# Finding Gale Strings and Orienting Room Partitionings of 1-Oiks

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**Do they always come in equal numbers?**

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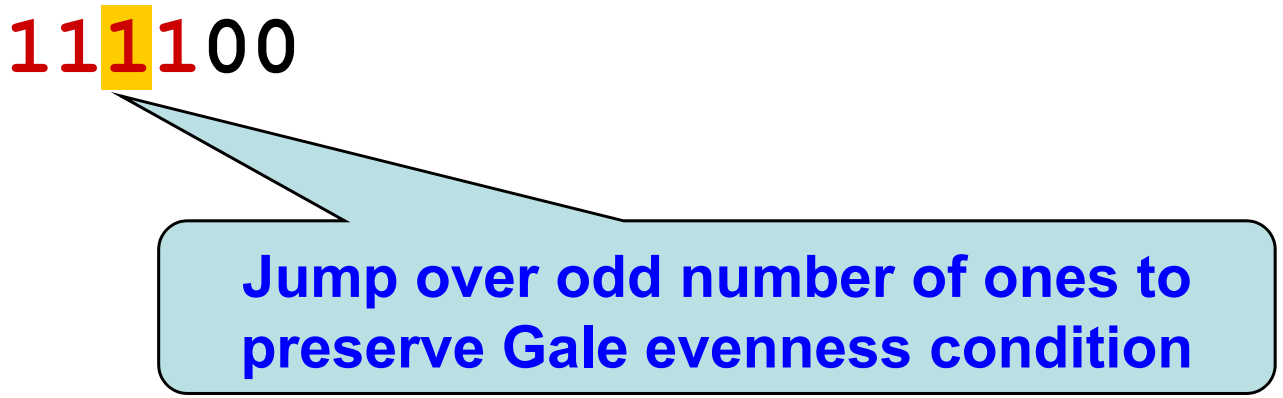

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1111100      0111110      0110111
1101101      1101101
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| | | |
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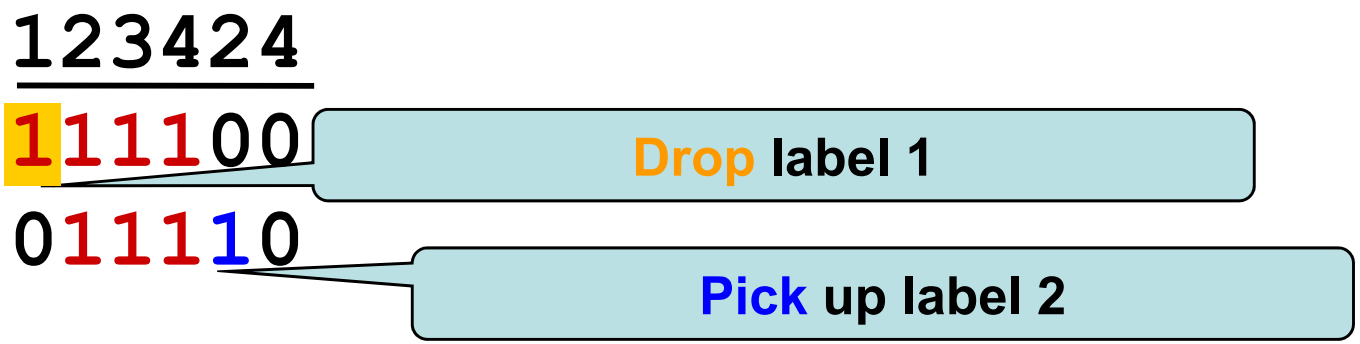

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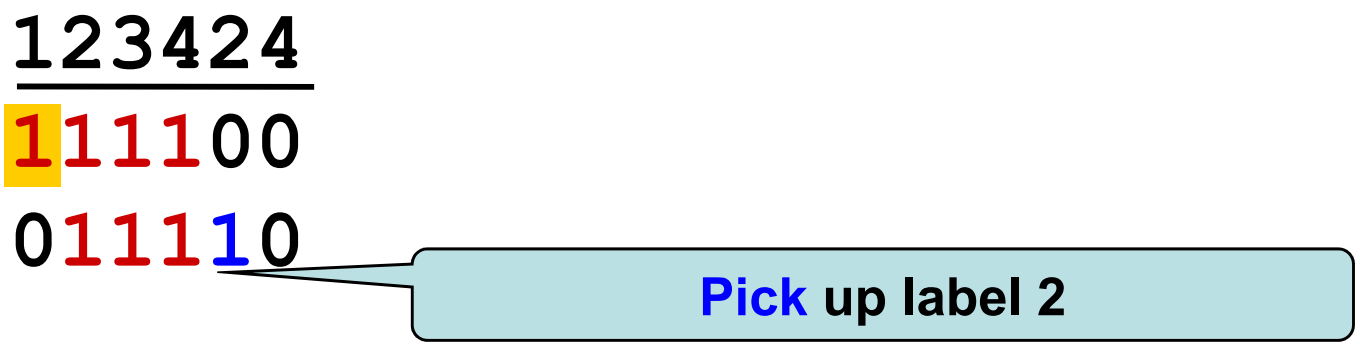

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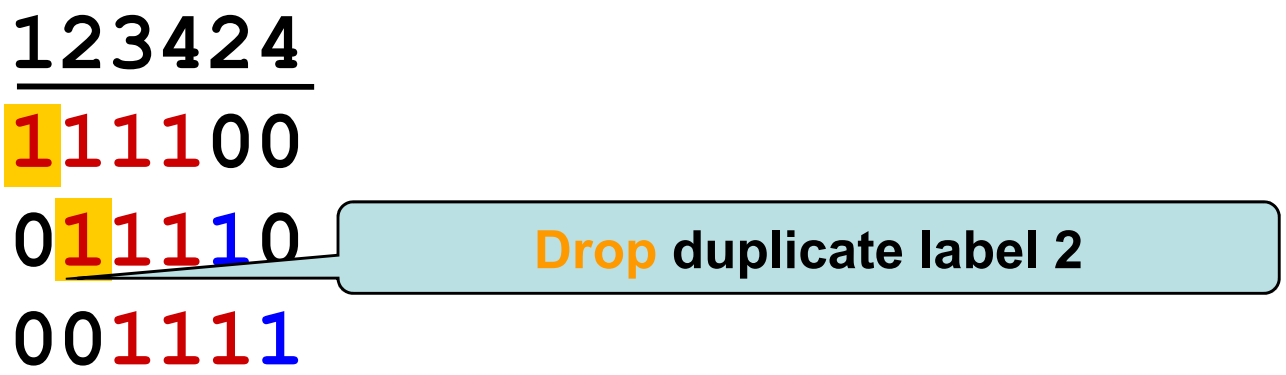

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111001
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```

**Missing label found again: second Gale string!**

# How good is this pivot algorithm?

**Example [Morris 1994]**

$$\begin{array}{r} 1234564523 \\ \hline 1111110000 \end{array}$$

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In general: exponentially long path in **d** for any missing label.

# Motivation from bimatrix games

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Labels by [Morris 1994] define **exponentially long** Lemke–Howson paths [McLennan &Tourky 2007; Savani & von Stengel 2006].

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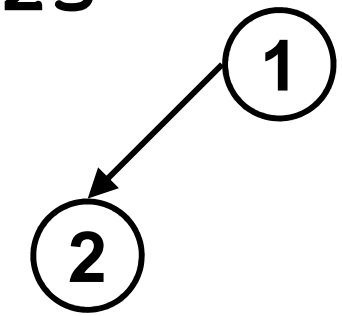


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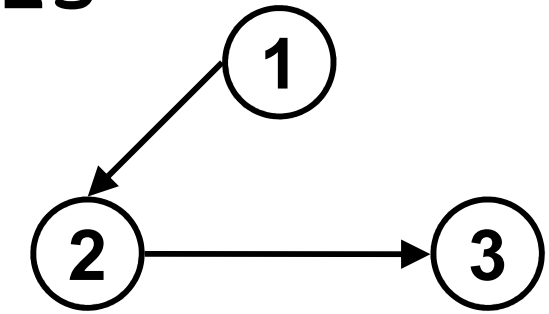

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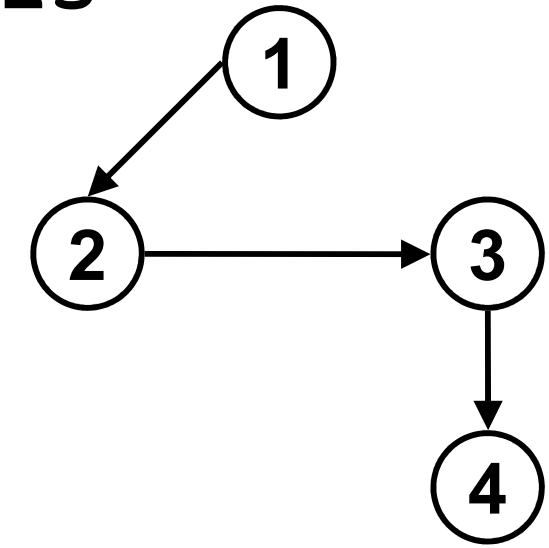

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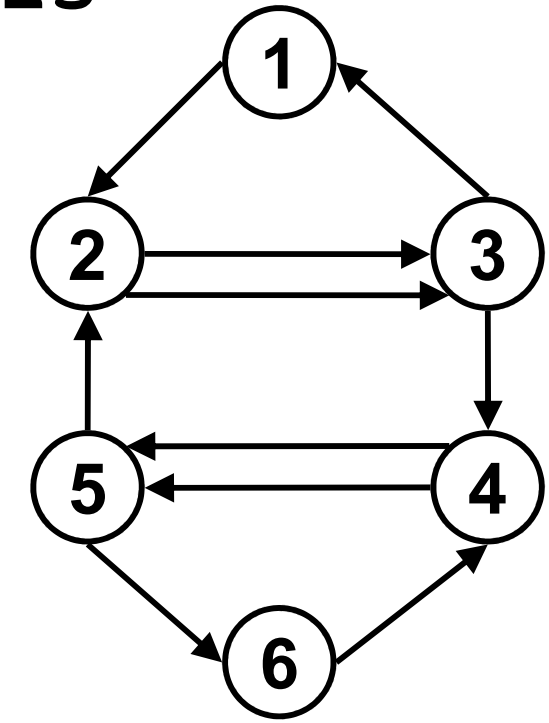

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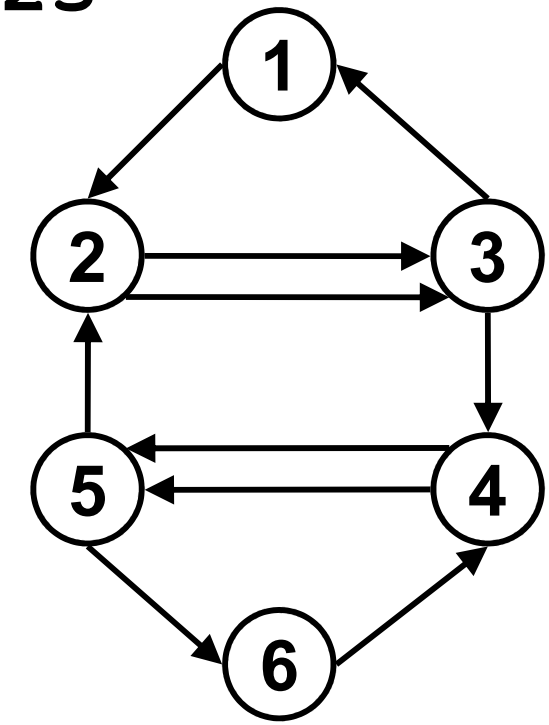

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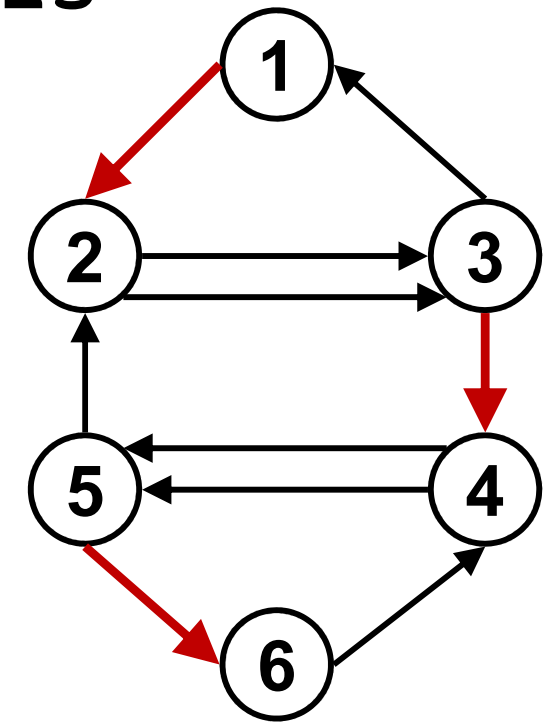


**Perfect Matching** =

**Completely labeled Gale String**

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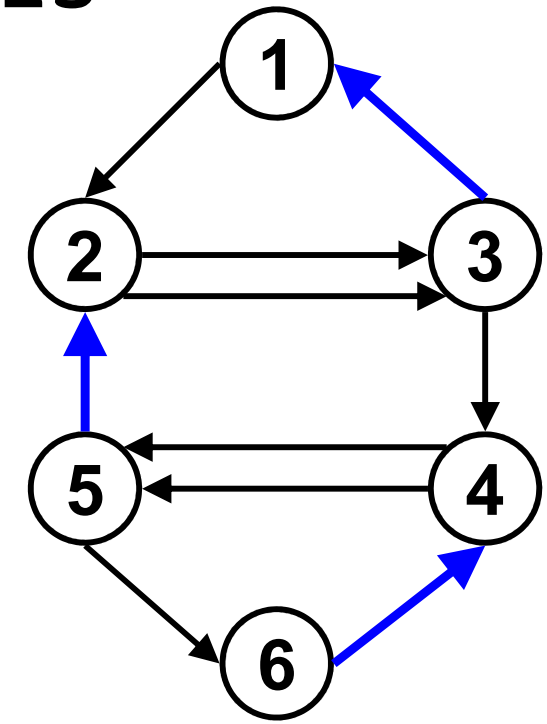


**Perfect Matching** =

**Completely labeled Gale String**

1234564523
1000011111

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|---|
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|---|---|---|
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|---|---|---|---|
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|---|---|---|---|
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Given a string of labels, any completely labeled Gale string has sign ⊕ or ⊖.

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Theorem

Number of completely labeled Gale strings of sign ⊕

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Define **signs** of an **almost completely labeled** Gale string, by replacing **duplicate** with missing label (**two** possibilities one **new** one **old**) ⇒ one ⊕, one ⊖ permutation.

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**Example**

$$\frac{123424}{011110}$$

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|---|---|
| sign ⊕ | 011110 |

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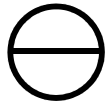
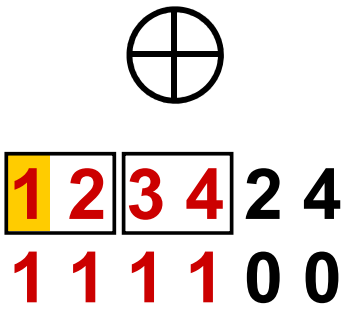
**Example**

| 1342 | 123424 | 2341 |
|---|---|---|
| sign ⊕ | 011110 | sign ⊖ |

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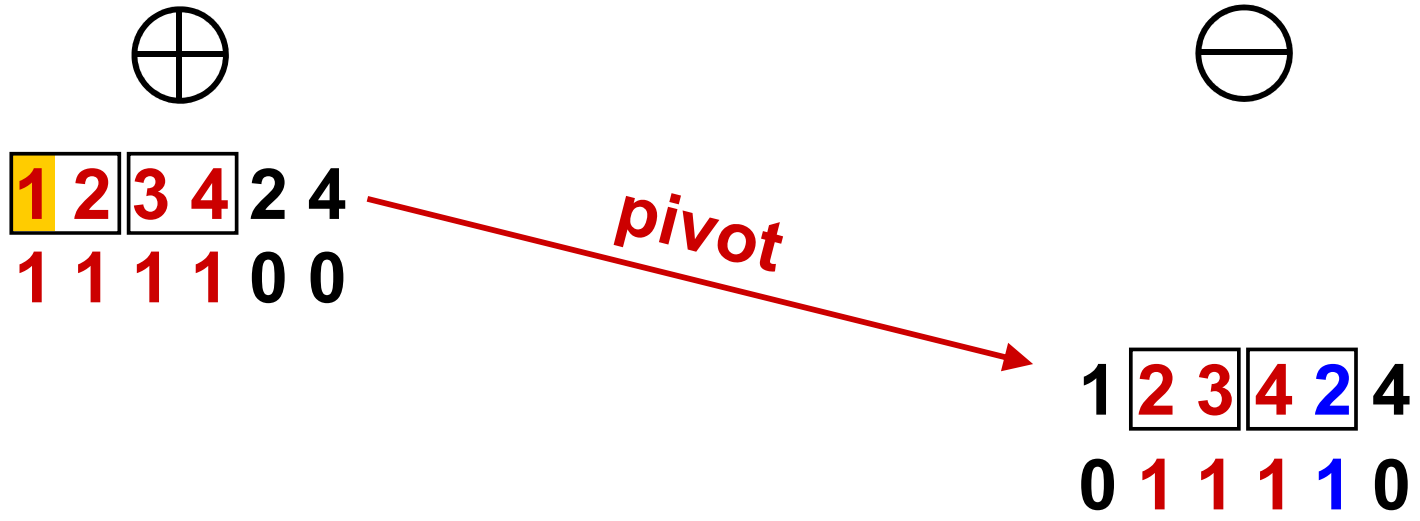
1 1 1 1 0 0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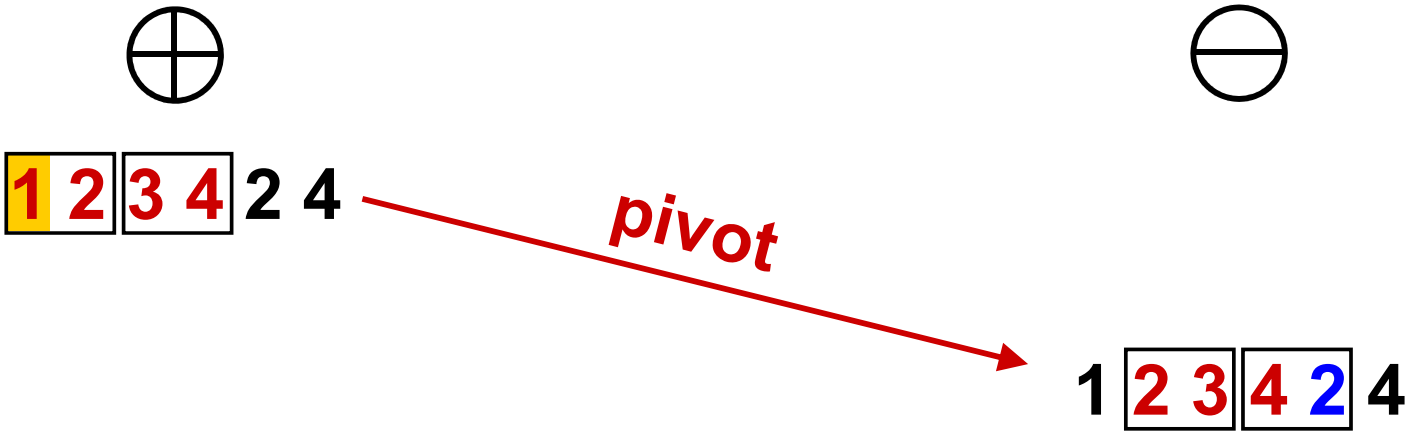
pivot
1 2 3 4 2 4
0 1 1 1 1 0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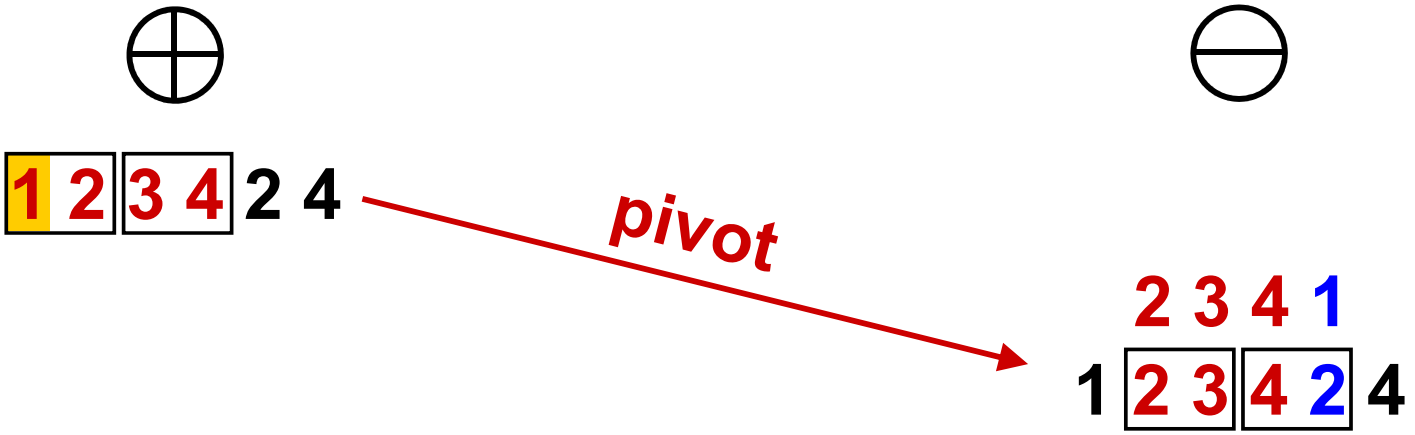
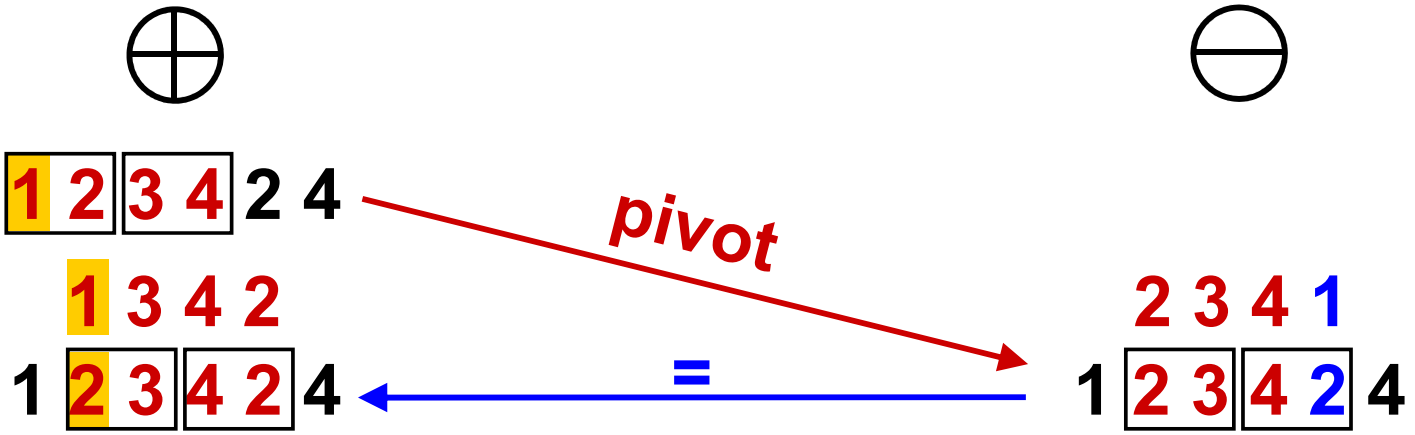
pivot
1 2 3 4 2 4


1 2 3 4 2 4
pivot
2 3 4 1
1 2 3 4 2 4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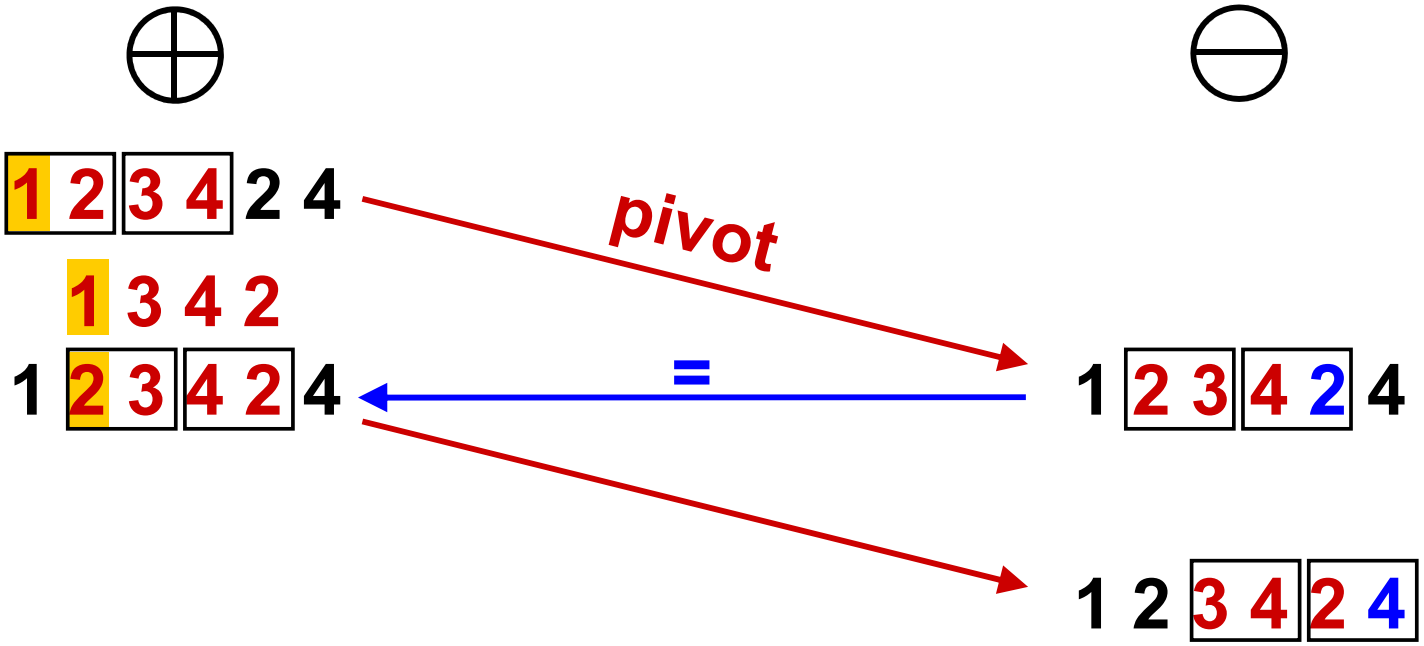
pivot
1 3 4 2
2 3 4 1
1 2 3 4 2 4
=
1 2 3 4 2 4


⊕
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1 2 3 4 2 4
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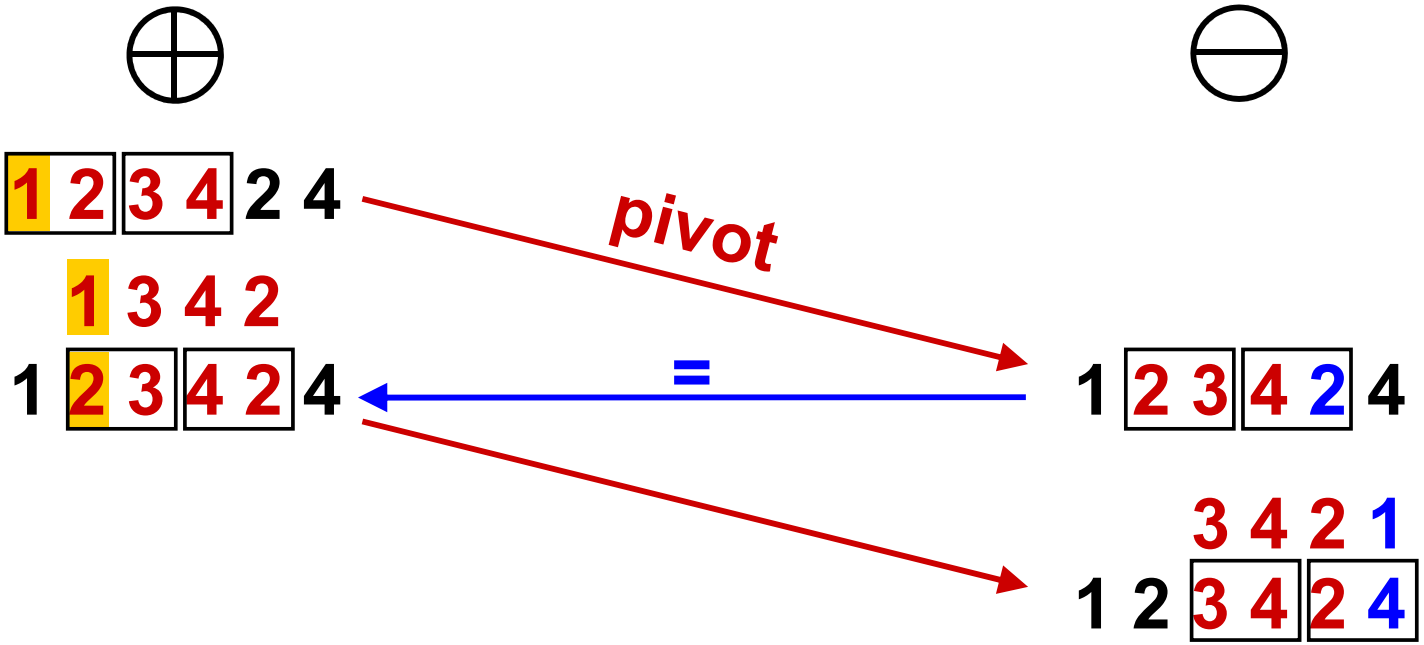
⊕
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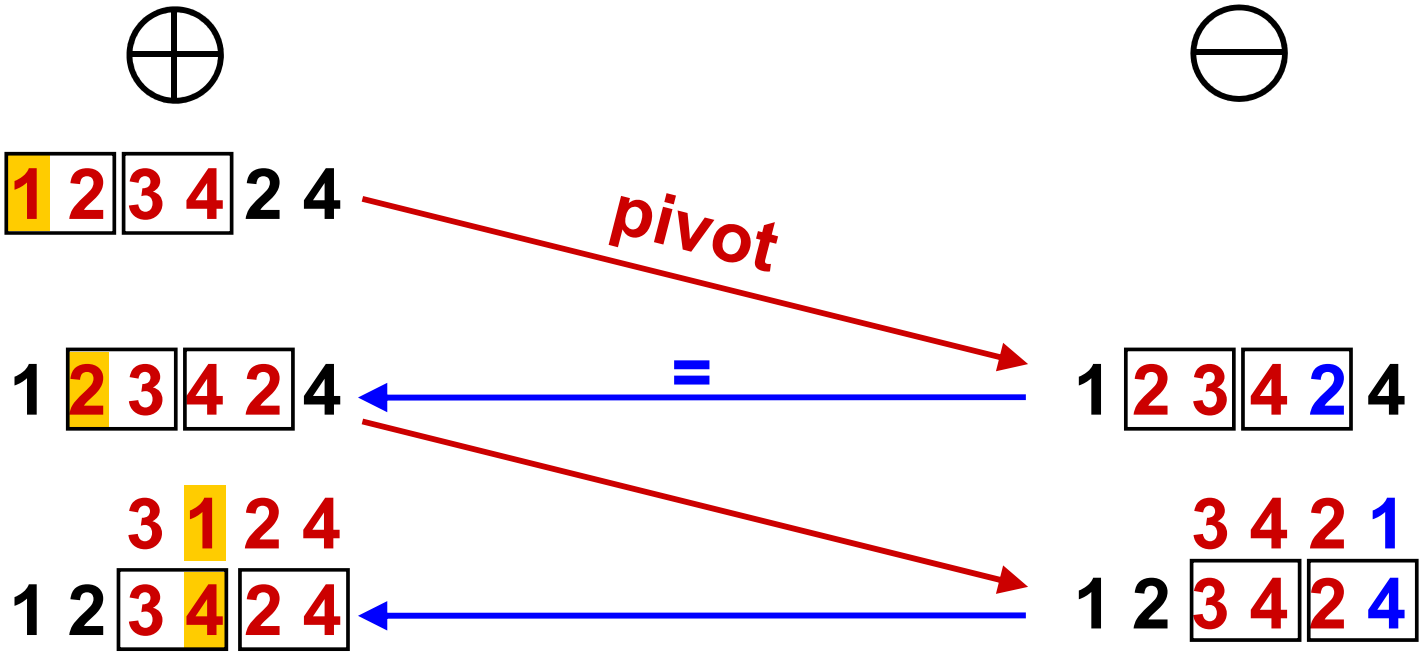
3 4 2 1
1 2 3 4 2 4


⊕
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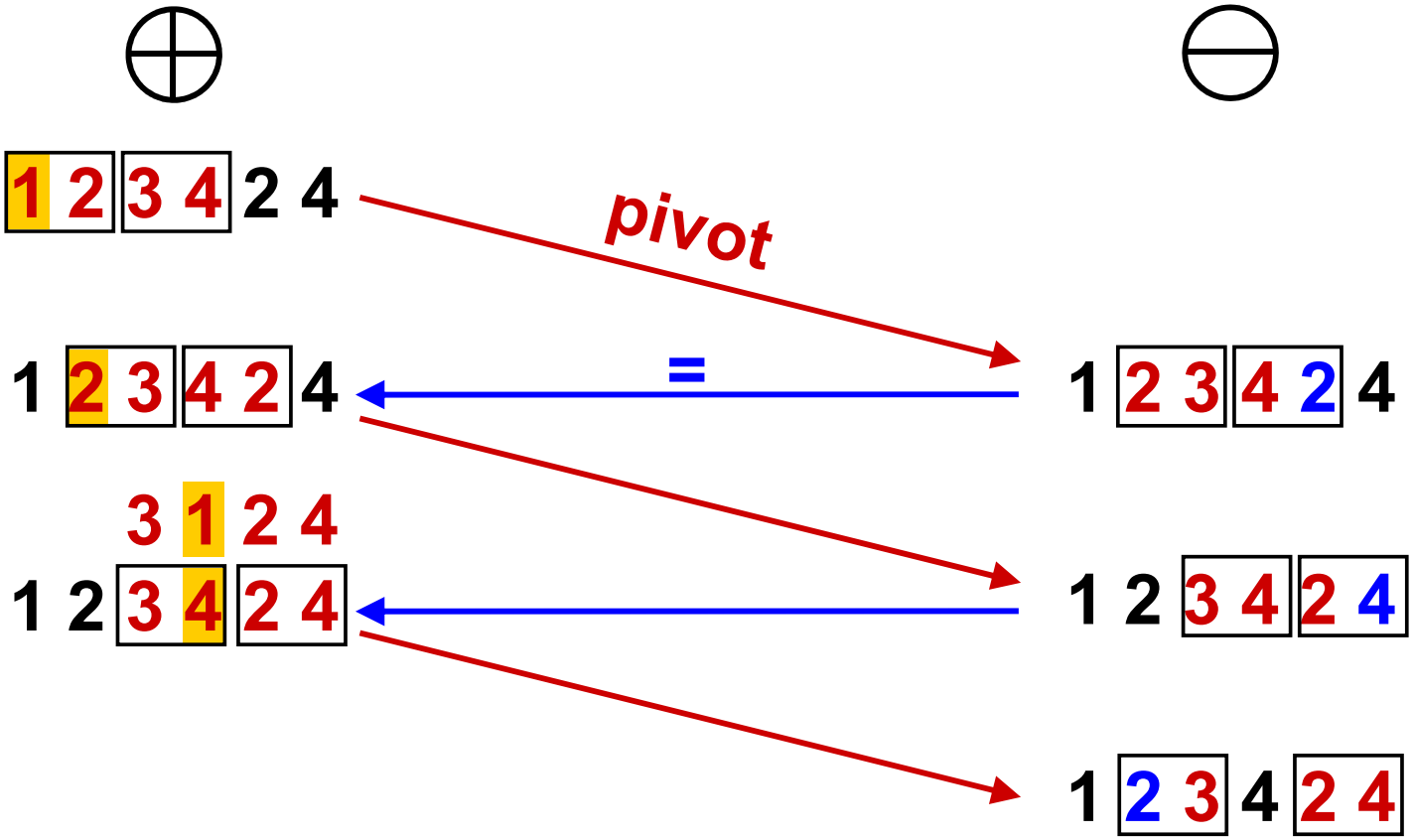
1 2 3 4 2 4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1 2 3 4 2 4
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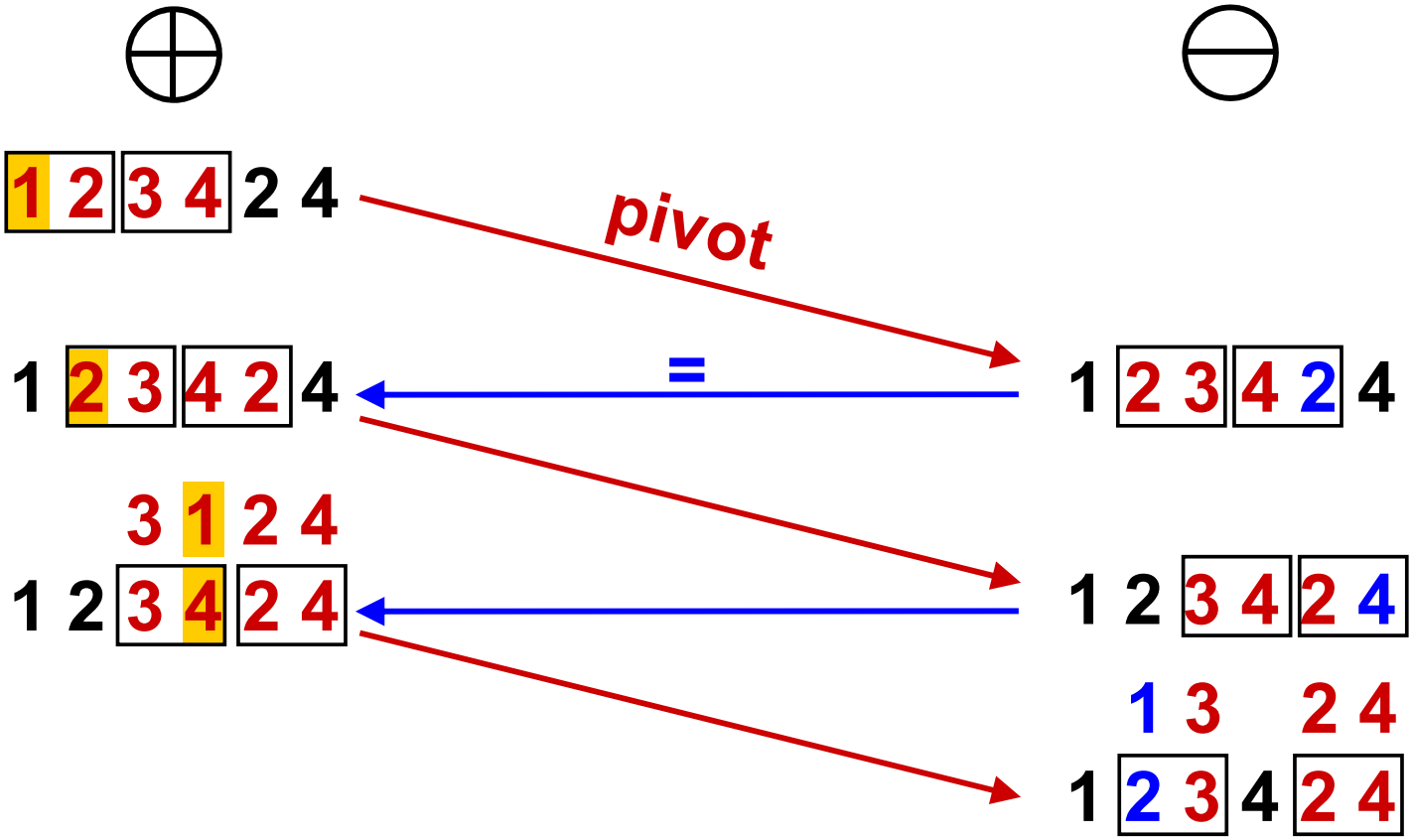
1 2 3 4 2 4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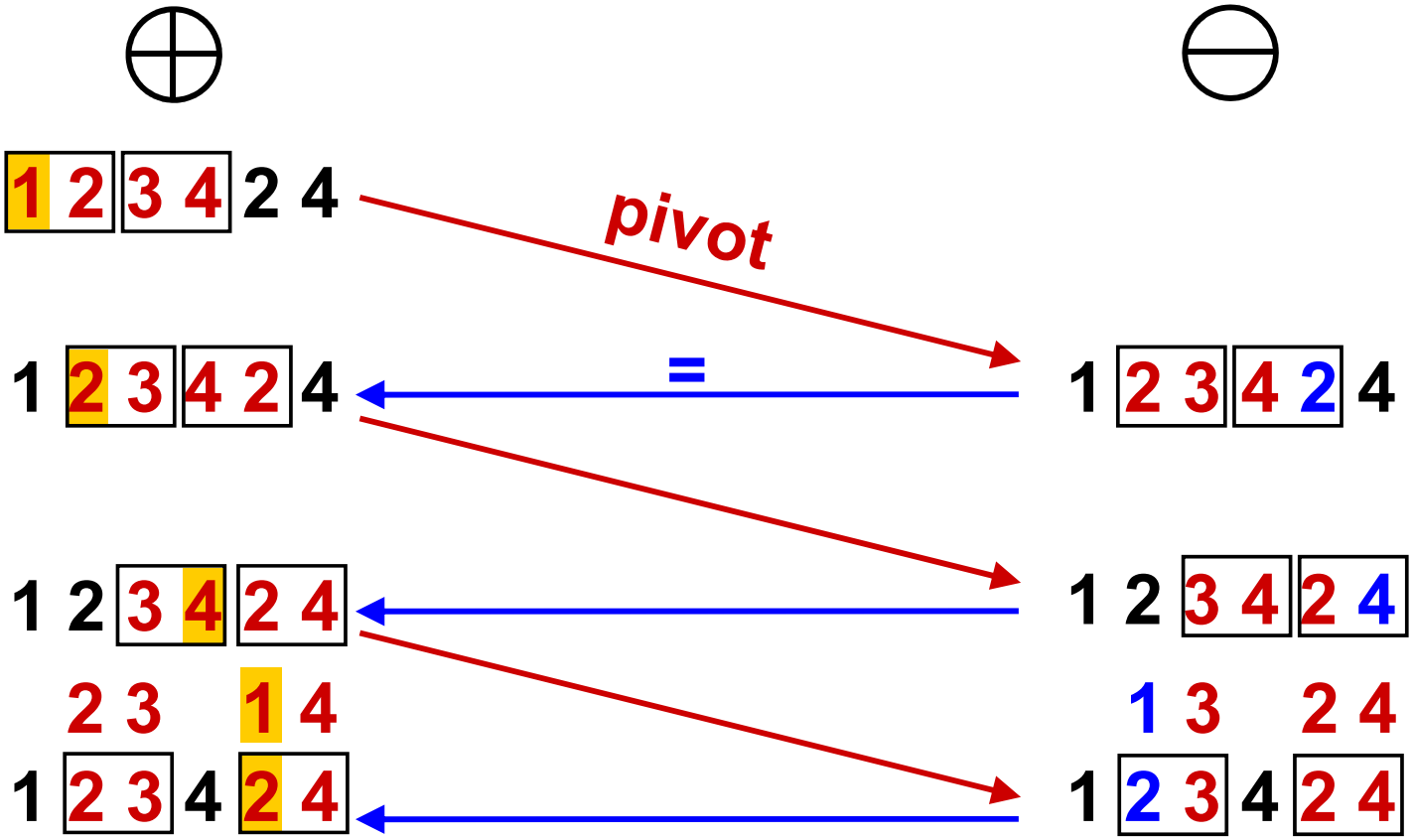
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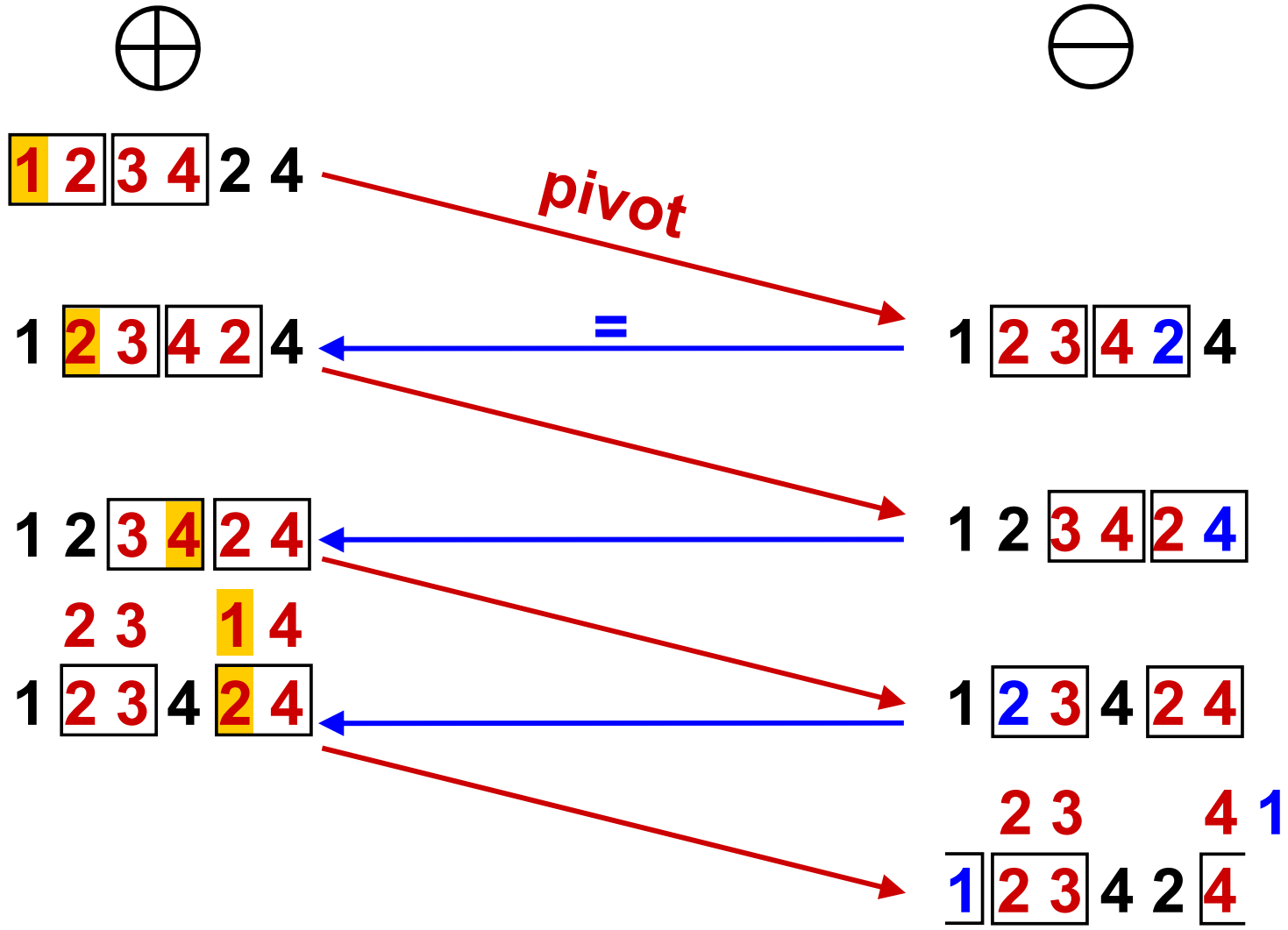
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pivot
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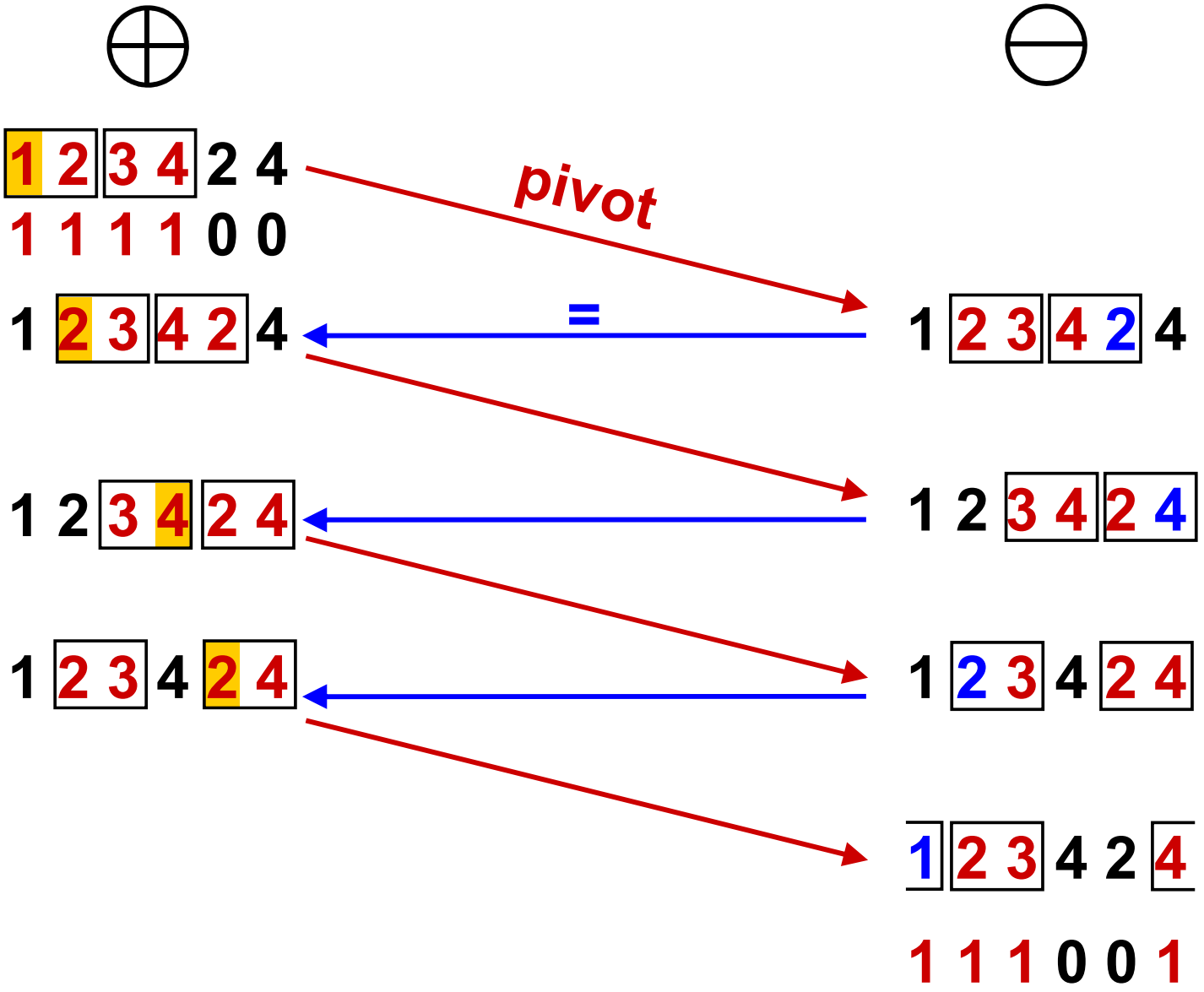
1 2 3 4 2 4
1 2 3 4 2 4
2 3 4 1
1 2 3 4 2 4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1 2 3 4 2 4
1 1 1 1 0 0
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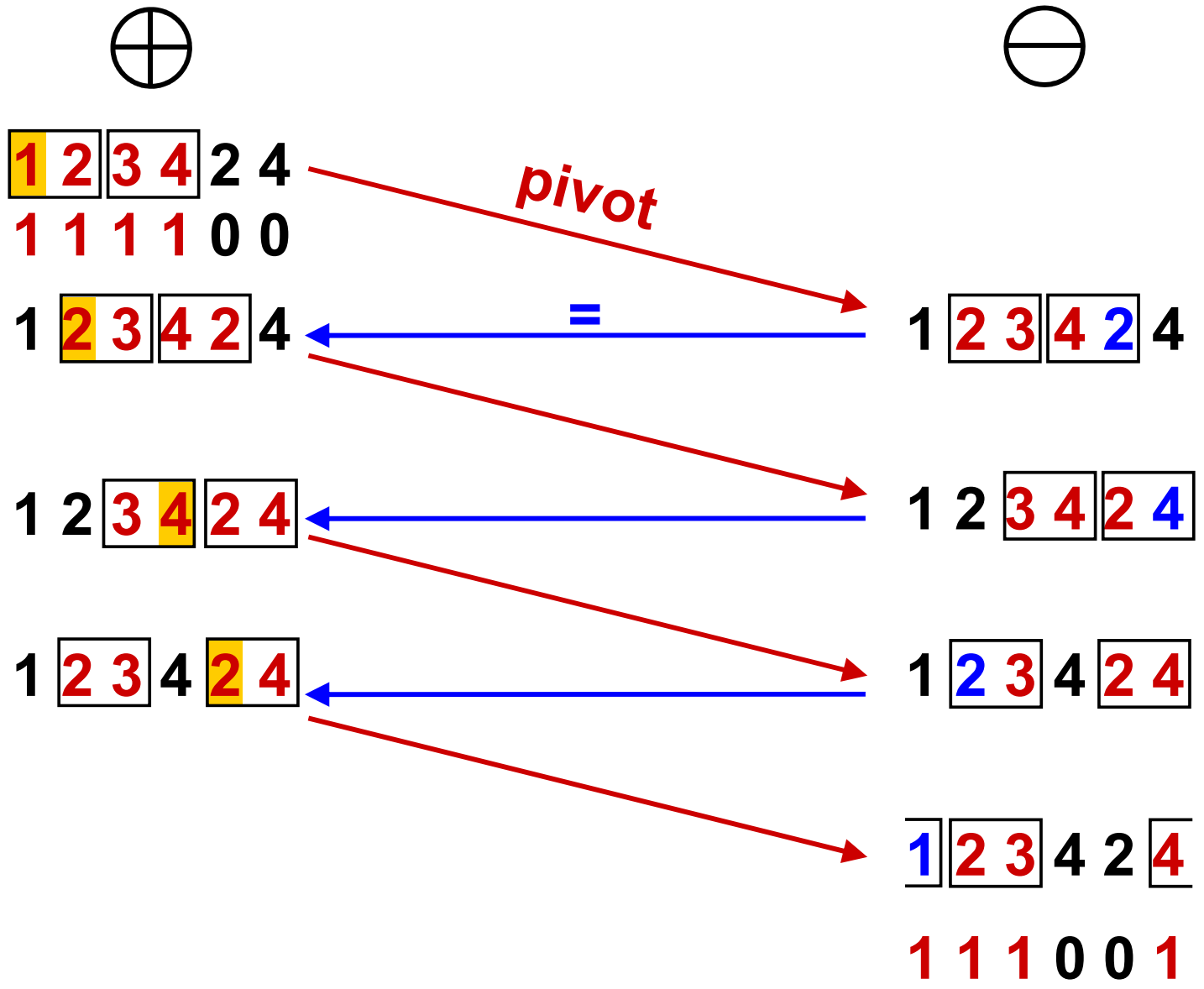
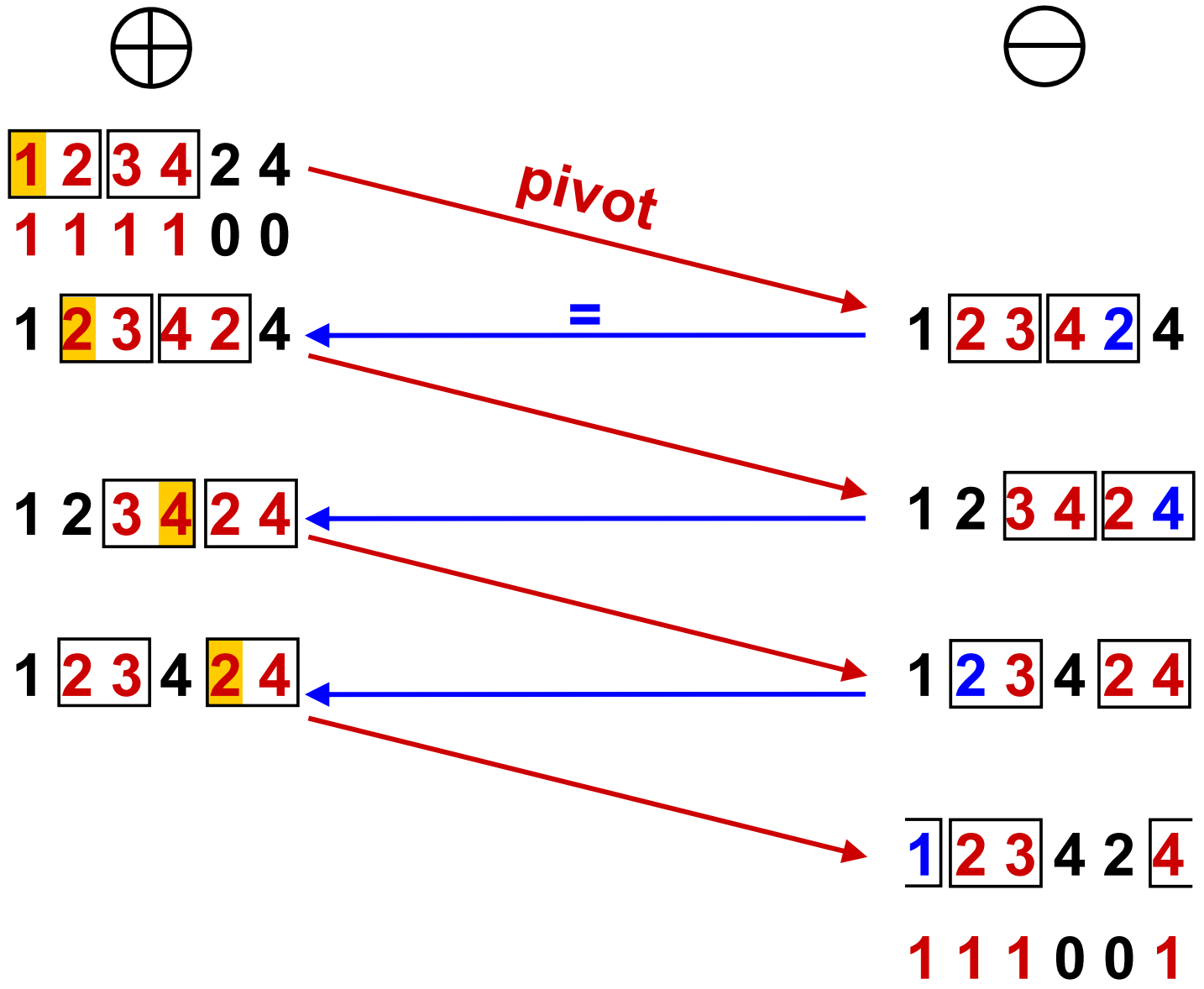
1 2 3 4 2 4
1 2 3 4 2 4
1 1 1 0 0 1




**Directed path: PPAD membership.**

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**Euler complex** or **Oik** of dimension **d−1** (or **(d−1)-oik**)
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**Example** **1 2 3 4 2 4**

**Let R be 2 3 and R – {2} = {3}.**
**The wall 3 is in two rooms, 2 3 and 3 4.**

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Idea:
1-oiks are Euler graphs, Euler tour = label string,
completely labeled Gale string = perfect matching
= room partitioning.

# Sign for room partitionings if d > 2?

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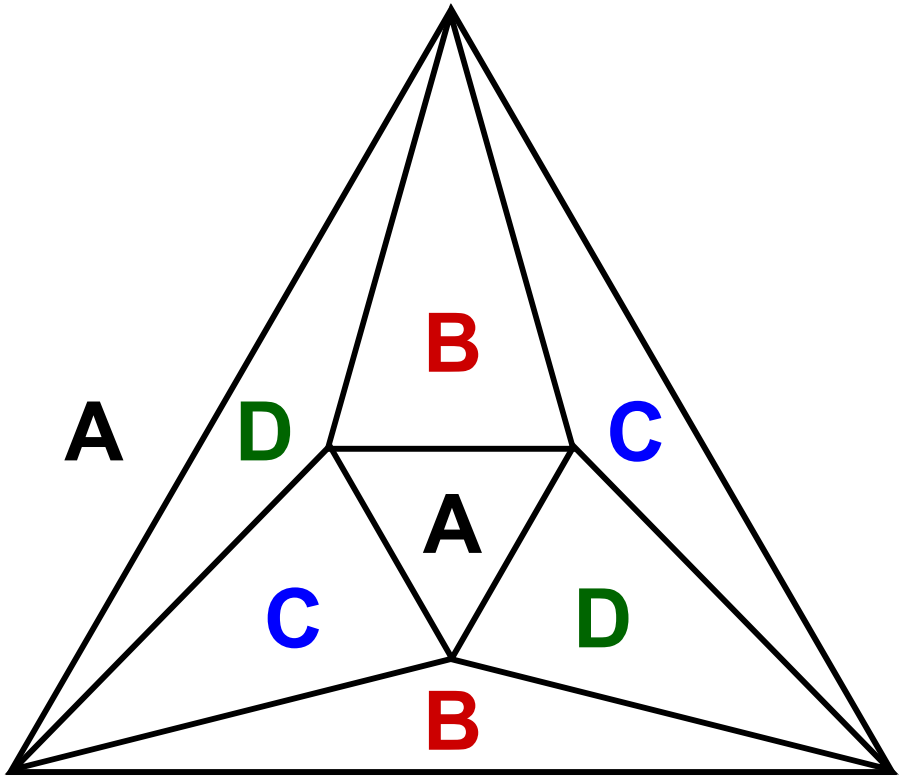

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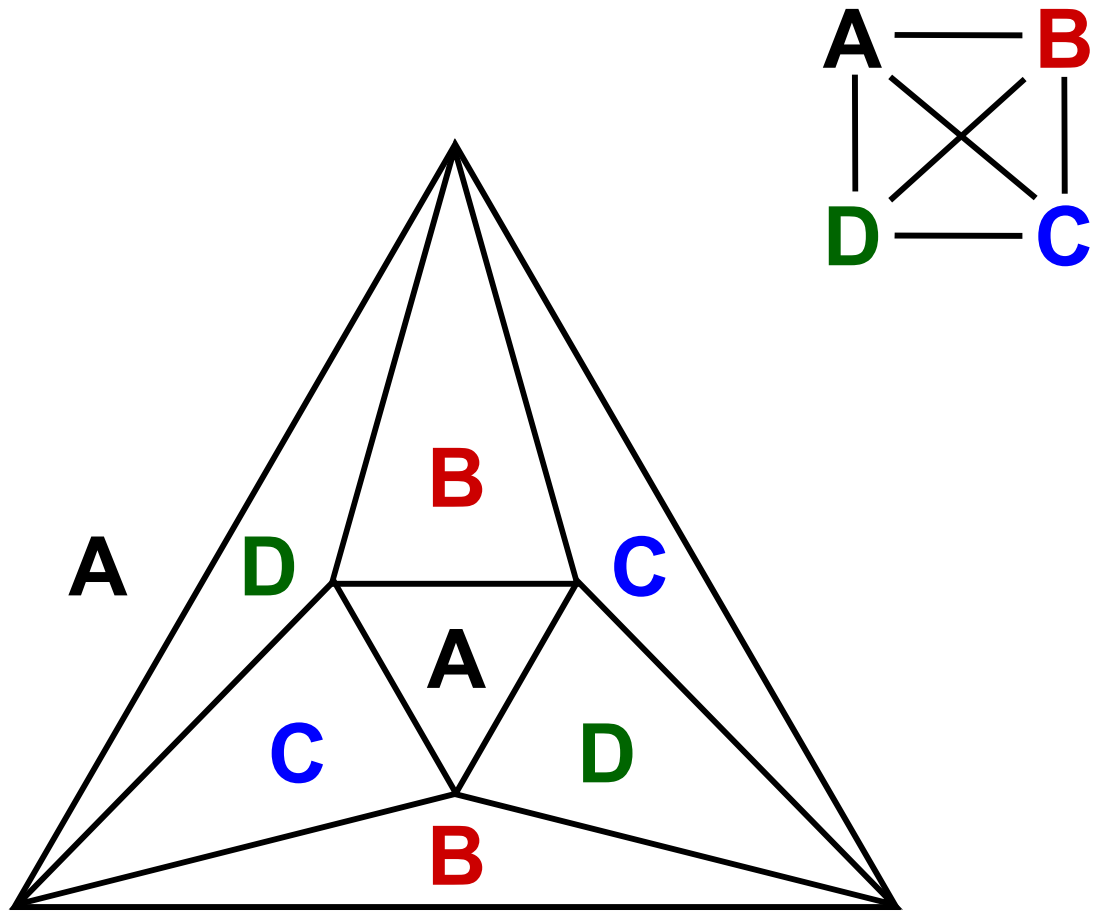

# Thank you