Conditional vs. Unconditional Expectations

Michele Piffer, LSE

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You are in t=0 and you want to form an expectation of X, which is a random variable whose realization occurs in t=2. The distribution of X is

$$X = \begin{cases} 1 & \text{with probability } 1/4 \\ 3 & \text{with probability } 1/4 \\ 5 & \text{with probability } 1/4 \\ 7 & \text{with probability } 1/4 \end{cases}$$

What is your best guess in t=0 of X in t=2?

$$E(X) = \frac{1}{4}[1+3+5+7] = 4$$

Suppose now that there is a random variable observed in t=1 which takes value 1 or 2 with equal probability. Call this variable Z. Suppose also that Z is linked to X in the following way: if Z=1, then X is still random, but only values 1 and 3 can occur. Instead, if Z=2, then X can only take values 5 or 7.

We saw that your best guess of X formed in t=0 is 4. What is instead your best guess of X in t=1 if you observe that Z =1? What is instead your best guess if you observe Z=2?

$$E(X \mid Z=1) = \frac{1}{2}1 + \frac{1}{2}3 = 2$$

$$E(X \mid Z=2) = \frac{1}{2}5 + \frac{1}{2}7 = 6$$

or in general

$$E(X \mid Z) = 2 + 4(Z - 1)$$

This is the conditional expectation of X, i.e., conditioning on the realization of Z. Note that what matters is the information set on which you form your expectation.

Last, what is your best guess in t=0 of the best guess you will form in t=1, when you will observe Z? Well, we just saw that your guess will be either 2 or 6 depending on which Z you observe, and we are also told that Z=1 and Z=2 are equally likely. So

$$E(X \mid Z = 1)\text{Prob}(Z = 1) + E(X \mid Z = 2)\text{Prob}(Z = 2) = \frac{1}{2}2 + \frac{1}{2}6 = 4$$

Note that 4 was also the unconditional expectation. This makes sense: your best guess today of your best guess tomorrow is your best guess today. If this was not the case you could refine your best guess today and form a better expectation. This is the law of iterated expectations:

$$E\Big[E(X \mid Z)\Big] = E(X)$$

To convince yourself, do the same exercise assuming

$$X = \begin{cases} 1 & \text{with probability } 3/24 \\ 3 & \text{with probability } 3/24 \\ 5 & \text{with probability } 9/24 \\ 7 & \text{with probability } 9/24 \end{cases}$$

Note that the probability is unevenly allocated, which implies that now Z takes value 1 or 2 with probability 1/4 and 3/4. Show that the conditional expectations remain unchanged, but that the unconditional expectation changes. Convince yourself that the law of iterated expectations actually works.