Notes for students of EC413 (Macroeconomics for MSc)
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Proof that under, certain conditions, one can decentralize an economy by a competitive equilibrium.

SOCIAL PLANNER

When doing social planner, no price system is required

$$\max_{\{c_t, i_t, k_{t+1}, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-l_t)$$

$$\text{s.t.} \quad c_t + i_t = f(k_t, l_t) \quad (1) \quad \forall t$$

$$k_{t+1} = i_t + (1-\delta) k_t \quad (2) \quad \forall t$$

$$k_0 \text{ given}$$

i.e., the economy starts with a given level of capital, has access to a production technology and has to choose how to allocate the resources produced.

Substitute (2) in (1), simplify maximization problem and solve for optimal consumption, capital accumulation and labour/leisure choice:

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-l_t)$$

$$\text{s.t. } c_t + k_{t+1} - (1-\delta)k_t = f(k_t, l_t) \quad \forall t \ (3)$$

k_0 given

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \cdot \{u(c_t, 1-l_t) + \lambda_t [f(k_t, l_t) - c_t - k_{t+1} + (1-\delta)k_t]\}$$

with λ_t = current value multiplier

FOC c_t : $\beta^t \{u_1(c_t, 1-l_t) - \lambda_t\} = 0 \qquad u_1(c_t, 1-l_t) = \lambda_t$

FOC l_t : $\beta^t \{-u_2(c_t, 1-l_t) + \lambda_t f_l(k_t, l_t)\} = 0$

FOC k_{t+1}: $-\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} [f_k(k_{t+1}, l_{t+1}) + 1 - \delta] = 0$

combine, get

$$u_2(c_t, 1-l_t) = f_l(k_t, l_t) \cdot u_1(c_t, 1-l_t)$$

labour/leisure choice

$$u_1(c_t, 1-l_t) = \beta [f_k(k_{t+1}, l_{t+1}) + 1 - \delta] \cdot u_1(c_{t+1}, 1-l_{t+1})$$

Euler equation

Note, we don't have an interest rate since we don't have any bank nor market: the only way of postponing today's consumption is using investment in physical (not financial) capital

DECENTRALIZED ECONOMY

Assume now that there is no social planner who is choosing the allocations, but agents interact freely on markets through a price system.

As for property rights, assume that capital is owned by households and rent to firms. Firms produce a homogeneous good, which consumers have to allocate into consumption or investment. Consumers can postpone consumption via accumulation in physical capital or via investing in financial assets that yield r_t

There is no money, so prices are in real terms, where the numeraire is consumption good at time zero

Firm maximization problem:

$$\max_{\{c_t^s, i_t^s, l_t^d, k_t^d\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} Q_t \left[p_t^c c_t^s + p_t^i i_t^s - w_t l_t^d - R_t k_t^d \right]$$

$$\text{s.t.} \quad c_t^s + i_t^s = f(k_t, l_t)$$

where Q_t is the discounting factor (formally it is given by $Q_t = \frac{1}{1+R_1} \cdot \frac{1}{1+R_2} \cdot \ldots \cdot \frac{1}{1+R_t}$)

Since the problem has no intertemporal link it is essentially static. Substitute the constraint, get

$$\max_{c_t^s, i_t^s, l_t^d, k_t^d} (p_t^c - p_t^i) c_t^s + p_t^i f(k_t, l_t) - w_t l_t^d - R_t k_t^d$$

FOC c_t^s : $\quad p_t^c = p_t^i \qquad (4)$

FOC l_t^d : $\quad p_t^i f_l(k_t, l_t) = w_t \qquad (5)$

FOC k_t^d : $\quad p_t^i f_k(k_t, l_t) - R_t \qquad (6)$

Household maximization problem:

$$\max_{\{c_t^d, i_t^d, l_t^s, k_{t+1}^s, b_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-l_t)$$

$$\text{s.t. } p_t^c c_t^d + p_t^i i_t^d + b_{t+1} = w_t l_t^s + R_t k_t + (1+r_t) b_t \quad \forall t$$

$$k_{t+1}^s = i_t^d + (1-\delta) k_t^s \qquad \forall t$$

k_0^s given

b_0 given

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{ u(c_t, 1-l_t) + \lambda_t [w_t l_t^s + R_t k_t + (1+r_t) b_t - p_t^c c_t^d +$$
$$- p_t^i (k_{t+1}^s - (1-\delta) k_t^s) - b_{t+1}] \}$$

FOC c_t: $u_1(c_t, 1-l_t) = p_t^c \lambda_t$

FOC l_t: $-u_2(c_t, 1-l_t) + \lambda_t w_t = 0$

FOC b_{t+1}: $-\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1+r_{t+1}) = 0$

FOC k_{t+1}: $-\beta^t \lambda_t p_t^i + \beta^{t+1} \lambda_{t+1} [R_{t+1} + p_{t+1}^i (1-\delta)] = 0$

Combine, get

$$u_2(c_t, 1-l_t) = \frac{w_t}{p_t^c} \cdot u_1(c_t, 1-l_t) \qquad (7)$$

labour/leisure choice

We have 2 Euler equations, given that the consumer can postpone consumption in 2 ways. Derive first an arbitrage condition. Combine FOC b_{t+1} with FOC k_{t+1}, get

$$p_t^i = \frac{R_{t+1} + p_{t+1}^i (1-\delta)}{1+r_{t+1}} \quad (8)$$

A competitive equilibrium is a price system $\{p_t^c, p_t^i, w_t, R_t, r_t\}_{t=0}^{\infty}$ so that agent optimization is satisfied and markets clear. Combine FOCs and market clearing conditions, and show that the allocations pinned down by the decentralized economy coincide with the one of the social planner

(5) and (4) in (7):

$$u_2(c_t, 1-l_t) = \frac{p_t^i f_l(k_t, l_t)}{p_t^c} u_1(c_t, 1-l_t)$$

$$\Rightarrow \boxed{u_2(c_t, 1-l_t) = f_l(k_t, l_t) u_1(c_t, 1-l_t)}$$

(8) and (4) with (6)

$$p_t^c = \frac{p_{t+1}^c [f_k(k_{t+1}, l_{t+1}) + 1 - \delta]}{1+r_{t+1}} \quad (9)$$

FOC b_{t+1} with FOC c_t:

$$\frac{u_1(c_t, 1-l_t)}{p_t^c} = \beta \frac{u_1(c_{t+1}, 1-l_{t+1})}{p_{t+1}^c} (1+r_{t+1})$$

substitute (9)

$$\boxed{u_1(c_t, 1-l_t) = \beta [f_k(k_{t+1}, l_{t+1}) + 1 - \delta] u_1(c_{t+1}, 1-l_{t+1})}$$

Given that the optimality conditions coincide, it means that decentralized markets lead to the same allocations that would be chosen by the social planner,