Econometrics, PS 3 MT:

Complementary solutions

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3 November 2011

Answer to question 1

In class

Answer to question 2

Assume DGP is

$$y_t = \beta_1 + \beta_2 x_t + \epsilon_t$$

Use data $\{y_t, x_t\}_{t=1}^T$ to estimate β_1 and β_2. OLS estimators minimize $\sum \big(y_t - \beta_1 - \beta_2 x_t\big)^2$.

FOC in β_1 gives $-2\sum \big(y_t - \beta_1 - \beta_2 x_t\big) = 0$, or $\sum y_t - T\beta_1 - \beta_2 \sum x_t = 0$, which yields

$$\hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \bar{x}$$

FOC in β_2 gives $-2\sum \big(y_t - \beta_1 - \beta_2 x_t\big)x_t = 0$, or $\sum y_t x_t - \beta_1 \sum x_t - \beta_2 \sum x_t^2 = 0$. Substitute $\hat{\beta}_1$ and remember that

$$\frac{\sum (x_t - \bar{x})^2}{T} = \frac{\sum x_t^2}{T} - \big(\frac{\sum x_t}{T}\big)^2 \tag{1}$$

get

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$$\hat{\beta}_2 = \frac{(\sum y_t x_t)/T - (\sum x_t)/T(\sum y_t)/T}{\sum (x_t - \bar{x})^2/T} = \frac{\text{sample } cov(x, y)}{\text{sample } V(x)}$$

Alternatively, $y = X\beta + \epsilon$, with $\beta = (\beta_1 \ , \ \beta_2)'$ and $X = [\iota \ x]$, so

$$X'X = \begin{bmatrix} \iota' \\ x' \end{bmatrix} \begin{bmatrix} \iota & x \end{bmatrix} = \begin{bmatrix} T & \sum x_t \\ \sum x_t & \sum x_t^2 \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{T \sum x_t^2 - (\sum x_t)^2} \begin{bmatrix} \sum x_t^2 & -\sum x_t \\ -\sum x_t & T \end{bmatrix}$$

$$X'y = \begin{bmatrix} \iota' \\ x' \end{bmatrix} y = \begin{bmatrix} \sum y_t \\ \sum y_t x_t \end{bmatrix}$$

Substitute them into $\hat{\beta} = (X'X)^{-1}X'y$ and use equation (1), get the same solution for $\hat{\beta}_1$ and $\hat{\beta}_2$.

To obtain variance covariance matrix of $\hat{\beta}$, use formula

$$V(\hat{\beta}) = \sigma_\epsilon^2 (X'X)^{-1} = \sigma_\epsilon^2 \frac{1}{T \sum x_t^2 - (\sum x_t)^2} \begin{bmatrix} \sum x_t^2 & -\sum x_t \\ -\sum x_t & T \end{bmatrix}$$

which, using again equation (1), gives

$$V(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{T} + \frac{\sigma_\epsilon^2 \bar{x}^2}{\sum (x_t - \bar{x})^2}$$

$$V(\hat{\beta}_2) = \frac{\sigma_\epsilon^2}{\sum (x_t - \bar{x})^2}$$

$$Cov(\hat{\beta}_1, \hat{\beta}_2) = -\frac{\sigma_\epsilon^2 \bar{x}}{\sum (x_t - \bar{x})^2}$$

To compute the expected value of RSS, note that $\hat{\epsilon} = y - X\hat{\beta} = M_x y = M_x \epsilon$, so

$$E[RSS] = E[\hat{\epsilon}'\hat{\epsilon}] = E[y' M_x y] = E[\epsilon' M_x \epsilon]$$

Given that 1) trace of a scalar is the trace itself, 2) $tr(AB) = tr(BA)$ if conformable, 3) A4GM and that 4) the trace is a linear operator,

$$
\begin{aligned}
E[RSS] &= E[tr(\epsilon' M_x \epsilon)] = E[tr(M_x \epsilon\epsilon')] = tr(M_x E[\epsilon\epsilon']) = tr(M_x \sigma_\epsilon^2 I_T) \\
&= \sigma_\epsilon^2 tr(M_x) = \sigma_\epsilon^2 tr(I_T - X(X'X)^{-1}X') = \sigma_\epsilon^2 (T - tr(X(X'X)^{-1}X')) = \\
&= \sigma_\epsilon^2 (T - tr(X'X(X'X)^{-1})) = \sigma_\epsilon^2 (T - trI_k) = \sigma_\epsilon^2 (T - k)
\end{aligned}
$$

Here $k = 2$

Answer to question 3

From A1 and A2, get $\hat{\beta} - \beta = (X'X)^{-1}\epsilon$. Under A4GM and A5N, $\epsilon \sim N(0, \sigma_\epsilon^2 I_T)$, so under A1-A5N $\hat{\beta} \sim N\left(\beta, \sigma_\epsilon^2 (X'X)^{-1}\right)$, which means that

$$\hat{\beta}_2 \sim N\left(\beta_2, \frac{\sigma_\epsilon^2}{TV(x)}\right)$$

(Vx is sample variance, not population variance)

normalize, get

$$\sqrt{T}\frac{\hat{\beta}_2 - \beta_2}{\sigma_\epsilon / \sqrt{V(x)}} \sim N(0, 1)$$

Use $S_{ols}^2 = \hat{\epsilon}'\hat{\epsilon}/(T-2)$, get distribution

$$\tau = \sqrt{T}\frac{\hat{\beta}_2 - \beta_2}{s / \sqrt{V(x)}} \sim t(T-2)$$

a) Confidence interval: find c_1, c_2 so that $1 - \alpha = P[c_1 < \beta_2 < c_2]$, get

$$CI = \left[\hat{\beta}_2 - t_{\frac{\alpha}{2}, T-2}\frac{s}{\sqrt{TV(x)}}; \hat{\beta}_2 + t_{\frac{\alpha}{2}, T-2}\frac{s}{\sqrt{TV(x)}}\right]$$

c_1 and c_2 are stochastic. With $1 - \alpha$ probability they enclose unknown parameter β_2

b) Test $\beta_2 = 0$, i.e. find d_1, d_2 so that $1 - \alpha = P[d_1 < \hat{\beta}_2 < d_2]$ (remember, under null hypothesis). You reject if $\hat{\beta}_2 < 0 - t_{\frac{\alpha}{2},T-2}\frac{s}{\sqrt{TV(x)}}$ or $\hat{\beta}_2 > 0 + t_{\frac{\alpha}{2},T-2}\frac{s}{\sqrt{TV(x)}}$

Note that the event *I fail to reject* means

$$-t_{\frac{\alpha}{2},T-2}\frac{s}{\sqrt{TV(x)}} < \hat{\beta}_2 < t_{\frac{\alpha}{2},T-2}\frac{s}{\sqrt{TV(x)}}$$

while event *zero lies inside my CI* means

$$\hat{\beta}_2 - t_{\frac{\alpha}{2},T-2}\frac{s}{\sqrt{TV(x)}} < 0 < \hat{\beta}_2 + t_{\frac{\alpha}{2},T-2}\frac{s}{\sqrt{TV(x)}}$$

which can be rewritten as $-t_{\frac{\alpha}{2},T-2}\frac{s}{\sqrt{TV(x)}} < \hat{\beta}_2 < t_{\frac{\alpha}{2},T-2}\frac{s}{\sqrt{TV(x)}}$. If one occurs, the other occurs as well

Answer to question 4

$y = X\hat{\beta} + \hat{\epsilon}$. Pre-multiply both sides by M_ι, where M_ι transforms into deviation from the mean:

$$M_\iota y = M_\iota X\hat{\beta} + M_\iota\hat{\epsilon}$$

Since sum of residuals is zero (from FOC in first OLS estimator), its mean is zero, so $M_\iota\hat{\epsilon} = \hat{\epsilon}$. This means that

$$M_\iota y = M_\iota X\hat{\beta} + \hat{\epsilon}$$

This $Tx1$ vector can be turned into a scalar by the inner product with itself. Remembering that M_ι is symmetric and idempotent,

$$y'M_\iota y = y'M_\iota'M_\iota y = [M_\iota X\hat{\beta} + \hat{\epsilon}]'[M_\iota X\hat{\beta} + \hat{\epsilon}] = [\hat{\epsilon}' + \hat{\beta}'(M_\iota X)'][M_\iota X\hat{\beta} + \hat{\epsilon}]$$

Do the multiplication. Use result $M_\iota\hat{\epsilon} = \hat{\epsilon}$ and (from FOC in second OLS estimator) $X'\hat{\epsilon} = 0$. Substitute $X\hat{\beta} = \hat{y}$, get

$$y'M_\iota y = \hat{y}'M_\iota \hat{y} + \hat{\epsilon}'\hat{\epsilon}$$

which is what you had to show.

Last,

$$RSS = \frac{ESS}{TSS} = \frac{\sum(\hat{y}_t - \bar{\hat{y}})^2}{\sum(y_t - \bar{y})^2} = \frac{\left[\sum(\hat{y}_t - \bar{\hat{y}})^2\right]^2}{\sum(y_t - \bar{y})^2 \sum(\hat{y}_t - \bar{\hat{y}})^2}$$

Since $y = \hat{y} + \hat{\epsilon}$, and remembering that with an intercept the mean of fitted y coincides with the mean of y, you can rewrite the numerator as

$$\begin{aligned}(\hat{y}' - \bar{\hat{y}}\iota')(\hat{y} - \bar{\hat{y}}\iota) &= (\hat{y}' - \bar{\hat{y}}\iota')(y - \bar{y}\iota - \hat{\epsilon}) = (\hat{y}' - \bar{\hat{y}}\iota')(y - \bar{y}\iota) - \hat{y}'\hat{\epsilon} + \bar{\hat{y}}\iota'\hat{\epsilon} \\ &= (\hat{y}' - \bar{\hat{y}}\iota')(y - \bar{y}\iota)\end{aligned}$$

(for the last equality, remember that $X'\hat{\epsilon} = \hat{\epsilon}'\iota = 0$).

This means that the numerator of R^2 can be written as T times the (sample) covariance of true and fitted y. Since you have variances in the denominator, R^2 becomes the square of the correlation coefficient.