar{w} - \frac{\varepsilon}{n-k}\right) < cX = \frac{kcA\left(\bar{w} + \frac{\varepsilon}{k}\right)}{b+kc}$$

Consider the derivative in Eq. (7) evaluated at  $\varepsilon^*$ . The second and third terms are clearly positive. Because of constant returns to scale,  $f_1(\lambda w, \lambda z)=f_1(w, z)$  for  $\lambda > 0$ . Notice also that from the definition of  $\varepsilon^*$ :

$$\frac{kc}{b+kc} \left(\bar{w} + \frac{\varepsilon^*}{k}\right) = \bar{w} - \frac{\varepsilon^*}{n-k}.$$

From this equation we can solve for  $\varepsilon^*$ :

$$\varepsilon^* = \frac{b\bar{w}(n-k)}{cn+b}.$$

But then the first term is:

$$\begin{aligned} & f_1\left(\bar{w} + \frac{\varepsilon^*}{k}, g\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\right) - f_1\left(\bar{w} - \frac{\varepsilon^*}{n-k}, \frac{kc}{b+kc}g\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\right) \\ &= f_1\left(\bar{w} + \frac{\varepsilon^*}{k}, g\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\right) - f_1\left(\frac{kc}{b+kc}\left(\bar{w} + \frac{\varepsilon^*}{k}\right), \frac{kc}{b+kc}g\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\right) = 0 \end{aligned}$$

This implies that  $\Pi^I(\varepsilon)$  is increasing in a neighborhood of  $\varepsilon^*$  and thus it achieves a maximum for some  $\bar{\varepsilon} > \varepsilon^*$ , i.e.  $\max_{\varepsilon \in [\varepsilon^*, \bar{w}(n-k)]} \Pi^I(\varepsilon) > \Pi^I(\varepsilon^*)$ . Now let us prove that inequality is always joint profit maximizing. Joint profits under perfect equality when  $f(w, z)$  is homogeneous of degree one is

$$\Pi^E(\bar{w}) = n \left[ f(\bar{w}, g(\bar{w})) - \frac{g(\bar{w})}{b+nc} \right] = n\bar{w} \left[ f(1, A) - \frac{A}{b+nc} \right].$$

Also, joint profits under an unequal wealth distribution such that there are  $k$  agents with wealths  $\bar{w} + \frac{\varepsilon^*}{k}$  and  $n-k$  agents with wealths  $\bar{w} + \frac{\varepsilon^*}{n-k}$  in which only the former group contributes is:

$$\begin{aligned} \Pi^E(\varepsilon^*) &= k \left[ f\left(\bar{w} + \frac{\varepsilon^*}{k}, g\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\right) - \frac{g\left(\bar{w} + \frac{\varepsilon^*}{k}\right)}{b + kc} \right] \\ &\quad + (n - k)f\left(\bar{w} - \frac{\varepsilon^*}{n - k}, \frac{kc}{b + kc}g\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\right) = kf\left(\bar{w} + \frac{\varepsilon^*}{k}, A\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\right) \\ &\quad + (n - k)f\left(\bar{w} - \frac{\varepsilon^*}{n - k}, A\left(\bar{w} - \frac{\varepsilon^*}{n - k}\right)\right) - \frac{kA\left(\bar{w} + \frac{\varepsilon^*}{k}\right)}{b + kc} \\ &= k\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\left(f(1, A) - \frac{A}{b + kc}\right) + (n - k)\left(\bar{w} - \frac{\varepsilon^*}{n - k}\right)f(1, A) \\ &= n\bar{w}f(1, A) - k\left(\bar{w} + \frac{\varepsilon^*}{k}\right)\frac{A}{b + kc}. \end{aligned}$$

Using the value of  $\varepsilon^*$  we get:

$$\begin{aligned} \frac{k\bar{w} + \frac{b\bar{w}(n-k)}{cn+b}}{b + kc} &= \frac{k(b + cn) + b(n - k)}{(b + kc)(b + cn)}\bar{w} = \frac{n(b + kc)}{(b + kc)(b + cn)}\bar{w} \\ &= \frac{n}{(b + cn)}\bar{w}. \end{aligned}$$

*i.e.*  $\Pi^I(\varepsilon^*) = \Pi^E$ . Therefore,  $\max_{\varepsilon} \Pi^I(\varepsilon) > \Pi^I(\varepsilon^* = \Pi^E)$  and thus some degree of inequality (with  $\varepsilon > \frac{b(n-k)}{b+nc}\bar{w}$ ) is joint profit maximizing.

For the second part of Proposition 6 (a), it is sufficient to provide an example. Suppose  $f(w, z)$  has the Cobb–Douglas, constant returns to scale form  $f(w, z) = w^\alpha z^{1-\alpha}$  and there are two agents in the economy<sup>31</sup> with endowments of the private input  $w + \varepsilon$  and  $w - \varepsilon$ , where  $\varepsilon \in [0, w]$ . By Lemma 3 we have  $g(w) = Aw$  and  $f(w, g(w)) = A^{1-\alpha}w$ , where  $A = [(b + c)(1 - \alpha)]^{\frac{1}{1-\alpha}}$ . Under perfect equality each player obtains a surplus of:

$$\pi^E = wA^{1-\alpha} \left[ 1 - \frac{A^\alpha}{b + 2c} \right] = wA^{1-\alpha} \left[ \frac{c + \alpha(b + c)}{b + 2c} \right] \tag{10}$$

Let, as in the proof of Proposition 5,  $\varepsilon^* = \frac{bw}{b+2c}$  be the degree of inequality at which the poorer player is just indifferent between contributing and not contributing. Thus, for  $\varepsilon \in (\varepsilon^*, w]$  only the richer player (*i.e.* the one with endowment  $w + \varepsilon$ ) would contribute and her profits would be given by:

$$\begin{aligned} \pi^{\text{rich}}(\varepsilon) &= A^{1-\alpha}(w + \varepsilon) - \frac{A(w + \varepsilon)}{b + c} = A^{1-\alpha}\alpha(w + \varepsilon) \\ \pi^{\text{poor}}(\varepsilon) &= (w - \varepsilon)^\alpha(w + \varepsilon)^{1-\alpha} \left( \frac{Ac}{b + c} \right)^{1-\alpha} \end{aligned}$$

<sup>31</sup> We have actually proven the proposition for any  $f(w, z)$  satisfying Assumptions 1–3 and any redistribution in which  $k$  agents obtain  $w + \varepsilon$  and  $n - k$  obtain  $w - \varepsilon$  but the expressions are much less tractable which is why we chose to present the result for a Cobb–Douglas function.

using the expression for  $\pi_i$  obtained previously. First notice that, evaluating the above expressions at  $\varepsilon = \varepsilon^*$  it is possible to have:

$$\pi^{\text{rich}}(\varepsilon^*) < \pi^{\text{poor}}(\varepsilon^*) \tag{11}$$

as it is equivalent to  $\alpha < \frac{c}{b+c}$ . Clearly, for  $\varepsilon > \varepsilon^*$   $\pi^{\text{rich}}(\varepsilon)$  is increasing in  $\varepsilon$ . We can verify directly that  $\frac{\partial \pi^{\text{poor}}(\varepsilon)}{\partial \varepsilon} < 0$  for  $\varepsilon > \varepsilon^*$  if  $\alpha > \frac{c}{b+2c}$ . In addition as  $\varepsilon \rightarrow w$ ,  $\pi^{\text{rich}}(\varepsilon)$  goes to some positive value, while  $\pi^{\text{poor}}(\varepsilon)$  goes to 0. Combining these results with Eq. (11) we see that there can exist some level of inequality  $\varepsilon_0 \in (\varepsilon^*, w)$  such that:

$$\pi^{\text{rich}}(\varepsilon_0) = \pi^{\text{poor}}(\varepsilon_0) \tag{12}$$

Using the expressions obtained above we can solve for  $\varepsilon_0$  to get:

$$\varepsilon_0 = \frac{w(1 - B)}{1 + B}$$

where  $B = \alpha^{\frac{1}{2}} \left( \frac{b+c}{c} \right)^{\frac{1-\alpha}{2}}$ . Finally, it is easy to verify that the condition

$$\pi^{\text{rich}}(\varepsilon_0) = \pi^{\text{poor}}(\varepsilon_0) > \pi^E$$

is equivalent to  $\frac{2\alpha(b+2c)}{c+\alpha(b+c)} > 1 + B$ . As long as this condition, and  $\frac{c}{b+2c} < \alpha < \frac{c}{b+c}$  hold simultaneously, we have an example where inequality Pareto-dominates perfect equality. For the case  $\alpha = 1/2$  the first condition is equivalent to  $c^2 > b^2$ , and the second one is equivalent to  $c > b > 0$ , i.e. if the latter is true inequality is Pareto dominating.

- (b) The result follows immediately from Lemma 5 and the fact that for  $c \leq 0$  all agents are contributors (see Observations 2–3).  $\square$

**Stability of equilibrium**

The stability condition in Assumption 2 that  $b+nc \geq 0$  can be derived from a simple adjustment mechanism of the following form:

$$\frac{dx_i}{dt} = \mu_i(\hat{x}_i - x_i(t)), i = 1, 2, \dots, m$$

where  $\mu_i$  are positive constants,  $x_i(t)$  is the actual value of  $x_i$  at time  $t$ , and  $\hat{x}_i$  is the reaction function as given by Eq. (1). Given that reaction functions are linear in our model, the condition for stability is equivalent to the following determinant of order  $m$

$$\begin{vmatrix} b+c & c & & c \\ c & b+c & & c \\ & & \dots & \\ \cdot & \cdot & & \cdot \\ c & c & & b+c \end{vmatrix}$$

being positive definite. Performing some simple operations to make all elements in the first row (or column) except for the first two to be equal to zero, we can prove by induction that the value of this determinant is equal to  $b^{m-1}(b+mc)$ .

### The CES example

For the CES production function:

$$f(w, z) = (\delta w^\rho + (1 - \delta)z^\rho)^{\frac{k}{\rho}}$$

we show that if  $0 < \rho < k \leq 1$  then  $\gamma(w)$  is increasing and concave. First we need to ensure that  $f$  is concave and  $w$  and  $z$  are complements. The condition for non-increasing returns is  $k \leq 1$ , since  $f(\lambda w, \lambda z) = \lambda^k f(w, z)$ . The condition for  $f_{12} > 0$  is  $k > \rho$ . The first order condition of maximization is:

$$(\delta w^\rho + (1 - \delta)\gamma(w)^\rho)^{\frac{k}{\rho}-1} \gamma(w)^{\rho-1} = \frac{1}{k(1 - \delta)(b + c)}.$$

Differentiating with respect to  $w$  and using the first order condition we get:

$$\gamma'(w) = \frac{(k - \rho)\delta w^{\rho-1} \gamma(w)}{(1 - k)(1 - \delta)(\gamma(w))^\rho + \delta(1 - \rho)w^\rho}.$$

As  $k > \rho$  by assumption the numerator is positive. Also, the denominator is positive as  $1 - k \geq 0$  and  $\rho \in (0, 1)$  and  $\delta \in (0, 1)$ . Therefore  $\gamma(\cdot)$  is increasing. Observe that  $\frac{w\gamma'(w)}{\gamma(w)} = \frac{(k - \rho)\delta w^\rho}{(1 - k)(1 - \delta)(\gamma(w))^\rho + \delta(1 - \rho)w^\rho} \leq 1$  since the numerator is less than the second term in the denominator (which follows from  $k \leq 1$ ). Differentiating the expression for  $\gamma'(w)$ , the sign of  $\gamma''(w)$  turns out to be the same as that of the following expression:

$$(1 - \rho)\{(1 - k)(1 - \delta)w^{\rho-2}\gamma(w)^{\rho+1} + \delta w^{2\rho-2}\gamma(w)\} \left\{ \frac{w\gamma'(w)}{\gamma(w)} - 1 \right\}.$$

This expression is non-negative under our assumptions, and the fact that  $\frac{w\gamma'(w)}{\gamma(w)} \leq 1$ . For  $k = 1$ ,  $\frac{w\gamma'(w)}{\gamma(w)} = 1$  and so the expression is equal to 0. Therefore  $\gamma(w)$  is concave, and strictly so for  $k < 1$ .  $\square$

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