The Probability of Inconsistencies in Complex Collective Decisions

Christian List[1]

Forthcoming in *Social Choice and Welfare*

Abstract. Many groups make decisions over multiple interconnected propositions. The "doctrinal paradox" or "discursive dilemma" shows that propositionwise majority voting can generate inconsistent collective sets of judgments, even when individual sets of judgments are all consistent. I develop a simple model for determining the probability of the paradox, given various assumptions about the probability distribution of individual sets of judgments, including impartial culture and impartial anonymous culture assumptions. I prove several convergence results, identifying when the probability of the paradox converges to 1, and when it converges to 0, as the number of individuals increases. Drawing on the Condorcet jury theorem and work by Bovens and Rabinowicz (2001, 2003), I use the model to assess the "truth-tracking" performance of two decision procedures, the premise- and conclusion-based procedures. I compare the present results with existing results on the probability of Condorcet's paradox. I suggest that the doctrinal paradox is likely to occur under plausible conditions.

1 Introduction

A new paradox of aggregation, the "doctrinal paradox" or "discursive dilemma", has been the subject of a growing body of literature in law, economics and philosophy (Kornhauser and Sager 1986, 1993; Kornhauser 1992; Chapman 1998, 2001, 2002; Brennan 2001; Pettit 2001; List and Pettit 2002, 2003; Bovens and Rabinowicz 2001, 2003; List 2003). An example illustrates the problem. A three-member court has to decide on whether a defendant is liable under a charge of breach of contract. Legal doctrine requires that the court should find that the defendant is liable (proposition *R*) if and only if it finds, first, that the defendant did some action X (proposition *P*), and, second, that the defendant had a contractual obligation not to do action X (proposition *Q*). Thus legal doctrine stipulates the connection rule $(R\leftrightarrow(P\wedge Q))$. Suppose the opinions of the three judges are as in Table 1.

Table 1: The doctrinal paradox (conjunctive version)

	P	Q	$(R \leftrightarrow (P \wedge Q))$	R
Judge 1	Yes	Yes	Yes	Yes
Judge 2	Yes	No	Yes	No
Judge 3	No	Yes	Yes	No
Majority	Yes	Yes	Yes	No

[1] The author wishes to express his gratitude to Luc Bovens, Matthew Braham, Steven Brams, Bruce Chapman, Philip Pettit, Wlodek Rabinowicz and two anonymous reviewers for helpful comments or discussion. A previous version of this paper was presented at the 2002 Annual Meeting of the European Public Choice Society, Belgirate, Lago Maggiore, Italy, April 2002, and at the Sixth International Meeting of the Society for Social Choice and Welfare, held at the California Institute of Technology, Pasadena, California, July 2002. Address for correspondence: SPT Program, Research School of Social Sciences, Australian National University, Canberra ACT 0200, Australia; and Department of Government, London School of Economics, London WC2A 2AE, U.K.; E-mail c.list@lse.ac.uk.

All judges accept the connection rule, $(R\leftrightarrow(P\wedge Q))$. Judge 1 accepts both P and Q and, by implication, R. Judges 2 and 3 each accept only one of P or Q and, by implication, they both reject R. If the court applies majority voting on each proposition (including $(R\leftrightarrow(P\wedge Q))$), it faces a paradoxical outcome. A majority accepts P, a majority accepts Q, a majority (unanimity) accepts $(R\leftrightarrow(P\wedge Q))$, and yet a majority rejects R. Propositionwise majority voting thus yields an inconsistent collective set of judgments, namely $\{P, Q, (R\leftrightarrow(P\wedge Q)), \neg R\}$ (corresponding to the last row of Table 1). This set is inconsistent in the standard sense of propositional logic: there exists no assignment of truth-values to P, Q and R that makes all the propositions in the set simultaneously true. This outcome occurs even though the sets of judgments of individual judges (corresponding to the first three rows of Table 1) are all consistent. The doctrinal paradox is related to Anscombe's paradox, or Ostrogorski's paradox (Anscombe 1976; Kelly 1989; Brams, Kilgour and Zwicker 1997). Like the doctrinal paradox, these paradoxes are concerned with aggregation over multiple propositions. Unlike the doctrinal paradox, they do not incorporate explicit logical connections between the propositions.

Pettit (2001) has argued that the doctrinal paradox occurs not only in the aggregation of judgments according to legal doctrine, but that it poses a more general "discursive dilemma", potentially facing any group that seeks to form collective judgments on the basis of reasons. Further, the paradox illustrates an impossibility theorem (List and Pettit 2002, 2003): there exists no function for aggregating consistent individual sets of judgments over multiple interconnected propositions into consistent collective ones which satisfies some minimal conditions (unrestricted domain, anonymity, systematicity). The relation between the paradox and the impossibility theorem is somewhat analogous to the relation between Condorcet's paradox and Arrow's impossibility theorem.

Aggregation problems over multiple interconnected propositions may arise, for example, when a committee makes a decision that involves the resolution of several premises; or when a political party or interest group chooses a policy package that consists of multiple interconnected components. Although we use the label "doctrinal paradox", we should keep the more general nature of the problem in mind.

How serious is the threat posed by this paradox? It is one thing to recognize that a given paradox of aggregation is logically possible. It is another to claim that the paradox is of practical significance. There are at least two possible reasons why a particular paradox might not (seem to) occur in practice. One is that many decision procedures that are used in practice do not *explicitly reveal* the paradox, even when individual views have the pattern that *would* give rise to the paradox. Two such decision procedures are discussed below, the so-called premise-based and conclusion-based procedures. These procedures do not produce inconsistent collective sets of judgments, even when a pattern of individual views as in Table 1 occurs. But it will become evident that the question of how frequently such patterns occur is relevant for assessing the performance of the two procedures. A second possible reason why the paradox might not occur in practice is that the patterns of individual views that generate the paradox might themselves be rare.

How likely is the occurrence of this paradox, or more precisely, how likely is the occurrence of patterns of individual views that generate the paradox? This paper seeks to give a theoretical answer to this question. Inevitably, many other important questions raised by the doctrinal paradox cannot be addressed here. In Section 2, I identify necessary and sufficient conditions for the occurrence of the paradox. In Section 3, I develop a model for determining the probability of its occurrence, given various assumptions about the probability distribution of individual sets of judgments, including the so-called impartial culture assumption. In Section 4, I determine the expected probability of the paradox under the assumption that all logically possible probability distributions of the form discussed in Section 3 are equally probable. That expected probability coincides with the probability of the paradox under the so-called impartial anonymous culture assumption. In Section 5, I discuss two escape-routes from the paradox, the premise- and conclusion-based decision procedures, and drawing on the Condorcet jury theorem and recent work by Bovens and Rabinowicz (2001, 2003), I assess their performance in terms of "tracking the truth". The present model yields alternative proofs and extensions of some of Bovens and Rabinowicz’s results. I show that, under certain conditions, if each individual is better than random, but not very good, at tracking the “truth” on each premise, then the probability of the doctrinal paradox (and of a discrepancy between the premise- and conclusion-based procedures) converges to 1 as the number of individuals increases. In Section 6, I address generalizations of the present results. In Section 7, I briefly compare the present results with existing results on the probability of Condorcet’s paradox.

We should address one objection. Many results of this paper concern the convergence of certain probabilities as the number of individuals increases. Since those groups that have to make decisions over multiple interconnected propositions are typically small, it is not obvious why such convergence results, or any results about large groups, are relevant. Examples of the groups in question are courts, committees, panels of experts, or parliaments, with between a handful and a few hundred members. In response, note four points. First, the present framework allows calculating the relevant probabilities for finite (and thus small) numbers of individuals too. Second, the convergence behaviour of the probabilities of various voting outcomes has received attention since Condorcet's classical work, and it is therefore interesting to address Condorcet's classical questions in the new context of aggregation over multiple interconnected propositions. Third, as Table 3 illustrates, convergence results may be relevant even to situations of a few dozen or a few hundred decision-makers, as the convergence speed is often high. Finally, the results may illuminate some questions in democratic theory, such as (i) whether it is desirable to introduce large-scale political participation on complex issues by running referenda over multiple propositions and (ii) what the optimal group size for complex decisions is. In Section 7, I cite some anecdotal evidence from referenda in California.

2 Necessary and Sufficient Conditions for the Occurrence of the Paradox

Let there be n individuals and three propositions, P, Q and R. We assume that all individuals accept the connection rule ($R \leftrightarrow (P \wedge Q)$). To avoid complications raised by majority ties, we assume that n is odd. We admit only consistent individual sets of judgments over P, Q and R. There are 4 such sets, as shown in Table 2.

Table 2: All logically possible consistent sets of judgments over P, Q and R, given ($R \leftrightarrow (P \wedge Q)$)

Label	Judgment on P	Judgment on Q	Judgment on R
TT	Yes	Yes	Yes
TF	Yes	No	No
FT	No	Yes	No
FF	No	No	No

I make no claims as to whether it is *empirically* plausible to assume that individuals hold consistent sets of judgments. Admitting only consistent individual sets seems to make collective inconsistencies less rather than more likely. If we can still show that collective inconsistencies are plausible, the argument will have been strengthened rather than weakened by the exclusion of inconsistent individual sets of judgments.

Let n_{TT}, n_{TF}, n_{FT}, n_{FF} be the numbers of individuals holding the sets of judgments *TT*, *TF*, *FT*, *FF*, respectively. A vector $<n_{TT}, n_{TF}, n_{FT}, n_{FF}>$ is called an *anonymous profile*. Then $N := \{<n_{TT}, n_{TF}, n_{FT}, n_{FF}> : \ n_{TT}, n_{TF}, n_{FT}, n_{FF} \geq 0 \textit{ and } n_{TT}+n_{TF}+n_{FT}+n_{FF} = n\}$ is the set of all logically possible anonymous profiles.

If ($R \leftrightarrow (P \wedge Q)$) is unanimously accepted, a collective inconsistency (a "doctrinal paradox") occurs if and only if there are majorities for each of P and Q, and there is a majority against R.

Proposition 1. *Given the connection rule* ($R \leftrightarrow (P \wedge Q)$), *there is a collective inconsistency under propositionwise majority voting if and only if* ($n_{TT} + n_{TF} > n/2$) *and* ($n_{TT} + n_{FT} > n/2$) *and* ($n_{TT} < n/2$).

Define

$$N^* := \{<n_{TT}, n_{TF}, n_{FT}, n_{FF}> \in N : (n_{TT}+n_{TF}>n/2) \text{ and } (n_{TT}+n_{FT}>n/2) \text{ and } (n_{TT}<n/2)\}.$$

Then N^* is the set of all anonymous profiles for which propositionwise majority voting generates a collective inconsistency.

3 A Probability-Theoretic Framework

We assume that (i) each individual has probabilities p_{TT}, p_{TF}, p_{FT}, p_{FF} of holding the sets of judgments *TT*, *TF*, *FT*, *FF*, respectively (where $p_{TT}+p_{TF}+p_{FT}+p_{FF} = 1$); and (ii) the judgments of different individuals are independent from each other.

The simplifications of these assumptions follow the classical Condorcet jury theorem. Specifically, we assume (i) identical probabilities for different individuals, and (ii) independence between different individuals. It is known in the literature on the Condorcet jury theorem that the types of convergence mechanisms based on the law of large numbers invoked here apply, with certain modifications, also when probabilities vary across individuals or when there are certain dependencies between individuals (Grofman, Owen and Feld 1983; Boland 1989; Berg 1993; Ladha 1995).

Let X_{TT}, X_{TF}, X_{FT}, X_{FF} be the random variables whose values are the numbers of individuals holding the sets of judgments *TT*, *TF*, *FT*, *FF*, respectively. The joint distribution of the *X*s is a multinomial distribution with the probability function

$$P(X_{TT}=n_{TT}, X_{TF}=n_{TF}, X_{FT}=n_{FT}, X_{FF}=n_{FF}) = \frac{n!}{n_{TT}!\, n_{TF}!\, n_{FT}!\, n_{FF}!}\; p_{TT}^{\,n_{TT}}\, p_{TF}^{\,n_{TF}}\, p_{FT}^{\,n_{FT}}\, p_{FF}^{\,n_{FF}}.$$

Let P^*_n denote the probability of a collective inconsistency under propositionwise majority voting, where the connection rule is ($R \leftrightarrow (P \wedge Q)$) and where there are n individuals. Using proposition 1, we see that

$$P^*_n := P((X_{TT} + X_{TF} > n/2) \text{ and } (X_{TT} + X_{FT} > n/2) \text{ and } (X_{TT} < n/2)).$$

We can easily infer the following proposition, recalling that n is odd.

Proposition 2.

$$P^*_n := \sum_{<n_{TT}, n_{TF}, n_{FT}, n_{FF}> \in N^*} \frac{n!}{n_{TT}!\, n_{TF}!\, n_{FT}!\, n_{FF}!} p_{TT}^{n_{TT}} p_{TF}^{n_{TF}} p_{FT}^{n_{FT}} p_{FF}^{n_{FF}}.$$

$$= \sum_{n_{TT}=1}^{(n-1)/2} \; \sum_{n_{TF}=(n+1)/2-n_{TT}}^{(n-1)/2} \; \sum_{n_{FT}=(n+1)/2-n_{TT}}^{n-n_{TT}-n_{TF}} \frac{n!}{n_{TT}!\, n_{TF}!\, n_{FT}!\, n_{FF}!} p_{TT}^{n_{TT}} p_{TF}^{n_{TF}} p_{FT}^{n_{FT}} p_{FF}^{n_{FF}}.$$

The probabilities of other logically possible combinations of majorities for or against P, Q and R can be calculated analogously. An *impartial culture* is the situation in which $p_{TT}=p_{TF}=p_{FT}=p_{FF}$. An impartial culture is to be distinguished from an *impartial anonymous culture*, as defined in Section 4. Table 3 shows some sample calculations of P^*_n for various values of n, p_{TT}, p_{TF}, p_{FT}, p_{FF}.

Table 3: P^*_n (probability of a collective inconsistency under propositionwise majority voting, given $(R \leftrightarrow (P \wedge Q))$), for various scenarios

	Scenario 1 $p_{TT} = 0.25$ $p_{TF} = 0.25$ $p_{FT} = 0.25$ $p_{FF} = 0.25$	Scenario 2 $p_{TT} = 0.26$ $p_{TF} = 0.25$ $p_{FT} = 0.25$ $p_{FF} = 0.24$	Scenario 3 $p_{TT} = 0.3$ $p_{TF} = 0.25$ $p_{FT} = 0.25$ $p_{FF} = 0.2$	Scenario 4 $p_{TT} = 0.24$ $p_{TF} = 0.27$ $p_{FT} = 0.25$ $p_{FF} = 0.24$	Scenario 5 $p_{TT} = 0.49$ $p_{TF} = 0.2$ $p_{FT} = 0.2$ $p_{FF} = 0.11$	Scenario 6 $p_{TT} = 0.51$ $p_{TF} = 0.2$ $p_{FT} = 0.2$ $p_{FF} = 0.09$	Scenario 7 $p_{TT} = 0.55$ $p_{TF} = 0.2$ $p_{FT} = 0.2$ $p_{FF} = 0.05$	Scenario 8 $p_{TT} = 0.33$ $p_{TF} = 0.33$ $p_{FT} = 0.33$ $p_{FF} = 0.01$
$n = 3$	0.0938	0.0975	0.1125	0.0972	0.1176	0.1224	0.1320	0.2156
$n = 11$	0.2157	0.2365	0.3211	0.2144	0.3570	0.3432	0.2990	0.6188
$n = 31$	0.2487	0.2946	0.4979	0.2409	0.5183	0.4420	0.2842	0.9104
$n = 51$	0.2499	0.3101	0.5815	0.2405	0.5525	0.4414	0.2358	0.9757
$n = 71$	≈ 0.2500	0.3216	0.6417	0.2393	0.5663	0.4327	0.1983	0.9930
$n = 101$	≈ 0.2500	0.3362	0.7113	0.2375	0.5798	0.4201	0.1562	0.9989
$n = 201$	≈ 0.2500	0.3742	0.8511	0.2317	0.6118	0.3882	0.0774	≈ 1.0000
$n = 501$	≈ 0.2500	0.4527	0.9754	0.2149	0.6729	0.3271	0.0124	≈ 1.0000
$n = 1001$	≈ 0.2500	0.5426	0.9985	0.1897	0.7366	0.2634	0.0008	≈ 1.0000
$n = 1501$	≈ 0.2500	0.6097	0.9999	0.1676	0.7808	0.2192	0.0001	≈ 1.0000

Small differences in p_{TT}, p_{TF}, p_{FT}, p_{FF} correspond to substantial differences in P^*_n. Under an impartial culture (Scenario 1), P^*_n converges to 0.25 as n increases. Small deviations from an impartial culture lead to a completely different convergence pattern. This is confirmed by the following proposition, proved in Appendix 2.

Proposition 3.

(a) If $(p_{TT} + p_{TF} > 1/2)$ *and* $(p_{TT} + p_{FT} > 1/2)$ *and* $(p_{TT} < 1/2)$, *then* $P^*_n \to 1$ *as* $n \to \infty$.

(b) If $(p_{TT} + p_{TF} < 1/2)$ *or* $(p_{TT} + p_{FT} < 1/2)$ *or* $(p_{TT} > 1/2)$, *then* $P^*_n \to 0$ *as* $n \to \infty$.

Table 3 illustrates the convergence results of Proposition 3. Scenarios 2, 3, 5 and 8 satisfy the conditions of 3a, and Scenarios 4, 6 and 7 satisfy the conditions of 3b. The convergence results follow from the law of large numbers. If each individual holds the sets of judgments *TT*, *TF*, *FT*, *FF* with probabilities p_{TT}, p_{TF}, p_{FT}, p_{FF}, respectively, then np_{TT}, np_{TF}, np_{FT}, np_{FF} are the expected numbers of these sets of judgments across n individuals, and p_{TT}, p_{TF}, p_{FT}, p_{FF} are the expected frequencies (i.e. the expected numbers divided by n). If n is small, the actual frequencies may differ substantially from the expected ones, but as n increases, the actual frequencies will approximate the expected ones increasingly closely. If the probabilities p_{TT}, p_{TF}, p_{FT}, p_{FF} satisfy a set of strict inequalities, the actual frequencies (and by implication the actual numbers) are increasingly likely to satisfy a matching set of strict inequalities. Therefore, if these inequalities correspond to the occurrence or absence of a collective inconsistency (compare Proposition 1), the probability of the occurrence or absence of such an inconsistency will converge to 1. This mechanism is also used to prove other convergence results below. Lemma 1 in Appendix 2 captures the mechanism formally.

The present results are also relevant to the impossibility theorem on the aggregation of judgments over multiple propositions (List and Pettit 2002). The impossibility result uses an unrestricted domain condition. The present results allow us to determine, under various assumptions, how likely it is that a profile of sets of judgments across individuals falls into a problematic domain (where propositionwise majority voting generates inconsistencies), and how likely it is that it falls into an unproblematic one (where propositionwise majority voting generates consistent outcomes).

4 The Expected Probability of the Paradox and the Impartial Anonymous Culture[2]

The results of Section 3 concern the probability of a collective inconsistency for a specific vector $<p_{TT}, p_{TF}, p_{FT}, p_{FF}>$. An impartial culture, where $p_{TT}=p_{TF}=p_{FT}=p_{FF}$, is a benchmark case of such a probability vector. An alternative approach is to assume that all logically possible vectors of the form $<p_{TT}, p_{TF}, p_{FT}, p_{FF}>$ are equally likely to occur (see Gehrlein 1981 on this approach in the context of Condorcet's paradox). Let $E(P^*_n)$ denote the expected value of P^*_n under this assumption. This can be interpreted as the expected probability of a collective inconsistency if we have no information about the "correct" vector $<p_{TT}, p_{TF}, p_{FT}, p_{FF}>$ and assign equal probability to every possible such vector.

---

[2] I am greatly indebted to an anonymous reviewer for most helpful suggestions that led to the results of this section.

Let $f(p_{TT}, p_{TF}, p_{FT}, p_{FF})$ denote the probability density function corresponding to this equiprobability assumption. Define $p_{FF} := 1\text{-}p_{TT}\text{-}p_{TF}\text{-}p_{FT}$. Then

$$E(P^*_n) = \int_{p_{TT}=0}^{1} \int_{p_{TF}=0}^{1-p_{TT}} \int_{p_{FT}=0}^{1-p_{TT}-p_{TF}} P^*_n f(p_{TT}, p_{TF}, p_{FT}, p_{FF}) dp_{FT}\, dp_{TF}\, dp_{TT}.$$

To determine $f(p_{TT}, p_{TF}, p_{FT}, p_{FF})$, note that

$$\int_{p_{TT}=0}^{1} \int_{p_{TF}=0}^{1-p_{TT}} \int_{p_{FT}=0}^{1-p_{TT}-p_{TF}} 1\, dp_{FT}\, dp_{TF}\, dp_{TT} = 1/6;$$

thus the equiprobability assumption implies $f(p_{TT}, p_{TF}, p_{FT}, p_{FF}) = 6$. Following the logic in Gehrlein (1981), it can then be shown (for odd n) that

$$\begin{aligned}
E(P^*_n) &= 6 \int_{p_{TT}=0}^{1} \int_{p_{TF}=0}^{1-p_{TT}} \int_{p_{FT}=0}^{1-p_{TT}-p_{TF}} P^*_n\, dp_{FT}\, dp_{TF}\, dp_{TT}. \\
&= \sum_{<n_{TT}, n_{TF}, n_{FT}, n_{FF}>\in N^*} \big(6/((n+3)(n+2)(n+1))\big) \\
&= \Big(\sum_{n_{TT}=1}^{(n-1)/2} \sum_{n_{TF}=(n+1)/2-n_{TT}}^{(n-1)/2} \sum_{n_{FT}=(n+1)/2-n_{TT}}^{n-n_{TT}-n_{TF}} 1\Big) \big(6/((n+3)(n+2)(n+1))\big) \\
&= \big({}^1/_{48}(n-1)(n+1)(n+3)\big) \big(6/((n+3)(n+2)(n+1))\big) \\
&= \frac{n-1}{8(n+2)}.
\end{aligned}$$

An *impartial culture* was defined as the situation in which every logically possible individual set of judgments is equally likely to be held by an individual, i.e. $p_{TT}=p_{TF}=p_{FT}=p_{FF}$. An *impartial anonymous culture* is the situation in which every logically possible anonymous profile $<n_{TT}, n_{TF}, n_{FT}, n_{FF}>$ is equally likely to occur; i.e. every such anonymous profile occurs with probability $1/|N|$. Since N^* is the set of all anonymous profiles for which there is a collective inconsistency, the probability of a collective inconsistency under an impartial anonymous culture is therefore $|N^*|/|N|$. Now

$$|N^*| = \sum_{<n_{TT}, n_{TF}, n_{FT}, n_{FF}>\in N^*} 1 = {}^1/_{48}(n-1)(n+1)(n+3), \quad \text{and}$$

$$|N| = \sum_{n_{TT}=0}^{n} \sum_{n_{TF}=0}^{n-n_{TT}} \sum_{n_{FT}=0}^{n-n_{TF}-n_{FT}} 1 = (n+3)(n+2)(n+1)/6; \quad \text{hence}$$

$$|N^*|/|N| = \left({}^1/_{48}(n\text{-}1)(n+1)(n+3)\right) / \left((n+3)(\mathrm{n}+2)(\mathrm{n}+1)/6\right) = \frac{n\text{-}1}{8(n+2)}.$$

So the probability of a collective inconsistency under an impartial anonymous culture is equal to $E(P^*_n)$, the expected value of P^*_n under the assumption that all logically possible probability vectors of the form $<p_{TT}, p_{TF}, p_{FT}, p_{FF}>$ are equally likely to occur. This corresponds to Gehrlein's analogous result on the Condorcet paradox (1981), as discussed in Section 7 below (see also Gehrlein and Fishburn 1976a). Note that $E(P^*_n) = |N^*|/|N|$ converges to 1/8 as n tends to infinity.

5 Voting for the Premises Versus Voting for the Conclusion

Premise- and conclusion-based decision procedures have been proposed as escape-routes from the doctrinal paradox (e.g. Pettit 2001). According to the *premise-based procedure* (hereafter *PBP*), the group applies majority voting on P and Q, the "premises", but not on R, the "conclusion", and lets the connection rule, $(R\leftrightarrow(P\wedge Q))$, dictate the collective judgment on R, ignoring the majority verdict on R. In the example of Table 1, the PBP leads to the acceptance of P and Q and, by implication, R. According to the *conclusion-based procedure* (hereafter *CPB*), the group applies majority voting only on R, but not on P and Q, ignoring the majority verdicts on P and Q. In Table 1, the CBP leads to the rejection of R. Thus the PBP and CBP may produce divergent outcomes.

Pettit (2001) and Chapman (2002) have argued that the PBP is particularly attractive from the perspective of deliberative democracy: it prioritises, and "collectivises", the reasons underlying a given collective decision. A key concern of deliberative democracy is to make collective decisions based on *publicly* defensible reasons. The CBP, by contrast, focuses solely on the conclusions that individuals *privately* reach, ignoring individual views on the premises. The CBP thus fails to make the underlying reasons for a decision explicit at the collective level.

Pettit's and Chapman's arguments are concerned mainly with the *procedural* merits of the PBP and CBP. Bovens and Rabinowicz (2001, 2003) (hereafter *B&R*) have compared the PBP and CBP from an *epistemic* perspective, drawing on the Condorcet jury theorem (see also Pettit and Rabinowicz 2001; on procedural versus epistemic conceptions of democracy, see List and Goodin 2001). Assuming an independent fact on the truth-values of P, Q and R, B&R determine the probability that the PBP and CBP reach the correct decision on R. I here connect the B&R framework with the present framework and discuss the implications of the Condorcet jury assumptions for the probability of collective inconsistencies. In Section 6, I generalize the results to a disjunctive version of the doctrinal paradox and to cases of more than two premises.

We assume that there are four possible states of the world: both P and Q are true (TT), P is true and Q is false (TF), P is false and Q is true (FT), both P and Q are false (FF). Given the connection rule ($R \leftrightarrow (P \wedge Q)$), each state of the world determines the truth-value of R. In the spirit of the Condorcet jury theorem, we assume the following:

(i) Each individual has probabilities ("competence") p and q of making correct judgments on P and Q, respectively, where p, $q > 0.5$; i.e. for each individual,

- the probability that the individual judges P to be true, given that P is true, equals p;
- the probability that the individual judges P to be false, given that P is false, equals p; and likewise for Q.[3]

(ii) Each individual's judgment on P and the *same* individual's judgment on Q are independent from each other, conditional on the state of the world.

(iii) The judgments of *different* individuals are independent from each other, conditional on the state of the world.[4]

Below we briefly address dependencies between the same individual's judgments on P and on Q, i.e. a relaxation of assumption (ii).

For each state of the world, the values of p and q induce corresponding values of p_{TT}, p_{TF}, p_{FT}, p_{FF}, as shown in Table 4; i.e. from the probabilities corresponding to each individual's judgments on P and Q under a given state of the world, we can infer the probabilities corresponding to each individual's holding each of the sets of judgments TT, TF, FT, FF.

Table 4: p_{TT}, p_{TF}, p_{FT}, p_{FF} as induced by p and q, for different states of the world

State of the world	p_{TT}	p_{TF}	p_{FT}	p_{FF}
TT	pq	$p(1-q)$	$(1-p)q$	$(1-p)(1-q)$
TF	$p(1-q)$	pq	$(1-p)(1-q)$	$(1-p)q$
FT	$(1-p)q$	$(1-p)(1-q)$	pq	$p(1-q)$
FF	$(1-p)(1-q)$	$(1-p)q$	$p(1-q)$	pq

[3] We assume that all individuals have the same pair of competence parameters p and q. If different individuals have different pairs of competence parameters, say p_i and q_i for individual i, then assuming [for all i, p_i, $q_i > 0.5$] is *not* in general sufficient for a Condorcet jury result on each of P and Q. Paroush (1998, Section 2) provides a counterexample. Following Paroush, we must then assume that there exist $\varepsilon_1,\varepsilon_2>0$ such that, for *all* individuals i, $p_i \geq 0.5+\varepsilon_1$ and $q_i \geq 0.5+\varepsilon_2$.

[4] The independence requirement given by (ii) and (iii) can be stated more precisely as follows. For any individual-proposition pair <i, P>, let V_{iP} denote the judgment of individual i on proposition P. Then the requirement is that all distinct V_{iP}s should be independent from each other, conditional on the state of the world.

Proposition 4. *Suppose that* p, $q > 0.5$.

(a) Suppose both P and Q are true.

- *If* $pq < 0.5$, then $P^*_n \to 1$ *as* $n \to \infty$.
- *If* $pq > 0.5$, then $P^*_n \to 0$ *as* $n \to \infty$.

(b) Suppose not both P and Q are true. Then $P^*_n \to 0$ *as* $n \to \infty$.

See Appendix 2 for a proof. By Proposition 4, P^*_n converges to 1 as n tends to infinity when all premises are true and individual competence is better than random but not high, e.g. when $0.5 < p, q < \sqrt{(0.5)}$; P^*_n converges to 0 when either at least one premise is false or individual competence is high, e.g. when $p, q > \sqrt{(0.5)}$. As the PBP and CBP produce divergent outcomes precisely when a collective inconsistency occurs, Proposition 4 implies that, when all premises are true and individual competence is low (but better than random), the probability of a discrepancy between the two procedures converges to 1 as n tends to infinity. If cases of true premises and low competence are frequent, discrepancies between the two procedures may also be frequent. Which of the two procedures should we use if our aim is to make a correct decision?

B&R distinguish between reaching the truth for the right reasons, and reaching it regardless of reasons. *Reaching the truth for the right reasons* requires deducing a correct judgment on the conclusion from correct judgments on each premise. *Reaching the truth regardless of reasons* includes the possibility of reaching the correct judgment on the conclusion accidentally, while making a wrong judgment on at least one premise. Which of the two criteria we consider more compelling depends on our view on democracy. Deliberative democrats or lawyers in the common law tradition stress the importance of giving public reasons for collective decisions (Pettit 2001 and Chapman 2002), and may therefore endorse the criterion of reaching the truth for the right reasons. Pure epistemic democrats or pure consequentialists focus primarily on reaching correct outcomes reliably, irrespective of the underlying reasoning *process*, and may therefore endorse the criterion of reaching the truth regardless of reasons.

Table 5 shows the conditions under which the PBP and CBP reach the correct decision on R (i) regardless of reasons and (ii) for the right reasons, for different states of the world. B&R show that the PBP is always better at reaching the truth on *R for the right reasons*, while the CBP is sometimes better at reaching it *regardless of reasons*. Some of these results can be derived from Table 5.

- *Reaching the correct decision on R for the right reasons:* To compare the PBP and CBP, we need to compare the relevant conditions corresponding to the four possible states of the world. In Table 5, (2) implies (1); (8) implies (4); (9) implies (5); and (10) implies (6). Hence the PBP is always at least as good as the CBP.

- *Reaching the correct decision on R regardless of reasons:* We distinguish two cases.
  - *P* and *Q* are both true: Since (2) implies (1), the PBP is always at least as good as the CBP.
  - Not both *P* and *Q* are true: Since (3) implies (7), the CBP is always at least as good as the PBP.

These results are consistent with a result by Grofman (1985): when a group decision on a conjunctive composite proposition can be disaggregated into separate group decisions on each conjunct, disaggregation is superior in terms of reaching the correct decision (regardless of reasons) for true propositions, but not for false propositions.

Table 5: Conditions under which the PBP and CBP reach the correct decision on R (given $(R \leftrightarrow (P \wedge Q))$) (i) regardless of reasons and (ii) for the right reasons, for different states of the world

<table>
<tr><td rowspan="2">State of the world</td><td colspan="2">PBP reaches a correct decision on R</td><td colspan="2">CBP reaches a correct decision on R</td></tr>
<tr><td>regardless of reasons if and only if …</td><td>for the right reasons if and only if …</td><td>regardless of reasons if and only if …</td><td>for the right reasons if and only if …</td></tr>
<tr><td>TT</td><td colspan="2">there are majorities for each of P and Q: $(n_{TT}+n_{TF}>n/2)$ and $(n_{TT}+n_{FT}>n/2)$ (1)</td><td colspan="2">there is a single majority supporting both P and Q: $n_{TT}>n/2$ (2)</td></tr>
<tr><td>TF</td><td rowspan="3">there are not majorities for each of P and Q: $(n_{TT}+n_{TF}<n/2)$ or $(n_{TT}+n_{FT}<n/2)$ (3)</td><td>there is a majority for P and a majority against Q: $(n_{TT}+n_{TF}>n/2)$ and $(n_{TF}+n_{FF}>n/2)$ (4)</td><td rowspan="3">there is not a single majority supporting both P and Q: $n_{TT}<n/2$ (7)</td><td>there is a single majority supporting P and rejecting Q: $n_{TF}>n/2$ (8)</td></tr>
<tr><td>FT</td><td>there is a majority against P and a majority for Q: $(n_{FT}+n_{FF}>n/2)$ and $(n_{TT}+n_{FT}>n/2)$ (5)</td><td>there is a single majority rejecting P and supporting Q: $n_{FT}>n/2$ (9)</td></tr>
<tr><td>FF</td><td>there are majorities against each of P and Q: $(n_{FT}+n_{FF}>n/2)$ and $(n_{TF}+n_{FF}>n/2)$ (6)</td><td>there is a single majority rejecting both P and Q: $n_{FF}>n/2$ (10)</td></tr>
</table>

Appendix 1 shows how to calculate, for a given *n* and a given state of the world, the probabilities that the PBP and the CBP reach the correct decision on *R* (i) regardless of reasons and (ii) for the right reasons.

The B&R results imply several convergence results. The present framework provides alternative proofs of some of these results, given in Appendix 2. Recall that p, $q > 0.5$.

Proposition 5. *Let the connection rule be* ($R \leftrightarrow (P \wedge Q)$). *The probabilities, as n tends to infinity, that the PBP and CPB reach a correct decision on R (i) regardless of reasons and (ii) for the right reasons, under various scenarios, are as shown in Table* 6.

Table 6: Probability, as n tends to infinity, of a correct decision on R (given ($R \leftrightarrow (P \wedge Q)$)) under the PBP and CBP (i) regardless of reasons and (ii) for the right reasons, under various scenarios

<table>
<tr><th rowspan="2">State of the world</th><th rowspan="2">Competence</th><th colspan="2">PBP:
Probability, as n tends to infinity, of …</th><th colspan="2">CBP:
Probability, as n tends to infinity, of …</th></tr>
<tr><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th></tr>
<tr><td rowspan="2">TT</td><td>$pq < 0.5$</td><td colspan="2" rowspan="4">1
(a)</td><td colspan="2">0
(b)</td></tr>
<tr><td>$pq > 0.5$</td><td colspan="2">1
(e)</td></tr>
<tr><td rowspan="2">TF,
FT,
or FF</td><td>$pq < 0.5$</td><td>1
(c)</td><td>0
(d)</td></tr>
<tr><td>$pq > 0.5$</td><td colspan="2">1
(e)</td></tr>
</table>

For many conditions, the performance of the CBP is poor. Unless individual competence is high ($pq > 0.5$), the probability that the CBP reaches the truth on R for the right reasons converges to 0 as n tends to infinity; and the probability that the CPB reaches the truth regardless of reasons converges to 0 unless at least one premise is false. By contrast, the probability that the PBP reaches the truth, both for the right reasons and regardless of reasons, converges to 1 whenever p, $q > 0.5$. But, when P and Q are not both true, then the probability that the CBP reaches the correct decision regardless of reasons converges to 1 *faster* than the corresponding probability for the PBP. This follows from the fact (remarked above) that condition (3) in Table 5 implies condition (7), whereas the converse implication does not hold.

The results of this section require that each individual's judgment on P and the *same* individual's judgment on Q are independent from each other. If there is a high *dependency*, the probability of a collective inconsistency (and of a discrepancy between the PBP and CBP) is drastically reduced. In the limiting case, if each individual makes a correct judgment on P *if and only if* they make a correct judgment on Q, the individual's competence on R equals their competence on each of P and Q. Then, if p, q>0.5, by the classical Condorcet jury theorem, the probability of a correct decision under the CBP converges to 1 as n tends to infinity. In this case of *perfect* dependency, the PBP and CBP will always coincide and there will be no collective inconsistencies.

6 Extensions and Generalizations

So far we have addressed only the *conjunctive* version of the doctrinal paradox. Generalizations have been discussed, such as disjunctive versions and extensions to more than two premises (e.g. Chapman 1998 and Pettit 2001). This section applies the present probability-theoretic framework to some of these generalizations.

6.1 The Disjunctive Version of the Doctrinal Paradox

Consider a tenure decision in an academic department. A candidate is assessed in terms of teaching and research.[5] We assume that the candidate is good, but not necessarily outstanding, in *both* teaching *and* research. The relevant propositions are P ("the candidate is outstanding in teaching") and Q ("the candidate is outstanding in research"). According to the department's constitution, the candidate should be given tenure (R) *if and only if* he is outstanding in *at least one* of teaching *or* research (assuming that he is at least good in both); i.e. the propositions are connected by $(R \leftrightarrow (P \vee Q))$.

Table 7: The doctrinal paradox (disjunctive version)

	P	Q	$(R \leftrightarrow (P \vee Q))$	R
Individual 1	Yes	No	Yes	Yes
Individual 2	No	Yes	Yes	Yes
Individual 3	No	No	Yes	No
Majority	No	No	Yes	Yes

Given the individual judgments in Table 7, majorities *reject* each of P and Q, a majority (unanimity) *accepts* $(R \leftrightarrow (P \vee Q))$, and yet a majority *accepts* R. A majority holds that the candidate is *not* outstanding in teaching; a majority holds that he is *not* outstanding in research, and yet a majority holds that he should be given tenure.

Again, let n_{TT}, n_{TF}, n_{FT}, n_{FF} be the numbers of individuals holding the 4 possible sets of judgments, as shown in Table 8, and p_{TT}, p_{TF}, p_{FT}, p_{FF} the corresponding probabilities.

Table 8: All logically possible consistent sets of judgments over P, Q and R, given $(R \leftrightarrow (P \vee Q))$

Label	Judgment on P	Judgment on Q	Judgment on R
TT	Yes	Yes	Yes
TF	Yes	No	Yes
FT	No	Yes	Yes
FF	No	No	No

[5] I am indebted to Bruce Chapman for this example.

As $(R\leftrightarrow(P\vee Q))$ is logically equivalent to $(\neg R\leftrightarrow(\neg P\wedge\neg Q))$, our results on the conjunctive version of the paradox have direct corollaries for the disjunctive version. We can simply swap T and F in all the propositions and proofs. If $(R\leftrightarrow(P\vee Q))$ is unanimously accepted, a collective inconsistency occurs if and only if $(n_{FF}+n_{FT} > n/2)$ and $(n_{FF}+n_{TF} > n/2)$ and $(n_{FF} < n/2)$. This corresponds to the majority acceptance of the (inconsistent) set of propositions $\{\neg P, \neg Q, (R \leftrightarrow (P \vee Q)), R\}$. Let Q^*_n denote the probability of a collective inconsistency under propositionwise majority voting, where the connection rule is $(R \leftrightarrow (P \vee Q))$ and where there are n individuals.

Proposition 6.

(a) If $(p_{FF} + p_{FT} > 1/2)$ *and* $(p_{FF} + p_{TF} > 1/2)$ *and* $(p_{FF} < 1/2)$, *then* $Q^*_n \to 1$ *as* $n \to \infty$.

(b) If $(p_{FF} + p_{FT} < 1/2)$ *or* $(p_{FF} + p_{TF} < 1/2)$ *or* $(p_{FF} > 1/2)$, *then* $Q^*_n \to 0$ *as* $n \to \infty$.

Table 9: Scenarios corresponding to Q^*_n (probability of a collective inconsistency under propositionwise majority voting, given $R \leftrightarrow (P \vee Q)$))

Scenario 1*	Scenario 2*	Scenario 3*	Scenario 4*	Scenario 5*	Scenario 6*	Scenario 7*	Scenario 8*
$p_{TT} = 0.25$	$p_{TT} = 0.24$	$p_{TT} = 0.2$	$p_{TT} = 0.24$	$p_{TT} = 0.11$	$p_{TT} = 0.09$	$p_{TT} = 0.05$	$p_{TT} = 0.01$
$p_{TF} = 0.25$	$p_{TF} = 0.25$	$p_{TF} = 0.25$	$p_{TF} = 0.25$	$p_{TF} = 0.2$	$p_{TF} = 0.2$	$p_{TF} = 0.2$	$p_{TF} = 0.33$
$p_{FT} = 0.25$	$p_{FT} = 0.25$	$p_{FT} = 0.25$	$p_{FT} = 0.27$	$p_{FT} = 0.2$	$p_{FT} = 0.2$	$p_{FT} = 0.2$	$p_{FT} = 0.33$
$p_{FF} = 0.25$	$p_{FF} = 0.26$	$p_{FF} = 0.3$	$p_{FF} = 0.24$	$p_{FF} = 0.49$	$p_{FF} = 0.51$	$p_{FF} = 0.55$	$p_{FF} = 0.33$

If Scenarios 1 to 8 in Table 3 are replaced with Scenarios 1* to 8* in Table 9, Q^*_n can be read off from Table 3. The conditions of 6a hold in 2*, 3*, 5* and 8*; the conditions of 6b hold in 4*, 6* and 7*. We use the Condorcet jury framework of Section 5 again.

Proposition 7. *Suppose that* $p, q > 0.5$.

(a) Suppose both P and Q are false.

- *If* $pq < 0.5$, then $Q^*_n \to 1$ *as* $n \to \infty$.
- *If* $pq > 0.5$, then $Q^*_n \to 0$ *as* $n \to \infty$.

(b)Suppose not both P and Q are false. Then $Q^*_n \to 0$ *as* $n \to \infty$.

The PBP and CBP provide escape-routes from the disjunctive version of the paradox too. A discrepancy between the PBP and CBP occurs precisely when the paradox occurs. By proposition 7, the probability of such a discrepancy therefore converges to 1 (as n tends to infinity) when both premises are false and individual competence is better than random, but not high, e.g. when $0.5 < p, q < \sqrt{(0.5)}$. Which procedure is better at reaching correct decisions? As before, we distinguish between reaching the truth for the right reasons and reaching the truth regardless of reasons, as detailed in Table 10.

Table 10: Conditions under which the PBP and CBP reach the correct decision on R (given $(R \leftrightarrow (P \vee Q))$) (i) regardless of reasons and (ii) for the right reasons, for different states of the world

<table>
<tr><th rowspan="2">State of the world</th><th colspan="2">PBP reaches a correct decision on R</th><th colspan="2">CBP reaches a correct decision on R</th></tr>
<tr><th>regardless of reasons if and only if …</th><th>for the right reasons if and only if …</th><th>regardless of reasons if and only if …</th><th>for the right reasons if and only if …</th></tr>
<tr><td>TT</td><td rowspan="3">there is a majority for at least one of P or Q: $(n_{TT} + n_{TF} > n/2)$ or $(n_{TT} + n_{FT} > n/2)$ (3)</td><td>there are majorities for each of P and Q: $(n_{TT} + n_{TF} > n/2)$ and $(n_{TT} + n_{FT} > n/2)$ (6)</td><td rowspan="3">there is not a single majority against both P and Q: $n_{FF} < n/2$ (7)</td><td>there is a single majority supporting both P and Q: $n_{TT} > n/2$ (10)</td></tr>
<tr><td>TF</td><td>there is a majority for P and a majority against Q: $(n_{TT} + n_{TF} > n/2)$ and $(n_{FF} + n_{TF} > n/2)$ (4)</td><td>there is a single majority supporting P and rejecting Q: $n_{TF} > n/2$ (8)</td></tr>
<tr><td>FT</td><td>there is a majority against P and a majority for Q: $(n_{FT} + n_{FF} > n/2)$ and $(n_{FT} + n_{TT} > n/2)$ (5)</td><td>there is a single majority rejecting P and supporting Q: $n_{FT} > n/2$ (9)</td></tr>
<tr><td>FF</td><td colspan="2">there are majorities against each of P and Q: $(n_{FF} + n_{FT} > n/2)$ and $(n_{FF} + n_{TF} > n/2)$ (1)</td><td colspan="2">there is a single majority rejecting both P and Q: $n_{FF} > n/2$ (2)</td></tr>
</table>

In analogy with Table 5, Table 10 allows us to deduce the following:

- *Reaching the correct decision on R for the right reasons:* The PBP is always at least as good as the CBP.
- *Reaching the correct decision on R regardless of reasons:* We distinguish two cases:
    - *P* and *Q* are both false: The PBP is always at least as good as the CBP.
    - At least one of *P* or *Q* is true: The CBP is always at least as good as the PBP.

These results are also consistent with Grofman's results (1985): when a group decision on a disjunctive composite proposition can be disaggregated into separate group decisions on each disjuncts, disaggregation is superior in terms of reaching the correct decision (regardless of reasons) for false propositions, but not for true propositions.

Proposition 8. *Let the connection rule be* $(R \leftrightarrow (P \vee Q))$. *The probabilities, as n tends to infinity, that the PBP and CPB reach a correct decision on R (i) regardless of reasons and (ii) for the right reasons, under various scenarios, are as shown in Table* 11.

Table 11: Probability, as n tends to infinity, of a correct decision on R (given $(R \leftrightarrow (P \vee Q))$) under the PBP and CBP (i) regardless of reasons and (ii) for the right reasons, under various scenarios

<table>
<tr><th rowspan="2">State of the world</th><th rowspan="2">Competence</th><th colspan="2">PBP:
Probability, as n tends to infinity, of …</th><th colspan="2">CBP:
Probability, as n tends to infinity, of …</th></tr>
<tr><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th></tr>
<tr><td rowspan="2">TT, TF, or FT</td><td>$pq < 0.5$</td><td colspan="2" rowspan="4">1
(a)</td><td>1
(c)</td><td>0
(d)</td></tr>
<tr><td>$pq > 0.5$</td><td colspan="2">1
(e)</td></tr>
<tr><td rowspan="2">FF</td><td>$pq < 0.5$</td><td colspan="2">0
(b)</td></tr>
<tr><td>$pq > 0.5$</td><td colspan="2">1
(e)</td></tr>
</table>

As in the conjunctive case, the CBP performs poorly for many conditions, particularly at reaching the truth on R for the right reasons. Unlike in the conjunctive case, the probability that the CBP reaches the truth regardless of reasons converges to 1 if at least one premise is true. When P and Q are not both false, the probability that the CBP reaches the correct decision on R regardless of reasons converges to 1 *faster* than the corresponding probability for the PBP. Condition (3) in Table 10 implies condition (7), but not vice-versa (compare the remarks on the conjunctive case in Section 5).

6.2 The Conjunctive Version of the Doctrinal Paradox with More than Two Premises

We first generalize Propositions 1 and 3 to the case of three premises. We then generalize Propositions 4 and 5 to the case of k premises.

Table 12: The doctrinal paradox (conjunctive version) with three premises

	P	Q	R	$(S \leftrightarrow (P \wedge Q \wedge R))$	S
Individual 1	Yes	Yes	No	Yes	No
Individual 2	No	Yes	Yes	Yes	No
Individual 3	Yes	No	Yes	Yes	No
Majority	Yes	Yes	Yes	Yes	No

If the individual judgments are as in Table 12, there are majorities for each of P, Q and R; the connection rule, $(S \leftrightarrow (P \wedge Q \wedge R))$, is unanimously accepted; and yet S is unanimously rejected. Let n_{TTT}, n_{TTF}, n_{TFT}, n_{TFF}, n_{FTT}, n_{FTF}, n_{FFT}, n_{FFF} be the numbers of individuals holding the 8 possible sets of judgments, as shown in Table 13, and let p_{TTT}, p_{TTF}, p_{TFT}, p_{TFF}, p_{FTT}, p_{FTF}, p_{FFT}, p_{FFF} be the corresponding probabilities.

Table 13: All logically possible consistent sets of judgments over P, Q, R and S, given $(S \leftrightarrow (P \wedge Q \wedge R))$

	P	Q	R	S
TTT	Yes	Yes	Yes	Yes
TTF	Yes	Yes	No	No
TFT	Yes	No	Yes	No
TFF	Yes	No	No	No
FTT	No	Yes	Yes	No
FTF	No	Yes	No	No
FFT	No	No	Yes	No
FFF	No	No	No	No

If $(S\leftrightarrow(P\wedge Q\wedge R))$ is unanimously accepted, a collective inconsistency occurs if and only if $(n_{TTT}+n_{TTF}+n_{TFT}+n_{TFF} > n/2)$ *and* $(n_{TTT}+n_{TTF}+n_{FTT}+n_{FTF} > n/2)$ *and* $(n_{TTT}+n_{TFT} +n_{FTT}+n_{FFT} > n/2)$ *and* $(n_{TTT} < n/2)$. This corresponds to the majority acceptance of the (inconsistent) set of propositions $\{P, Q, R, (S\leftrightarrow(P\wedge Q\wedge R)), \neg S\}$. Let P^{**}_n denote the probability of a collective inconsistency under propositionwise majority voting, where the connection rule is $(S\leftrightarrow(P\wedge Q\wedge R))$ and where there are n individuals.

Proposition 9.

(a) If $(p_{TTT}+p_{TTF}+p_{TFT}+p_{TFF} > n/2)$ *and* $(p_{TTT}+p_{TTF}+p_{FTT}+p_{FTF} > n/2)$ *and* $(p_{TTT}+p_{TFT} +p_{FTT}+p_{FFT} > n/2)$, $P^{**}_n \rightarrow 1$ *as* $n \rightarrow \infty$.

(b) If $(p_{TTT}+p_{TTF}+p_{TFT}+p_{TFF} < n/2)$ *or* $(p_{TTT}+p_{TTF}+p_{FTT}+p_{FTF} < n/2)$ *or* $(p_{TTT}+p_{TFT} +p_{FTT}+p_{FFT} < n/2)$, *then* $P^{**}_n \rightarrow 0$ *as* $n \rightarrow \infty$.

A proof is given in Appendix 2. To illustrate, the conditions of 9a hold when the vector $<p_{TTT}, p_{TTF}, p_{TFT}, p_{TFF}, p_{FTT}, p_{FTF}, p_{FFT}, p_{FFF}>$ equals <0.126, 0.125, …, 0.125, 0.124> or <0.49, 0.08, …, 0.08, 0.03>. The conditions of 9b hold for <0.124, 0.125, …, 0.125, 0.126> or <0.51, 0.08, …, 0.08, 0.01>. This confirms that small changes in the probabilities can correspond to large changes in the corresponding convergence pattern.

Now consider a decision problem with k premises, $P_1, P_2, ..., P_k$, and a conclusion, R, where the connection rule is $(R\leftrightarrow(P_1\wedge P_2\wedge\ldots\wedge P_k))$. We assume that there are 2^k possible states of the world: each of $P_1, P_2, ..., P_k$ can be either true or false. Given the connection rule $(R\leftrightarrow(P_1\wedge P_2\wedge\ldots\wedge P_k))$, each state of the world determines the truth-value of R. We assume that each individual has probabilities $p_1, p_2, \ldots, p_k$ of making correct judgments on $P_1, P_2, ..., P_k$, respectively. Our assumptions on the individual judgments on $P_1, P_2, ..., P_k$ are perfectly analogous to assumptions (i), (ii) and (iii) in Section 5. Let P^{***}_n denote the probability of a collective inconsistency under propositionwise majority voting, with the connection rule $(R\leftrightarrow(P_1\wedge P_2\wedge\ldots\wedge P_k))$ and for n individuals. Propositions 10 and 11 can be proved analogously to Propositions 3 and 4. The probability that an individual holds a correct judgment on *every* proposition is $p_1 p_2 \dots p_k$. If $p_1, p_2, \dots, p_k < {}^k\sqrt{(0.5)}$, then $p_1 p_2 \dots p_k < 0.5$. If $p_1, p_2, \dots, p_k > {}^k\sqrt{(0.5)}$, then $p_1 p_2 \dots p_k > 0.5$.

Proposition 10. *Suppose that* $p_1, p_2, \dots, p_k > 0.5$.
(a) Suppose both $P_1, P_2, \dots, P_k$ *are all true.*

- *If* $p_1 p_2 \dots p_k < 0.5$, then $P^{***}_n \to 1$ *as* $n \to \infty$.
- *If* $p_1 p_2 \dots p_k > 0.5$, then $P^{***}_n \to 0$ *as* $n \to \infty$.

(b) Suppose not all of $P_1, P_2, \dots, P_k$ *are true. Then* $P^{***}_n \to 0$ *as* $n \to \infty$.

Proposition 11. *Let the connection rule be* $(R \leftrightarrow (P_1 \wedge P_2 \wedge \dots \wedge P_k))$. *The probabilities, as n tends to infinity, that the PBP and CBP reach a correct decision on R (i) regardless of reasons and (ii) for the right reasons, under various scenarios, are as shown in Table* 14.

Table 14: Probability, as n tends to infinity, of a correct decision on R (given $(R \leftrightarrow (P_1 \wedge P_2 \wedge \dots \wedge P_k))$) under the PBP and CBP (i) regardless of reasons and (ii) for the right reasons, under various scenarios

<table>
<tr><th rowspan="2">State of the world</th><th rowspan="2">Competence</th><th colspan="2">PBP:
Probability, as n tends to infinity, of …</th><th colspan="2">CBP:
Probability, as n tends to infinity, of …</th></tr>
<tr><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th></tr>
<tr><td rowspan="2">all P_i true</td><td>$p_1 p_2 \dots p_k <$ 0.5</td><td rowspan="4" colspan="2">1
(a)</td><td colspan="2">0
(b)</td></tr>
<tr><td>$p_1 p_2 \dots p_k >$ 0.5</td><td colspan="2">1
(e)</td></tr>
<tr><td rowspan="2">not all P_i true</td><td>$p_1 p_2 \dots p_k <$ 0.5</td><td>1
(c)</td><td>0
(d)</td></tr>
<tr><td>$p_1 p_2 \dots p_k >$ 0.5</td><td colspan="2">1
(e)</td></tr>
</table>

When k is large and all premises are true, a high level of individual competence is required for avoiding collective inconsistencies, namely $p_1 p_2 \dots p_k > 0.5$. If $p_1 = p_2 = \dots = p_k$, the required lower bound on each of $p_1, p_2, \dots, p_k$ is ${}^k\sqrt{(0.5)}$, which converges to 1 as k tends to infinity. If individual competence is below that bound, the CBP performs poorly in terms of reaching a correct decision for the right reasons. The CBP also performs poorly in terms of reaching a correct decision regardless of reasons, unless competence is high or at least one premise is false. The PBP, by contrast, reaches a correct decision more reliably, both for the right reasons and regardless of reasons (a caveat on the speed of convergence applies as in Section 5). Again, the results depend on the mutual independence of each individual's judgments on *different* premises.

7 The Probability of the Doctrinal Paradox compared with the Probability of Condorcet's Paradox

The doctrinal paradox invites comparison with Condorcet's paradox (on the parallels between the two paradoxes see List and Pettit 2003). Condorcet's paradox concerns the aggregation of preferences over multiple alternatives rather than the aggregation of sets of judgments over multiple propositions. Suppose there are three individuals, where one prefers alternative x_1 to alternative x_2 to alternative x_3, the second prefers x_2 to x_3 to x_1, and the third prefers x_3 to x_1 to x_2. Then there is a majority for x_1 against x_2, a majority for x_2 against x_3, and a majority for x_3 against x_1, a cycle. The probability of Condorcet's paradox has been studied under various assumptions (e.g. Niemi 1969; Gehrlein and Fishburn 1976a, 1976b; Gehrlein 1981, 1983, 1997). While earlier work focussed primarily on impartial culture or impartial anonymous culture conditions, several recent papers have addressed deviations from such conditions (e.g. Tangian 2000; Tsetlin, Regenwetter and Grofman 2003; List and Goodin 2001; Gehrlein 2002).

I now compare the present results on the doctrinal paradox with existing results on Condorcet's paradox. The paradoxes under comparison are the conjunctive version of the doctrinal paradox with two premises (as discussed in Sections 1-5) and Condorcet's paradox over three alternatives. I compare the probability of the two paradoxes under (i) an impartial culture, (ii) systematic deviations from an impartial culture, (iii) an impartial anonymous culture, and (iv) Condorcet jury assumptions.

We begin with some definitions. Let there be n individuals, and three alternatives, x_1, x_2 and x_3. Let n_{123}, n_{132}, n_{213}, n_{231}, n_{312}, n_{321} be the numbers of individuals holding the 6 logically possible strict preference orderings shown in Table 15, respectively.

Table 15: All logically possible strict preference orderings over three options

Label	1st preference	2nd preference	3rd preference
123	x_1	x_2	x_3
132	x_1	x_3	x_2
213	x_2	x_1	x_3
231	x_2	x_3	x_1
312	x_3	x_1	x_2
321	x_3	x_2	x_1

A vector $<n_{123}, n_{132}, n_{213}, n_{231}, n_{312}, n_{321}>$ is an *anonymous profile*. Let p_{123}, p_{132}, p_{213}, p_{231}, p_{312}, p_{321} be the probabilities that an individual holds each of the 6 orderings ($p_{123}+p_{132}+p_{213}+p_{231}+p_{312}+p_{321}=1$). An *impartial culture* is the situation in which $p_{123}=p_{132}=p_{213}=p_{231}=p_{312}=p_{321}$. An *impartial anonymous culture* is the situation in which every logically possible anonymous profile $<n_{123}, n_{132}, n_{213}, n_{231}, n_{312}, n_{321}>$ is equally probable.

There is a cyclical collective preference ordering under pairwise majority voting *if and only if* [more than $^n/_2$ individuals prefer x_1 to x_2, more than $^n/_2$ prefer x_2 to x_3, and more than $^n/_2$ prefer x_3 to x_1] or [more than $^n/_2$ prefer x_3 to x_2, more than $^n/_2$ prefer x_2 to x_1, and more than $^n/_2$ prefer x_1 to x_3], formally *if and only if*

$$((n_{123} > n_{321} \textit{ and } n_{312} > n_{213} \textit{ and } n_{231} > n_{132}) \textit{ or } (n_{321} > n_{123} \textit{ and } n_{213} > n_{312} \textit{ and } n_{132} > n_{231})) \\ \textit{and } |n_{123}-n_{321}|<n'/2 \textit{ and } |n_{231}-n_{132}|<n'/2 \textit{ and } |n_{312}-n_{213}|<n'/2,$$

where $n' := |n_{123}-n_{321}|+|n_{231}-n_{132}|+|n_{312}-n_{213}|$ (for a simple proof see Elsholtz and List 2002).

Let $P^C{}_n$ be the probability of a cyclical collective preference ordering under pairwise majority voting, for n individuals. Recall that $P^*{}_n$ is the probability of a collective inconsistency under propositionwise majority voting with connection rule $(R\leftrightarrow(P\wedge Q))$, for n individuals.

Impartial culture. Under an impartial culture, both $P^C{}_n$ and $P^*{}_n$ increase as n increases, and they both converge to a value strictly between 0 and 1: $P^C{}_n \rightarrow 0.08774$ and $P^*{}_n \rightarrow 0.25$ as $n \rightarrow \infty$ (see Gehrlein 1983).

Systematic deviations from an impartial culture. We have seen that, under systematic, however small, deviations from an impartial culture, $P^*{}_n$ converges to either 0 or 1 as n tends to infinity, depending on the nature of the deviation. A similar result holds for $P^C{}_n$ (see Tangian 2000; Tsetlin, Regenwetter and Grofman 2003; List 2001).

Proposition 12 (List 2001). *Let* $p' := |p_{123}-p_{321}| + |p_{231}-p_{132}| + |p_{312}-p_{213}|$.

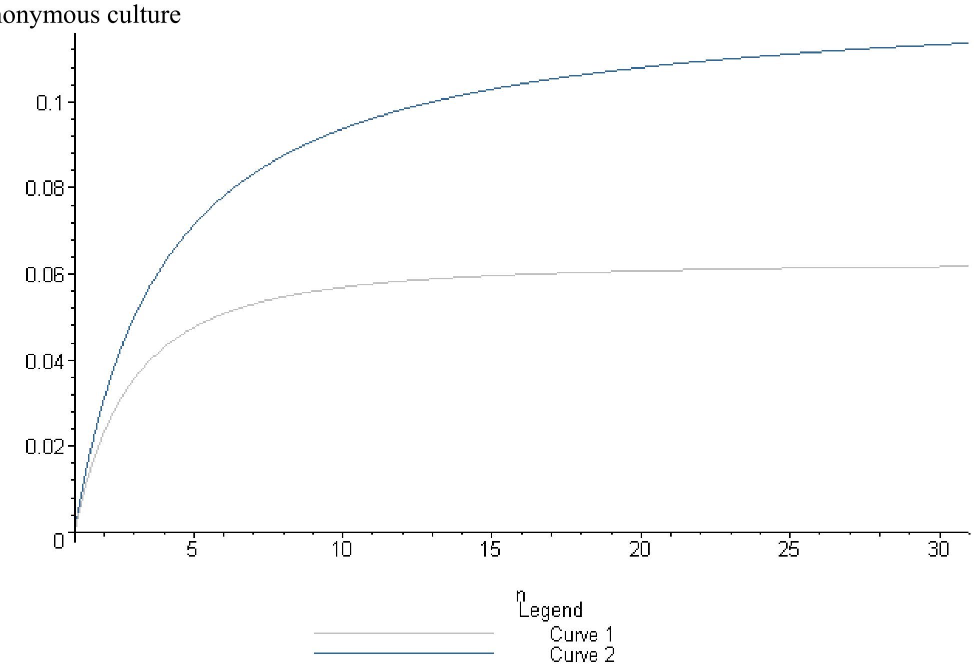
(a) If $((p_{123}>p_{321}$ *and* $p_{312}>p_{213}$ *and* $p_{231}>p_{132})$ *or* $(p_{321}>p_{123}$ *and* $p_{213}>p_{312}$ *and* $p_{132}>p_{231}))$ *and* $|p_{123}-p_{321}|<p'/2$ *and* $|p_{231}-p_{132}|<p'/2$ *and* $|p_{312}-p_{213}|<p'/2$, *then* $P^C{}_n \rightarrow 1$ *as* $n \rightarrow \infty$.

(b) If $((p_{123}<p_{321}$ *or* $p_{312}<p_{213}$ *or* $p_{231}<p_{132})$ *and* $(p_{321}<p_{123}$ *or* $p_{213}<p_{312}$ *or* $p_{132}<p_{231}))$ *or* $|p_{123}-p_{321}|>p'/2$ *or* $|p_{231}-p_{132}|>p'/2$ *or* $|p_{312}-p_{213}|>p'/2$, *then* $P^C{}_n \rightarrow 0$ *as* $n \rightarrow \infty$.

Parts (a) and (b) of Proposition 12 corresponds to parts (a) and (b) of Proposition 3, respectively. Propositions 3 and 12 show that, in any ε-neighbourhood of an impartial culture (under the appropriate definition), there exist probability vectors $<p_{TT}, p_{TF}, p_{FT}, p_{FF}>$ and $<p_{123}, p_{132}, p_{213}, p_{231}, p_{312}, p_{321}>$ for which $P^*{}_n$ and $P^C{}_n$ each converge to 0, and ones for which they each converge to 1, as n tends to infinity. Which kinds of deviations from an impartial culture are more ‘typical’? In the case of Condorcet’s paradox, Tangian (2000) and Tsetlin, Regenwetter and Grofman (2003) have offered arguments that the deviations for which $P^C{}_n$ converges to 0 are the more typical ones, and consequently that cycles are improbable in a large electorate. List and Goodin (2001) have investigated deviations from an impartial culture that result when the individuals satisfy the minimal competence assumptions of a Condorcet jury framework and shown that, under such deviations, P^C_n converges to 0 as n tends to infinity. Below we compare P^C_n and P^*_n from the perspective of such deviations from an impartial culture.

Impartial anonymous culture. As shown in Section 4, under an impartial anonymous culture, the probability of a collective inconsistency equals $(n\text{-}1)/(8(n+2))$, where there are n individuals. This probability is equal to $E(P^*_n)$, the expected value of P^*_n under the assumption that all logically possible $<p_{TT}, p_{TF}, p_{FT}, p_{FF}>$ vectors are equally probable. Thus $E(P^*_n)$ converges to 1/8 as n tends to infinity. Gehrlein and Fishburn (1976a) have shown that, under an impartial anonymous culture, the probability of Condorcet's paradox over three alternatives equals $(1/16)(n+7)(n\text{-}1)/((n+2)(n+4))$. Further, this probability is equal to $E(P^C_n)$, the expected value of P^C_n under the assumption that all logically possible $<p_{123}, p_{132}, p_{213}, p_{231}, p_{312}, p_{321}>$ vectors are equally probable (Gehrlein 1981). Thus $E(P^C_n)$ converges to 1/16 as n tends to infinity. Graph 1 shows $E(P^C_n)$ and $E(P^*_n)$. The numbers are meaningful only when n is odd.

Graph 1: The probabilities of Condorcet's paradox (Curve 1) and the doctrinal paradox (Curve 2) under an impartial anonymous culture



Condorcet jury assumptions. Both under an impartial culture and under an impartial anonymous culture the doctrinal paradox is more likely to occur than Condorcet's paradox. For systematic deviations from an impartial culture, we have found a parallel between the two paradoxes: in any ε-neighbourhood of an impartial culture, there exist probability vectors for which P^*_n and P^C_n each converge to 0, and ones for which they each converge to 1, as n tends to infinity. We now consider deviations from an impartial culture that result when the individuals satisfy the minimal "competence" assumptions of a Condorcet jury framework. In decisions over multiple interconnected propositions, our minimal competence assumption follows from our discussion in Section 5.

Minimal Competence Assumption C1: Each individual has probabilities p and q of making a correct judgment on P and Q, respectively, where $0.5 < p, q < \sqrt{(0.5)}$.

As we have seen in Section 5, for each state of the world, the values of p and q induce values of p_{TT}, p_{TF}, p_{FF}, p_{FF}, and we can compute the corresponding value of P^*_n for any n. By assumption C1, the probability distribution over all logically possible individual sets of judgments is skewed, however slightly, in favour of the "correct" judgment on each premise; C1 is "minimal" in that there exist probability vectors in any ε-neighbourhood of an impartial culture that satisfy C1.

In the framework of preference orderings, we can state the minimal competence assumption as follows. We assume that each individual has probabilities p_1, p_2, p_3 of choosing x_1, x_2, x_3 as their *first choice*, respectively; and that the first choice preferences of different individuals are independent from each other. We assume that there are three possible states of the world: the "correct" alternative is either x_1 or x_2 or x_3.

Minimal Competence Assumption C2: If x_i is the "correct" alternative, then, for all alternatives $j \neq i$, $p_i > p_j$.

We can give a simple definition by which, for each state of the world, the values of p_1, p_2 and p_3 determine corresponding values of p_{123}, p_{132}, p_{213}, p_{231}, p_{312}, p_{321}: i.e. from the probabilities corresponding to each individual's first choice under a given state of the world, we calculate probabilities corresponding to each individual's holding each of the orderings shown in Table 15. We define the probability for the strict ordering $x_h > x_i > x_j$ (where $h, i, j \in \{1, 2, 3\}$) to be $p_h({p_i}/{(p_i+p_j)})$ (see List and Goodin 2001). This corresponds to the way in which, for each state of the world, the values of p and q induce values of p_{TT}, p_{TF}, p_{FF}, p_{FF}. Given p_1, p_2, p_3, we can then compute the corresponding value of P^C_n for any n. By assumption C2, the probability distribution over all logically possible preference orderings is skewed, however slightly, in favour of a preference for the "correct" alternative; C2 is also "minimal" in that there exist probability vectors in any ε-neighbourhood of an impartial culture that satisfy C2.

What are the implications of C1 and C2? By Proposition 4, if the state of the world is TT, C1 implies that P^*_n converges to 1 as n tends to infinity. By contrast, for any state of the world, C2 implies that the probability that the "correct" alternative will beat all other alternatives in pairwise majority voting converges to 1, and thus that P^{C}_{n} converges to 0, as n tends to infinity (List and Goodin 2001). While the probability of Condorcet's paradox always converges to 0 under C2 (which can be satisfied in any ε-neighbourhood of an impartial culture), no such general result holds for the doctrinal paradox under C1. We require $pq > 0.5$ to get general convergence of P^{*}_{n} to 0.

We may expect this effect to be even greater in conjunctive decision tasks with more than two premises. If there are k premises (supposing, for our argument, all are true), any individual competence above 0.5 but below $\sqrt[k]{(0.5)}$ implies that P^{***}_{n} – the probability of a collective inconsistency – converges to 1 as the n tends to infinity.

These considerations suggest that there exist plausible conditions under which the doctrinal paradox is more probable than Condorcet's paradox. This hypothesized discrepancy between the probability of cycles and the probability of inconsistent collective sets of judgments is consistent with two pieces of anecdotal evidence. The predicted low probability of cycles in a large electorate (so long as we are not in an impartial culture) is consistent with the striking lack of empirical evidence for cycles (see Mackie 2000 for a critique of several purported empirical examples of cycles). The predicted higher probability of doctrinal paradoxes in a large electorate (even when we are not in an impartial culture) is consistent with the findings of an empirical study of voting on referenda (Brams, Kilgour and Zwicker 1997). The study showed that, for three related propositions on the environment in a 1990 referendum in California, less than 6% of the (sampled) electorate individually endorsed the particular combination of these three propositions (acceptance of two, rejection of the third) that won under propositionwise majority voting. Of course, there were no explicit logical connections in the referendum. However, if the winning combination of propositions had been used as a jointly necessary and sufficient condition for some further conclusion, or if a separate vote had been taken on the particular winning *combination* as a single proposition (which would have presumably failed to get majority support), we would have had an instance of an inconsistent collective set of judgments.

8 Concluding Remarks

We have developed a model for determining the probability of collective inconsistencies under propositionwise majority voting, and applied the model to conjunctive and disjunctive versions of the doctrinal paradox with two premises, and also to the conjunctive version of the paradox with more than two premises.

We have identified conditions under which the probability of a collective inconsistency converges to 1 and ones under which it converges to 0. Both sets of conditions can occur in plausible circumstances. In the case of the conjunctive version of the doctrinal paradox, convergence of the probability of the paradox to 1 is implied by standard competence assumptions in a Condorcet jury framework when all premises are true and individual competence is low (but better than random). Convergence of the probability of the paradox to 0 occurs when either at least one of the premises is false or individual competence is high. In the disjunctive case, convergence of the probability of the paradox to 1 occurs when all premises are false and individual competence is low. Convergence of the probability of the paradox to 0 occurs when either at least one of the premises is true or individual competence is high.

Since decision problems with medium individual competence seem empirically plausible, the occurrence of the doctrinal paradox may be quite likely. This reinforces the importance of identifying escape-routes from the paradox and of asking what methods groups can and do employ to avoid the paradox (see also List and Pettit 2002).

With regard to possible escape-routes, following Bovens and Rabinowicz (2001, 2003), we have seen that, for a large class of cases, the PBP is superior to the CBP in terms of reaching correct decisions (where there is an independent standard of "correctness"), especially when we care about the underlying reasons. This suggests a happy coincidence between epistemic and procedural perspectives on the PBP and CBP. While the arguments offered by Pettit (2001) and Chapman (2002) are mainly procedural arguments in favour of the PBP, we have here seen that, in a large class of cases, the PBP will also be preferred on epistemic grounds. Finally, we have compared the present results with existing results on the probability of Condorcet's paradox.

The results of this paper are initial results, not the final word, on the probability of collective inconsistencies under propositionwise majority voting. More sophisticated probability-theoretic models could be constructed, for instance allowing different probabilities corresponding to different individuals, and certain dependencies between the judgments of different individuals or between the same individual's judgments on different propositions (compare the discussion at the end of Section 5). But even the present results support one conclusion. The occurrence of the doctrinal paradox is not implausible, and the paradox deserves attention.

Appendix 1: Calculating the Probability of the Various Scenarios in Table 5

For each of the 10 scenarios in Table 5, let M be the set of all $<n_{TT}, n_{TF}, n_{FT}, n_{FF}>\in N$ for which the condition of the given scenario is satisfied. Using the probability function for the joint distribution of X_{TT}, X_{TF}, X_{FT}, X_{FF} (see Section 3), the desired probability is

$$\sum_{<n_{TT}, n_{TF}, n_{FT}, n_{FF}>\in M} \frac{n!}{n_{TT}!\, n_{TF}!\, n_{FT}!\, n_{FF}!} p_{TT}^{n_{TT}} p_{TF}^{n_{TF}} p_{FT}^{n_{FT}} p_{FF}^{n_{FF}}.$$

For example, if the state of the world is FF and we are interested in the probability that the CBP reaches the correct decision on R for the right reasons (Scenario 10), then we put $M := \{<n_{TT}, n_{TF}, n_{FT}, n_{FF}>\in N : (n_{FF} > n/2)\}$.

Appendix 2: Proofs

A *condition* ϕ *on a set of k probabilities*, $p_1, p_2, \ldots, p_k$, is a mapping whose domain is the set of all logically possible assignments of probabilities to $p_1, p_2, \ldots, p_k$ and whose co-domain is the set {*true*, *false*}. Whenever $\phi(p_1, p_2, \ldots, p_k)$ = *true*, we say that the probabilities $p_1, p_2, \ldots, p_k$ *satisfy* ϕ; and whenever $\phi(p_1, p_2, \ldots, p_k)$ = *false*, we say the probabilities $p_1, p_2, \ldots, p_k$ *violate* ϕ. Examples of conditions ϕ for $p_{TT}, p_{TF}, p_{FT}, p_{FF}$ are
(i) $(p_{TT} + p_{TF} > 1/2)$ *and* $(p_{TT} + p_{FT} > 1/2)$ *and* $(p_{TT} < 1/2)$;
(ii) $(p_{TT} \geq 1/2)$;
(iii) $(p_{TT} > 1/2)$ *and* $(p_{TF} > 1/2)$.

A condition ϕ is *consistent* if there exists at least one logically possible assignment of probabilities to $p_1, p_2, \ldots, p_k$ satisfying ϕ. A condition ϕ is *strict* if, for every assignment of probabilities $p_1, p_2, \ldots, p_k$ satisfying ϕ, there exists an $\varepsilon > 0$ such that, if the probabilities $p^*_1, p^*_2, \ldots, p^*_k$ lie inside a sphere in $\mathbf{R}^k$ with centre $p_1, p_2, \ldots, p_k$ and radius ε, then the probabilities $p^*_1, p^*_2, \ldots, p^*_k$ also satisfy ϕ. It is easily seen that the condition in example (i) is both consistent and strict; the condition in (ii) is consistent, but not strict; and the condition in (iii) is not consistent.

Let $X_1, X_2, \ldots, X_k$ be a set of k random variables whose joint distribution is a multinomial distribution with the following probability function:

$$P(X_1{=}n_1, X_2{=}n_2, \ldots, X_k{=}n_k) = \frac{n!}{n_1!\, n_2!\, \ldots\, n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k},$$

where $n_1+n_2+\ldots+n_k = n$. The weak law of large numbers implies the following lemma.

Lemma 1 (Convergence lemma). *Let ϕ be any consistent strict condition on a set of k probabilities. Suppose the probabilities p_1, p_2, …, p_k satisfy ϕ. Then $P(X_1/n, X_2/n, \ldots, X_k/n$ satisfy $\phi) \rightarrow 1$* as $n \rightarrow \infty$.

Proof of Proposition 3.

(a) $(p_{TT} + p_{TF} > 1/2)$ *and* $(p_{TT} + p_{FT} > 1/2)$ *and* $(p_{TT} < 1/2)$ is a consistent strict condition. By Lemma 1, $P((X_{TT}+X_{TF}>n/2)$ *and* $(X_{TT}+X_{FT}>n/2)$ *and* $(X_{TT} <n/2)) \rightarrow 1$ as $n \rightarrow \infty$. The result then follows from Proposition 1.

(b) The result follows from Lemma 1 and Proposition 1 analogously. Note that $(p_{TT} + p_{TF} < 1/2)$ *or* $(p_{TT} + p_{FT} < 1/2)$ *or* $(p_{TT} > 1/2)$ is a consistent strict condition.

Proof of Proposition 4.

(a) The state of the world is *TT*.

For the first part, it is sufficient to show that p_{TT}, p_{TF}, p_{FT}, p_{FF} (as defined in Table 4) satisfy the conditions of Proposition 3a. Suppose p, $q > 0.5$ and $pq < 0.5$. Then

$p_{TT} + p_{TF} = pq + p(1\text{-}q) = p > 0.5$;

$p_{TT} + p_{FT} = pq + (1\text{-}p)q = q > 0.5$;

$p_{TT} = pq < 0.5$, as required.

For the second part, it is sufficient to show that p_{TT}, p_{TF}, p_{FT}, p_{FF} satisfy the conditions of Proposition 3b. If $pq > 0.5$, then $p_{TT} = pq > 0.5$, as required.

(b) The state of the world is *TF*, *FT* or *FF*.

It is sufficient to show that p_{TT}, p_{TF}, p_{FT}, p_{FF} (as defined in Table 4) satisfy the conditions of Proposition 3b. Suppose p, $q > 0.5$.

Under *TF*, $p_{TT} + p_{FT} = p(1\text{-}q) + (1\text{-}p)(1\text{-}q) = 1\text{-}q < 1/2$, as required.

Under *FT*, $p_{TT} + p_{TF} = (1\text{-}p)q + (1\text{-}p)(1\text{-}q) = 1\text{-}p < 1/2$, as required.

Under *FF*, $p_{TT} + p_{TF} = (1\text{-}p)(1\text{-}q) + (1\text{-}p)q = 1\text{-}p < 1/2$, as required.

Proof of Proposition 5.

There are four possible states of the world: *TT*, *TF*, *FT*, or *FF*. The probabilities p_{TT}, p_{TF}, p_{FT}, p_{FF} are defined as shown in Table 4. Suppose p, $q > 0.5$.

(a) It is sufficient to show that the probability that the PBP reaches a correct decision on *R* for the right reasons (implying also that it reaches a correct decision regardless of reasons) converges to 1 as n tends to infinity.

Under *TT*: $p_{TT}+p_{TF}=pq+p(1\text{-}q)=p > 0.5$ *and* $p_{TT}+p_{FT}=pq+(1\text{-}p)q = q > 0.5$, a consistent strict condition. By Lemma 1, $P((X_{TT}+X_{TF} > n/2)$ *and* $(X_{TT}+X_{FT} > n/2)) \rightarrow 1$ as $n \rightarrow \infty$. Now compare condition (1) in Table 5. The result follows.

All other cases are analogous. In each case, we identify the relevant consistent strict condition, and the result follows from Lemma 1.

Under *TF*: $p_{TT}+p_{TF}=p(1\text{-}q)+pq=p>0.5$ *and* $p_{TF}+p_{FF}=pq+(1\text{-}p)q=q>0.5$. $P((X_{TT}+X_{TF} > n/2)$ *and* $(X_{TF}+X_{FF} > n/2)) \rightarrow 1$ as $n \rightarrow \infty$. See condition (4) in Table 5.

Under *FT*: $p_{FT}+p_{FF}=pq+p(1\text{-}q)=p>0.5$ *and* $p_{TT}+p_{FT}=(1\text{-}p)q+pq=q>0.5$. $P((X_{FT}+X_{FF} > n/2)$ *and* $(X_{TT}+X_{FT}>n/2)) \rightarrow 1$ as $n \rightarrow \infty$. See condition (5) in Table 5.

Under *FF*: $p_{FT}+p_{FF}=p(1\text{-}q)+pq=p>0.5$ *and* $p_{TF}+p_{FF}=(1\text{-}p)q+pq=q>0.5$. $P((X_{FT}+X_{FF} > n/2)$ *and* $(X_{TF}+X_{FF} > n/2)) \rightarrow 1$ as $n \rightarrow \infty$. See condition (6) in Table 5.

(b) Suppose $pq < 0.5$, and we have *TT*. Then $p_{TT}=pq<0.5$. $P(X_{TT}<n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (2) in Table 5.

(c) Suppose $pq < 0.5$, and we have *TF*, *FT*, or *FF*. By part (a), the probability that there are not majorities for *P* and for *Q* converges to 1 as *n* tends to infinity. This implies that $P(X_{TT} < n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (7) in Table 5.

(d) Suppose $pq < 0.5$.

Under *TF*: $p_{TF} = pq < 0.5$. $P(X_{TF} < n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (8) in Table 5.

Under *FT*: $p_{FT} = pq < 0.5$. $P(X_{FT} < n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (9) in Table 5.

Under *FF*: $p_{FF} = pq < 0.5$. $P(X_{FF} < n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (10) in Table 5.

(e) Suppose $pq > 0.5$. It is sufficient to show that the probability that the CBP reaches a correct decision on *R* for the right reasons (implying also that it reaches a correct decision regardless of reasons) converges to 1 as *n* tends to infinity.

Under *TT*: $p_{TT} = pq > 0.5$. $P(X_{TT} > n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (2) in Table 5.

Under *TF*: $p_{TF} = pq > 0.5$. $P(X_{TF} > n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (8) in Table 5.

Under *FT*: $p_{FT} = pq > 0.5$. $P(X_{FT} > n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (9) in Table 5.

Under *FF*: $p_{FF} = pq > 0.5$. $P(X_{FF} > n/2) \rightarrow 1$ as $n \rightarrow \infty$. See condition (10) in Table 5.

Proof of Proposition 9.

The conditions in (a) and (b) are consistent and strict. The results then follow from Lemma 1 and the necessary and sufficient condition for the doctrinal paradox (conjunctive version) with three premises stated in Section 6.2.

References

Anscombe GEM (1976) On Frustration of the Majority by Fulfillment of the Majority's Will. Analysis 36: 161-168

Berg S (1993) Condorcet's Jury Theorem: Dependency among Jurors. Social Choice and Welfare 10: 87-96

Boland P (1989) Majority Systems and the Condorcet Jury Theorem. Statistician 38(3): 181-189

Brams SJ, Kilgour DM, Zwicker WS (1997) Voting on Referenda: the Separability Problem and Possible Solutions. Electoral Studies 16(3): 359-377

Brennan G (2001) Collective Coherence? International Review of Law and Economics 21(2): 197-211

Bovens L, Rabinowicz W (2001) Democracy and Argument - Tracking Truth in Complex Social Decisions. Paper presented at the conference on Deliberation and Decision, held in Wittenberg, Germany, September 2001

Bovens L, Rabinowicz W (2003) Democratic Answers to Complex Questions - An Epistemic Perspective. Synthese, forthcoming

Chapman B (1998) More Easily Done than Said: Rules, Reason and Rational Social Choice. Oxford Journal of Legal Studies 18: 293-329

Chapman B (2001) Public Reason, Social Choice, and Cooperation. Paper presented at the Eighth Conference on Theoretical Aspects of Rationality and Knowledge, University of Siena, held at Certosa di Pontignano, Italy, July 2001

Chapman B (2002) Rational Aggregation. Politics, Philosophy and Economics 1: 337-354

Elsholtz C, List C (2002) A Simple Proof of Sen's Possibility Theorem on Majority Decision. Nuffield College Working Paper in Politics 2002-W18

Gehrlein W, Fishburn PC (1976a) Condorcet's Paradox and Anonymous Preference Profiles, Public Choice 26: 1-18

Gehrlein W, Fishburn PC (1976b) The probability of the paradox of voting: A computable solution. Journal of Economic Theory 13: 14-25

Gehrlein W (1981) The expected probabiltiy of Condorcet's paradox. Economics Letters 7: 33-37

Gehrlein W (1983) Condorcet's paradox. Theory and Decision 15: 161-197

Gehrlein W (1997) Condorcet's Paradox and the Condorcet Efficiency of Voting Rules. Mathematica Japonica 45: 173-199

Gehrlein W (2002) Consistency in Measures of Social Homogeneity: A Connection with Proximity to Single Peaked Preferences. Unpublished manuscript, Department of Business Administration, University of Delaware

Grofman B (1985) Research Note: The Accuracy of Group Majorities for Disjunctive and Conjunctive Decision Tasks. Organizational Behavior and Human Decision Processes 35: 119-123

Grofman B, Owen G, Feld SL (1983) Thirteen theorems in search of the truth. Theory and Decision 15: 261-278

Kelly JS (1989) The Ostrogorski Paradox. Social Choice and Welfare 6: 71-76

Kornhauser LA, Sager LG (1986) Unpacking the Court. Yale Law Journal 96: 82-117

Kornhauser LA (1992) Modelling Collegial Courts. II. Legal Doctrine. Journal of Law, Economics and Organization 8: 441-470

Kornhauser LA, Sager LG (1993) The One and the Many: Adjudication in Collegial Courts. California Law Review 91: 1-51

Ladha KK (1995) Information pooling through majority-rule voting: Condorcet's jury theorem with correlated votes. Journal of Economic Behavior and Organization 26: 353-372

List C (2001) An implication of the *k*-option Condorcet jury mechanism for the probability of cycles. Appendix to List and Goodin 2001

List C (2003) Research Note: A Possibility Theorem on Aggregation over Multiple Interconnected Propositions, Mathematical Social Sciences, forthcoming

List C, Goodin RE (2001) Epistemic Democracy: Generalizing the Condorcet Jury Theorem. Journal of Political Philosophy 9(3): 277-306

List C and Pettit P (2002) Aggregating Sets of Judgments: An Impossibility Result. Economics and Philosophy 18: 89-110

List C and Pettit P (2003) Aggregating Sets of Judgments: Two Impossibility Results Compared. Synthese, forthcoming

Mackie G (2000) The Coherence of Democracy. Unpublished manuscript, St. John's College, Oxford

Niemi RG (1969) Majority Decision-Making with Partial Unidimensionality. American Political Science Review 63: 488-497

Paroush J (1998) Stay away from fair coins: A Condorcet jury theorem. Social Choice and Welfare 15: 15-20.

Pettit P (2001) Deliberative Democracy and the Discursive Dilemma. Philosophical Issues 11: 268-299.

Pettit P, Rabinowicz W (2001) The Jury Theorem and the Discursive Dilemma. Appendix to Pettit (2001)

Tangian AS (2000) Unlikelihood of Condorcet's paradox in a large society. Social Choice and Welfare 17: 337-365

Tsetlin I, Regenwetter M, Grofman B (2003) The Impartial Culture maximizes the probability of majority cycles. Social Choice and Welfare, forthcoming