MACROECONOMICS FOR MSc

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$$\max_{\{c_{t+s}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s \log(c_{t+s})$$

$$B_{t+s+1} = y_{t+s} - c_{t+s} + (1+r) B_{t+s}$$

$\{y_{t+s}\}_{s=0}^{\infty}$ known with certainty

1) BC^t $\quad B_{t+1} = y_t - c_t + (1+r) B_t$

BC^{t+1} $\quad B_{t+2} = y_{t+1} - c_{t+1} + (1+r) B_{t+1}$

$$\to B_{t+1} = \frac{y_{t+1}}{1+r} - \frac{c_{t+1}}{1+r} + \frac{B_{t+2}}{1+r}$$

where B_{t+s} is asset left at $t+s-1$, only the principal with which you start period $t+s$

$$c_t + \frac{c_{t+1}}{1+r} + \frac{B_{t+2}}{1+r} = y_t + \frac{y_{t+1}}{1+r} + (1+r) B_t$$

BC^{t+2} $\quad B_{t+3} = y_{t+2} - c_{t+2} + (1+r) B_{t+2}$

$$\to B_{t+2} = \frac{y_{t+2}}{1+r} - \frac{c_{t+2}}{1+r} + \frac{B_{t+3}}{1+r}$$

$$c_t + \frac{c_{t+1}}{1+r} + \frac{c_{t+2}}{(1+r)^2} + \frac{B_{t+3}}{(1+r)^2} = y_t + \frac{y_{t+1}}{1+r} +$$

$$+ \frac{y_{t+2}}{(1+r)^2} + (1+r) B_t$$

Suppose economy starts at t:

$$\sum_{s=0}^{q} \frac{c_{t+s}}{(1+r)^s} + \frac{B_{t+q+1}}{(1+r)^q} = \sum_{s=0}^{q} \frac{y_{t+s}}{(1+r)^s} + (1+r) B_t$$

Take $\lim_{q \to \infty}$

$$\sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} + \underbrace{\lim_{q \to \infty} \frac{B_{t+q+1}}{(1+r)^q}}_{} = \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} + \overbrace{(1+r) B_t}^{=0}$$

= 0 under no Ponzi scheme
Don't die in debt + don't die with some assets not spent

2). $$\max_{\{c_{t+s}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s \log(c_{t+s})$$

$$\text{s.t. } \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} \leq \underbrace{\sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s}}_{W}$$

$$L = \sum_{s=0}^{\infty} \beta^s \log(c_{t+s}) + \lambda \cdot \left\{ W - \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} \right\}$$

FOC c_{t+s} $$\frac{\beta^s}{c_{t+s}} = \frac{\lambda}{(1+r)^s}$$

FOC c_{t+s+1} $$\frac{\beta^{s+1}}{c_{t+s+1}} = \frac{\lambda}{(1+r)^{s+1}}$$

$$\frac{1}{c_{t+s}} = \beta(1+r)\frac{1}{c_{t+s+1}}$$ Euler Equation

$$c_{t+s+1} = \beta(1+r)\, c_{t+s}$$

3) $c_{t+1} = \beta(1+r)\, c_t$

$$c_{t+2} = \beta(1+r)\cdot\beta(1+r)\, c_t = \beta^2 (1+r)^2 c_t$$

$$c_{t+s} = [\beta(1+r)]^s \cdot c_t \quad , \forall s$$

4) Combine EE and IBC HH

$$\begin{cases} \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} = W \\ c_{t+s} = \beta^s (1+r)^s c_t \end{cases}$$

$$c_t \cdot \sum_{s=0}^{\infty} \frac{\beta^s \cancel{(1+r)^s}}{\cancel{(1+r)^s}} = W$$

$$c_t \cdot \frac{1}{1-\beta} = W$$

$$c_t^* = (1-\beta)\cdot W$$

$$c_t^* = (1-\beta)\cdot \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s}$$

$c_t = (1-\beta) W$

$c_{t+1} = \beta(1+r) c_t = \beta(1-\beta)(1+r) W$

$c_{t+2} = \beta^2(1+r)^2 c_t = \beta^2(1-\beta)(1+r)^2 W$

So, $c^*_{t+s} = (1-\beta)\cdot\beta^s(1+r)^s \cdot W$

$$= (1-\beta)\beta^s(1+r)^s \cdot \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s}$$

5) Given $y_{t+s} = \bar{y} + \varepsilon_{t+s}$,

$$W = \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} = \sum_{s=0}^{\infty} \frac{\bar{y} + \varepsilon_{t+s}}{(1+r)^s} =$$

$$= \bar{y} \cdot \frac{1}{1 - \frac{1}{1+r}} + \sum_{s=0} \frac{\varepsilon_{t+s}}{(1+r)^s} =$$

$$= \bar{y} \cdot \frac{1+r}{r} + \sum_{s=0}^{\infty} \frac{\varepsilon_{t+s}}{(1+r)^s}$$

$$c^*_t = (1-\beta) W = (1-\beta) \cdot \left\{ \frac{1+r}{r} \cdot \bar{y} + \sum_{s=0}^{\infty} \frac{\varepsilon_{t+s}}{(1+r)^s} \right\}$$

$$\frac{dc_t}{d\bar{y}} = (1-\beta)\frac{1+r}{r}$$

$$\frac{dc_{t+s}}{d\bar{y}} = (1-\beta)\beta^s(1+r)^s \cdot \frac{1+r}{r}$$

Note, if $r = \delta$, ~~~~ i.e., $\beta(1+r) = 1$, then all levels of consumption are scaled up by the same proportion. Here $\delta = r$, so levels are scaled up not. EE still satisfied

$$\frac{\partial C_t}{\partial \varepsilon_{t+s}} = (1-\beta) \cdot \frac{1}{(1+r)^s}$$

so, C_t is scaled up the ~~less~~ the further away is the temporary increase in years.

$$\frac{\partial C_{t+s}}{\partial \varepsilon_{t+s}} = \beta^s \cancel{(1+r)^s} \cdot (1-\beta) \frac{1}{\cancel{(1+r)^s}} = \beta^s (1-\beta)$$

where C_{t+s} is the level of consumption corresponding exactly to the period where the extra output increase lives.

$$\frac{\partial C_{t+q}}{\partial \varepsilon_{t+s}} = \beta^q (1+r)^q \cdot (1-\beta) \frac{1}{(1+r)^s}$$

where C_{t+q} is for future levels of consumption that are not necessarily in the period when extra output will temporarily increase

5) Given $c_{t+1} = \beta^{1}(1+r)^{1} \cdot c_t$

with $c_t = (1-\beta)\,W$

given $\beta(1+r) > 1$, consumption increases over time while $\beta(1+r) < 1$ will imply consumption decreasing over time

As for the impact of $\bar{y}$ on c_t:

$$\frac{\partial c_t}{\partial \bar{y}} = (1-\beta)\,\frac{1+r}{r} = \frac{1+r-\beta(1+r)}{r} =$$

$$\begin{cases} = 1 & \text{if } \beta(1+r) = 1, \ \delta = r \\ > 1 & \text{if } \beta(1+r) < 1, \ \delta > r \\ < 1 & \text{if } \beta(1+r) > 1, \ \delta < r \end{cases}$$

The more you care about the future and the less $\bar{y}$ impacts on c_t relative to future flows of consumption

7.) $\max_{\{c_{t+s}\}_{s=0}^{\infty}} \quad E_t \sum_{s=0}^{\infty} \beta^s u(c_{t+s})$

$$\text{s.t.} \quad \sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} = W$$

Euler equation:

$$u'(c_{t+s}) = \beta(1+r) \cdot E_{t+s} \cdot u'(c_{t+s+1})$$

But $u(c) = c - \frac{\alpha}{2} c^2$, so

$$u'(c) = 1 - \alpha c$$

$$\Rightarrow \quad 1 - \alpha c_{t+s} = \underbrace{\beta(1+r)}_{=1 \text{ under assumption}} \overset{E_{t+s}}{\vee} [1 - \alpha c_{t+s+1}]$$

$$1 - \alpha c_{t+s} = 1 - \alpha E_{t+s}(c_{t+s+1})$$

$$\Rightarrow \quad c_{t+s} = E_{t+s}[c_{t+s+1}] \quad \text{constant}$$

Euler is then $c_t = E_t(c_{t+1}) =$

$$= E_t(c_{t+s}), \quad \forall s$$

From constraint

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$$\begin{cases} E_t \sum_{s=0}^{\infty} \frac{C_{t+s}}{1+r} = E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} \\ E_t(C_{t+s}) = C_t \end{cases}$$

$$C_t = \frac{r}{1+r} \cdot E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s}$$

Assume $y_{t+s} = \bar{y} + \eta_{t+s}$

$$\eta_{t+s} = \rho \cdot \eta_{t+s-1} + \varepsilon_{t+s}$$

$$\Rightarrow \quad y_{t+s} = \bar{y} + \rho \eta_{t+s-1} + \varepsilon_{t+s}$$

more convenient to write η_{t+s} as

$$\eta_t = \varepsilon_t$$
$$\eta_{t+1} = \rho \varepsilon_t + \varepsilon_{t+1}$$
$$\eta_{t+2} = \rho^2 \varepsilon_t + \rho \varepsilon_{t+1} + \varepsilon_{t+2}$$
$$\eta_{t+s} = \sum_{i=0}^{s} \rho^{s-i} \varepsilon_{t+i}$$

$$y_{t+s} = \bar{y} + \eta_{t+s} =$$
$$" \quad = \bar{y} + \rho^s \varepsilon_t + \rho^{s-1} \varepsilon_{t+1} + \dots + \rho \varepsilon_{t+s-1} + \varepsilon_{t+s}$$
$$" \quad = \bar{y} + \underbrace{\sum_{i=0}^{s} \rho^{s-i} \varepsilon_{t+i}}_{\text{all known at } t+s}$$

Here he just wants effect of ε_t on c_t.
It will increase all y_{t+s} via its persistence on ρ:

$$y_t = \underbrace{\bar{y}}_{=y} + \eta_t = \bar{y} + \rho\eta_{t-1} + \varepsilon_t$$

$$y_{t+1} = \bar{y} + \rho\eta_t \overset{+\varepsilon_{t+1}}{\vee} = \bar{y} + \rho^2\eta_{t-1} + \rho\varepsilon_t + \varepsilon_{t+1}$$

$$y_{t+s} = \bar{y} + \rho^{s+1}\eta_{t-1} + \sum_{i=0}^{s} \rho^{s-i}\varepsilon_{t+i}$$

The only thing we know at t is ε_t!

$$E_t y_{t+s} = \bar{y} + \rho^{s+1}\eta_{t-1} + E_t \sum_{i=0}^{s} \rho^{s-i}\varepsilon_{t+i} =$$

$$= \bar{y} + \rho^{s+1}\eta_{t-1} + E_t \rho^s \varepsilon_t$$

$$= \bar{y} + \rho^{s+1}\eta_{t-1} + \rho^s \varepsilon_t$$

Hence $E_t W = E_t \sum_{s=0}^{\infty} \dfrac{y_{t+s}}{(1+r)^s} =$

$$= \sum_{s=0}^{\infty} \frac{E_t y_{t+s}}{(1+r)^s} =$$

with $E_t(y_{t+s}) = \bar{y} + \rho^s \varepsilon_t$

(assume $\eta_{t+1} = 0$)

$$E_t W = \sum_{s=0}^{\infty} \frac{\bar{y} + \rho^s \varepsilon_t}{(1+r)^s} =$$

$$= \frac{1+r}{r} \cdot \bar{y} + \varepsilon_t \cdot \sum_{s=0}^{\infty} \left(\frac{\rho}{1+r}\right)^s =$$

$$= \frac{1+r}{r} \bar{y} + \varepsilon_t \frac{1}{1 - \frac{\rho}{1+r}} =$$

$$= \frac{1+r}{r} \bar{y} + \frac{1+r}{1+r-\rho} \varepsilon_t$$

$$c_t = \frac{r}{1+r} \cdot E_t \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} = \frac{r}{1+r} \cdot \left\{ \frac{1+r}{r} \bar{y} + \frac{1+r}{1+r-\rho} \varepsilon_t \right\} =$$

$$= \bar{y} + \frac{r}{1+r-\rho} \cdot \varepsilon_t$$

$\rho = 1$, shock ε_t remains in $\{y_{t+s}\}_{s=0}^{\infty}$ permanently, so sole c_t up by the same amount:

$$\left.\frac{\partial c_t}{\partial \varepsilon_t}\right|_{\rho=1} = 1$$

$\rho = 0$, shock ε_t remains only for one period, so spread its effect over time according to annuity value

$$\left.\frac{\partial c_t}{\partial \varepsilon_t}\right|_{\rho=0} = \frac{r}{1+R}$$