JULLIARD, PROBLEM SET 2, MICHELE PIFFER

$$\max_{\{c_1, c_2\}} \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}$$

Endowment y at $t=1$. There are 2 ways in which the agent can transform wealth at $t=1$ into wealth in $t=2$:

a) lend on international financial markets

b) use technology that produces I^α out of I

Note: "small open economy" means that we can borrow/lend at r as much as we like

Note: if ~~we~~ borrow 1 we owe $1+r$. Since assume that bonds has 1 period maturity the contract is closed at time $t=2$. But if we invest I and produce I^α at $t=2$ we still have I units of capital invested (no full depreciation is assumed here). This means that you disinvest I at $t=2$, given that there is no $t=3$ in this economy.

$$BC_1^{HH}: \quad c_1 + I + a = y$$

a → save on bonds

I → invest in Technology

BC_2^{HH}: $\quad c_2 - I = I^{\alpha} + (1+r)a$

$\lfloor$ disinvest

left from previous period

or

$$c_2 = I^{\alpha} + I + (1+r)a$$

$$\Rightarrow \quad a = \frac{c_2}{1+r} - \frac{I^{\alpha}+I}{1+r}$$

Substitute into BC_1^{HH}, get IBC^{HH}

$$\underbrace{c_1 + \frac{c_2}{1+r} + I}_{\text{OUTFLOW}} = \underbrace{y + \frac{I^{\alpha}+I}{1+r}}_{\text{INFLOW}}$$

Maximization problem requires computing

$$\max_{\{c_1, c_2, I\}} \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}$$

$$\text{st} \quad c_1 + \frac{c_2}{1+r} + I \leq y + \frac{I^{\alpha}+I}{1+r}$$

$$L = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma} + \lambda \left\{ y + \frac{I^{\alpha}+I}{1+r} - c_1 - \frac{c_2}{1+r} - I \right\}$$

1) Solve for I^*

FOC I: $\lambda \cdot \left\{ \frac{1}{1+r} \cdot [\alpha I^{\alpha-1} + 1] - 1 \right\} \overset{!}{=} 0$

$$\alpha I^{\alpha-1} + 1 \overset{!}{=} 1 + r$$

This is an arbitrage condition, linking returns on the 2 solutions for postponing consumption: RHS is what you get by investing 1 € in financial markets; LHS is what you get by investing the same amount in your technology. Note, LHS depends on the level of investment since marginal product of investment is not constant in the overall quantity you invest in $\overline{\pi}$.

$$I^* = \left(\frac{\alpha}{r}\right)^{\frac{1}{1-\alpha}}$$

optimal level of investment is independent on y.

This is because you can borrow/lend as much as you like on fin. markets and this won't affect r.

if $\alpha I^{\alpha-1} + 1 > 1 + r$, borrow from fin markets to $I \uparrow$, this will reduce ~~the~~ LHS ~~till~~ till equality is reached

$\alpha \in (0,1)$

if $\alpha I^{\alpha-1} + 1 < 1 + r$, disinvest from I till equality is reached

2) FOC c_1: $\quad c_1^{-\gamma} = \lambda$

FOC c_2: $\quad \beta c_2^{-\gamma} = \dfrac{\lambda}{1+r}$

$$\Rightarrow c_1^{-\gamma} = \beta(1+r)\, c_2^{-\gamma}$$

$$\frac{c_2}{c_1} = \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1}{\gamma}}$$ Euler Equation

Combine ~~this~~ with optimal level of I and IBC to pin down c_1^*, c_2^*

$$\begin{cases} \dfrac{c_2}{c_1} = \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1}{\gamma}} \\[2mm] c_1 + \dfrac{c_2}{1+r} + I = y + \dfrac{I^{\alpha} + I}{1+r} \\[2mm] I = \left(\dfrac{\alpha}{r}\right)^{\frac{1}{1-\alpha}} \end{cases}$$

$$c_1 + \frac{c_2}{1+r} = \underbrace{y + \frac{I^{\alpha} + I}{r} - I \,\Big|_{I = I^*}}_{W(r)}$$

(don't substitute I^* if you don't need it)

$$\begin{cases} \dfrac{c_2}{c_1} = \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1}{\gamma}} \\[2ex] c_1 + \dfrac{c_2}{1+r} = W(r) \end{cases}$$

$$c_1 + \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1}{\gamma}-1} c_1 = W(r)$$

$$c_1^* = \underbrace{\frac{1}{1+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}}}_{mpc(r)} W(r)$$

mpc(r) marginal propensity of consumption

$$c_2^* = \frac{\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}} \cdot W(r)$$

$$a^* = y - I^* - c_1^*$$

Now, study what happens to $W(r)$, $mpc(r)$, c_1^* and a^* when r changes

__$W(r)$:__ $$W(r) = y + \frac{I^\alpha + I}{1+r} - I \Big|_{I=I^*} =$$

$$= y + \underbrace{\frac{I^\alpha}{1+r}}_{\text{decreasing in } r} - \underbrace{\frac{r}{1+r}}_{\text{increasing in } r} I \Big|_{I=I^*}$$

decreasing in r

decreasing in r

Given that I^* is decreasing in r, $W(r)$ is decreasing in r. As $r\uparrow$, it is more expensive to borrow, so domestic agents borrow less in order to invest in technology I, Hence they will produce less and $W(r)$ decreases.

This means that the cake to divide between c_1 and c_2 is smaller. As for the way in which the cake is divided, check marginal propensity to consume

mpc(r) =

$$mpc(r) = \frac{1}{1+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}}$$

Note,
$$\frac{\partial\, mpc(r)}{\partial r} = \begin{cases} >0 & \text{if } \gamma>1 \\ =0 & \text{if } \gamma=1 \\ <0 & \text{if } \gamma<1 \end{cases}$$

This is because γ pins down the elasticity of intertemporal substitution (more on this later)

$\Rightarrow c_1^* = mpc(r)\cdot W(r)$

$$\frac{\partial c_1^*}{\partial r} = \begin{cases} \leq 0 & \text{if } \gamma \leq 1 \\ \text{undetermined} & \text{if } \gamma > 1 \end{cases}$$

$$a^* = y - c_1^* - I^* =$$

$$= y - mpc(r) \cdot W(r) - I^* = CA^*$$

As $r\uparrow$, $I^*\downarrow$ so economy runs a current account surplus, i.e. saves more and accumulates assets a^* against the rest of the world. But at the same time optimal level of consumption in $t=1$ changes. If $c_1^*\downarrow$, then for sure the economy will run an even bigger current account surplus $\Rightarrow a^*\uparrow$.

(By differentiation, one can also check mathematically what happens if $\gamma = 1$, i.e. if $\gamma = 1$ one effect perfectly offsets the other s.t. a^* remains unchanged)

3) given $g_t = T_t$, $t = 1, 2$, rederive IBC^{HH}

$t=1$ $\quad c_1 + I + a = y - T_1$

$t=2$ $\quad c_2 = I^\alpha + I + (1+r)a - T_2$

$$IBC^{HH} \quad c_1 + \frac{c_2}{1+r} + I = y + \frac{I^\alpha + I}{1+r} - T_1 - \frac{T_2}{1+r}$$

This is the budget constraint that the Household faces in the maximization problem. Under $g_t = T_t$, gets

$$c_1 + \frac{c_2}{1+r} + I = y + \frac{I^\alpha + I}{1+r} - g_1 - \frac{g_2}{1+r}$$

$$\mathcal{L} = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma} + \lambda \left\{ y + \frac{I^\alpha + I}{1+r} - g_1 - \frac{g_2}{1+r} + \right.$$

$$\left. - c_1 - \frac{c_2}{1+r} - I \right\}$$

FOC I : $I^* = \left(\frac{\alpha}{r}\right)^{\frac{1}{1-\alpha}}$, $\{g_t\}$ doesn't affect ~~this~~ this since taxes are lump sum and we are still in small open economy

Again,

$$c_1^* = \frac{1}{1+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}} W(r)$$

$$c_2^* = \frac{\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}} W(r)$$

$$\text{with } W(r) = y + \frac{I^\alpha + I}{1+r} - I - g_1 - \frac{g_2}{1+r} \Bigg|_{I=I^*}$$

$$\frac{\partial c_1^*}{\partial g_1} = \frac{1}{1+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}} \frac{\partial W(r)}{\partial g_1} =$$

$$= - \frac{1}{1+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1}}$$

$$CA^* = a^* = y - c_1^* - I^* - g_1$$

$$\frac{\partial CA}{\partial g_2} = + \frac{1}{1+\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}} - 1 =$$

$$= - \frac{\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}}{1+\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}}$$

As g_2, government subtracts resources to HH, so economy runs a current account deficit. But at the same time the HH reoptimizes and decides to consume less, s.t. deficit is smaller

4) If government borrows / lends at rate r

$$BC_1^G : \quad T_1 + D_1 = g_1$$

$$BC_2^G : \quad T_2 + D_2 = g_2 + (1+r) D_1$$

Note, define D_i as total deficit, not as primary deficit. Hence, $D_2 = 0$ since no extra period on which to postpone budget balancing. Alternatively, call $\tilde{D}_2$ as primary deficit, hence, $\tilde{D}_2 = g_2 - T_2$. Note, need

$$\tilde{D}_2 = -(1+r) \tilde{D}_1$$

From BC_2^G get $D_1 = \frac{T_2}{1+r} - \frac{g_2}{1+r}$, substitute out

$$T_1 + \frac{T_2}{1+r} = g_1 + \frac{g_2}{1+r} \qquad IBC^G$$

From IBC^{HH} we had

$$c_1 + \frac{c_2}{1+r} + I = y + \frac{I^\alpha + I}{1+r} \underbrace{- t_2 - \frac{T_2}{1+r}}_{\text{via } IBC^G}$$

$$" \qquad = y + \frac{I^\alpha + I}{1+r} - g_1 - \frac{g_2}{1+r}$$

If $\{g_t\}$ is constant and the only ~~thing~~ thing that changes is $\{T_t\}$, then no effect on c_1^*, c_2^*. Hence CA is unaffected.

Formally, $T_1 + D_1 = g_1$, $D_1 \uparrow$ means $T_1 \downarrow$:

$$\Delta T_1 = -\Delta D_1$$

But $T_2' = g_2 + (1+r) D_1'$

$$= \underbrace{g_2 + (1+r) D_1}_{T_2} + (1+r) \Delta D_1$$

$$T_2' = T_2 + (1+r) \Delta D_1$$

$$\Delta T_2 = -(1+r) \Delta T_1$$

$$T_1' + \frac{T_2'}{1+r} = \underbrace{T_1 + \Delta T_1}_{\text{so given } D, P} + \frac{T_2 + \Delta T_2}{1+r}$$

$$= T_1 + \Delta T_1 + \frac{T_2}{1+r} - \frac{(1+r)}{1+r}\Delta T_1$$

$$= T_1 + \frac{T_2}{1+r} \quad , \text{IBC}^{\text{HH}} \text{ unchanged}$$

Ricardian Equivalence

5) Under this setting government spending has some purpose

$$\max_{c_1, c_2, I} \quad \log(c_1 + g_1) + \beta \log(c_2 + g_2)$$

$$\text{s.t.} \quad c_1 + \frac{c_2}{1+r} + I = y + \frac{I^\alpha + I}{1+r} - g_1 - \frac{g_2}{1+r}$$

(he is assuming balanced budget)

$$L = \log(c_1 + g_1) + \beta \log(c_2 + g_2) + \lambda \left\{ y + \frac{I^\alpha + I}{1+r} - g_1 - \frac{g_2}{1+r} - c_1 - \frac{c_2}{1+r} - I \right\}$$

FOC I : $I^* = \left(\frac{\alpha}{r}\right)^{\frac{1}{1-\alpha}}$

FOC c_1 : $\frac{1}{c_1 + g_1} = \lambda$

FOC c_2 : $\frac{\beta}{c_2 + g_2} = \frac{\lambda}{1+r}$

$$\Rightarrow \frac{c_2 + g_2}{c_1 + g_1} = \beta(1+r)$$

substitute EE into IBC^{HH},

$$c_1 + \beta(c_1 + g_1) - g_2 = \underbrace{y + \frac{I^\alpha + I}{1+r} - I - g_1 - \frac{g_2}{1+r}}_{W(r)}$$

$$c_1^* = \frac{1}{1+\beta}\left[W(r) + g_2 - \beta g_1\right]$$

$$\frac{\partial c_1^*}{\partial g_1} = \frac{1}{1+\beta}\left[-1 - \beta\right] = -1$$

Since private and public consumption are perfect substitutes, HH responds by decreasing c_1.

$$c_1^* = a^* = y - c_1^* - I^* - g_1$$

$$\frac{\partial a^*}{\partial g_1} = 1 - 1 = 0 \quad \text{So, no change}$$

PRELIMINARY ISSUES

Define the elasticity of intertemporal substitution EIS as

$$EIS = \varepsilon_{\frac{c_{t+1}}{c_t}, 1+r}$$

where $\frac{c_{t+1}}{c_t}$ is the gross rate of growth of consumption and $1+r$ is the gross interest rate

Remember that $\varepsilon_{A,B} = \frac{\partial \log A}{\partial \log B}$, since B_2 $A = B^c$, $\varepsilon_{A,B} = c B^{c-1} \cdot \frac{B}{A} = c$, while $\frac{\partial \log B^c}{\partial \log B} =$

$$= \frac{\partial c \log B}{\partial \log B} = c \quad \text{Hence}$$

$$EIS := \varepsilon_{\frac{c_{t+1}}{c_t}, 1+r} = \frac{\partial \log \frac{c_{t+1}}{c_t}}{\partial \log (1+r)}$$

under $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, get Euler equation

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\gamma}}$$

which yields

$$EIS = \varepsilon_{\frac{c_{t+1}}{c_t}, 1+r} = \frac{\partial \log c_{t+1}/c_t}{\partial \log(1+r)} = \frac{1}{\gamma}$$

This means that if gross interest rate increases by one percent, gross rate of growth of consumption increases by $1/\gamma$ percent

But note,

$\log(1+r) \simeq$ (Taylor expansion around $r=0$)

$\simeq r$

$\log \frac{c_{t+1}}{c_t} \simeq$ (Taylor expansion around $c_{t+1} = c_t$)

$\simeq \frac{c_{t+1} - c_t}{c_t}$

Hence, from operative definition of EIS you get that, if net interest rate r increases by 1 (not by 1 percent since it is already in percentage), then the net growth rate of consumption increases by $1/\gamma$ (not by $1/\gamma$ percent since again it is already a percentage variation)

In short, how much consumption growth reacts to variations in the interest rate.

Note, EIS is a different concept from relative risk aversion, even though under this utility function it is the same parameter pinning down both

1) EIS has to do with your willingness to take advantage of high interest rates, so does not require uncertainty (it is actually computed under certainty equivalence)

2) RRA is related to risk, hence needs uncertainty, and is unrelated to interest rate and to the motive of taking advantage of how the market is pricing the future

Risk aversion is given by the fact that the utility is concave in the level of consumption. The more concave is $u()$ and the more risk averse is the agent. This is equivalent to saying, the faster is the fall in marginal utility as consumption increases and the more risk averse is the agent. Hence, need a measure of how fast $u'(c)$ decreases as c increases. Use elasticities:

$$RRA = \left| \varepsilon_{u'(c), c} \right| = -u''(c) \cdot \frac{c}{u'(c)}$$

3) Note, this is an even different concept from prudence, which has to do with HOW marginal utility decreases (instead of how fast), i.e., if it decrease linearly or in a convex way. This is important since, with

convex marginal utility the Euler Equation prescribes a marginal utility in t that is increasing in the level of uncertainty on t+1. This is equivalent to prescribing a lower level of consumption in time t, that is to say, Precautionary Saving.

(See Bagliano - Bertola, chapter 1)