General Equilibrium Theory*

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How do markets work? Do they organize production well? Are market outcomes fair? If not, how can we improve them? These questions have been around since the days of Adam Smith, who talked about the "invisible hand" of markets, where self-interested individuals create efficient outcomes.

The idea behind the invisible hand metaphor is that the price system plays the crucial role of coordinating decisions: a high price signals scarcity, encouraging consumers to consume less and producers produce more. This is easily seen in models of a single good ("partial equilibrium"), where a price that equates supply and demand ensures that there is neither excess production nor shortage, maximizing "total surplus".

With multiple goods, things are less clear. In general, changing the price of one good affects the demand for other goods. Are there prices that balance supply and demand simultaneously in all markets? And if there are, do they produce desirable outcomes? To explore these questions, we need a model that takes into account multiple markets simultaneously. This is known as a model of "general equilibrium".

The first general equilibrium model was formulated by Walras (1874). Unfortunately, the mathematics of the time did not allow him or his contemporaries to answer the questions above. They were answered by authors such as Lerner (1934), Wald (1936), Lange (1942), Allais (1943), Arrow and Debreu (1954), McKenzie (1954), Gale (1955), Nikaido (1956), Negishi (1959), and Aumann (1966), with progressively greater generality and abstraction. The Walrasian model of general equilibrium played a key role in the so-called "socialist calculation debate", which discussed the relative merits of market and planned economies.

General equilibrium models are now widely used in economics. They allowed economists to formalize arguments from Ricardo (1817) and constitute the benchmark models of trade. General equilibrium models have also become workhorse models in macroeconomics, finance,

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and public finance. More recently, the field of market design has rediscovered and built on many old results from general equilibrium theory. In particular, even in markets without prices (such as the allocation of organs or school admissions), many intuitions from market economies can be used.

In these notes, I present the main ideas from general equilibrium. We will start with the basic model and progressively discuss different interpretations and applications.[1]

Contents

1 Walrasian Model **3**

2 Efficiency **4**

2.1 Pareto Efficiency 4

2.2 Social Welfare 7

2.3 Differentiable Approach 11

3 Risk **15**

4 Competitive Equilibrium: Definition and Walras' Law **17**

4.1 Gross Substitutes, Uniqueness, and Stability 22

4.2 Differentiable Approach to Competitive Equilibrium 27

4.3 "Anything Goes" Results 33

5 Welfare Theorems **34**

5.1 Differentiable Approach to Welfare Theorems 36

6 Financial Securities **38**

6.1 Basic Model 38

6.2 Equilibrium 41

7 Appendix: Proof of Existence **44**

[1]Each section begins with formal abstract definitions, followed by examples and exercises. I will post solutions to the exercises, but it is important that you work on them before looking at the solutions. When linear algebra notation is used (especially in Subsection 2.3 and Section 6), it may be helpful for you to write down each coordinate of vectors and matrices.

1 Walrasian Model

We will consider pure exchange economies, meaning that we will abstract from the production side of the economy. This setting is general enough to highlight the main insights from the theory, although our analysis can be extended to include production (and you will see simple general equilibrium models with production in your macro classes). The physical description of the economy has two components: consumers and goods. There are $I \in \mathbb{N}$ consumers (sometimes referred to as individuals) and $N \in \mathbb{N}$ goods (sometimes called commodities).

Individual i's consumption of good n is denoted x_n^i. Consumption cannot be negative: $x_n^i \geq 0$ for each i, n. The vector of individual i's consumption of each good, $\mathbf{x}^i = (x_1^i, x_2^i ..., x_N^i) \in \mathbb{R}_+^N$, is called a *consumption bundle*. An *allocation* is a matrix $\mathbf{x} = \left(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^I\right) \in \mathbb{R}_+^{N \times I}$ that gives a consumption bundle for each consumer.

Each consumer i has preferences over consumption bundles $\succsim_i$ that can be represented by a continuous utility function $u_i : \mathbb{R}_+^N \to \mathbb{R}$. Note that we are assuming the domain of each individual's preferences is the individual's own consumption only. In principle, and perhaps more realistically, one could extend the domain to include the consumption of others, capturing different forms of externalities, altruism, and envy.[2]

Consumer i's preferences are (weakly) *monotone* if $\hat{\mathbf{x}}^\mathbf{i} \succ \mathbf{x}^\mathbf{i}$ whenever $\hat{\mathbf{x}}^\mathbf{i} >> \mathbf{x}^\mathbf{i}$.[3] That is, preferences are monotone if consuming more of every good makes the individual strictly better off. With monotonicity the consumer may not care about some goods. Consumer i's preferences are *strongly monotone* if $\hat{\mathbf{x}}^\mathbf{i} \succ \mathbf{x}^\mathbf{i}$ whenever $\hat{\mathbf{x}}^\mathbf{i} > \mathbf{x}^\mathbf{i}$. That is, preferences are strongly monotone if consuming more of any good makes the consumer strictly better off. Preferences are *locally non-satiated* if for any $\mathbf{x}^\mathbf{i} \in \mathbb{R}_+^N$ and any $\varepsilon > 0$, there exists $\hat{\mathbf{x}}^i$ in an ε-neighborhood of $\mathbf{x}^i$ such that $\hat{\mathbf{x}}^\mathbf{i} \succ \mathbf{x}^\mathbf{i}$. That is, local non-satiation says that for any consumption bundle $\mathbf{x}^i$, there exists another bundle that is close to $\mathbf{x}^i$ and that consumer i prefers over $\mathbf{x}^i$. Local non-satiation is a weak assumption that rules out bliss points.

We use a bar to denote aggregate quantities. The *aggregate consumption* of good n is the sum of the amounts of that good consumed by all individuals:

$$\bar{x}_n = \sum_{i=1}^{I} x_n^i.$$

The *aggregate consumption vector* is $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_N) \in \mathbb{R}_+^N$.

Aggregate consumption is limited by the total amount of resources available in the economy. The *aggregate endowment* of good n, $\bar{e}_n > 0$, gives the total amount of good n available.[4]

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[2]For externalities, see Exercise 13. For altruism and envy, see Dufwenberg, Heidhues, Kirchsteiger, Riedel, and Sobel (2011).

[3]For two vectors $\hat{x}^i$ and x^i, we write $\hat{x}^i >> x^i$ if $\hat{x}_n^i > x_n^i$ for all n. We write $\hat{x}^i > x^i$ if $\hat{x}_n^i \geq x_n^i$ for all n and $\hat{x}_n^i \neq x_n^i$.

[4]Although individuals may consume zero amounts of some goods, the aggregate endowment of each good

The *aggregate endowment vector* is $\bar{\mathbf{e}} = (\bar{e}_1, \bar{e}_2, ..., \bar{e}_N) \in \mathbb{R}^N_{++}$. We say that an allocation is *feasible* if, for each n,

$$\bar{x}_n \leq \bar{e}_n.$$

This notion of feasibility implicitly assumes free disposal: it is possible to consume less than the endowment.

Exercise 1. Determine if the preferences represented by the utility functions below are (i) monotonic, (ii) strongly monotonic, and (iii) locally-non-satiated.

a. Cobb-Douglas: $u(x_1, x_2) = x_1^a x_2^b$, where $a > 0$ and $b > 0$.

b. Leontief: $u(x_1, x_2) = \min\{x_1; x_2\}$.

c. One good, one bad: $u(x_1, x_2) = x_1 - x_2$.

d. Two bads: $u(x_1, x_2) = -x_1 - x_2$.

Exercise 2. Show that local non-satiation is strictly weaker than monotonicity, which is itself strictly weaker than strong monotonicity. That is, show that

a. monotonicity implies local non-satiation but the converse is not true, and

b. strong monotonicity implies monotonicity but the converse is not true.

2 Efficiency

2.1 Pareto Efficiency

Definition 1. Let $\mathbf{x}$ and $\hat{\mathbf{x}}$ be two allocations.

- $\mathbf{x}$ Pareto dominates $\hat{\mathbf{x}}$ if $\mathbf{x}^i \succsim_i \hat{\mathbf{x}}^i$ for all i with $\mathbf{x}^i \succ_i \hat{\mathbf{x}}^i$ for at least one i.
- $\mathbf{x}$ strictly Pareto dominates $\hat{\mathbf{x}}$ if $\mathbf{x}^i \succ_i \hat{\mathbf{x}}^i$ for all i.

The Pareto criterion captures the idea of unanimity.

Definition 2. Let $\mathbf{x}$ be a feasible allocation.

- $\mathbf{x}$ is Pareto efficient if it is not Pareto dominated by any feasible allocation.
- $\mathbf{x}$ is weakly Pareto efficient if it is not strictly Pareto dominated by any feasible allocation.

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is always strictly positive. A good with zero aggregate endowment is irrelevant for our purposes and can be removed from the model.

In words: a feasible allocation is Pareto efficient if we cannot make anyone better off without making someone else worse off. A feasible allocation is weakly Pareto efficient if we cannot make everyone better off. Since $\mathbf{x}$ Pareto dominates $\hat{\mathbf{x}}$ whenever $\mathbf{x}$ strictly Pareto dominates $\hat{\mathbf{x}}$, efficiency implies weak efficiency. The converse is not true in general.

Pareto efficiency can be equivalently stated in terms of maximization programs. An allocation $\mathbf{x}$ is Pareto efficient if and only if, for each consumer i, $\mathbf{x}$ solves the *Pareto program of consumer* i:

$$\max_{\hat{\mathbf{x}} \in \mathbb{R}_+^{NI}} u_i(\hat{\mathbf{x}}^i)$$

subject to

$$u_j(\hat{\mathbf{x}}^j) \geq u_j(\mathbf{x}^j) \quad \forall j \neq i, \tag{1}$$

$$\sum_{j=1}^{I} \hat{x}_n^j \leq \bar{e}_n \quad \forall n. \tag{2}$$

In words: the allocation $\mathbf{x}$ maximizes the utility of every consumer i among feasible allocations (equation 2) that give all other consumers $j \neq i$ a utility of at least $u_j(\mathbf{x}^j)$ (equation 1).

Exercise 3. Suppose there are two individuals (Ann and Bob, indexed by A and B) and two goods (1 and 2). Preferences of each individual are represented by the utility functions:

$$u_A(x_1^A, x_2^A) = x_1^A + x_2^A \quad \text{and} \quad u_B(x_1^B, x_2^B) = x_1^B.$$

Let $\mathbf{x}^A = (0, 0)$ and $x^B = (1, 1)$ (that is, A consumes zero of both goods whereas B consumes 1 unit of both goods).

a. Is this allocation Pareto efficient?

b. Is this allocation weakly Pareto efficient?

c. Do preferences of consumer B satisfy local non-satiation? Do they satisfy monotonicity? Do they satisfy strong monotonicity?

Proposition 1. *Suppose preferences of all consumers satisfy strong monotonicity. The following are equivalent:*

1. $\mathbf{x}$ *is Pareto efficient;*

2. $\mathbf{x}$ *is weakly Pareto efficient;*

3. $\mathbf{x}$ *solves the Pareto program for some consumer* i.

Proof. As argued previously, Pareto efficiency implies weak Pareto efficiency. We now show that weak Pareto efficiency implies Pareto efficiency under strong monotonicity. The proof follows by contraposition: if $\hat{\mathbf{x}}$ is feasible but is not Pareto efficient then it is not weakly Pareto efficient.

Suppose $\hat{\mathbf{x}}$ is feasible but not Pareto efficient. By the definition of Pareto efficiency, there exists a feasible allocation $\mathbf{x}$ that Pareto dominates $\hat{\mathbf{x}}$. That is, $u_i(\mathbf{x}) \geq u_i(\hat{\mathbf{x}})$ for all i with strict inequality for some i (to simplify notation, let $i = 1$ denote the consumer for which the inequality is strict). Since

$$u_1(\mathbf{x}^1) > u_1(\hat{\mathbf{x}}^1),$$

monotonicity implies $x^1_{n^*} > 0$ for some good n^* (no bundle can be worse than getting zero of all goods). Let $\tilde{\mathbf{x}}(\varepsilon)$ be an allocation that subtracts ε from consumer 1's consumption of good n^* and splits it evenly among all other consumers, while leaving the consumption of other goods unchanged. That is, $\tilde{\mathbf{x}}^1_{n^*}(\varepsilon) = \mathbf{x}^1_{n^*} - \varepsilon$, $\tilde{\mathbf{x}}^i_{n^*}(\varepsilon) = \mathbf{x}^i_{n^*} + \frac{\varepsilon}{N-1}$, and $\tilde{\mathbf{x}}_n(\varepsilon) = \mathbf{x}_n$ for $n \neq n^*$. By strong monotonicity,

$$u_j\left(\tilde{\mathbf{x}}^j(\varepsilon)\right) > u_j\left(\mathbf{x}^j\right)$$

for all $j > 1$. Moreover, by continuity, for $\varepsilon > 0$ sufficiently small

$$u_1(\tilde{\mathbf{x}}^1\,(\varepsilon)) > u_1(\hat{\mathbf{x}}^1).$$

Thus, $\tilde{\mathbf{x}}(\varepsilon)$ is feasible and strictly Pareto dominates $\hat{\mathbf{x}}$, implying that $\hat{\mathbf{x}}$ is not weakly efficient.

The equivalence between claims 2 and 3 follows directly from the definition of Pareto efficiency and the Pareto program. □

Part 3 of Proposition 1 shows that under strong monotonicity, it suffices to consider the Pareto program of only one consumer. When preferences are not strongly monotone, one often needs to consider the Pareto programs of all consumers.

Pareto efficiency is a weak condition. For example, it may be efficient for one individual to consume everything and everyone else to consume nothing. There are usually many efficient allocations, so Pareto efficiency is not enough for us to compare many different allocations. Since an allocation that is not Pareto efficient can be unambiguously improved upon, Pareto efficiency is often considered minimal requirement for its desirability. But a Pareto efficient allocation may not satisfy other desirability criteria (e.g. fairness). In fact, all but one person may strictly prefer a Pareto inefficient allocation to a Pareto efficient one, such as when one person owns consumes the entire aggregate endowment of all goods.

Exercise 4. Suppose there are two individuals (A and B) and two goods (1 and 2). Preferences of each individual are represented by the utility functions:

$$u_A(x^A_1, x^A_2) = x^A_1 + x^A_2 \quad \text{and} \quad u_B(x^B_1, x^B_2) = \min\left\{x^B_1, x^B_2\right\}.$$

The aggregate endowment of each good is $\bar{e}_1 = 1$ and $\bar{e}_2 = 2$. Calculate all Pareto efficient allocations of this economy.

2.2 Social Welfare

Recall from consumer theory that pairwise comparisons of all bundles can be a complex task. Instead, we usually prefer to represent preferences using a utility function, so we can use optimization methods to represent an individual's behavior. Studying efficiency directly in terms of preferences can be even more complex, since it would require us to perform pairwise comparisons for each consumer and verify that we cannot make one consumer better off without making someone else worse off.

As with utility functions in individual decision making, it is often convenient to represent the preferences of a set of individuals in terms of a "social welfare function." In this section, we will see that under some conditions, an allocation is Pareto efficient if and only if maximizes such a function. Because of this connection between Pareto efficiency and optimization, Pareto efficiency is often called Pareto optimality.

The *utility possibility set* U is the set of possible utilities for each consumer that are consistent with feasibility:

$$U = \left\{ v \in \mathbb{R}^I : \quad v^i = u(\mathbf{x}^i) \text{ and } \sum_{i=1}^{I} x_n^i \leq \bar{e}_n \right\}.$$

Since the set of feasible allocations is compact and utility functions are continuous, U is compact. It can be shown that U is convex if the utility functions u_i are concave.

Example 1. Suppose there are two individuals (A and B) and two goods (1 and 2). The preferences of both individuals are represented by the utility function:

$$u_i(x_1, x_2) = x_1 + x_2, \quad i = A, B.$$

Both goods have an aggregate endowment of $\bar{e}_1 = \bar{e}_2 = \frac{1}{2}$. The utility possibility set is $U = \left\{ (v_1, v_2) \in \mathbb{R}_+^2 : \quad v^1 + v^2 \leq 1 \right\}$, depicted in Figure 1.

Fix a consumer i and fix a vector of utilities $\mathbf{v} \in U$. Consumer i's *Pareto program* $P_i(\mathbf{v})$ is:

$$\max_{\mathbf{x} \in \mathbb{R}_+^{N \cdot I}} u_i(\mathbf{x}^i)$$

subject to

$$u_j(\mathbf{x}^j) \geq v_j \quad \forall j \neq i,$$

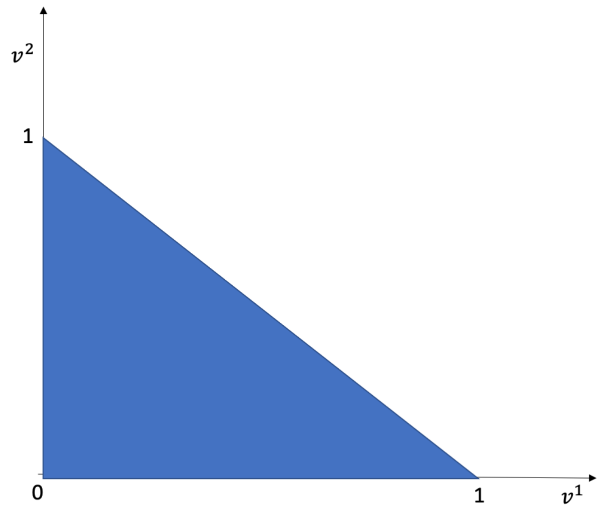

Figure 1: Utility Possibility Set from Example 1

$$\sum_{i=1}^{I} x_n^i \leq \bar{e}_n \quad \forall n.$$

The Pareto program maximizes consumer i's utility over feasible allocations that give each other consumer j a utility of at least v_j. As shown in Proposition 1, if preferences are strongly monotone, an allocation $\mathbf{x}$ is Pareto efficient if and only if there exists $\mathbf{v} \in U$ for which $\mathbf{x}$ solves $P_i(\mathbf{v})$.

Exercise 5. Suppose there are two goods (1 and 2) and two individuals (A and B) with the same utility function:

$$u(x_1, x_2) = \log x_1 + \log x_2.$$

The aggregate endowment of both goods is $\bar{e}_1 = \bar{e}_2 = 1$. Calculate all Pareto efficient allocations.

Exercise 6. Consider the economy from Exercise 3.

a. Does the allocation $\mathbf{x}_A = (0, 0)$ and $\mathbf{x}_B = (1, 1)$ solve Bob's Pareto program $P_B(\mathbf{v})$ for some $\mathbf{v} \in U$?

b. Based on your answer to (a), what can you conclude about the validity of Proposition 1 part 3 when preferences are not strongly monotone?

Given a vector of utility weights $\alpha = (\alpha^1, ..., \alpha^I) > 0$, let

$$W_\alpha(\mathbf{x}) = \sum_{i=1}^{I} \alpha^i u_i\left(x^i\right)$$

denote the *social welfare* from allocation $\mathbf{x}$.[5] A social welfare function "aggregates" preferences of individuals by attributing a weight α^i to each of them (we will return to the interpretation of weights in Remark 1 below).

The *welfare maximization program* $\tilde{P}(\alpha)$ maximizes social welfare over all feasible allocations:

$$\max_{\mathbf{x}} W_\alpha(\mathbf{x})$$

subject to $\sum_{i=1}^{I} x_n^i \leq \bar{e}_n \ \forall n$.

Proposition 2. *Fix an aggregate endowment* $\bar{\mathbf{e}}$*.*

1. *For any* $\alpha \in \mathbb{R}^I_{++}$*, if* x^* *solves* $\tilde{P}(\alpha)$*, then* x^* *is Pareto efficient.*
2. *For any* $\alpha \in \mathbb{R}^I_{+} \backslash \{\mathbf{0}\}$*, if* x^* *solves* $\tilde{P}(\alpha)$ *then* x^* *is weakly Pareto efficient.*

Proof. (1) If $\mathbf{x}^*$ is not Pareto efficient, then there is a feasible $\mathbf{x}$ such that $u_i(x^i) \geq u_i(x^{i^*})$, with strict inequality for at least one i. Multiplying each inequality by $\alpha^i > 0$ and adding them, we obtain

$$W_\alpha(\mathbf{x}) = \sum_{i=1}^{I} \alpha^i u_i(x^i) > \sum_{i=1}^{I} \alpha^i u_i(x^{i^*}) = W_\alpha(\mathbf{x}^*),$$

so $\mathbf{x}^*$ does not solve $\tilde{P}(\alpha)$.

(2) Similarly to part (1), if $\mathbf{x}^*$ is not weakly Pareto efficient then there is a feasible $\mathbf{x}$ such that $u_i(x^i) > u_i(x^{i^*})$ for all i. Multiplying each inequality by $\alpha^i \geq 0$ with at least one strict inequality and adding, we again obtain

$$W_\alpha(\mathbf{x}) = \sum_{i=1}^{I} \alpha^i u_i(x^i) > \sum_{i=1}^{I} \alpha^i u_i(x^{i^*}) = W_\alpha(\mathbf{x}^*).$$

□

Corollary 1. *There exists a Pareto efficient allocation.*

Proof. Fix any strictly positive vector of utility weights α. By assumption, all utility functions are continuous, so W_α is also continuous (it is the sum of continuous functions). Since the set of feasible allocations is non-empty and compact, there exists a solution to Program $\tilde{P}(\alpha)$. By Proposition 2, this solution is efficient. □

Proposition 2 shows how to find a Pareto efficient allocation: pick a vector of utility weights α and solve the welfare maximization program $\tilde{P}(\alpha)$. Different weights generally

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[5] This is a special case of a Bergson-Samuelson social welfare function, due to Bergson (1938) and Samuelson (1947). In general, a Bergson-Samuelson social welfare function is an increasing function $U(u_1, u_2, ...u_I)$. The social welfare function considered above is a linear Bergson-Samuelson function.

lead to different allocations. But does this procedure give us *all* Pareto efficient allocations? In other words, for each Pareto efficient allocation, is there always a vector of utility weights, α, such that the allocation solves $\tilde{P}(\alpha)$?

In general, the answer is negative: there may be some Pareto efficient allocations that do not solve the welfare maximization program for any α. The proposition below gives sufficient conditions for the welfare maximization program to map all Pareto efficient allocations:

Proposition 3. *Suppose preferences are strongly monotone and utility functions are concave. Then, x^* is Pareto efficient if and only if it solves $\tilde{P}(\alpha)$ for some $\alpha \in \mathbb{R}^I_+ \backslash \{\mathbf{0}\}$.*

That is, if U is convex and preferences are strongly monotone, we can trace all Pareto efficient allocations by solving the program for all non-negative utility weights. The general proof uses the Supporting Hyperplane Theorem, but I will omit it here because an easier proof can be obtained with some additional assumptions, which we will consider later.

Exercise 7. Consider an economy with two goods (1 and 2) and two individuals (A and B) with the same utility function:

$$u_i(x_1^i, x_2^i) = \log(x_1^i) + x_2^i.$$

The aggregate endowment of both goods is $\bar{e}_1 = \bar{e}_2 = 1$. Calculate the set of Pareto efficient allocations.

Remark 1. The solution to welfare maximization program $\tilde{P}(\alpha)$ depends on both the utility function u_i and on the vector of weights α. But you saw previously that utility functions are ordinal representations of preferences: if u_i represents $\succsim_i$, so does any increasing transformation of u_i. And if we represent the preferences of consumer 1 using a monotone transformation of the utility function (say, $2 \cdot u_i$ instead of u_i), we would typically obtain a different allocation.

How can this observation be reconciled with a welfare function corresponding to a weighted sum of individual utility functions? The answer is that weights cannot be interpreted separately from the particular cardinalization of the utility function. If we multiply the utility of consumer i and divide that consumer's weight by 2, we end up with the same allocation. Therefore, even when utility functions are ordinal, social welfare functions can be used to obtain the Pareto frontier by varying the vector of weights α.

Picking one particular vector requires one to choose a particular cardinalization of the utility function, so as to allow for interpersonal comparisons. This is often done in public finance, when one picks a particular utility representation and considers a "utilitarian" criterion that sets $\alpha = (1, 1, ..., 1)$.

Remark 2. In our analysis so far, we have not described an ownership structure over the resources in the economy, which will be introduced later when we discuss individual endowments. This means that the set of Pareto efficient allocations does not depend on individual endowments, only on the aggregate endowment $\bar{\mathbf{e}}$.

Exercise 8. Consider an economy with two goods (1 and 2) and two individuals (A and B) with the same utility function:

$$u_A(x_1, x_2) = u_B(x_1, x_2) = \log(x_1) + \log(x_2).$$

The aggregate endowment of both goods is $\bar{e}_1 = \bar{e}_2 = 1$.

a. Calculate the allocation that maximizes the social welfare function with equal weights $\alpha = (1, 1)$ (i.e. the "utilitarian" allocation).

In the items below, suppose individual A's utility function is $\tilde{u}_A(x_1, x_2) = \frac{\log(x_1)+\log(x_2)}{2}$, whereas B's utility function is $\tilde{u}_B = 2\log(x_1) + 2\log(x_2)$.

b. Show that the new utility functions $\tilde{u}_A$ and $\tilde{u}_B$ represent the same preferences as u_A and u_B.

c. Calculate the "utilitarian" allocation for the new utility functions ($\tilde{u}_A$ and $\tilde{u}_B$).

c. Show that the allocation found in (a) maximizes a social welfare function for some vector of weights $\alpha \neq (1, 1)$.

2.3 Differentiable Approach

Suppose preferences are strongly monotone and the utility functions are twice differentiable and concave (so the utility possibility set U is convex). Recall the Pareto program $P_i(\mathbf{v})$:

$$\max_{\mathbf{x} \in \mathbb{R}_+^{N \cdot I}} u_i(\mathbf{x}^i)$$

subject to

$$u_j(\mathbf{x}^j) \geq v_j \quad \forall j \neq i,$$

$$\sum_{i=1}^{I} x_n^i \leq \bar{e}_n \quad \forall n.$$

The Lagrangian associated with this program is:

$$L = u_i(\mathbf{x}^i) + \sum_{j \neq i} \lambda_j \left[u_j(\mathbf{x}^j) - v_j\right] + \sum_{n=1}^{N} \xi_n \sum_{i=1}^{I} \left(\bar{e}_n - x_n^i\right). \tag{3}$$

The multiplier $\lambda_j \geq 0$ is the shadow price associated with the utility left to consumer j. It measures how much utility consumer i has to give up if we increased the utility of consumer

j. The multiplier ξ_n is the shadow price of the aggregate endowment of good n. It measures how much individual i would gain if we increased the aggregate endowment of good n.

Next, consider the welfare maximization program $\tilde{P}(\alpha)$ with utility weights $\alpha = (\alpha^1, ..., \alpha^I) > 0$. Since $\alpha > 0$, there exists a consumer i for which $\alpha^i > 0$. Note that dividing the welfare function $W_\alpha(\mathbf{x})$ by $\alpha^i > 0$ does not affect its solution. Therefore, we can normalize $\alpha^i = 1$, and the welfare maximization program becomes:

$$\max_{\mathbf{x}} u_i\left(x^i\right) + \sum_{j \neq i} \alpha^j u_j\left(x^j\right)$$

subject to

$$\sum_{i=1}^{I} x_n^i \leq \bar{e}_n \ \forall n.$$

The Lagrangian associated with this program is

$$L = u_i(\mathbf{x}^i) + \sum_{j \neq i} \alpha_j u_j(\mathbf{x}^j) + \sum_{n=1}^{N} \xi_n \sum_{i=1}^{I} \left(\bar{e}_n - x_n^i\right). \tag{4}$$

Note that Lagrangians (3) and (4) coincide when $\alpha_j = \lambda_j$. That is, the Pareto program and the welfare maximization program coincide when the shadow cost of the utility left to each consumer is equal to the utility weight of that consumer.

Proposition 4. *Suppose preferences are strongly monotone and the utility functions are differentiable and concave. The following are equivalent:*

1. *x is Pareto efficient,*
2. *$\mathbf{x}$ solves $P_i(\mathbf{v})$ for some $\mathbf{v} \in U$,*
3. *$\mathbf{x}$ solves $\tilde{P}(\alpha)$ for some $\alpha > 0$.*

Proof. The equivalence between (1) and (2) follows from Proposition 1. By point 2 in Proposition 2 and the equivalence between Pareto and weak Pareto efficiency for strongly monotone preferences, (3) implies (1). By the argument above, the Lagrangian associated with $\tilde{P}(\alpha)$ coincides with the Lagrangian associated with $P_i(\mathbf{v})$ when we pick utility weights $\alpha_j = \lambda_j$. Therefore, any allocation that solves $P_i(\mathbf{v})$ must solve $\tilde{P}(\alpha)$ for some suitably chosen weights $\alpha > 0$, showing that (2) implies (3). $\square$

To understand the structure of efficient allocations, it helps to consider $\alpha >> 0$ and assume that the solution is interior (it is straightforward, but notationally cumbersome to allow for corner solutions by adding the complementary slackness conditions). We can ensure that the solution is indeed interior by assuming the following "Inada" condition:

$\lim_{x_n^i \searrow 0} \frac{\partial u_i}{\partial x_n^i}(\mathbf{x}^i) = +\infty$ for all n. This condition implies that indifference curves never touch the axes, so that it is not Pareto efficient to consume a positive amount of some good and zero of another good.

Since utility functions are concave, the following first-order conditions are necessary and sufficient for a solution to $\tilde{P}(\alpha)$:

$$\frac{\partial u_i}{\partial x_n}(\mathbf{x_i}) = \alpha_j \frac{\partial u_j}{\partial x_n}(\mathbf{x}_j) = \xi_n \quad \forall n, \forall j \neq i,$$

and

$$\sum_{i=1}^{I} x_n^i = \bar{e}_n \quad \forall n.$$

Rearranging the first condition, gives:

$$\frac{\frac{\partial u_i}{\partial x_n}(\mathbf{x_i})}{\frac{\partial u_j}{\partial x_n}(\mathbf{x}_j)} = \alpha_j \quad \forall n, \ \forall j \neq i.$$

Since α_j depends on the consumer but not on the good n the expression on the LHS must be the same for all goods. Rearranging, we obtain:

$$\frac{\frac{\partial u_i}{\partial x_n}(\mathbf{x}_i)}{\frac{\partial u_i}{\partial x_{\tilde{n}}}(\mathbf{x}_i)} = \frac{\frac{\partial u_j}{\partial x_n}(\mathbf{x}_j)}{\frac{\partial u_j}{\partial x_{\tilde{n}}}(\mathbf{x}_j)} \quad \forall i, j, n, \tilde{n}.$$

In words: any (interior) Pareto efficient allocation equalizes the marginal rate of substitution (MRS) between all goods and among all consumers. Suppose, for example, that individual 1 has an MRS of two apples per orange whereas individual 2 has an MRS of one apple for each orange. If we transfer oranges from consumer 1's bundle to 2's bundle and transfer apples in the opposite direction, we can strictly improve both of them. These gains from reallocating resources exist whenever individuals have different marginal rates of substitution between any two goods, so Pareto efficient allocations must equalize MRSs.

With two goods (1 and 2) and two consumers (A and B), a useful way to graphically represent efficient allocations is through Edgeworth boxes, originally introduced in Edgeworth (1881). The bottom left corner is the origin for consumer 1, whereas the top right corner is the origin for consumer 2. Each point in the box represents a feasible (non-wasteful) allocation:

$$\left\{ \mathbf{x} \in \mathbb{R}_+^4 : \ \mathbf{x}^1 + \mathbf{x}^2 = \bar{\mathbf{e}} \right\}.$$

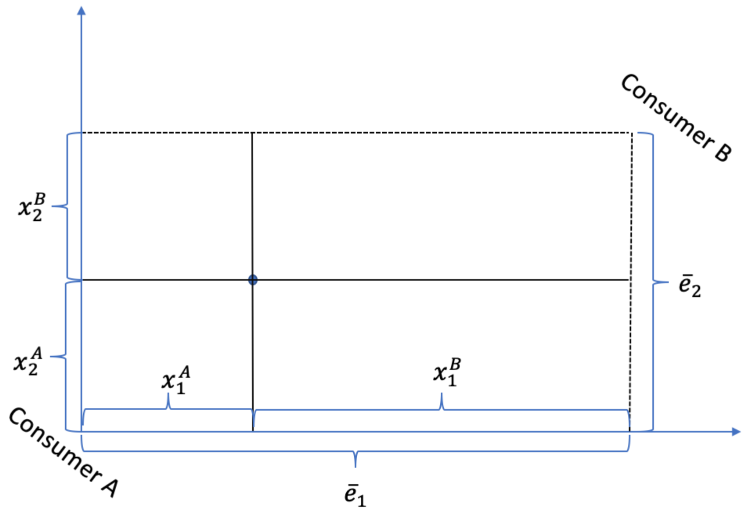

Figure 2: Allocation in the Edgeworth Box

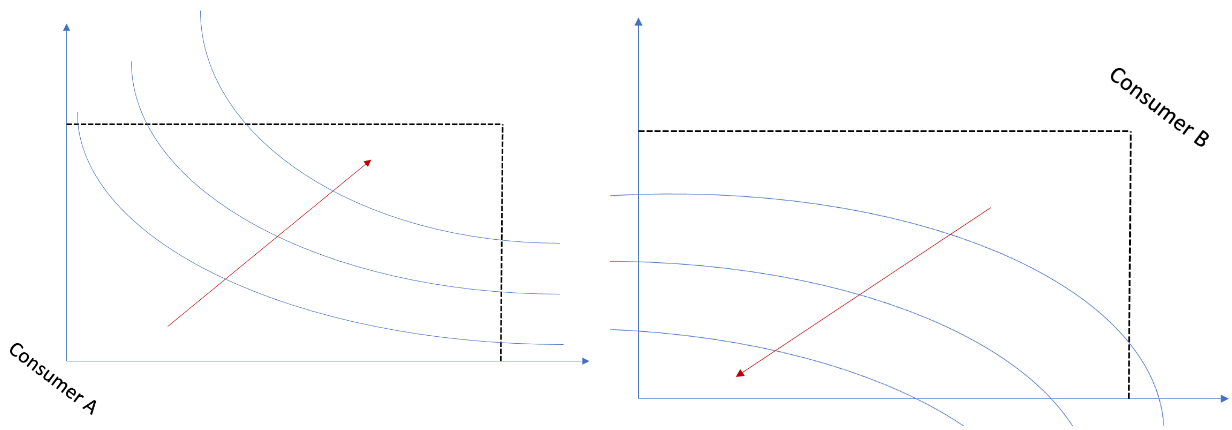

Figure 3: Preferences in the Edgeworth Box

Figure 2 represents a generic allocation $\mathbf{x}$ in the Edgeworth box. Good 1 is represented in the horizontal axis and good 2 is represented in the vertical axis.

Figure 3 represents preferences in the Edgeworth box in terms of the indifference curves of each consumer (arrows point in the direction of preferred bundles). Because preferences are also well defined by bundles that are not feasible, indifference curves extend beyond the box. (This will be important when we consider competitive equilibria, since we must ensure that individuals do not wish to purchase different bundles under existing prices, whether or not those bundles are not feasible.)

Figure 4 represents Pareto efficient allocations in the Edgeworth box. Under our assumptions above (concavity and the Inada conditions), Pareto efficient allocations are those for which the indifference curves are tangent. That is, the allocations are feasible and consumers have the same MRS between the two goods.

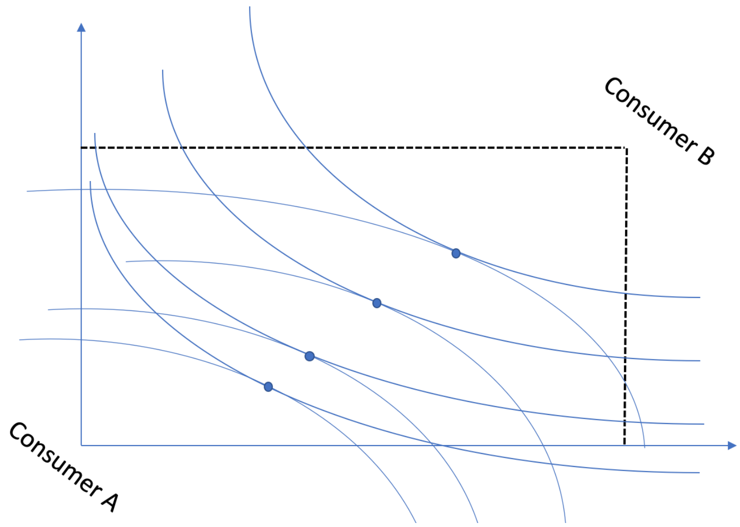

Figure 4: Pareto Efficient Allocations in the Edgeworth Box

3 Risk

The description of the model so far suggests a deterministic environment. One may think that uncertainty would significantly complicate matters, but Arrow (1953; 1963) showed that a simple reinterpretation of the commodity space to incorporate risk.

Recall that economists treat the same physical good in different situations as different. A bottle of water on the beach and the same bottle of water in the supermarket are conceptually different goods. An apple today is a different from an apple in a year. And an umbrella on a sunny day is a different good from the same umbrella on a rainy day. This observation allowed the standard general equilibrium model to become workhorse models in international trade, macroeconomics, and finance. International trade can be thought of as the shipment of bottles of water from the supermarket to the beach. Savings can be thought of as transferring an apple from today until a year from now. And risk smoothing is analogous to ensuring you have an umbrella when it rains.

Arrow's insight was to introduce "states of nature" similarly to Savage's decision theory. A *state of nature* is a complete description of all uncertainty until consumption occurs. Importantly, we assume here that all individuals agree on the possible states of nature before trading.

We now consider an *exchange economy with uncertainty*. There are $I \in \mathbb{N}$ consumers $N \in \mathbb{N}$ goods.[6] To simplify notation, we assume that there are only two periods: 0 ("first

[6]The assumption that the set of goods is the same in all states of nature simplifies notation, but there is

period") and 1 ("second period"). Consumers are uncertain about the second period. This uncertainty can be exogenous – concerning endowments or preferences –, or endogenous – concerning allocations and prices (to be described below). We represent this uncertainty by considering $S \in \mathbb{N}$ states of nature in the second period.

In general, an agent can consume different amounts in different periods and in different states. In particular, there is no reason for us to assume that consumers treat goods in different periods or states as equivalent. Formally, this general formulation is obtained by considering an economy with $M \equiv (S+1) \cdot N$ goods (N in the first period, and N in each state of nature in the second period). Consumer i's consumption of good n in the first period is denoted $x^i_{n,0}$. Consumer i's consumption of good n in state s in the second period is denoted $x^i_{n,s}$, where $s = 0$ corresponds to the first period. The vector of consumer i's (state-dependent) consumption of each good,

$$\mathbf{x}^i = \left(x^i_{1,0}, x^i_{2,0} ..., x^i_{N,0}, x^i_{1,1}, x^i_{2,1} ..., x^i_{N,1}, ..., x^i_{1,S}, x^i_{2,S} ..., x^i_{N,S}\right) \in \mathbb{R}^M_+$$

is called a *consumption bundle.* An *allocation* is a vector $\mathbf{x} = \left(\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^I\right) \in \mathbb{R}^{N \cdot M}_+$ giving a consumption bundle for each consumer.

Each consumer i has preferences over consumption bundles represented by a continuous utility function $u_i\left(\mathbf{x}^i\right)$. While this utility function allows for very general risk preferences, it is often convenient to assume that preferences can be represented by an expected utility function:

$$u_i\left(\mathbf{x}^i\right) = V_i(x^i_0) + \beta \sum_{s=1}^{S} \pi^i_s V_i\left(x^i_s\right), \tag{5}$$

where $\pi^i_s > 0$ corresponds to person i's subjective probability associated with state $s \in S$.

With this formulation, an environment with uncertainty is equivalent to a deterministic environment with an enlarged space of goods (so that the same good in each state is treated as a different good). The results from deterministic environments therefore generalize to environment with uncertainty.

The simplest model with uncertainty was originally developed by Borch (1962). Suppose there are $I = 2$ consumers with represented by subjective expected utility and a single consumption good in each state ($N = 1$). They assign the same probability to each state of nature: $\pi^1_s = \pi^2_s =: \pi_s$ and have preferences represented by expected utility (5), where V_i is differentiable, strictly increasing, and strictly concave. The aggregate endowment in each state is $\bar{e}_{1,0} > 0$ and $\bar{e}_{2,s} > 0$ for all s.

Exercise 9. Let $\mathbf{x} >> 0$ be a Pareto efficient allocation.

no conceptual difficulty in allowing the consumption bundles to depend on the state of nature.

a. Show that the following "Borch rule" must hold:

$$\frac{V_1'\left(x_s^1\right)}{V_2'\left(\bar{e}_s - x_s^1\right)} = \frac{V_1'\left(x_{\tilde{s}}^1\right)}{V_2'\left(\bar{e}_{\tilde{s}} - x_{\tilde{s}}^1\right)} \; \forall s, \tilde{s}.$$

b. Suppose individual 2 is risk neutral and individual 1 is risk averse and satisfies the Inada condition: $\lim_{x \searrow 0} V_2'(x) = +\infty$ for all n. Show that, in any Pareto efficient allocation, individual 2 fully insures individual 1.

c. Suppose there is no aggregate uncertainty ($\bar{e}_s = \bar{e}_{\tilde{s}}$ for all $s, \tilde{s}$) and at least one individual is risk averse and satisfies the Inada condition. Show that any Pareto efficient allocation has constant consumption to all individuals.

The Borch rule illustrates some important empirical implications of efficient risk sharing. The first one is that with efficient risk sharing, idiosyncratic income shocks should not affect consumption. Only aggregate income shocks should matter for individual consumption. See Cochrane (1991) and Townsend (1994) for empirical applications.

Exercise 10. Suppose there are two states of nature in period 1: $S = \{1, 2\}$. Both individuals have utility function:

$$u_i\left(\mathbf{x}^i\right) = \log(x_0^i) + \sum_{s=1}^{2} \frac{\log\left(x_s^i\right)}{2},$$

meaning that they have a logarithmic Bernoulli utility and attribute probability 50% to each state. The aggregate endowment is $\bar{\mathbf{e}} = (2, 2, 2)$. Calculate the set of Pareto efficient allocations.

4 Competitive Equilibrium: Definition and Walras' Law

To consider a market economy, we need to introduce a notion of private property. This is done by assuming that each consumer i has an individual endowment $e_n^i \geq 0$ of good n. The *allocation of endowments* is a vector $\mathbf{e} = \left(\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^I\right) \in \mathbb{R}_+^{N \times I}$. We say that an allocation of endowments is consistent with aggregate endowment $\bar{e}$ if

$$\bar{e} = \sum_{i=1}^{I} e^i.$$

We now present the main concept in general equilibrium. A competitive equilibrium is a vector of prices and a consumption bundle for each consumer, such that each consumer chooses her favorite bundle given prices and markets clear.

Definition 3. A competitive equilibrium is a set of prices $\mathbf{p} \in \mathbb{R}_+^N$ and an allocation $\mathbf{x} \in \mathbb{R}_+^{I\times N}$ such that:

1. Each consumer i picks an optimal bundle $\mathbf{x}^i$ given prices $\mathbf{p}$: $\mathbf{x}^i \in \arg\max u_i(\hat{\mathbf{x}}^i)$ subject to $\mathbf{p} \cdot \hat{\mathbf{x}}^i \leq \mathbf{p} \cdot \mathbf{e}^i$.

2. Markets clear: For each n, $\sum_{i=1}^I x_n^i \leq \sum_{i=1}^I e_n^i$ with equality if $p_n > 0$.

The consumer's budget constraint is slightly different than the one you saw in consumer theory. There, we considered an individual with wealth w, so the budget constraint was $p \cdot \hat{\mathbf{x}}^i \leq w$. Here, each individual's wealth is determined by the value of all goods that the individual has,

$$w = \mathbf{p} \cdot \mathbf{e}^i,$$

giving rise to the budget constraints above.

The market clearing condition also slightly different than the one you may have seen before. It states that if a good is sold at a positive price, aggregate demand must equal the aggregate endowment (supply). However, goods may be in excess supply if they are available for free. Arrow and Debreu (1954) note that this more general definition is needed if one wishes to allow for goods such as clean air, which are available for free.

When at least one consumer has strongly monotone preferences, prices must be strictly positive (otherwise, this consumer would demand infinite amounts, which is inconsistent with market clearing). In that case, the market clearing condition becomes the standard requirement that supply equals demand:

$$\sum_{i=1}^I \mathbf{x}^i = \sum_{i=1}^I \mathbf{e}^i \ \forall i.$$

In applications, it is easier to check this market clearing condition without having to worry about zero prices. This is justified if we can rule out prices at the boundary. However, when preferences are not strongly monotone, a competitive equilibrium in which all prices are strictly positive may not exist.

Note that budget constraints are homogenous of degree zero: if we multiply the price vector by any positive constant $\gamma \in \mathbb{R}_{++}$, the budget constraint remains unchanged. This means that *only relative prices matter* in a competitive equilibrium. If $\mathbf{p}^*$ is a competitive equilibrium, so is $\gamma\mathbf{p}^*$ for any scalar $\gamma > 0$. In particular, the Walrasian model cannot provide a theory inflation. We can therefore normalize one good n with a positive price to $p_n = 1$.[7]

[7] We can only normalize the price of a good if that good has strictly positive prices in the competitive equilibrium. However, in most situations below we will consider only equilibria with strictly positive prices, so we choose any good to have its price set equal to 1.

Exercise 11. Consider the economy from Exercise 4. Suppose the endowments are $e_A = e_B = \left(\frac{1}{2}, 1\right)$. Calculate the competitive equilibrium.

Exercise 12. Consider an economy with $I \geq 1$ identical consumers and $N = 2$ goods. All consumers have utility function:

$$u(x_1, x_2) = \log x_1 + \log x_2,$$

and endowment $\mathbf{e} = (1, 1)$. Calculate the competitive equilibrium.

Exercise 13. Consider an economy with $I \geq 1$ identical consumers and $N = 2$ goods. All consumers have utility function:

$$u(x_1, x_2) = \min\left\{x_1, x_2\right\},$$

and endowment $\mathbf{e} = (2, 1)$.

a. Is there a competitive equilibrium with strictly positive prices? Explain.

b. Calculate all competitive equilibria of this economy.

Suppose preferences are locally non-satiated, so individuals consume their entire budget:

$$\mathbf{p} \cdot \left(\mathbf{x}^i - \mathbf{e}^i\right) = 0 \quad i = 1, ..., I. \tag{6}$$

Summing this condition for all individuals, we obtain:

$$\sum_{i=1}^{I} \mathbf{p} \cdot \left(\mathbf{x}^i - \mathbf{e}^i\right) = 0. \tag{7}$$

That is, the value of all consumer's "net demands" (their demands net of their endowments) must add up to zero. This expression, which Lange (1942) called *Walras' law*, must hold for any price vector $\mathbf{p}$. It plays an important role in the proof of equilibrium existence.

Suppose the price vector is strictly positive $\mathbf{p} >> 0$, in which case the market clearing conditions become:

$$\sum_{i=1}^{I} \left(\mathbf{x}^i_m - \mathbf{e}^i_m\right) = 0 \quad m = 1, ..., M. \tag{8}$$

The system of linear equations given by all budget constraints (6) as well as all market clearing conditions (8) has one degree of freedom. Therefore, **if all but one markets clear and all budget constraints hold, the remaining market must also clear**. This is a stronger version of Walras' Law, which holds when all prices are positive.

To derive this stronger version of Walras' Law, suppose the markets for goods $n = 2, 3, ..., N$ clear. Rewrite (7) as:

$$p_1 \sum_{i=1}^{I} \left(x_1^i - e_1^i\right) + p_2 \sum_{i=1}^{I} \left(x_2^i - e_2^i\right) + ... + p_N \sum_{i=1}^{I} \left(x_N^i - e_N^i\right) = 0.$$

Divide both sides by $p_1 > 0$ and rearrange to obtain:

$$\sum_{i=1}^{I} \left(x_1^i - e_1^i\right) = -\frac{p_2}{p_1} \underbrace{\sum_{i=1}^{I} \left(x_2^i - e_2^i\right)}_{0} - ... - \frac{p_N}{p_1} \underbrace{\sum_{i=1}^{I} \left(x_N^i - e_N^i\right)}_{0} = 0,$$

where the last equality follows from the assumption that market n clears, so $\sum_{i=1}^{I} \left(x_n^i - e_n^i\right) = 0$ for $n > 1$. (We picked good $n = 1$, but the same calculation can be done for any good.)

The two most important questions in general equilibrium are: (1) under what conditions does a competitive equilibrium exist, and (2) what is the relationship between competitive equilibria and Pareto efficiency? Walras noted that his system had the same number of equations as unknowns, interpreting this as a "proof" of the existence of equilibrium. Hopefully you understand that counting equations is not enough to establish existence of a solution to a system of equations. However, the mathematics required for studying the existence of solutions to systems of nonlinear equations was not available at Walras' time.

The first attempts to formally establish existence build on a model due to Cassel (1918) and were mostly written in German. They take as primitives an aggregate demand for each good as a function of all prices (no consumer optimization) and a supply function for each good based on linear production functions. To map these early approaches into our model, let $\bar{x}_n(\mathbf{p})$ denote the aggregate demand for good n as a function of all prices. To show existence of an equilibrium amounted to show the existence of a price vector $\mathbf{p}$ such that:

$$\bar{x}_n(\mathbf{p}) = \bar{e}_n \quad \forall n \in N.$$

If the aggregate demand is linear, this corresponds to a simple linear algebra exercise. However, even in this linear case, it was not clear what to conclude if the solution involved negative prices or quantities.[8] Another issue with this approach is that ignored consumer optimization. Wald (1936) presented the first existence theorem for very specific demand and supply functions, as well as a counterexample if some individuals had zero endowment of some goods. Under his strong assumptions, a unique equilibrium exists.

[8] Neisser (1932) first showed that such systems often generate negative prices or quantities and Schlesinger (1933) suggested the introduction of boundary conditions in these cases. In a series of papers, Wald showed that, when one takes such boundary conditions into consideration, an equilibrium always exists.

The main breakthrough to this agenda was Nash (1950), who used a fixed point theorem to prove the existence of Nash equilibrium in games. Soon afterwards, in the same conference in 1952, Arrow and Debreu (1954) and McKenzie (1954) both showed existence using a related approach. Soon afterwards, Gale (1955) and Nikaido (1956) presented other existence theorems.

Theorem 1. *Suppose preferences are locally non-satiated and convex, and suppose* $\mathbf{e} >> 0$*. Then, there exists a competitive equilibrium.*

Proof. See Section 7. □

Local non-satiation is a very weak condition. The theorem makes two substantial assumptions: convexity and positive individual endowments. We discuss the role of convexity in Example 3 below.

The assumption that each individual's endowment is strictly positive is quite strong. However, as Arrow and Debreu (1954) show (building on Wald's example), if an individual has a zero endowment of some goods, the quantity demanded can be undefined when some prices equal zero. This happens because the value of that individual's endowment would reach zero as the price of the goods that the individual owns approaches zero. As a result, the demand for goods available at zero prices may jump discontinuously when the individual's wealth reaches zero, preventing the existence of competitive equilibria.[9] The exercise below asks you to work through an economy with no competitive equilibrium:

Exercise 14. As in Exercise 3, suppose there are two individuals (Ann and Bob) with preferences represented by the utility functions:

$$u_A(x_1^A, x_2^A) = x_1^A + x_2^A \quad \text{and} \quad u_B(x_1^B, x_2^B) = x_1^B.$$

Their endowments are $\mathbf{e}^A = (0, 1)$ and $\mathbf{e}^B = (1, 1)$.

a. Show that there is no competitive equilibrium with $p_2 = 0$.

b. Show that there is no competitive equilibrium with $p_2 > 0$.

Exercise 15. (Shapley-Shubik) Consider an economy with two consumers (Ann and Bob) and two goods (1 and 2). Ann's preferences are represented by

$$u_A(x_1^A, x_2^A) = x_1^A - \frac{1}{2\,(x_2^A)^{20}},$$

[9] It is possible to weaken the assumption that endowments are all strictly positive while ensuring that each individual's wealth is always bounded away from zero. I don't think these are crucial issues to understand the existence result and therefore do not pursue this approach here.

whereas Bob's preferences are represented by:

$$u_B(x_1^B, x_2^B) = -\frac{1}{2\left(x_1^B\right)^{20}} + x_2^B.$$

Both consumers have endowment $\mathbf{e} = (1, 1)$.

a. Are preferences locally non-satiated? Are preferences convex?

b. Does this economy have a competitive equilibrium? If so, how many? [Hint: you can use a computer to obtain a numerical answer.]

c. Suppose instead that the endowments are $e_n^i = 1 + \epsilon_n^i$ for $i = A, B$ and $n = 1, 2$ (that is, consider a small perturbation to the endowments from part a). Show that your answer to (b) remains unchanged if the perturbation is small enough, that is, if ϵ_n^i is close enough to zero for all i, n.

Exercise 16. (Quasi-Linear Preferences) Consider an economy with $I \geq 1$ consumers and $N = 2$ goods. Consumer i's preferences are represented by

$$u_i(x_1^i, x_2^i) = x_1^i + \phi_i(x_2^i),$$

where ϕ_i is differentiable, strictly increasing, strictly concave, and $\lim_{x \searrow 0} \phi_i'(0) = +\infty$. Consumer i's endowment is $e^i = (e_1^i, e_2^i)$ (consumers may have different endowments).

a. Obtain an expression for each consumer's demand for each good.

b. Show that the competitive equilibrium is unique (up to a standard normalization of a price).

c. Explain why the economy from Exercise 15 is not a special case of this one.

d. Calculate the Pareto efficient allocations.

Exercise 17. Consider an economy with two consumers and two goods. Both consumers have utility function

$$u_i(x_1^i, x_2^i) = \min\left\{x_1^i, x_2^i\right\}.$$

Consumer 1 has individual endowment $\mathbf{e}^1 = (1, \delta)$ whereas 2 has individual endowment $\mathbf{e}^2 = (\delta, 1)$, where $\delta \geq 0$. Is there a competitive equilibrium? If so, is it unique?

4.1 Gross Substitutes, Uniqueness, and Stability

As we have seen in exercise 15, some economies have multiple competitive equilibria. The economy considered there is quite standard: preferences are strongly monotone and convex and the endowment is bounded away from zero. Moreover, the multiplicity result is robust to small perturbations in the initial endowment, indicating that the result is not knife edge. Given this observation, it is natural to ask whether there exist reasonable conditions that guarantee a unique equilibrium. One example of economies with unique equilibria was given in Exercise 16 (quasi-linearity in the same good). We now consider a more general class of preferences that ensure that the equilibrium is unique.

Before proceeding with formal definitions and results, let us step back and try to understand why there may be multiple equilibria. In partial equilibrium analysis, multiplicity is typically not an issue: a weakly decreasing demand function and a weakly increasing supply function cannot intercept multiple times. With general equilibrium, prices can have non-monotonic effects on the excess demand for that good, by making consumers who own that good wealthier. Moreover, adjusting the price of one good can affect the demand for all other goods.

Throughout this subsection, we will obtain results in terms of *aggregate demand functions* $\bar{\mathbf{x}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_+$. When using these results in applications, you may have to write down the preferences of each consumer, calculate the individual demands, and sum them to obtain the aggregate demand. To do so, you may want to assume that individual preferences are strictly convex (so individual demand curves are functions rather than correspondences) and that at least one consumer has strongly monotone preferences (so the restriction of the domain of $\bar{\mathbf{x}}(\cdot)$ to strictly positive prices is justified). You may also want to impose the assumptions of Theorem 1 to ensure that a competitive equilibrium exists.

It is helpful to introduce some notation. The aggregate excess demand for good n is the difference between its aggregate demand and supply of that good: $\bar{d}_n(\mathbf{p}) \equiv \bar{x}_n(\mathbf{p}) - \bar{e}_n$. Combining the aggregate excess demands of all goods, we obtain the aggregate excess demand function $\bar{\mathbf{d}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_+$ defined as $\bar{\mathbf{d}}(\mathbf{p}) \equiv \bar{\mathbf{x}}(\mathbf{p}) - \bar{\mathbf{e}}$. The market clearing condition can be equivalently written either as $\bar{\mathbf{x}}(\mathbf{p}) = \bar{\mathbf{e}}$ or as $\bar{\mathbf{d}}(\mathbf{p}) = 0$.

Definition 4. An aggregate demand function $\bar{\mathbf{x}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_+$ satisfies the *gross substitutes* property if for any $\mathbf{p} \in \mathbb{R}^N_{++}$ and any $n \in \{1, ..., N\}$,

$$\bar{x}_{\tilde{n}}(p_1, ..., p_{n-1}, p_n + \varepsilon, p_{n+1}, ..., p_N) \geq \bar{x}_{\tilde{n}}(p_1, ..., p_{n-1}, p_n, p_{n+1}, ..., p_N) \tag{9}$$

for all $\tilde{n} \neq n$ with strict inequality for at least one $\tilde{n}$.

In words, an aggregate demand function satisfies the gross substitutes property if increasing the price of good n weakly raises the aggregate demand for all other good, with a strict increase for at least one good.

Remark 3. Since the aggregate demand is the sum of the individual demands of all consumers, if each individual demand satisfies the gross substitutes property, so does the aggregate demand. Subtracting the aggregate endowment of good $\tilde{n}$ from both sides of (9), we find that the aggregate demand $\bar{\mathbf{x}}(\cdot)$ satisfies gross substitutes if and only if the aggregate excess demand function $\bar{\mathbf{d}}(\cdot) \equiv \bar{\mathbf{x}}(\cdot) - \bar{\mathbf{e}}$ satisfies gross substitutes.

Gross substitutes is a very restrictive assumption, although is satisfied by many utility functions used in applications.

Exercise 18. Consider an individual with a Cobb-Douglas utility function,

$$u\left(x_1, ..., x_N\right) = \sum_{n=1}^{N} \alpha_n \ln\left(x_n\right),$$

and endowment $\mathbf{e} = (e_1, ..., e_N)$.

a. Show that this individual's demand function satisfies the gross substitutes property if $\mathbf{e} >> 0$.

b. Show that this individual's demand function does not satisfy the gross substitutes property if $e_n = 0$ for some n.

We now show that economies with gross substitutes have unique competitive equilibria (when an equilibrium exists):

Proposition 5. *Suppose the aggregate demand function $\bar{\mathbf{x}}$ satisfies the gross substitutes property. Then, the economy has at most one competitive equilibrium (up to a normalization).*

Proof. Let $\mathbf{p} \in \mathbb{R}^N_{++}$ be the price vector in a competitive equilibrium and consider a different price vector $\tilde{\mathbf{p}} \in \mathbb{R}^N_{++}$ that is not collinear with $\mathbf{p}$. Let n be the good with the lowest ratio of prices:

$$n \in \arg\min_k \left\{ \frac{\tilde{p}_k}{p_k} : \; k = 1, ..., N \right\}.$$

Without loss, we can normalize the price of one good to 1 in both $\mathbf{p}$ and $\tilde{\mathbf{p}}$. Pick good n, so that $\tilde{p}_n = p_n = 1$ and $\tilde{p}_k \geq p_k$ for all k. Since $\mathbf{p}$ and $\tilde{\mathbf{p}}$ are not collinear, it follows that $\tilde{p}_l > p_l$ for some l. By gross substitutes and the fact that markets clear under price vector $\mathbf{p}$, we find that $\bar{x}_k(\tilde{\mathbf{p}}) \geq \bar{x}_k(\mathbf{p}) = \bar{e}_k$ for all k with strict inequality for at least one k, so markets do not clear under $\tilde{\mathbf{p}}$. $\square$

Exercise 19. Show that the aggregate demand from the economy in Exercise 15 does not have the gross substitutes property. Interpret your result in light of Proposition 5. *[Hint: A numerical calculation is enough.]*

We now consider a generalization of the gross substitutes property:

Definition 5. (Weak Axiom for Aggregate Demand) An aggregate demand function $\bar{\mathbf{x}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_+$ satisfies the weak axiom if $\bar{\mathbf{x}}(\mathbf{p}^*) = \bar{\mathbf{e}}$ implies $\mathbf{p}^* \cdot \bar{\mathbf{x}}(\mathbf{p}) > \mathbf{p}^* \cdot \bar{\mathbf{e}}$ for all $\mathbf{p}$ not collinear with $\mathbf{p}^*$.

The definition above is best understood by considering an economy with a single individual. In that case, the weak axiom becomes

$$\mathbf{p}^* \cdot \mathbf{x}(\mathbf{p}) > \mathbf{p}^* \cdot \bar{\mathbf{e}} = \mathbf{p}^* \cdot \mathbf{x}(\mathbf{p}^*),$$

where the equality follows from the fact that $\mathbf{x}(\mathbf{p}^*) = \mathbf{e}$. Therefore, the optimal bundle under the new prices is unaffordable under the old prices.

Beyond the single-individual case, it is much harder to make a case for the weak axiom. Apart from very special cases, one should not expect aggregate demand have the same properties as individual demand functions. One justification of the weak axiom for aggregate demand is that it is implied by the gross substitutes property:

Lemma 1. *Suppose the aggregate demand function $\bar{\mathbf{x}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_+$ satisfies the gross substitutes property. Then, it satisfies the weak axiom.*

Proof. I will only consider the case of two goods – see Arrow and Hahn (1971) for a proof of the general case. Let $\mathbf{p}^*$ be an equilibrium price and normalize $p_1 = p_1^* = 1$. Then,

$$\begin{array}{rcl} \mathbf{p}^* \cdot \bar{\mathbf{d}}(\mathbf{p}) & = & \mathbf{p}^* \cdot \bar{\mathbf{d}}(\mathbf{p}) - \mathbf{p} \cdot \bar{\mathbf{d}}(\mathbf{p}) - (\mathbf{p}^* - \mathbf{p}) \cdot \bar{\mathbf{d}}(\mathbf{p}^*) \\ & = & (\mathbf{p}^* - \mathbf{p}) \cdot \big(\bar{\mathbf{d}}(\mathbf{p}) - \bar{\mathbf{d}}(\mathbf{p}^*)\big) \\ & = & (p_2^* - p_2) \cdot \big(\bar{d}_2(\mathbf{p}) - \bar{d}_2(\mathbf{p}^*)\big) > 0 \end{array},$$

where the first equality follows from Walras' Law (so that $\mathbf{p} \cdot \bar{\mathbf{d}}(\mathbf{p}) = 0$) and the fact that $\mathbf{p}^*$ is an equilibrium price (so $\bar{\mathbf{d}}(\mathbf{p}^*) = 0$), the second equality regroups terms, the last equality follows from the price normalization ($p_2^* - p_2 = 0$), and the inequality at the end uses the gross substitutes property. □

We now show that uniqueness can be generalized for aggregate demand functions that satisfy the weak axiom:

Proposition 6. *Suppose the aggregate demand function $\bar{\mathbf{x}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_+$ satisfies the weak axiom. Then, the economy has at most one competitive equilibrium (up to a normalization).*

Proof. Let $\mathbf{p}^*$ be the price vector in a competitive equilibrium ($\bar{\mathbf{x}}(\mathbf{p}^*) = \bar{\mathbf{e}}$) and suppose $\mathbf{p}$ is not collinear with $\mathbf{p}^*$. By the weak axiom, we have $\mathbf{p}^* \cdot (\bar{\mathbf{x}}(\mathbf{p}) - \bar{\mathbf{e}}) > 0$, so $\bar{\mathbf{x}}(\mathbf{p}) \neq \bar{\mathbf{e}}$. Therefore, $\mathbf{p}$ is not a competitive equilibrium price vector. □

In addition to uniqueness, the economies considered here have two convenient properties: stability and nice comparative statics.

To study whether a competitive equilibrium is stable, we need to think about a dynamical system in which prices evolve depending on supply and demand. In the remainder of this subsection, we normalize prices so that $\sum_{n=1}^{N} p_n = 1$. Suppose at time 0, we start with some arbitrary price vector $\mathbf{p}(0) >> 0$. If this price happens to balance supply and demand in all markets, we have found a competitive equilibrium and the price is kept constant. If $\mathbf{p}(0)$ does not balance supply and demand, the price needs to be adjusted.

The idea of a "tatonnement" is to increase prices in markets with excess demand a reduce prices in markets with excess supply. Consider the adjustment process:

$$\dot{p}_n(t) = \alpha \times \bar{d}_n\left(\mathbf{p}(t)\right), \; i = 1, ..., N,$$

which changes the price of each good proportionally to its aggregate excess demand. Note that by Walras' law (equation 8), we have $\sum_{n=1}^{N} \dot{p}_n(t) = 0$, so this adjustment process preserves our normalization of prices: $\sum_{n=1}^{N} p_n(t) = 1$ for all t.

Proposition 7. *(Arrow and Hurwicz, 1958). Suppose the aggregate demand function* $\bar{\mathbf{x}} : \mathbb{R}^N_{++} \to \mathbb{R}^N_{+}$ *satisfies the weak axiom and suppose* $\mathbf{p}^*$ *is a competitive equilibrium price vector. Then,* $\lim_{t \nearrow +\infty} \mathbf{p}(t) = \mathbf{p}^*$ *for any initial condition* $\mathbf{p}(0) \in \mathbb{R}^N_{++}$.

Proof. Fix $\mathbf{p}(0) \in \mathbb{R}^N_{++}$ and let $D(\cdot)$ denote the squared Euclidean distance between $\mathbf{p}(t)$ and $\mathbf{p}^*$:

$$D(\mathbf{p}) \equiv \sum_{n=1}^{N} \left(p_n - p_n^*\right)^2,$$

so that

$$D(\mathbf{p}(t)) = \sum_{n=1}^{N} \left[p_n(t) - p_n^*\right]^2.$$

Note that $D(\mathbf{p}(t)) \geq 0$ with "=" if and only if $\mathbf{p}(t) = \mathbf{p}^*$. Note also that the distance between $\mathbf{p}(t)$ and $\mathbf{p}^*$ is decreasing in time:

$$\begin{aligned} \tfrac{d}{dt} D(\mathbf{p}(t)) &= 2\textstyle\sum_{n=1}^{N} \left[p_n(t) - p_n^*\right] \frac{dp_n(t)}{dt} \\ &= 2\alpha \textstyle\sum_{n=1}^{N} \left[p_n(t) - p_n^*\right] \bar{d}_n(\mathbf{p}(t)) \;, \\ &= -2\alpha \mathbf{p}^* \cdot \bar{\mathbf{d}}(\mathbf{p}(t)) \leq 0 \end{aligned}$$

where the second equality uses the formula for the adjustment process and the last equality uses Walras' Law ($\mathbf{p}(t) \cdot \bar{d}(\mathbf{p}(t)) = 0$). The inequality at the end follows from the weak axiom and is strict unless $\bar{\mathbf{d}}(\mathbf{p}(t)) = 0$. The result follows from the fact that D is a Lyapunov function. □

Since the proof above uses concepts from dynamical systems that you may not have seen, let's discuss what being a Lyapunov function means and why it implies that the dynamical system converges. The proof shows that there are two possible cases: (1) we are already at a competitive equilibrium ($\mathbf{p}(t) = \mathbf{p}^*$) and prices remain unchanged, or (2) we are not at a competitive equilibrium $\mathbf{p}(t) \neq \mathbf{p}^*$ and prices are adjusted so the distance between the current price and the equilibrium price vector decreases. A classic result in dynamical systems (Lyapunov's second method) shows that such systems are stable.

In addition to existence and stability, another desirable property of economies with the gross substitutes is that they have nice comparative statics properties. Any parameter that raises aggregate demand or reduces the endowment of one good (while keeping the aggregate demand and endowment of other goods unchanged) leads to an increase in the relative price of that good. This is not true in general. The next exercise asks you to prove this result for economies with two goods (a general proof can be constructed using the monotone methods that we will see later in the course).

Exercise 20. Consider an economy with $N = 2$ goods, with aggregate excess demands $\bar{d}_1(\mathbf{p}, \theta)$ and $\bar{d}_2(\mathbf{p})$ with the gross substitutes property. Let $\theta \in \mathbb{R}$ be a parameter that shifts the excess demand for good 1: $\bar{d}_1(\mathbf{p}, \theta_H) > \bar{d}_1(\mathbf{p}, \theta_L)$ for all $\theta_H > \theta_L$. Let $\mathbf{p}^*(\theta)$ denote a competitive equilibrium price vector. Show that $\frac{p_1^*(\theta)}{p_2^*(\theta)}$ is increasing in θ.

4.2 Differentiable Approach to Competitive Equilibrium

Suppose utility functions are twice differentiable, strictly increasing, strictly concave, and satisfy the Inada condition, so each consumer's utility maximization program is characterized by its first-order conditions.

$$\frac{\partial u_i}{\partial x_n^i}(\mathbf{x}^i) = \lambda^i p_n \ \forall i \in I, \ \forall n \in N$$

$$\sum_{n=1}^{N} p_n \left(x_n^i - e_n^i\right) = 0 \quad \forall i \in I.$$

A competitive equilibrium is then characterized by the first-order conditions above along with the $N-1$ market clearing conditions.

$$\sum_{i=1}^{I} \left(\mathbf{x}_n^i - \mathbf{e}_n^i\right) = 0 \quad n = 2, ..., N.$$

Note that I omitted the market clearing condition for good 1 in the previous equation, which is ensured by Walras' law.

Example 2. The simplest general equilibrium model has a single consumer $I = 1$.[10] Since

[10]The argument also holds with $I > 1$ identical consumers (with strictly increasing and strictly concave utility functions).

there is no one to trade with, the equilibrium allocation must coincide with the consumer's endowment for markets to clear. But even though no trade happens in equilibrium, we can still calculate equilibrium prices. To ensure that the consumer prefers to stick to her endowment, relative prices must be equal to consumer's MRS between goods. Due to its simplicity, this is a convenient model of asset prices (Lucas, 1978).

Figure 5 shows the equilibrium with of $N = 2$ goods. The orange line shows all consumption bundles that exhaust the individual's wealth ("budget line"). The budget line must go through the individual's endowment, since she can always afford to consumer her endowment. In fact, in equilibrium the individual must prefer to consume her endowment. Therefore, the budget line must be tangent to the consumer's indifference curve going through the endowment. Recall from consumer theory that the slope of the budget line equals the price ratio. Therefore, relative prices in the competitive equilibrium are given by the consumer's MRS between the goods (slope of indifference curve).

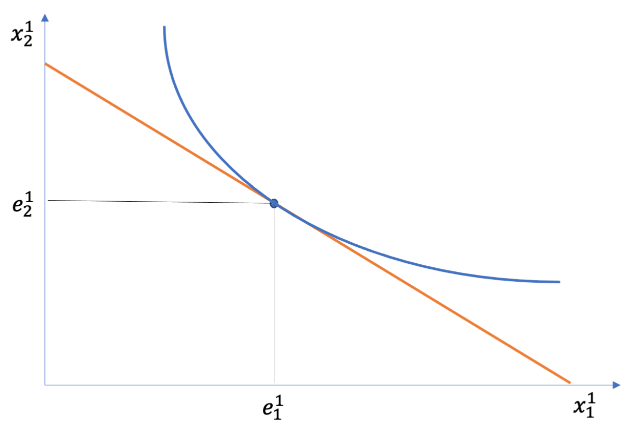

Figure 5: Equilibrium with One Consumer and Two Goods

Example 3. The model with a single consumer is also convenient to illustrate the role that convex preferences play on equilibrium existence. Consider again the model with $N = 2$ goods but suppose the utility function is strictly convex (so preferences are not convex). As Figure 6 shows, there are no prices that support an interior endowment as an optimal consumption bundle. The tangency point is actually a minimum: it gives the worst bundle that exhausts the individual's wealth. Since for any price vector, the individual prefers to consume only one good, we cannot support an interior endowment as a competitive allocation.[11,12]

[11]Note that the individual's demand is discontinuous: as the relative price of good 1 increases, the consumer'

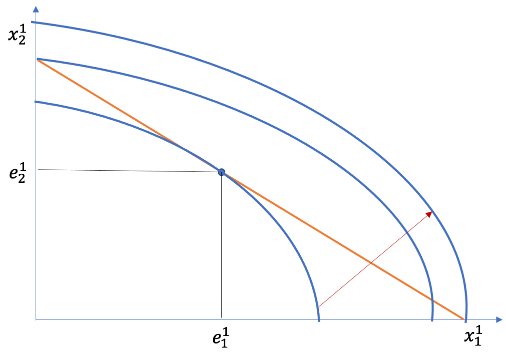

Figure 6: Economy with Non-Convex Preferences and no Equilibrium

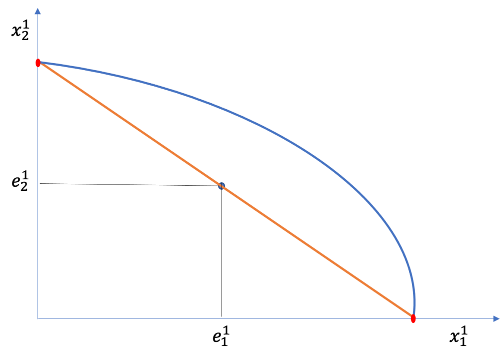

Figure 7: Existence with a Continuum of Consumers and Non-Convex Preferences

With a continuum of consumers, a competitive equilibrium exists even when preferences are not convex (Aumann, 1966). Figure 7 illustrates how an equilibrium can be restored with a continuum of consumers. As before, because the utility function is strictly convex, each of the identical individuals prefers to consume only one good. Pick relative prices that make them indifferent between spending their entire wealth in either of the two goods. (the red points on each axis). Each consumer is indifferent between these two points (but not by any other point on the budget line, which buys positive amounts of both goods).

By picking the mass of consumers who buy good 1 and good 2 appropriately, we can ensure that the market clears. For example, if half of the individuals consumes $(2, 0)$ and the other half consumes $(0, 2)$, aggregate consumption equals $(1, 1)$. To summarize, with a continuum of consumers and non-convex preferences, each of them can choose a bundle different from the endowment. By picking picking prices that make them indifferent between these points and choosing the proportions appropriately, we can ensure that markets clear.

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moves from spending the entire wealth in good 1 to spending the entire wealth in good 2. Continuity of demand plays an important role in the proof of equilibrium existence (see Section 7).

[12]Araujo et al. (2018) assumes that some individuals have strictly convex utility whereas others have strictly concave utility. An equilibrium exists if consumers with strictly concave utility are some “sufficiently rich”.

Example 4. Figure 8 depicts the equilibrium conditions in the 2x2 case in the Edgeworth box. The figure on the left represents the endowment of each consumer. The orange line represents, for this endowment and for a fixed price vector, all allocations in which both consumers exhaust their wealth. The figure on the right adds the indifference curves of both consumers going through their endowments. At the current price vector, the individuals would not be optimizing if they chose to consume their endowments. Since they have different marginal rates of substitution, A would like to trade good 2 for 1, whereas B would like to trade good 1 for 2.

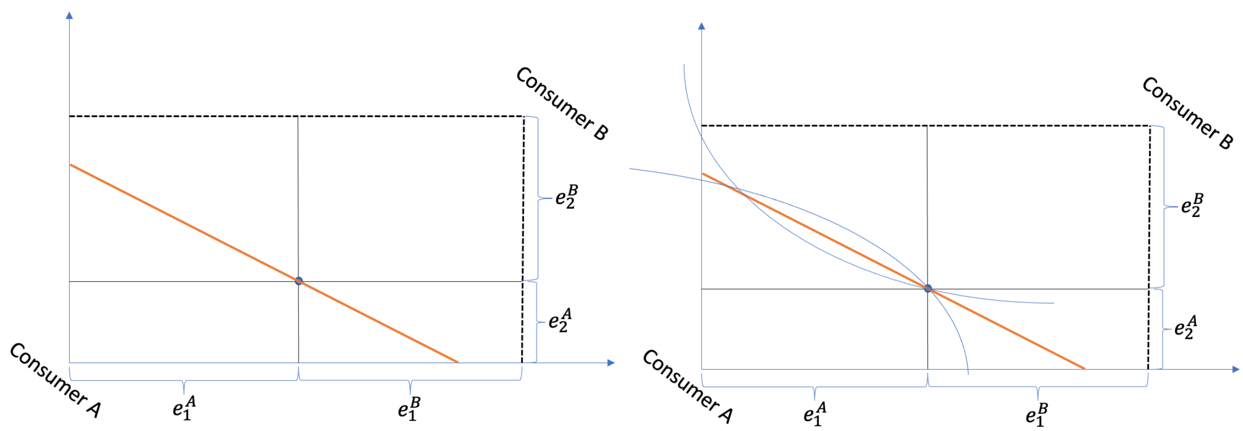

Figure 8: Indifference Curves going through the Endowment in the Edgeworth Box

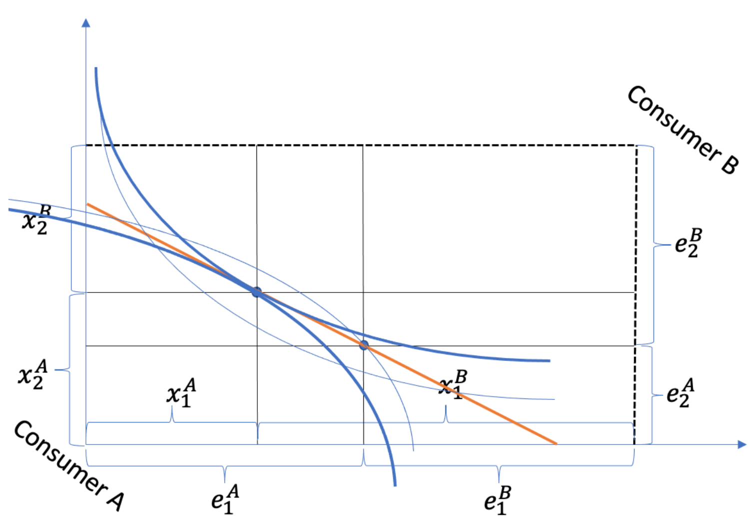

Figure 9: Competitive Equilibrium in the Edgeworth Box

Figure 9 shows the competitive equilibrium allocation $\mathbf{x}$. Note that in the competitive equilibrium, each consumer's MRS is equal to the price, so their indifference curves must be tangent to each other as well as to each other's budget constraints.

Exercise 21. As in Exercise 5, suppose there are two goods and two individuals with the same utility function:

$$u(x_1, x_2) = \log x_1 + \log x_2.$$

The individual endowments are $e^A = (1, 0)$ and $e^A = (0, 1)$. Calculate the competitive equilibrium allocation. Is this allocation Pareto efficient?

Exercise 22. Consider the economy from Exercise 10. Suppose the endowment of consumer 1 is $\left(e_0^1, e_{1,1}^1, e_{1,2}^1\right) = (1, 2, 0)$ and the endowment of consumer 2 is $\left(e_0^1, e_{1,1}^1, e_{1,2}^1\right) = (1, 0, 2)$. Calculate the competitive equilibrium. Is the competitive equilibrium Pareto efficient?

Exercise 23. Now suppose the two consumers may disagree about the likelihood of each state. Individual i has utility function:

$$u_i\left(\mathbf{x}^i\right) = \log(x_0^i) + \pi_1^i \log\left(x_1^i\right) + \left(1 - \pi_1^i\right) \log\left(x_2^i\right),$$

where $\pi_s^i \in (0, 1)$ denotes i's subjective probability of state s. Endowments are the same as in the previous exercise.

a. Suppose the consumers agree on the likelihood of each state $\pi_s^1 = \pi_s^2 = \pi_s \in (0, 1)$. Calculate the the competitive equilibrium.

   In the remainder of the question, suppose the consumers may disagree about the likelihood of each state.

b. Calculate the set of Pareto efficient allocations. Do Pareto efficient allocations fully insure each consumer? Justify.

c. Calculate the competitive equilibrium.

d. Is the competitive equilibrium Pareto efficient?

Consider first the economy with 2 consumers (labeled A and B) and 2 goods (labeled 1 and 2). Consider the function $F : \mathbb{R}^7_{++} \to \mathbb{R}^7$ given by:

$$F\left(x_1^A, x_2^A, x_1^B, x_2^B, \lambda^A, \lambda^B, p_2\right) = \begin{bmatrix} \frac{\partial u_A}{\partial x_1^A} - \lambda^A \\ \frac{\partial u_A}{\partial x_2^A} - \lambda^A p_2 \\ \frac{\partial u_B}{\partial x_1^B} - \lambda^B \\ \frac{\partial u_B}{\partial x_2^B} - \lambda^B p_2 \\ x_1^A - e_1^A + p_2\left(x_2^A - e_2^A\right) \\ x_1^B - e_1^B + p_2\left(x_2^B - e_2^B\right) \\ x_1^A - e_1^A + x_1^B - e_1^B \end{bmatrix}.$$

In words: the function F combines the first-order conditions for each consumer, the budget constraints for each consumer, and one of the two market clearing conditions. It imposes the $p_1 = 1$ normalization and omits one market clearing condition, which is redundant by Walras' law. The solution to the system of equations $F\left(x_1^A, x_2^A, x_1^B, x_2^B, \lambda^A, \lambda^B, p_2\right) = 0$ characterizes the competitive equilibrium along with the Lagrange multipliers of each consumer's program.

Exercise 24. Consider the economy from Exercise 21. Write down the system of equations $F\left(x_1^A, x_2^A, x_1^B, x_2^B, \lambda^A, \lambda^B, p_2\right) = 0$. Show that it has a unique solution.

More generally, consider the original economy with I consumers and N goods under the assumptions made above, and let $\tilde{\mathbf{p}} = (p_2, p_3, ..., p_N)$. Since we normalize the price of good 1 to 1, we have $\mathbf{p} = (1, \tilde{\mathbf{p}})$. Let $F : \mathbb{R}^{N \cdot I}_{++} \times \mathbb{R}^I_{++} \times \mathbb{R}^{N-1}_{++} \to \mathbb{R}^{NI+I+N-1}$ be given by

$$F(\mathbf{x}, \lambda, \tilde{\mathbf{p}}) = \begin{bmatrix} Du_1\left(\mathbf{x}^1\right) - \lambda^1 \cdot (1, \tilde{\mathbf{p}}) \\ (1, \tilde{\mathbf{p}}) \cdot \left(\mathbf{x}^{\mathbf{1}} - e^1\right) \\ \vdots \\ Du_I\left(\mathbf{x}^I\right) - \lambda^I \cdot (1, \tilde{\mathbf{p}}) \\ (1, \tilde{\mathbf{p}}) \cdot \left(\mathbf{x}^I - e^I\right) \\ \sum_{i=1}^I \left(x_2^i - e_2^i\right) \\ \vdots \\ \sum_{i=1}^I \left(x_N^i - e_N^i\right) \end{bmatrix}.$$

The solution to the system of equations $F(\mathbf{x}, \lambda, \tilde{\mathbf{p}}) = 0$ characterizes the competitive equilibrium along with the Lagrange multipliers of each consumer's program. It implicitly determines the values of the endogenous variables $\mathbf{x}, \lambda, \tilde{\mathbf{p}}$ as a function of parameters such as the endowments. Whenever the Jacobian matrix of F has full rank, we can apply the Implicit Function Theorem to study how the equilibrium changes with the parameters of the model.

Debreu (1970) shows that for "almost all" endowments $\mathbf{e} \in \mathbb{R}^I_+$, the matrix DF has full rank and that whenever DF has full rank the set of equilibria is finite:

Theorem 2. *Suppose utility functions are twice differentiable, strictly increasing, strictly concave, and satisfy the Inada condition. Then, except at a set of endowments with Lebesgue measure zero, there are finitely many equilibria.*

4.3 "Anything Goes" Results

Recall that any aggregate excess demand function $\bar{\mathbf{d}}$ must be continuous, homogeneous of degree zero, and satisfy Walras' law: $\mathbf{p}\cdot\bar{\mathbf{d}}(\mathbf{p}) = 0$. The theorem below shows that any function with these properties is an aggregate excess demand function of a "well behaved" economy:

Theorem 3. *(Sonnenschein-Mantel-Debreu) Let B be an open and bounded subset of $\mathbb{R}^N_{++}$ and let $f : B \to \mathbb{R}^N$ be a continuous, homogeneous of degree zero function satisfying Walras' law. There exists continuous, strictly convex, and monotone utility function and an initial endowment for each of $I \geq N$ consumers such that the aggregate excess demand function is $\bar{\mathbf{d}} = f$.*

The previous result shows that not much can be said about aggregate excess demand functions. We shouldn't expect uniqueness or stability to hold in general. We should also not expect aggregate demands to behave like individual demands. Since any continuous, homogeneous of degree zero function satisfying Walras' law is an aggregate demand function, there are no general aggregate counterparts for the weak axiom or Slutzky's equation. In particular, if we want to represent the aggregate demand in an economy to behave as if it was derived by the maximization of the utility of a single "representative agent", we would need stronger restrictions on preferences.[13]

What about the set of competitive equilibrium prices? Since the aggregate excess demand function is continuous, the set of its roots (i.e., the set of equilibrium prices) must be closed. The next result shows that this is essentially all that can be said. To simplify the statement of the theorem, it is convenient to normalize prices so they add up to 1, letting $P \equiv \{\mathbf{p} \in \mathbb{R}^N_{++} : \sum_{n=1}^N p_n = 1\}$ denote the set of interior (normalized) prices.

Theorem 4. *(Mas Colell, 1977). Let Q be a closed subset of P. There exist continuous, strictly convex, monotone utility functions and initial endowments for each of $I \geq N$ consumers, that has Q as the set of equilibrium equilibrium prices.*

These two theorems are referred to as "Anything Goes" theorems. They are usually interpreted as saying that competitive equilibrium has no testable implications. But this interpretation is wrong. Sonnenschein-Mantel-Debreu says that competitive equilibrium places almost no restrictions on aggregate excess demand. But aggregate excess demand is usually

[13] The most common restriction that allows for aggregation of preferences is that the preference of each consumer admits a "Gorman form". This is a strong condition and we won't study it here, but it's important in Macro.

not observable. Mas Colell's theorem says that if we only observe a set of prices, we can never reject that they are equilibrium prices from a competitive model. But a typical data set includes things beyond prices, such as individual consumption, individual endowments, sometimes even preferences. Can we reject the competitive model if we include additional data?

The answer is yes. For example, if we have data on everyone's endowments and we elicit everyone's preferences, we can directly calculate the competitive equilibrium. We can reject any price or consumption that doesn't match our calculations. Even if we don't elicit preferences, there are still testable predictions from competitive equilibrium. Brown and Matzkin (1996) show that if we observe prices and individual endowments, competitive equilibrium has testable implications. Additional restrictions can be placed if we also observe individual consumption.

So one should not conclude from these theorems that competitive equilibrium cannot be tested. Instead, the main empirical takeaway should be that we need micro data to test the model.

5 Welfare Theorems

We now turn to our second key question: what is the relationship between equilibrium and Pareto efficiency? The theorem below establishes that any competitive equilibrium is Pareto efficient:

Theorem 5 (First Welfare Theorem.)**.** *Suppose preferences are locally non-satiated. If x^* is a competitive equilibrium allocation, then x^* is efficient.*

Proof. Let $x \in \mathbb{R}^{N \cdot I}$ be an allocation that Pareto dominates x^*. I claim that x is not feasible.

Let e be the endowment. Since x^* is a competitive equilibrium allocation, there is a price vector $p^* \in \mathbb{R}^N_+$ such that (x^*, p^*) is a competitive equilibrium. Since x Pareto dominates x^*, we must have $x^i \succsim_i x^{i^*}$ with strict preference for some i (say, $i = 1$). Then, because x^1 maximizes consumer 1's utility, x^1 must not be affordable:

$$p^* \cdot x^1 > p^* \cdot e^1. \tag{10}$$

I claim that, for each other consumer i, x^i must cost at least as much as $i's$ endowment. The proof is by contraposition. Suppose $p^* \cdot x^i < p^* \cdot e^*$. Then, by continuity of the budget set and local non-satiation, there exists $\tilde{x}^i$ such that $\tilde{x}^i \succ_i x^i$ and $p^* \cdot \tilde{x}^i < p^* \cdot e^*$. But this contradicts the assumption that $x^{i*} \precsim_i x^i$ solves i's consumer program. Therefore, for each $i \neq 1$,

$$p^* \cdot x^i \geq p^* \cdot e^i. \tag{11}$$

Adding inequalities (10) and (11), we find that

$$\sum_{i=1}^{I} p^* \cdot x^i > \sum_{i=1}^{I} p^* \cdot e^i,$$

so that x is not feasible. □

The key argument in the proof of the theorem is deceptively simple: if a better bundle was available to someone at market-clearing prices, then it must cost more (otherwise, that person would be buying such a bundle). This proof is due to Arrow (1951) and Debreu (1951). The first proofs of the First Welfare Theorem are due to two LSE economists, Abba Lerner (1934) an Oscar Lange (1942), and by the French Economist Maurice Allais (1943). Their proofs use the tangency conditions for interior solutions that we will study in the next subsection (Lerner provides a graphical analysis, whereas Lange and Allais use Lagrange multipliers and first-order conditions).[14]

The assumption that preferences are locally non-satiated is a very weak assumption. So, for the theorem to fail, one of the following conditions must be true:

1. The observed allocation is not a Competitive Equilibrium. This can be because of market power (some consumers do not take prices as given), because prices are not equilibrium prices (say, because of restrictions on prices), or because consumers are not picking their preferred allocations (say, because of quotas).

2. The complete markets assumption that is implied by the formulation above fails. Complete markets says that every good that shows up as an argument in a utility function has a price and can be traded in a market, and the proceeds from the sale of each good can be used to purchase any other good. For example, if the music played by your neighbor lowers your utility, complete markets requires that you be able to control that music (in volume, quantity, and variety) through a market purchase. Moreover, both you are your neighbor must be price takers in such transaction (which is unlikely to be the case when you have very few neighbors).

While the First Welfare Theorem shows that every Competitive Equilibrium allocation is efficient, our next result, the Second Welfare theorem, shows that any efficient allocation can be interpreted as a Competitive Equilibrium allocation.

Theorem 6. *Suppose preferences are convex. If* $\mathbf{x}^*$ *is a Pareto efficient allocation, then there exists a price vector* $\mathbf{p}^*$ *such that* $(\mathbf{p}^*, \mathbf{x}^*)$ *is a competitive equilibrium of the economy with endowment* $\mathbf{e}^* = \mathbf{x}^*$*.*

[14] Arrow criticizes their approach, mentioning that the assumptions obfuscate the conditions required for the validity of the Welfare Theorems and suggest that the result would only hold at interior solutions.

We will provide a proof of Theorem 6 in the differentiable case, following Lange (1942). The general proof, due to Arrow (1951) and Debreu (1951), dispenses the differentiability conditions (but still requires convexity).

The First Welfare Theorem is often interpreted as supporting Adam Smith's idea that a society works efficiently when individuals act by selfishly maximizing their own interests, establishing the superiority of markets. But this is not the only possible interpretation of the theorems. In fact, marxist economists such as Taylor (1929), Lerner (1934b), and Lange (1936) have argued, based on the same welfare theorems, that governments can achieve the same outcome as any ideal competitive market. Given the difficulty in enforcing perfect competition, why rely on markets?

Note also that the Second Welfare Theorem states that, to implement a Pareto efficient allocation as a competitive equilibrium, one needs to transfer resources so as to give each individual an endowment equal to the efficient allocation. But if we can redistribute resources to an efficient allocation, why do we need markets? In fact, if this could be done, markets would be useless since no one would trade anyway.

The competitive model considered here cannot answer these questions. It does not include any of the difficulties that actual governments face when running planned economies or any market failures. And it doesn't consider the incentive issues associated with actual taxes, as studied in the Public Finance literature. Still, the theorems are conceptually important as a benchmark and can be useful in as modeling tricks as well.

Models of optimal taxation, for example, often start with a "first-best" benchmark which essentially replicates the Second Welfare Theorem, finding lump sum taxes that maximize a social welfare function. This benchmark is then contrasted with what can be achieve once incentive issues are introduced. As a modeling trick, in many models in Macro, it is hard to establish the existence and characterize a competitive equilibrium. It is sometimes easier to solve the Pareto program and show a version of the Second Welfare Theorem.

5.1 Differentiable Approach to Welfare Theorems

As in Subsection 4.2, suppose preferences each consumer's utility function is twice differentiable, strictly increasing, strictly concave, and satisfies the Inada condition. Then, the competitive equilibrium $(\mathbf{p}, \mathbf{x})$ is characterized by each consumer's first-order conditions and the market clearing conditions:

$$\frac{\partial u_i}{\partial x_m^i}(\mathbf{x}^i) = \lambda^i p_m \ \forall i, m, \tag{12}$$

$$\sum_{m=1}^{M} p_m \left(x_m^i - e_m^i \right) = 0 \ \ \forall i, \tag{13}$$

$$\sum_{i=1}^{I}\left(\mathbf{x}_m^i - \mathbf{e}_m^i\right) = 0 \quad m = 2, ..., M. \tag{14}$$

Consider the Welfare Maximization program associated with a vector of utility weights $\alpha = (\alpha^1, ..., \alpha^I)$. As shown in Subsection 2.3, that the Pareto efficient allocations are determined by the following first-order conditions:

$$\alpha_i \frac{\partial u_i}{\partial x_m}(\mathbf{x}_i) = \xi_m \quad \forall m \in M, \forall i \in I, \tag{15}$$

and

$$\sum_{i=1}^{I}\left(x_m^i - \bar{e}_m\right) = 0 \quad \forall m \in M. \tag{16}$$

We now use these conditions to obtain constructive proofs of the welfare theorems.

First, consider the Second Welfare Theorem. Let x^* be a solution to the welfare maximization program for some utility weight α and let ξ_m^* denote the Lagrange multiplier associated with this solution. Take $\mathbf{e}^* = \mathbf{x}^*$. Note that (13) and (14) are satisfied. Letting $p_m = \xi_m^*$ and $\lambda^i = \frac{1}{\alpha^i}$, it follows that (12) also holds. Therefore, we can support the Pareto efficient allocation $\mathbf{x}^*$ as a competitive equilibrium allocation of the economy with endowment $\mathbf{e}^* = \mathbf{x}^*$.

Next, consider the First Welfare Theorem. Suppose $(\mathbf{x}^*, \mathbf{p}^*, \lambda)$ solve (12), (13), and (14), so $(\mathbf{x}^*, \mathbf{p}^*)$ is a competitive equilibrium, and note that $\lambda >> 0$ and $\mathbf{p}^* >> 0$ (since $\frac{\partial u_i}{\partial x_m} > 0$ for all i, m). Letting $\alpha^i = \frac{1}{\lambda^i}$ and $\xi_m = p_m^*$, we find that equations (15) and (16) hold, so $(\mathbf{x}^*, \mathbf{p}^*)$ is a Pareto efficient allocation.

The constructive arguments above shed light on the utility weights associated with the competitive equilibrium that supports each Pareto efficient allocation. Note that λ^i in equation (12) is consumer i's marginal utility of income (the shadow cost associated with i's budget constraint). In particular, if all consumers have the same utility function, poorer consumers have higher marginal utility of income λ^i. Note also that the competitive equilibrium coincides with welfare maximization with utility weights $\alpha^i = \frac{1}{\lambda^i}$. So the construction above shows that the equilibrium allocation assigns consumer weights that are inversely proportional to their marginal utility of income. Again, if consumers have the same utility function, poorer consumers have a lower weight. This is not a statement about how things should be. It is a statement about how things work in a competitive equilibrium: someone with little wealth (because the value of his endowment is low) is treated in equilibrium as though he is less important.

Exercise 25. Consider the following economy with two consumers (A and B) and two goods (1 and 2). Each consumer has an endowment of one unit of each good: $\mathbf{e^i} = (1, 1)$. The utility functions are:

$$u_A = \ln x_1^A + \ln x_2^A, \text{ and } u_B = \ln x_1^B + \ln x_2^B + \ln x_1^A.$$

Note that individual B's utility also depends on A's consumption of good 1. That is, the consumption of good 1 by individual A generates a positive externality on individual B. Budget constraints are as defined previously, meaning that there is no market for this externality (B cannot "bribe" A to increase the consumption of good 1).

a. Calculate the competitive equilibrium of this economy.

b. Is the competitive equilibrium efficient? Explain.

6 Financial Securities

In our approach to risk in Section 3, we assumed that each consumer had a single budget constraint that incorporated all trades of state-dependent goods in the initial period. No trade happens after the initial period. This assumption seems at odds with trades in market economies, which often occur sequentially over time. Trades involving consumption in distant states may not be available at certain periods.

To deal with this shortcoming, Arrow (1964) and Radner (1972) considered models that introduce financial securities. In his model, in each period, only some goods and financial securities are traded. Each security is described by a vector of nominal payments, which specifies the amount that the seller of the security needs to pay to the buyer of the security in each period and state of nature.

A *financial equilibrium* is a price vector for each good and security and a set of consumption and portfolio choices for each individual such that each individual picks consumption and portfolio optimally given prices, and all markets (both of goods and securities) clear. Arrow shows that, when there are enough securities, the financial equilibrium allocations coincide with the competitive equilibrium allocations.

6.1 Basic Model

Consider an economy with the same assumptions as in Section 3. There are A financial securities (or "assets"), indexed by $a = 1, ..., A$. Each asset a consists of a price q_a in period 0 and a payment y_a^s in each state of nature s in period 1. Let $Y = (\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_A) \in \mathbb{R}^{S \times A}$ denote the matrix of asset payments.

In period 0, each individual has access to markets for period-0 goods and asset markets. Each consumer picks period-0 consumption and a portfolio. In period 1, uncertainty is resolved and a state of nature is observed. New markets are opened and agents finance their consumption using their period-1 endowment and the proceedings from their portfolios. In model with more periods, asset markets would then reopen in period 1 and individuals would choose a new portfolio, allowing them to transfer resources to a third period.

Given the prices of goods $\mathbf{p}$ and prices of financial assets $\mathbf{q}$, each individual i chooses a consumption path $\mathbf{x}^i$ and a portfolio $\mathbf{b}^i$ to solve:

$$\max_{\mathbf{x}^i,\mathbf{b}^i} u_i(\mathbf{x}^i) \text{ subject to } \begin{cases} \mathbf{p}_0\left(\mathbf{x}_0^i - \mathbf{e}_0^i\right) + \mathbf{q}\mathbf{b}^i = 0 \\ \mathbf{p}_s\left(\mathbf{x}_s^i - \mathbf{e}_s^i\right) - \mathbf{y}_s\mathbf{b}^i = 0, \quad s = 1, ..., S \end{cases}. \tag{17}$$

Let $\Pi \equiv \begin{bmatrix} -\mathbf{q} \\ Y \end{bmatrix}$ denote the matrix of "returns" from the financial assets.[15] If an individual purchases a portfolio $\mathbf{b} \in \mathbb{R}^A$, the first line of the vector $\Pi\mathbf{b}$, corresponds to the amount to be paid in period 0 (the "cost" of the portfolio$-\mathbf{qb}$). The s-th line of the vector, $\mathbf{y}_s\mathbf{b}$, corresponds to the payments the individual receives in state s.

Suppose there exists a portfolio $\mathbf{b}$ such that $\Pi b > 0$, meaning that each line of the vector of payments from the portfolio is non-negative with at least one of them being strictly positive. Then, the portfolio b yields strictly positive payments in some state $s \in \{0, 1, ..., S\}$ and never yields a negative payment. An individual can then obtain arbitrarily large incomes in state s at no cost by choosing portfolio λb where $\lambda > 0$ is large enough. If the individual has locally non-satiated preferences at state s, the optimization program 17 will have no solution: any fixed consumption bundle can be improved in state s without changing the consumption in other states.

Definition 6. A matrix of returns Π satisfies no arbitrage if there is no $b \in \mathbb{R}^A$ such that $\Pi b > 0$.

As argued previously, no arbitrage is necessary for the consumer program to have a solution. It plays an important role in asset pricing, as it allows us to price redundant assets using the price of a replicating portfolio (such as in the Black-Scholes formula for option prices).[16]

If $\text{rank}(Y) = S$ and there is no arbitrage, each consumer can obtain any consumption bundle by carefully picking a portfolio. Therefore, we say that the economy has *complete financial markets* if $\text{rank}(Y) = S$. We say that the economy has *incomplete financial markets if* $\text{rank}(Y) < S$. With complete financial markets, each consumer's program is "equivalent"

[15] Note that the model assumes the payments from each asset are specified in units of account (asset a pays y_s^a "dollars" in state s). In other words, this corresponds to a model with nominal assets. If, instead, assets specified an amount of goods to be delivered in each state, the matrix of returns Π would also depend on the prices of these goods. In this type of models, which we will not study in this course, assets are real.

[16] A replicating portfolio for a given asset is a portfolio of assets with the same payment in each state of nature.

to the one in Section 3 (where all trades of state-contingent consumption happened in period 0) in the following sense. By setting prices appropriately, the set of affordable consumption bundles in program 17 coincides with the ones in Section 3.

Exercise 26. Consider an economy with two states: $S = 2$. There are two assets: a bond that pays $y_1 = (1, 1)$ and a risky stock that pays $y_2 = (0, 1)$.

a. Show that each consumer can obtain any bundle of state-dependent consumption.

b. Show that there exists a price vector for the economy in Section 3 such that the consumption bundle chosen by the consumer in that economy coincides with the consumption bundle that solves program 17.

c. Show that there exist prices of goods $\mathbf{p}$ and of financial assets $\mathbf{q}$ such that the consumption bundle that solves program 17 coincides with the consumption bundle that solves the consumer program for the economy in Section 3.

It is sometimes convenient to assume that matrix Y has linearly independent columns. This assumption means that there are no redundant assets, so that we cannot replicate the payment of an asset by purchasing a bundle of other assets. If no arbitrage holds, the price of any redundant asset must equal the price of the bundle of other assets that replicates its payments. Therefore, under no arbitrage, removing redundant assets does not affect the set of feasible consumption by each individual.

Proposition 8. *Suppose Y has linearly independent columns. The matrix Π satisfies no arbitrage if and only if there exists $\lambda \in \mathbb{R}^S_{++}$ such that*

$$\sum_{s=1}^{S} \lambda_s y_s = \mathbf{q}.$$

Moreover, if rank$(Y) = S$, there exists another matrix of asset payments $\tilde{Y}$ that gives the same consumption bundle as the solution to each consumer's program as Y, and

$$\sum_{s=1}^{S} \lambda_s = 1.$$

This proposition follows from Farkas' Lemma. The vector λ allows us to price assets in this economy. In particular, λ_s is the price of an asset that pays 1 in state s and 0 elsewhere. This asset is known as an "Arrow security". A bond that pays 1 in every state costs $\sum_{s=1}^{S} \lambda_s$. Although the proposition above ensures the existence of state-price vector λ (often referred to as a "pricing kernel"), it does not tell us how to compute it. A lot of work in asset pricing involves obtaining the state-price vector λ.

The second part of Proposition 8 shows that when Y has full rank, we can work with a state-price vector that adds up to 1. This is sometimes interpreted as the probability of state s. But note that while this is a probability distribution (a positive vector that adds up to 1), it does not need to (and typically does not) coincide with the subjective probability that each individual attaches to state s. Under this probability distribution, the price of any financial asset can be obtained by taking expectations:

$$q = E_\lambda[y] \equiv \sum_{s=1}^{S} \lambda_s y_s,$$

where E_λ is the expectations operator according to probability λ. Therefore, an economy with no arbitrage and with sufficiently rich financial assets (i.e., when Y has rank S), there exists a probability vector such that the price of each asset equals the expected payments of this asset. This probability measure is known as the *equivalent martingale measure.*

6.2 Equilibrium

Note that, so far, we have only considered the consumer's problem. We now consider the equilibrium:

Definition 7. A financial equilibrium is a set of good prices $\mathbf{p}$, asset prices $\mathbf{q}$, and an allocation of goods $\mathbf{x}$ and portfolio $\mathbf{b}$ such that:

1. Each individual i picks $(\mathbf{x}^i, \mathbf{b}^i)$ to solve (17),
2. Goods market clear: $\mathbf{p} \cdot \left(\sum_{i=1}^{I} \mathbf{x}^i - \sum_{i=1}^{I} \mathbf{e}^i \right) = 0$ with $\sum_{i=1}^{I} \mathbf{x}_m^i = \sum_{i=1}^{I} \mathbf{e}_m^i$ if $p_m > 0$
3. Assets market clear: $\sum_{i=1}^{I} \mathbf{b}^i = 0$.

Each individual in this model faces $S+1$ budget constraints. However, when the economy has complete financial markets, we rewrite them as a single budget constraint involving state-depends goods only (rather than both goods and assets) and the consumer's optimization problem coincides with the one in the Walrasian model (Section 3). The theorem below states this result formally:

Theorem 7 (Arrow's equivalence theorem.)**.** *Suppose rank$(Y) = S$. Then, the set of allocations in financial equilibria is the same as the ones in competitive equilibria of the Walrasian model (Section 3).*

Proof. The details of the proof are left as an exercise. You should follow similar steps as Exercise 8. Fixing prices for state contingent goods in the Walrasian model, we can find prices for goods and commodities such that the set of affordable consumption bundles. Similarly, fixing prices for goods and commodities in the financial model, we can find prices for state contingent goods in the Walrasian model such that the set affordable consumption bundles coincide. □

Put differently, Arrow's equivalence theorem states that if $(\mathbf{p}, \mathbf{q}, \mathbf{x}, \mathbf{b})$ is a financial equilibrium of the economy with complete financial markets if and only if there exists a price vector $\tilde{\mathbf{p}}$ such that $(\tilde{\mathbf{p}}, \mathbf{x})$ is a competitive equilibrium of the economy. It implies that when financial markets are complete, a financial equilibrium exists and, moreover, every financial equilibrium is Pareto efficient.

Despite the equivalence of equilibrium allocations when there are complete financial markets, a financial economy operates quite differently from a Walrasian economy. The Walrasian model assumes the existence of markets for all goods at period 0, including those who will only be consumed in the future. The economy operates as if all trades happen simultaneously in period 0 and no new trades occur.

In the model with financial assets, trades occur sequentially. In each period, only commodities that will be consumed in that period are traded. When individuals make plans about what to consume in each state in future periods, they have conjectures about the price of each good in each future state. A financial equilibrium therefore requires all individuals to correctly anticipate those future prices. This concept in which individuals correctly anticipate the future prices in each state is known as a *rational expectations equilibrium*. The Walrasian model does not need to make any assumptions about expectations of future prices, since all markets are open at time zero.

Another important difference between the Walrasian model and the model with financial assets is the *timing* of decisions. In the Walrasian model, all consumption decisions are made in period 0. In the model with financial assets, individuals make their consumption decisions in each period. If consumers are *dynamically inconsistent*, they choose differently in different times. For example, a "present biased" individual prefers to consume more today and start saving tomorrow. When tomorrow arrives, he changes his mind and again prefers to consume more tomorrow and start saving on the following day.[17] Even with complete financial markets, the equilibrium consumption may be different in the Walrasian and the financial assets models if individuals are dynamically inconsistent. Dynamic inconsistency is ruled out in the model considered here because we assume that the individual maximizes the same utility function in both cases.

Exercise 27. Consider an economy with $S = 2$ states, and a single good in each state $(N = 1)$. There are two assets, a risk free bond that pays $y_1 = (1, 1)$ and a risky stock that pays $y_2 = (0, 1)$. There are I identical individuals with endowment (e_0, e_1, e_2). Their

[17] See Laibson (1997) and O'Donoghue and Rabin (1999) for models of dynamically inconsistent consumers.

(identical) utility functions are:

$$u(c_0, c_1, c_2) = \log(c_0) + \pi \log(c_1) + (1-\pi)\log(c_2).$$

Normalize the price of period-0 consumption to $p_0 = 1$. Calculate the equilibrium price of the risk free bond and the risky stock. [*Hint*: You can use the fact that, in equilibrium, all individuals will consume their endowment in each state.]

Exercise 28. Consider the same economy as before, except that there is only one asset: a risk free bond that pays $y_1 = (1,1)$. Normalize the price of period-0 consumption to $p_0 = 1$. Calculate the vector of equilibrium prices (i.e., both the price of the bond and the price of the good is states $s \neq 0$). Is the equilibrium price vector unique? [*Hint*: You can use the fact that, in equilibrium, all individuals will consume their endowment in each state.]

It is natural to ask whether the efficiency properties of financial equilibria continue to hold when financial markets are incomplete. Clearly, equilibria will generally not be Pareto efficient. For example, suppose there there are no financial markets whatsoever. Then, a financial equilibrium can only be efficient if, for some coincidence, each consumer's endowment in each state happens to be exactly what that consumer would have chosen to allocate to each state if with complete financial markets.

A less ambitious notion is that of constrained Pareto efficiency (Diamond, 1967): can a "social planner" with access to the same financial assets as the individuals choose an allocation that Pareto dominates the financial equilibrium? Hart (1975) shows that this can indeed happen. Therefore, when financial markets are incomplete, equilibria can be constrained Pareto inefficient. Moreover, Elul (1995) and Cass and Citanna (1998) show that when there are "more than one missing markets" (i.e., the rank of Y is less than $S-1$), it is possible to make all individuals strictly worse off by adding a new financial instrument. That is, reducing (but not eliminating) market incompleteness can lead to a Pareto dominated outcome.

These counterintuitive results belong to what is often referred to as the "general theory of the second best" after Lipsey and Lancaster (1956). Despite its name, this is not a general theory, but a list of results from different settings. The common theme is that when one variable is away from the social optimum, it may be desirable to move other variables away from the unconstrained optimum as well. In economic terms, it means that if it is impossible to remove a particular market distortion, there may exist new market distortions leading to Pareto improvements. This intuition plays an important role in Public Finance and International Trade.

7 Appendix: Proof of Existence

The proof of existence of a competitive equilibrium under weak conditions was one of the key results in 20th century economic theory. For almost a hundred years since Walras wrote down his model of general equilibrium, whether such an equilibrium existed was an open question. The first existence proofs under general enough conditions were due to Arrow and Debreu (1954) and another by McKenzie (1954). We consider a somewhat simplified version here.

To simplify the proof, I will assume that preferences are strongly monotone (although only local non-satiation is required for the result). To avoid necessary notation, suppose also that preferences are strictly convex so that individual demands are functions rather than correspondences. Generalizing it to correspondences does not require much additional work.

Let $x_n^i(\mathbf{p})$ denote consumer i's demand for good n. Let $d_n^i(\mathbf{p}) \equiv x_n^i(\mathbf{p}) - e_n^i$ denote i's *net demand* for good n. This is sometimes referred to as i's *individual excess demand* for n. Let

$$\bar{d}_n(\mathbf{p}) \equiv \sum_{i=1}^{I} d_n^i(\mathbf{p})$$

denote the *aggregate excess demand* for good n. The *aggregate excess demand* is the function $\bar{d} : \mathbb{R}_+^N \to \mathbb{R}^N$ given by $\left(\bar{d}_1(\mathbf{p}), ..., \bar{d}_N(\mathbf{p})\right)$.

Note that if there exists a price vector $\mathbf{p}^* \in \mathbb{R}_+^N$ such that $\bar{d}(\mathbf{p}^*) = 0$, then supply equals demand, so that $\mathbf{p}^*$is a competitive equilibrium price vector. Moreover, $\mathbf{p}^* \in \mathbb{R}_{++}^N$ is a competitive equilibrium price vector if and only if $\bar{d}(\mathbf{p}^*) = 0$. Our main result will establish the existence of a competitive equilibrium with strictly positive prices when preferences are strongly monotone, so that there will not be excess demand.[18]

Let B_{-0} denote the set of price vectors with at least one, but not all, coordinates equal to zero:

$$B_{-0} = \left\{\mathbf{p} \in \mathbb{R}_+^N \backslash \{0\} : \ p_m = 0 \text{ for some } m\right\}.$$

Lemma 2. *Let $\bar{d}$ be an aggregate excess demand function. Then:*

1. *$\bar{d}$ is continuous in $\mathbb{R}_{++}^N$.*

2. *$\mathbf{p} \cdot \bar{d}(\mathbf{p}) = 0$ for all $\mathbf{p}$.*

3. *$\bar{d}(\mathbf{p}) \geq -\bar{\mathbf{e}}$ for all $\mathbf{p}$*

4. *For any sequence $\{\mathbf{p}^t\}$ in $\mathbb{R}_{++}^N$ converging to $p \in B_{-0}$, we have that $\lim_{t\to\infty} \left\|\bar{d}(\mathbf{p^t})\right\| = +\infty$.*

[18] Note that this is stronger than needed for existence (as stated in Theorem 1). With local non-satiation, some economies may only have equilibria in which some prices are zero.

Proof. Property 1 follows from the continuity of Marshallian demand. Property 2 is Walras' Law. Property 3 follows from the fact that consumption is non-negative so excess consumption cannot be lower than the endowment.

For Property 4 note that if some, but not all, prices go to zero, some consumer's wealth must not be going to zero (since aggregate endowment is bounded away from zero). By strong monotonicity, this consumer's demand for the good whose price is going to zero must go to infinity as its price goes to zero. □

Lemma 3. *Let $\{\mathbf{p}^t\}$ be a sequence in $\mathbb{R}^N_{++}$ converging to $p \in B_{-0}$. Then:*

1. *For any strictly positive price for good n, $p_n > 0$, the excess demand for good n, $\bar{d}_n(\mathbf{p^t})$, is bounded.*

2. *For at least one good n with price converging to the boundary, $p_n = 0$, there is a subsequence $\{\mathbf{p}^{t_k}\}$ such that $\lim_{t\to\infty} \bar{d}_n(\mathbf{p^{t_k}}) = +\infty$.*

Proof. By Property 2, we have

$$p^t \cdot \bar{d}(p^t) = 0 \therefore p^t_n \bar{d}_n(\mathbf{p}^t) = -\sum_{l\neq n} p^t_l \bar{d}_l(p^t). \tag{18}$$

Since, by property (3), $p^t_n \bar{d}_n(p^t) \geq -p^t_n \bar{e}_n$ for any n, it follows that

$$-\sum_{l\neq n} p^t_l \bar{d}_l(p^t) \leq \sum_{l\neq n} p^t_n \bar{e}_n \leq \mathbf{p}^t \cdot \bar{e}. \tag{19}$$

Combine (18) and (19) to obtain:

$$p^t_n \bar{d}_n(\mathbf{p}^t) \leq \mathbf{p}^t \cdot \bar{e},$$

and, since $p^t_n > 0$,

$$\bar{d}_n(\mathbf{p}^t) \leq \frac{\mathbf{p}^t \cdot \bar{e}}{p^t_n}.$$

If $p_n > 0$, then, since $\mathbf{p}^t \to p$, it follows that there exits $T \in \mathbb{N}$ such that for all $t > T$,

$$\frac{\mathbf{p}^t \cdot \bar{e}}{p^t_n} < \frac{\mathbf{p} \cdot \bar{e}}{p_n} + 1,$$

which implies that $\bar{d}_n(\mathbf{p}^t)$ is bounded above (by the larger of $\frac{\mathbf{p}\cdot\bar{e}}{p_n} + 1$ and $\max_{t\leq T} \bar{d}_n(\mathbf{p}^t)$). By property (3), $\bar{d}_n(\mathbf{p}^t)$ is also bounded below. Thus, $\bar{d}_n(\mathbf{p}^t)$ is bounded, establishing claim 1.

We now turn to Claim 2. Since $\bar{d}_n(\mathbf{p}^t)$ is bounded below (property 3) and $\lim_{t\to\infty} \left\|\bar{d}(\mathbf{p^t})\right\| = +\infty$ (property 4), there exists at least one good n for which there is a subsequence $\mathbf{p}^{t_k}$ with

$\bar{d}_n(\mathbf{p^t}) \to +\infty$. Then, by the first claim, any such good must have zero price in the limit: $p_n = 0$. □

We now show the existence of a competitive equilibrium. The proof follows Hildenbrand and Kirman (1988):

Theorem 8. *Let $\bar{d}$ be an aggregate excess demand function. There exists $p^* >> 0$ such that* $\bar{d}(\mathbf{p}^*) = \mathbf{0}$.

Before presenting the proof, it is helpful to introduce some notation. Recall that what only relative prices matter, so we can always normalize prices as long as we do not "divide by zero" (meaning, we cannot set a zero price equal to a positive number).

In many applications, we choose a "numeraire" setting the price of a good equal to 1. This is usually a convenient normalization when calculating an equilibrium since it reduces the number of variables to be computed, but it has two drawbacks here. First, it requires us to know that the price of that good will not be zero in equilibrium. Second, the resulting space of prices is unbounded above and therefore not compact. So we use a different normalization here.

Let Δ denote the N-1 dimensional simplex, that is, the set of prices that add up to 1:

$$\Delta \equiv \left\{ \mathbf{p} \in \mathbb{R}_+^N : \sum_{n=1}^{N} p_n = 1 \right\}.$$

Let Δ_+ denote the interior of Δ:

$$\Delta_+ \equiv \left\{ \mathbf{p} \in \mathbb{R}_{++}^N : \sum_{n=1}^{N} p_n = 1 \right\},$$

and let $\Delta_0 \equiv \Delta \backslash \Delta_+$ denote the set of points at the boundary of the the simplex (i.e., those in which at least one price equals zero). Note that Δ_0 corresponds to B_{-0} after we normalize prices so they add up to one. The proof will show that there exists $p^* \in \Delta_+$ such that $\bar{d}(\mathbf{p}^*) = \mathbf{0}$.

Proof. We introduce the non-empty-valued correspondence $g : \Delta \to P(\Delta)$ defined as follows:

- For $\mathbf{p} \in \Delta_+$, let
$$g_n(\mathbf{p}) = \frac{p_n + \max\left\{0, \bar{d}_n(\mathbf{p})\right\}}{1 + \sum_{l=1}^{N} \max\left\{0, \bar{d}_l(\mathbf{p})\right\}}.$$
  Note that g is a continuous function (since it is a composition of continuous functions). Moreover, $\mathbf{p} \in \Delta_+$ implies $g(\mathbf{p}) \in \Delta_+$ (that is, $p_n > 0 \implies g_n(p) > 0$ and $\sum_n g_n(p) = 1$), so that g maps Δ_+ into itself.

- For any $\mathbf{p} \in \Delta_0$, let $g(\mathbf{p})$ be defined by the following set:

$$g(\mathbf{p}) = \{\mathbf{q} \in \Delta : \; p_n > 0 \implies q_n = 0\}.$$

  This set reassigns a price of zero ($q_n = 0$) for each good that had a positive price $p_n > 0$. Note that $g(\mathbf{p})$ is a set because, in general, there will be many positive prices $q_n > 0$ assigned to goods that had a zero price $p_n = 0$.

We claim that $g : \Delta \to P(\Delta)$ is convex-valued and has a closed graph. Convex-valued follows from the fact that Δ is convex.

To see that g has a closed graph, consider a convergent sequence $(\mathbf{p}^t, \mathbf{q}^t)$ in the graph of g and let $(\mathbf{p}, \mathbf{q})$ denote its limit. Since $(\mathbf{p}^t, \mathbf{q}^t)$ is a sequence in Δ^2, which is closed, it follows that $(\mathbf{p}, \mathbf{q}) \in \Delta^2$. We need to show that $\mathbf{q} \in g(\mathbf{p})$.

If $\mathbf{p} \in \Delta_+$ (so that $\mathbf{p} >> 0$), then $\mathbf{p}^t > 0$ (so $\mathbf{p}^t \in \Delta_+$) for t large enough . From the continuity of g on Δ_+, it follows that $g(\mathbf{p}) \in \Delta_+$.

Next, suppose $\mathbf{p} \in \Delta_0$. If there is a subsequence $(\mathbf{p^{t_k}}, \mathbf{q^{t_k}})$ for which $p^{t_k} \in \Delta_0$, then for any good n with $p_n > 0$, it must follow that $p_n^t > 0$ for t large enough. Therefore, for all such t, we must have $q_n^t = 0$. Since by assumption $\{q^{t_k}\}$ converges to $\mathbf{q}$, it must be the case that $\mathbf{q} = 0$.

The remaining case is that $\mathbf{p}^t \in \Delta_+$ for all but at most finitely many t. Lemma 3 implies that: (i) $\lim_{t\to\infty} \sum_{n=1}^N \max\left\{0, \bar{d}_n(\mathbf{p}^t)\right\} = +\infty$, and that (ii) for any good n with $p_n > 0$, $\bar{d}_n(\mathbf{p}^t)$ is bounded. Therefore, for any good n with $p_n > 0$, $g_n(\mathbf{p}^t)$ must converge to 0. Since by assumption $\{q^{t_k}\} \to q$, it follows that $q_n = 0$.

Having shown that $g : \Delta \to P(\Delta)$ is convex-valued and has a closed graph, the Kakutani Fixed Point Theorem ensures that it has a fixed point $\mathbf{p}^* \in \Delta$ with $\mathbf{p}^* \in g(\mathbf{p}^*)$. Note that, from the definition of g, if $\mathbf{p} \in \Delta_0$ then $\mathbf{p} \notin g(\mathbf{p})$. Thus, no points in Δ_0 can be a fixed point of g. Since p^* is a fixed point of g, it must then be the case that $\mathbf{p}^* \in \Delta_+$ (so that $\mathbf{p}^* >> 0$).

We now show that $\mathbf{p}^*$ is an equilibrium price vector. By the definition of g for $\mathbf{p} \in \Delta_+$, for any good n,

$$p_n^* = \frac{p_n^* + \max\left\{0, \bar{d}_n(\mathbf{p}^*)\right\}}{1 + \sum_{l=1}^N \max\left\{0, \bar{d}_l(\mathbf{p}^*)\right\}},$$

which can be rearranged as

$$p_n^* \sum_{l=1}^N \max\left\{0, \bar{d}_l(\mathbf{p}^*)\right\} = \max\left\{0, \bar{d}_n(\mathbf{p}^*)\right\}.$$

Multiply both sides by $\bar{d}_n(\mathbf{p}^*)$ to obtain:

$$p_n^* \bar{d}_n(\mathbf{p}^*) \sum_{l=1}^{N} \max\left\{0, \bar{d}_l(\mathbf{p}^*)\right\} = \bar{d}_n(\mathbf{p}^*) \max\left\{0, \bar{d}_n(\mathbf{p}^*)\right\}.$$

Sum this expression for all goods n:

$$\sum_{n=1}^{N} \left[p_n^* \bar{d}_n(\mathbf{p}^*)\right] \sum_{n=1}^{N} \max\left\{0, \bar{d}_n(\mathbf{p}^*)\right\} = \sum_{n=1}^{N} \bar{d}_n(\mathbf{p}^*) \max\left\{0, \bar{d}_n(\mathbf{p}^*)\right\}.$$

By Walras' Law, $\sum_{n=1}^{N} p_n^* \bar{d}_n(\mathbf{p}^*) = 0$ for any price vector p^*. Therefore, the term on the LHS equals zero and the previous equation becomes

$$\sum_{n=1}^{N} \bar{d}_n(\mathbf{p}^*) \max\left\{0, \bar{d}_n(\mathbf{p}^*)\right\} = 0. \tag{20}$$

We claim that this equation implies $\bar{d}_n(\mathbf{p}^*) \leq 0$ for all n. To see this, note that:

$$\bar{d}_n(\mathbf{p}^*) > 0 \implies \bar{d}_n(\mathbf{p}^*) \max\left\{0, \bar{d}_n(\mathbf{p}^*)\right\} > 0,$$

and

$$\bar{d}_n(\mathbf{p}^*) \leq 0 \implies \bar{d}_n(\mathbf{p}^*) \max\left\{0, \bar{d}_n(\mathbf{p}^*)\right\} = 0.$$

Therefore, if $\bar{d}_n(\mathbf{p}^*) > 0$ for some n, then the sum in the LHS of equation (20) would be strictly positive instead of zero. $\square$

Since $\mathbf{p}^* \bar{d}(\mathbf{p}^*) = 0$ (by Walras' Law), $\mathbf{p}^* >> 0$ and $\bar{d}(\mathbf{p}^*) \leq 0$, it follows that $\bar{d}(\mathbf{p}^*) = 0$.

References

[1] Allais, M. (1943; 1994). *Traité d'Economie Pure*. Third edition. Paris: Clément Juglar, 1994.

[2] Araujo, A., Chateauneuf, A., Gama, J. P., Novinski, R. (2018). "General Equilibrium With Uncertainty Loving Preferences," *Econometrica*, 86 (5), 1859–1871.

[3] Arrow, K. J. and Debreu, G. (1954). "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, 22, 265-292.

[4] Arrow, K.J (1953; 1963). "Le Role des Valeus Boursieres Pour la Repartition la Meilleure des Risqués," In: Cahiers du Séminaire d'Economètrie. English Translation: "The Role

of Securities in the Optimal Allocation of Risk Bearing." *Review of Economic Studies*, 31, 91- 96.

[5] Arrow, K.J. (1951). "An extension of the basic theorems of classical welfare economics," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics*, J. Neyman ed., 507-532, Berkeley: University of California Press.

[6] Arrow, K.J. (1964). "The Role of Securities in the Optimal Allocation of Risk-Bearing," *Review of Economics Studies*, 31, 91-96.

[7] Arrow, K.J. and F. H. Hahn (1971). *General Competitive Analysis*, Amsterdam: North-Holland.

[8] Aumann, R. J. (1966). "Existence of Competitive Equilibrium in Markets with a Continuum of Traders," *Econometrica*, 34, 39-50.

[9] Bergson, A. (1938). "A Reformulation of Certain Aspects of Welfare Economics," *Quarterly Journal of Economics*, 52, 310-334.

[10] Borch, K. (1962). "Equilibrium in a Reinsurance Market," *Econometrica*, 30, 424-444.

[11] Brown, D.J. and Rosa L. Matzkin (1996). "Testable Restrictions on the Equilibrium Manifold," *Econometrica*, 64(6), 1249-1262.

[12] Cass, D. and A. Citanna (1998). "Pareto Improving FinanciaI Innovation in Incomplete Markets," *Economic Theory*, 11, 467-494.

[13] Cassel, G. (1918; 1932). *The Theory of Social Economy*, English Translation by: S. L. Barron. NY: Harcourt Brace.

[14] Cochrane, J.H. (1991). "A Simple Test of Consumption Insurance," *Journal of Political Economy*, 99(5).

[15] Debreu, G. (1951). "The Coefficient of Resource Utilization," *Econometrica*, 19, 273-292.

[16] Debreu, G. (1970). "Economies with a Finite Set of Equilibria," *Econometrica*, 38, 387-392.

[17] Diamond, P.A. (1967). "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty," *American Economic Review*, 57, 759-776.

[18] Dufwenberg, M., P. Heidhues, G. Kirchsteiger, F. Riedel, and J. Sobel. (2011). "Other-Regarding Preferences in General Equilibrium," *Review of Economic Studies*, 78 (2), 613–639.

[19] Elul, R. (1995). "Welfare Effects of Financial Innovation in Incomplete Markets Economies with Several Consumption Goods," *Journal of Economic Theory*, 65 (1), 43-78.

[20] Gale, D. (1955). "The Law of Supply and Demand," *Mathematica Scandinavica*, 3(1), 155-169.

[21] Hart, O.D. (1975). "On the Optimality of Equilibrium When the Market Structure is Incomplete," *Journal of Economic Theory*, 11, 418- 443.

[22] Hildenbrand, W. and A. P. Kirman (1988). *Equilibrium Analysis*. Amsterdam: North-Holland.

[23] Laibson, D. (1997). "Golden Eggs and Hyperbolic Discounting," *Quarterly Journal of Economics*, 112(2), 443-478.

[24] Lange, O. (1936). "On the Economic Theory of Socialism," *Review of Economic Studies*, 4, 53-71.

[25] Lange, O. (1942). "The Foundations of Welfare Economics," *Econometrica*, 10, 215-228.

[26] Lerner, A. P. (1934). "The Concept of Monopoly and the Measurement of Monopoly Power," *Review of Economic Studies*, 1, 157-175.

[27] Lerner, A. P. (1934b). "Economic Theory and Socialist Economy," *Review of Economic Studies*, 2, 51-61.

[28] Lipsey, R.G. and K. Lancaster (1956). "The General Theory of Second Best," *Review of Economic Studies*, 24 (1), 11–32.

[29] Lucas, R.E. (1978). "Asset Prices in an Exchange Economy," *Econometrica*, 46(6), 1429-1445.

[30] Mas Colell, A. (1977). "On the Equilibrium Price Set of an Exchange Economy," Journal of Mathematical Economics, 4, 117-126.

[31] McKenzie, L. W. (1954). "On Equilibrium in Graham's Model of World Trade and Other Competitive Systems," *Econometrica*, 22, 147- 161.

[32] Nash, J.F. (1950). "Equilibriumn Points in N-Person Games," *Proceedings of the National Academy of Sciences*, 36, 48-49.

[33] Negishi, T. (1960). "Welfare Economics and Existence of an Equilibrium for a Competitive Economy," *Metroeconomica*, 12, 92-97.

[34] Neisser, H. (1932; 1990). “The Wage Rate and Employment in Market Equilibrium,” *Structural Change and Economic Dynamics*. 1 (1): 141-163.

[35] Nikaido, H. (1956). “On the Classical Multilateral Exchange Problem,” *Metroeconomica*, 8, 135-145.

[36] O’Donoghue, T. and M. Rabin (1999). “Doing It Now or Later,” *American Economic Review*, 89 (1), 103-124.

[37] Radner, R. (1972). “Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets,” *Econometrica*, 40(2), 289-303.

[38] Ricardo, D. (1817; 1917). *On the Principles of Political Economy and Taxation*, Repinted by: J.M. Dent & Sons, London and E.P. Dutton & Co., New York.

[39] Samuelson, P. A. (1947). *Fundamentals of Economic Analysis.* Cambridge MA: Harvard University Press.

[40] Schlesinger, K. (1933; 1968) “On the Production Equations of Economic Value Theory,” In: Baumol and Goldfeld. English Translation: W. J. Baumol, pp. 278-80.

[41] Taylor, F.M. (1929). “The Guidance of Production in a Socialist State,” *American Economic Review*, 19(1), 1-8.

[42] Townsend, R.M. (1994). “Risk and Insurance in Village India,” *Econometrica*, 62(3), 539-591.

[43] Wald, A. (1936; 1951). “Über einige Gleichungssysteme der Mathematis chen Okonomie,” Translated as: “On some Systems of Equations of Mathematical Economics,” *Econometrica*, 19, 368-403.

[44] Walras, L. (1874; 1954). *Eléments d’Economie Politique Pure*. English Translation: Elements of Pure Economics. Translated and annotated by William Jaffé, Homewood, Ill.: Richard D. Irwin; London: Allen & Unwin.