

# A Simple Proof of Sen's Possibility Theorem on Majority Decisions

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**Abstract.** Condorcet's voting paradox shows that pairwise majority voting may lead to cyclical majority preferences. In a famous paper, Sen (1966) identified a general condition on a profile of individual preference orderings, called *triplewise value-restriction*, which is sufficient for the avoidance of such cycles. This note aims to make Sen's result easily accessible. We provide an elementary proof of Sen's possibility theorem and a simple reformulation of Sen's condition. We discuss how Sen's condition is logically related to a number of precursors. Finally, we state a necessary and sufficient condition for the avoidance of cycles, and suggest that, although there is still some logical space between that condition and Sen's sufficient condition, Sen's condition cannot be further generalized in an appealing way.

## 1. INTRODUCTION

Condorcet's paradox shows that pairwise majority voting over three or more candidates can lead to cyclical majority preferences, even when the preferences of individual voters are transitive: Suppose there are three voters, labelled 1, 2 and 3, and three candidates, labelled  $x_1$ ,  $x_2$  and  $x_3$ , with the following preferences:

voter 1:  $x_1 > x_2 > x_3$

voter 2:  $x_2 > x_3 > x_1$

voter 3:  $x_3 > x_1 > x_2$ .

Then there are majorities of 2 out of 3 voters for  $x_1 > x_2$ , for  $x_2 > x_3$  and for  $x_3 > x_1$ . The resulting majority preference ordering is cyclical:  $x_1 > x_2 > x_3 > x_1$ .

Cyclical majority preferences (in short: *cycles*) are democratically undesirable, as they are unsuitable for reaching consistent democratic decisions. A large literature in social choice theory addresses the threat posed by cycles for the functioning of democratic decision mechanisms (for example, Riker 1982). Several sufficient conditions for the avoidance of cycles have been identified. Black (1948) showed that, if the  $n$ -tuple of individual preference orderings across  $n$  voters satisfies an appealing condition called *single-peakedness*, the resulting majority preference ordering will be transitive. Later, other sufficient conditions for transitive majority preference were found, amongst them *single-cavedness* (Inada 1964), *separability into two groups* (Inada 1964), and *latin-squarelessness* (Ward 1965). (See also section 3 below.)

In a famous paper, Sen (1966) generalized these results, showing that a condition that is less demanding than, but implied by, each of these conditions is already sufficient for avoiding cycles. Sen's condition is called *triplewise value-restriction*. However, Sen's condition and theorem are not intuitively straightforward. This note aims to make the mechanism underlying Sen's result easily accessible, by giving an elementary proof of Sen's theorem, together with a simple reformulation of the condition of triplewise value-restriction. In terms of our reformulation, we also discuss how Sen's condition is logically related to its precursors. Finally, we suggest that, although there is still some logical space between Sen's

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*sufficient* condition for the avoidance of cycles and a *necessary and sufficient* condition, this space may be too narrow to allow an appealing generalization of Sen's condition.

## 2. AN EASY PROOF OF A SLIGHTLY SIMPLIFIED VERSION OF SEN'S THEOREM

We first prove a slightly simplified variant of Sen's result. Suppose there are  $n$  voters, labelled  $1, 2, \dots, n$ , and  $k$  candidates, labelled  $x_1, x_2, \dots, x_k$ . To avoid ties under majority voting, we assume that  $n$  is odd. And suppose each voter holds a preference ordering over the candidates. We use the notation  $x_1 > x_2$  to mean that the voter strictly prefers candidate  $x_1$  to candidate  $x_2$ . Each voter's preference ordering is assumed to be complete and transitive.

**Completeness.** For any two candidates  $x_1, x_2$ , either  $x_1 > x_2$  or  $x_2 > x_1$  (but not both).

**Transitivity.** For any  $x_1, x_2, x_3$ , if  $x_1 > x_2$  and  $x_2 > x_3$ , then  $x_1 > x_3$ .

An  $n$ -tuple of individual preference orderings across  $n$  voters is called a *profile of individual preference orderings*, in short a *profile*. A *profile* is *triplewise value-restricted* if it satisfies the following property.

**Triplewise value-restriction.** For every triple of distinct candidates  $x_1, x_2, x_3$ , there exists  $x_i \in \{x_1, x_2, x_3\}$  and  $r \in \{1, 2, 3\}$  such that no voter ranks  $x_i$  as his or her  $r$ -th preference among  $x_1, x_2, x_3$ .

**Theorem 1. (Sen 1966)** *For every profile satisfying triplewise value-restriction, pairwise majority voting generates a transitive (hence acyclic) majority preference ordering.*

Sen's original result allows voters to be indifferent between two or more candidates. In this section, we assume that voters always order candidates in a strict ranking. In section 4, we explain how our method can be used to prove Sen's result in full generality.

Our proof is in three steps. In a first step, we identify a condition that is sufficient for avoiding cycles over *triples* of alternatives. In a second step, we show that, if this condition holds *for every triple* of alternatives, this is sufficient for the avoidance of *any* cycles. In a third step, we show that our condition (applied to *every triple* of alternatives) is logically equivalent to triplewise value-restriction, as stated above.

*Step 1.* Consider three candidates,  $x_1, x_2, x_3$ . There are six logically possible strict preference orderings over  $x_1, x_2, x_3$ :

$$\begin{array}{lll} 1: x_1 > x_2 > x_3 & 2: x_1 > x_3 > x_2 & 3: x_2 > x_1 > x_3 \\ 4: x_2 > x_3 > x_1 & 5: x_3 > x_1 > x_2 & 6: x_3 > x_2 > x_1. \end{array}$$

Let  $a_{hij}$  denote the number of voters holding the preference ordering  $x_h > x_i > x_j$  ( $a_{hij}$  is a non-negative integer). A preference ordering can be represented as a matrix  $M = (m_{ij})$ , where

$$m_{ij} = \begin{cases} 1 & \text{if } x_i > x_j, \\ 0 & \text{otherwise.} \end{cases}$$

The six orderings above thus correspond to the following matrices:

$$\begin{array}{lll} M_{123} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & M_{132} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & M_{213} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ M_{231} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & M_{312} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, & M_{321} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \end{array}$$

Pairwise majority voting corresponds to the following weighted sum:

$$S = a_{123} M_{123} + a_{132} M_{132} + a_{213} M_{213} + a_{231} M_{231} + a_{312} M_{312} + a_{321} M_{321}$$

$$= \begin{pmatrix} 0 & a_{123}+a_{132}+a_{312} & a_{123}+a_{132}+a_{213} \\ a_{213}+a_{231}+a_{321} & 0 & a_{123}+a_{213}+a_{231} \\ a_{231}+a_{312}+a_{321} & a_{132}+a_{312}+a_{321} & 0 \end{pmatrix}.$$

$S = (s_{ij})$  induces a majority preference ordering defined as follows. For each  $i, j$ ,  $x_i > x_j$  if and only if  $s_{ij} > s_{ji}$ .

When can a cycle occur under pairwise majority voting? There are two logically possible cycles:  $x_1 > x_2 > x_3 > x_1$  and  $x_1 > x_3 > x_2 > x_1$ .

Suppose we have a majority cycle of the first type,  $x_1 > x_2 > x_3 > x_1$ .

$$x_1 > x_2 \text{ means } s_{12} > s_{21}, \text{ i.e. } a_{123}+a_{132}+a_{312} > a_{213}+a_{231}+a_{321} \quad (1)$$

$$x_2 > x_3 \text{ means } s_{23} > s_{32}, \text{ i.e. } a_{123}+a_{213}+a_{231} > a_{132}+a_{312}+a_{321} \quad (2)$$

$$x_3 > x_1 \text{ means } s_{31} > s_{13}, \text{ i.e. } a_{231}+a_{312}+a_{321} > a_{123}+a_{132}+a_{213}. \quad (3)$$

We now add pairs of these inequalities.

$$(1)+(2) \text{ implies } a_{123} > a_{321}$$

$$(1)+(3) \text{ implies } a_{312} > a_{213}$$

$$(2)+(3) \text{ implies } a_{231} > a_{132}.$$

Analogously, a majority cycle of the second type,  $x_1 > x_3 > x_2 > x_1$ , implies the reverse inequalities,  $a_{321} > a_{123}$ ,  $a_{213} > a_{312}$  and  $a_{132} > a_{231}$ . Hence we have the following lemma:

**Lemma 1.** *If there is a majority cycle over  $x_1, x_2, x_3$ , then  $(a_{123} > a_{321}$  and  $a_{312} > a_{213}$  and  $a_{231} > a_{132})$  or  $(a_{321} > a_{123}$  and  $a_{213} > a_{312}$  and  $a_{132} > a_{231})$ .*

An obvious corollary of lemma 1 is the following:

**Lemma 2.** *If  $(a_{123} \leq a_{321}$  or  $a_{312} \leq a_{213}$  or  $a_{231} \leq a_{132})$  and  $(a_{321} \leq a_{123}$  or  $a_{213} \leq a_{312}$  or  $a_{132} \leq a_{231})$ , then there is no majority cycle over  $x_1, x_2, x_3$ .*

We can infer a corollary of lemma 2 which is suitable for proving Sen's theorem.

**Lemma 3.** *If  $(a_{123} = 0$  or  $a_{312} = 0$  or  $a_{231} = 0)$  and  $(a_{321} = 0$  or  $a_{213} = 0$  or  $a_{132} = 0)$ , then there is no majority cycle over  $x_1, x_2, x_3$ .*

*Step 2.* Now suppose there are  $k$  candidates.

**Lemma 4.** (Standard result) *If there is a cycle over  $m$  candidates  $(3 \leq m \leq k)$  in the majority preference ordering, then there is also a cycle over three candidates in that ordering.*

**Proof.** Suppose there is a cycle over  $m$  candidates,  $x_1, x_2, \dots, x_m$ , in the majority preference ordering, i.e.  $x_1 > x_2 > \dots > x_m > x_1$ . We have  $x_1 > x_2$  and  $x_2 > x_3$ . Either  $x_3 > x_1$  or  $x_1 > x_3$ . If  $x_3 > x_1$ , we have found a cycle over three candidates, namely  $x_1, x_2, x_3$ . If  $x_1 > x_3$ , we consider  $x_1 > x_3$  and  $x_3 > x_4$ . Again, either  $x_4 > x_1$ , in which case we have a cycle over  $x_1, x_3, x_4$ , or  $x_1 > x_4$ . We continue until we reach either a cycle over three candidates, or until we reach  $x_1 > x_{m-1}$ ,  $x_{m-1} > x_m$ . But  $x_m > x_1$ , and hence we have a cycle over  $x_1, x_{m-1}, x_m$ . ■

Note that any complete strict preference ordering is *either* transitive *or* cyclic (where the ordering is *cyclic* if there exists at least one cycle). Lemma 3 and lemma 4 imply the following theorem.

**Theorem 2.** Suppose, for every triple of distinct candidates  $x_1, x_2, x_3$ , we have ( $a_{123} = 0$  or  $a_{312} = 0$  or  $a_{231} = 0$ ) and ( $a_{132} = 0$  or  $a_{321} = 0$  or  $a_{213} = 0$ ). Then pairwise majority voting generates a transitive (hence acyclic) majority preference ordering.

*Step 3.* To see that Sen's theorem is an immediate corollary of theorem 2, we give a simple reformulation of triplewise value-restriction.

**Lemma 5.** A profile satisfies triplewise value-restriction if and only if, for every triple of distinct candidates  $x_1, x_2, x_3$ , ( $a_{123} = 0$  or  $a_{312} = 0$  or  $a_{231} = 0$ ) and ( $a_{132} = 0$  or  $a_{321} = 0$  or  $a_{213} = 0$ ).

**Proof.** For every triple of distinct candidates  $x_1, x_2, x_3$ , Sen's condition of triplewise value-restriction corresponds to a disjunction of nine cases, as detailed in the first two rows of table 1. Each of the nine cases is equivalent to a corresponding case in the third row, and the condition of lemma 5 is precisely the disjunction of these cases.

$x_1$	$x_2$	$x_3$	$x_1$	$x_2$	$x_3$	$x_1$	$x_2$	$x_3$
is not ranked 1 <sup>st</sup> by any voter if and only if			is not ranked 2 <sup>nd</sup> by any voter if and only if			is not ranked 3 <sup>rd</sup> by any voter if and only if		
$a_{123}=0$ & $a_{132}=0$	$a_{213}=0$ & $a_{231}=0$	$a_{312}=0$ & $a_{321}=0$	$a_{213}=0$ & $a_{312}=0$	$a_{123}=0$ & $a_{321}=0$	$a_{132}=0$ & $a_{231}=0$	$a_{231}=0$ & $a_{321}=0$	$a_{132}=0$ & $a_{312}=0$	$a_{123}=0$ & $a_{213}=0$

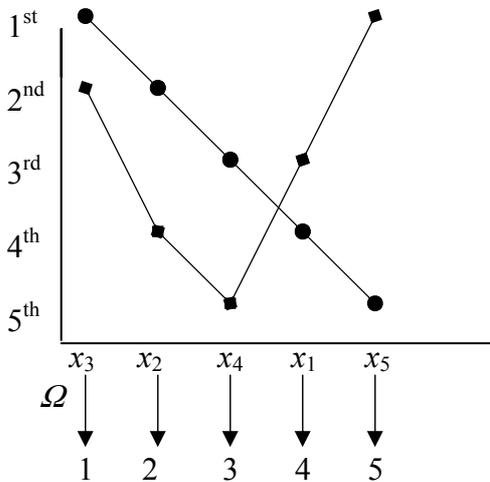
**Table 1**

■

### 3. SEN'S CONDITION AND ITS PRECURSORS

Table 1 is revealing in another respect. Each of the three sets of conditions ("is not ranked 1<sup>st</sup>", "is not ranked 2<sup>nd</sup>", "is not ranked 3<sup>rd</sup>") corresponds to one of the precursors of Sen's condition mentioned in section 1.

The first set of conditions ("is not ranked 1<sup>st</sup>") corresponds to *single-cavedness for every triple*. The general condition of *single-cavedness* requires the existence of a single ordering of *all* candidates from 'left'-most to 'right'-most such that each voter has a least preferred position on that 'left'/'right' ordering with increasing preference for candidates as they get increasingly distant from the least preferred position. Single-cavedness implies single-cavedness for every triple, but not the other way round.

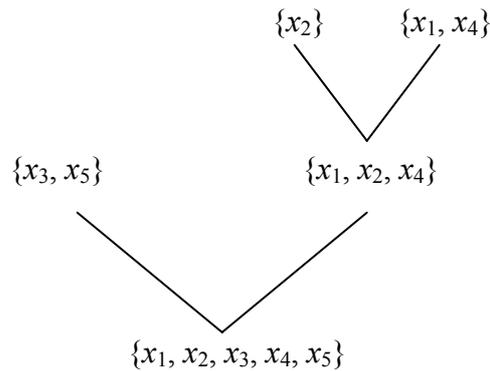


**Figure 1.** Two orderings which are single-caved.

**Single-Cavedness.** There exists a bijection  $\Omega: \{x_1, x_2, \dots, x_k\} \rightarrow \{1, 2, \dots, k\}$  such that, for every triple of candidates  $x_1, x_2, x_3$  and every voter  $i$ , if  $(\Omega(x_1) < \Omega(x_2) < \Omega(x_3))$  or  $(\Omega(x_3) < \Omega(x_2) < \Omega(x_1))$ , then  $[x_2 > x_1 \text{ implies } x_3 > x_1]$  for voter  $i$ .

Single-cavedness for triples permits a different bijection  $\Omega$  for every triple of candidates.

The second set of conditions (“is not ranked 2<sup>nd</sup>”) corresponds to *separability into two groups for every triple*. The general condition of *separability into two groups* requires that any subset of the set of all candidates can be partitioned into two parts such that each voter either prefers any candidate in one of the two parts to any candidate in the other. Separability into two groups implies separability into two groups for every triple, but not the other way round.

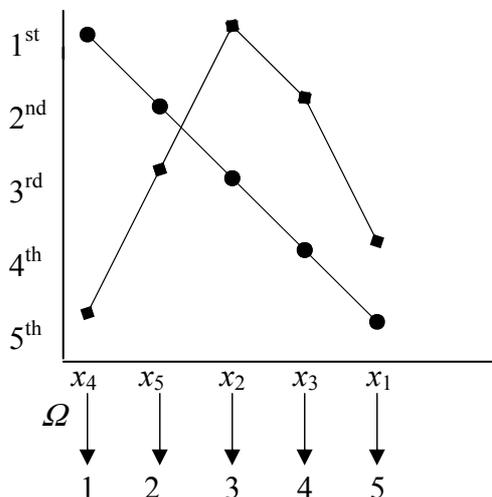


**Figure 3. Separability into two Groups.** For each vertex, each voter prefers any candidate in one branch to any candidate in the other.

**Separability into two Groups.** Any subset  $Y$  of the set of all candidates can be partitioned into two disjoint non-empty subsets  $Y_1$  and  $Y_2$  such that, for every voter  $i$ , either [for all  $x_1 \in Y_1$  and all  $x_2 \in Y_2, x_1 > x_2$ ] or [for all  $x_1 \in Y_1$  and all  $x_2 \in Y_2, x_2 > x_1$ ] for voter  $i$ .

*Separability into two groups for every triple* requires the existence of the required partition only for any triple of candidates.

The third set of conditions (“is not ranked 3<sup>rd</sup>”) corresponds to *single-peakedness for every triple*. The general condition of *single-peakedness* requires the existence of a single ordering of *all* candidates from 'left'-most to 'right'-most such that each voter has a most preferred position on that 'left'/'right' ordering with decreasing preference for candidates as they get increasingly distant from the most preferred position. Single-peakedness implies single-peakedness for every triple, but not the other way round.



**Figure 2. Two orderings which are single-peaked.**

**Single-Peakedness.** There exists a bijection  $\mathcal{Q}: \{x_1, x_2, \dots, x_k\} \rightarrow \{1, 2, \dots, k\}$  such that, for every triple of candidates  $x_1, x_2, x_3$  and every voter  $i$ , if  $(\mathcal{Q}(x_1) < \mathcal{Q}(x_2) < \mathcal{Q}(x_3))$  or  $(\mathcal{Q}(x_3) < \mathcal{Q}(x_2) < \mathcal{Q}(x_1))$ , then  $[x_1 > x_2 \text{ implies } x_1 > x_3]$  for voter  $i$ .

Single-peakedness for triples permits a different bijection  $\mathcal{Q}$  for every triple of candidates.

#### 4. PROVING SEN'S RESULT IN FULL GENERALITY

Finally, let us briefly sketch how our method can be used to allow the consideration of indifference in individual preference orderings, and thus to prove Sen's result in full generality. Allowing indifference means that there are three different ways in which a voter might rank two candidates,  $x_1$  and  $x_2$ . The voter might rank one strictly above the other, i.e.  $x_1 > x_2$  or  $x_2 > x_1$ ; the voter might have an equal preference for both, i.e.  $x_1 = x_2$ ; or the voter might not rank or compare them at all. The first possibility corresponds to a *strict ordering*, the second to *indifference*, and the third to an *incomplete ordering*. Following Sen's result, we will only consider the first two possibilities and rule out incomplete orderings (in principle, however, incomplete orderings are also representable in terms of the matrix formalism introduced in section 2).

Over triples of candidates, there are 7 possible preference orderings with indifference, in addition to the 6 strict orderings shown above:  $x_1 > x_2 = x_3$ ,  $x_1 = x_2 > x_3$ ,  $x_2 > x_1 = x_3$ ,  $x_1 = x_3 > x_2$ ,  $x_3 > x_1 = x_2$ ,  $x_2 = x_3 > x_1$ , and  $x_1 = x_2 = x_3$ . (Note that  $x_1 > x_2 = x_3$  and  $x_1 > x_3 = x_2$  are considered the same.)

We define a corresponding notation. Here  $a_{hij}$  has the same interpretation as before. For weak orderings, let (for example)  $a_{h>i=j}$  denote the number of voters holding the preference ordering  $x_h > x_i = x_j$ . The matrix corresponding to  $x_1 > x_2 = x_3$ , for example, is

$$M_{1>2=3} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Strict orderings correspond to matrices with three non-zero entries, as defined in section 2, weak orderings correspond to matrices with two non-zero entries, except  $x_1 = x_2 = x_3$ , which corresponds to the matrix consisting only of zeroes.

In analogy to the argument above, pairwise majority voting corresponds to the following weighted sum:

$$\begin{aligned} S &= a_{123} M_{123} + a_{132} M_{132} + a_{213} M_{213} + a_{231} M_{231} + a_{312} M_{312} + a_{321} M_{321} \\ &\quad + a_{1=2>3} M_{1=2>3} + a_{1=3>2} M_{1=3>2} + a_{2=3>1} M_{2=3>1} + a_{3>1=2} M_{3>1=2} + a_{2>1=3} M_{2>1=3} + a_{1>2=3} M_{1>2=3} \\ &= \begin{pmatrix} 0 & a_{123}+a_{132}+a_{312}+a_{1=3>2}+a_{1>2=3} & a_{123}+a_{132}+a_{213}+a_{1=2>3}+a_{1>2=3} \\ a_{213}+a_{231}+a_{321}+a_{2=3>1}+a_{2>1=3} & 0 & a_{123}+a_{213}+a_{231}+a_{1=2>3}+a_{2>1=3} \\ a_{231}+a_{312}+a_{321}+a_{2=3>1}+a_{3>1=2} & a_{132}+a_{312}+a_{321}+a_{1=3>2}+a_{3>1=2} & 0 \end{pmatrix}. \end{aligned}$$

For simplicity, we assume that the number of voters that are not indifferent about any particular pair of candidates is odd. This assumption is different from Sen's. Sen's own assumption is that the number of voters who are not indifferent about any particular triple of candidates is odd. The following two examples show that Sen's assumption and our assumption are logically independent. Case (i): Our assumption is satisfied, but Sen's is not. Consider 6 voters with the following preference orderings: (1)  $x_1=x_2>x_3>x_4$ ; (2)  $x_1=x_3>x_2>x_4$ ; (3)  $x_1=x_4>x_2>x_3$ ; (4)  $x_2=x_3>x_1>x_4$ ; (5)  $x_2=x_4>x_1>x_3$ ; (6)  $x_3=x_4>x_1>x_2$ . Case (ii): Sen's assumption is satisfied, but ours is not. Consider 3 voters with the following preference orderings: (1)  $x_1>x_2=x_3$ ; (2)  $x_2>x_3=x_1$ ; (3)  $x_3>x_1=x_2$ . Our assumption has the possible advantage of being slightly easier to verify than Sen's: verifying whether the number of voters who are

not indifferent over a *pair* of candidates is odd is simpler than verifying whether the number not indifferent over a *triple* of candidates is odd.

Given that the number of voters that are not indifferent about any particular pair of candidates is odd, there will not be any ties under majority voting, and, as before, the only two logically possible cycles are  $x_1 > x_2 > x_3 > x_1$  and  $x_1 > x_3 > x_2 > x_1$ .

The first cycle corresponds to the following inequalities:

$$x_1 > x_2 \text{ means } s_{12} > s_{21}, \text{ i.e. } a_{123} + a_{132} + a_{312} + a_{1=3>2} + a_{1>2=3} > a_{213} + a_{231} + a_{321} + a_{2=3>1} + a_{2>1=3} \quad (1)$$

$$x_2 > x_3 \text{ means } s_{23} > s_{32}, \text{ i.e. } a_{123} + a_{213} + a_{231} + a_{1=2>3} + a_{2>1=3} > a_{132} + a_{312} + a_{321} + a_{1=3>2} + a_{3>1=2} \quad (2)$$

$$x_3 > x_1 \text{ means } s_{31} > s_{13}, \text{ i.e. } a_{231} + a_{312} + a_{321} + a_{2=3>1} + a_{3>1=2} > a_{123} + a_{132} + a_{213} + a_{1=2>3} + a_{1>2=3}. \quad (3)$$

Adding pairs of these inequalities leads to:

$$(1)+(2) \text{ implies } a_{123} + a_{1=2>3} + a_{1>2=3} > a_{321} + a_{2=3>1} + a_{3>1=2}$$

$$(1)+(3) \text{ implies } a_{312} + a_{1=3>2} + a_{3>1=2} > a_{213} + a_{1=2>3} + a_{2>1=3}$$

$$(2)+(3) \text{ implies } a_{231} + a_{2=3>1} + a_{2>1=3} > a_{132} + a_{1=3>2} + a_{1>2=3}.$$

Analogously, the second cycle implies the reverse inequalities. Using (a version of) lemma 4 as before, we can state the following theorem.

**Theorem 3.** *Suppose, for every triple of distinct candidates  $x_1, x_2, x_3$ , we have  $(a_{123} + a_{1=2>3} + a_{1>2=3} = 0$  or  $a_{312} + a_{1=3>2} + a_{3>1=2} = 0$  or  $a_{231} + a_{2=3>1} + a_{2>1=3} = 0)$  and  $(a_{321} + a_{2=3>1} + a_{3>1=2} = 0$  or  $a_{213} + a_{1=2>3} + a_{2>1=3} = 0$  or  $a_{132} + a_{1=3>2} + a_{1>2=3} = 0)$ . Then pairwise majority voting generates a transitive (hence acyclic) majority preference ordering.*

To see that the condition of theorem 3 is equivalent to Sen's condition of triplewise value-restriction, we just need to use the interpretation, in the case of indifference, that each candidate can have more than one rank within a voter's preference ordering, i.e. if  $x_h = x_i > x_j$ , then  $x_h$  and  $x_i$  would each be regarded as both first and second within the given preference ordering among  $x_h, x_i, x_j$ . Using this interpretation and stating the condition of theorem 3 in a form similar to table 1 then yields a version of Sen's general result.

## 5. A NECESSARY AND SUFFICIENT CONDITION FOR AVOIDING CYCLES

As we have pointed out, triplewise value-restriction is a *sufficient* condition for avoiding cycles. It is a generalization of its precursors, which are themselves *sufficient* conditions. But it is still not a *necessary and sufficient* condition. Can triplewise value-restriction itself be further generalized? How 'close' is it to a *necessary and sufficient* condition? We will now see that there is still some logical space between Sen's condition and a *necessary and sufficient* condition, but we suggest that this space may be too narrow to allow an appealing generalization of triplewise value-restriction.

We first state a *necessary and sufficient* condition for the *occurrence* of cycles. The result is a version of a result by Miller (2000).

**Theorem 4.** *Pairwise majority voting generates a cycle if and only if, for some triple of distinct candidates  $x_1, x_2, x_3$ , we have*

$$((a_{123} > a_{321} \text{ and } a_{312} > a_{213} \text{ and } a_{231} > a_{132}) \text{ or } (a_{321} > a_{123} \text{ and } a_{213} > a_{312} \text{ and } a_{132} > a_{231})) \\ \text{and } |a_{123} - a_{321}| < n'/2 \text{ and } |a_{231} - a_{132}| < n'/2 \text{ and } |a_{312} - a_{213}| < n'/2,$$

where  $n' := |a_{123} - a_{321}| + |a_{231} - a_{132}| + |a_{312} - a_{213}|$ .

A proof is given in an appendix. Negating both sides of the *if-and-only-if* equivalence yields the following corollary.

**Corollary of theorem 4.** *Pairwise majority voting generates a transitive (hence acyclic) majority preference ordering if and only if, for every triple of distinct candidates  $x_1, x_2, x_3$ , we have*

$$((a_{123} \leq a_{321} \text{ or } a_{312} \leq a_{213} \text{ or } a_{231} \leq a_{132}) \text{ and } (a_{321} \leq a_{123} \text{ or } a_{213} \leq a_{312} \text{ or } a_{132} \leq a_{231})) \\ \text{or } |a_{123}-a_{321}| \geq n'/2 \text{ or } |a_{231}-a_{132}| \geq n'/2 \text{ or } |a_{312}-a_{213}| \geq n'/2,$$

where  $n' := |a_{123}-a_{321}| + |a_{231}-a_{132}| + |a_{312}-a_{213}|$ .

Sen's condition implies, but is not implied by, the condition of the corollary of theorem 4. However, to see why it may nonetheless be impossible to find an appealing generalization of Sen's condition, let us introduce a criterion for describing a condition on a profile as *simple*. A condition (on a profile) is *simple* if it is dependent, for each logically possible preference ordering, only on whether or not that ordering occurs in the profile, but not on the number of voters holding the given ordering. Thus a condition is simple if it consists only of propositions of the forms  $a_{hij} = 0$  and  $a_{hij} \neq 0$ , as well as conjunctions or disjunctions of such propositions. The condition of triplewise value-restriction as defined in section 2 satisfies the criterion of simplicity (leaving aside the requirement that  $n$  be odd). Technically, triplewise value-restriction in its full generality already violates the criterion, as it requires the number of voters who are not indifferent about any particular triple (or pair) of alternatives to be odd.

The task of finding a *simple* sufficient condition for the avoidance of cycles, then, is to find a condition with the following two properties: (i) the condition's basic components are only propositions of the forms  $a_{hij} = 0$  and  $a_{hij} \neq 0$ ; (ii) the condition implies

$$((a_{123} \leq a_{321} \text{ or } a_{312} \leq a_{213} \text{ or } a_{231} \leq a_{132}) \text{ and } (a_{321} \leq a_{123} \text{ or } a_{213} \leq a_{312} \text{ or } a_{132} \leq a_{231})) \\ \text{or } |a_{123}-a_{321}| \geq n'/2 \text{ or } |a_{231}-a_{132}| \geq n'/2 \text{ or } |a_{312}-a_{213}| \geq n'/2,$$

where  $n' := |a_{123}-a_{321}| + |a_{231}-a_{132}| + |a_{312}-a_{213}|$ . Sen's solution – ( $a_{123} = 0$  or  $a_{312} = 0$  or  $a_{231} = 0$ ) and ( $a_{132} = 0$  or  $a_{321} = 0$  or  $a_{213} = 0$ ) – seems to be the most general one we can get.

#### APPENDIX: PROOF OF THEOREM 4.

*Step 1.* Let  $x_1, x_2, x_3$  be any triple of candidates.

$$\begin{aligned} & \text{There is a majority cycle of type } x_1 > x_2 > x_3 > x_1 \\ \Leftrightarrow & \quad (a_{123}+a_{132}+a_{312} > a_{213}+a_{231}+a_{321}) \\ & \text{and } (a_{123}+a_{213}+a_{231} > a_{132}+a_{312}+a_{321}) \\ & \text{and } (a_{231}+a_{312}+a_{321} > a_{123}+a_{132}+a_{213}) \\ \Leftrightarrow & \quad (a_{123}-a_{321}+a_{231}-a_{132}+a_{312}-a_{213} > 2(a_{231}-a_{132})) \\ & \text{and } (a_{123}-a_{321}+a_{231}-a_{132}+a_{312}-a_{213} > 2(a_{312}-a_{213})) \\ & \text{and } (a_{123}-a_{321}+a_{231}-a_{132}+a_{312}-a_{213} > 2(a_{123}-a_{321})). \end{aligned}$$

Similarly,

$$\begin{aligned} & \text{There is a majority cycle of type } x_1 > x_3 > x_2 > x_1 \\ \Leftrightarrow & \quad (a_{321}-a_{123}+a_{132}-a_{231}+a_{213}-a_{312} > 2(a_{132}-a_{231})) \\ & \text{and } (a_{321}-a_{123}+a_{132}-a_{231}+a_{213}-a_{312} > 2(a_{213}-a_{312})) \\ & \text{and } (a_{321}-a_{123}+a_{132}-a_{231}+a_{213}-a_{312} > 2(a_{321}-a_{123})). \end{aligned}$$

*Step 2.* Suppose there is some majority cycle. By lemma 4, this implies that there is a cycle over three candidates, say  $x_1, x_2, x_3$ . The cycle must be of type 1 ( $x_1 > x_2 > x_3 > x_1$ ) or of type 2 ( $x_1 > x_3 > x_2 > x_1$ ). As we have seen in section 2, a cycle of type 1 implies ( $a_{123} > a_{321}$  and  $a_{312} > a_{213}$  and  $a_{231} > a_{132}$ ). By step 1, we then also have

$$\begin{aligned} & (|a_{123}-a_{321}|+|a_{231}-a_{132}|+|a_{312}-a_{213}| > 2|a_{231}-a_{132}|) \\ \text{and} & (|a_{123}-a_{321}|+|a_{231}-a_{132}|+|a_{312}-a_{213}| > 2|a_{312}-a_{213}|) \\ \text{and} & (|a_{123}-a_{321}|+|a_{231}-a_{132}|+|a_{312}-a_{213}| > 2|a_{123}-a_{321}|). \end{aligned}$$

Also, as we have seen in section 2, a cycle of type 2 implies ( $a_{321} > a_{123}$  and  $a_{213} > a_{312}$  and  $a_{132} > a_{231}$ ). By step 1, we then also have

$$\begin{aligned} & (|a_{123}-a_{321}|+|a_{231}-a_{132}|+|a_{312}-a_{213}| > 2|a_{231}-a_{132}|) \\ \text{and} & (|a_{123}-a_{321}|+|a_{231}-a_{132}|+|a_{312}-a_{213}| > 2|a_{312}-a_{213}|) \\ \text{and} & (|a_{123}-a_{321}|+|a_{231}-a_{132}|+|a_{312}-a_{213}| > 2|a_{123}-a_{321}|). \end{aligned}$$

Hence a majority cycle implies

$$\begin{aligned} & ((a_{123} > a_{321} \text{ and } a_{312} > a_{213} \text{ and } a_{231} > a_{132}) \text{ or } (a_{321} > a_{123} \text{ and } a_{213} > a_{312} \text{ and } a_{132} > a_{231})) \\ & \text{and } |a_{123}-a_{321}| < n'/2 \text{ and } |a_{231}-a_{132}| < n'/2 \text{ and } |a_{312}-a_{213}| < n'/2 \end{aligned} \quad (*)$$

(with  $n'$  as defined above).

Suppose, conversely, there exists a triple of candidates,  $x_1, x_2, x_3$ , such that (\*) holds. We must have either ( $a_{123} > a_{321}$  and  $a_{312} > a_{213}$  and  $a_{231} > a_{132}$ ) or ( $a_{321} > a_{123}$  and  $a_{213} > a_{312}$  and  $a_{132} > a_{231}$ ).

If ( $a_{123} > a_{321}$  and  $a_{312} > a_{213}$  and  $a_{231} > a_{132}$ ), we have

$$\begin{aligned} & (a_{123}-a_{321}+a_{231}-a_{132}+a_{312}-a_{213} > 2(a_{231}-a_{132})) \\ \text{and} & (a_{123}-a_{321}+a_{231}-a_{132}+a_{312}-a_{213} > 2(a_{312}-a_{213})) \\ \text{and} & (a_{123}-a_{321}+a_{231}-a_{132}+a_{312}-a_{213} > 2(a_{123}-a_{321})), \end{aligned}$$

which implies a majority cycle of type 1, by step 1.

If ( $a_{321} > a_{123}$  and  $a_{213} > a_{312}$  and  $a_{132} > a_{231}$ ), we have

$$\begin{aligned} & (a_{321}-a_{123}+a_{132}-a_{231}+a_{213}-a_{312} > 2(a_{132}-a_{231})) \\ \text{and} & (a_{321}-a_{123}+a_{132}-a_{231}+a_{213}-a_{312} > 2(a_{213}-a_{312})) \\ \text{and} & (a_{321}-a_{123}+a_{132}-a_{231}+a_{213}-a_{312} > 2(a_{321}-a_{123})), \end{aligned}$$

which implies a majority cycle of type 2, by step 1. ■

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