JULLARD, PROBLEM SET 5 MICHELE PIFFER

$$V_t = u(c_{1t}, c_{2t+1}) + \theta V_{t+1}$$

In each period, young receive bequests from old

a) BC_t^{HH} $c_{1t} + a = y + B_t$

BC_{t+1}^{HH} $c_{2,t+1} + B_{t+1} = (1+r)a$

$\hookrightarrow a = \frac{c_{2,t+1}}{1+r} + \frac{B_{t+1}}{1+r}$

$$\Rightarrow c_{1,t} + \frac{c_{2,t+1}}{1+r} + \frac{B_{t+1}}{1+r} = y + B_t \qquad BC_t^{HH}$$

b) Maximization problem for the agent is

$$\max_{\{c_{1,t},\, c_{2,t+1},\, B_{t+1}\}} V_t = u(c_{1,t}, c_{2,t+1}) + \theta V_{t+1}$$

$$\text{s.t. } c_{1,t} + \frac{c_{2,t+1}}{1+r} + \frac{B_{t+1}}{1+r} = y + B_t$$

Rewrite utility function:

$$V_t = u(c_{1,t}, c_{2,t+1}) + \theta u(c_{1,t+1}, c_{2,t+2}) + \theta^2 V_{t+2} =$$

$$= \sum_{j=0}^{\infty} \theta^j u(c_{1,t+j}, c_{2,t+j+1}) +$$

$$+ \lim_{j\to\infty} \theta^j V_{t+j} \Big\} \text{ assume } = 0$$

Iterate Budget constraint:

$$c_{1,t} + \frac{c_{2,t+1}}{1+r} + \frac{B_{t+1}}{1+r} = y + B_t$$

$$c_{1,t+1} + \frac{c_{2,t+2}}{1+r} + \frac{B_{t+2}}{1+r} = y + B_{t+1}$$

$$\Rightarrow c_{1,t} + \frac{c_{1,t+1}}{1+r} + \frac{c_{2,t+1}}{1+r} + \frac{c_{2,t+2}}{(1+r)^2} + \frac{B_{t+2}}{(1+r)^2} =$$

$$= y + \frac{y}{1+r} + B_t$$

iterate, get

$$\sum_{j=0}^{N} \frac{c_{1,t+j}}{(1+r)^j} + \sum_{j=0}^{N} \frac{c_{2,t+j+1}}{(1+r)^{j+1}} + \frac{B_{t+N+1}}{(1+r)^{N+1}} =$$

$$= y \cdot \sum_{j=0}^{N} \frac{1}{(1+r)^j} + B_t$$

take $N \to \infty$

$$\sum_{j=0}^{\infty} \frac{c_{1,t+j}}{(1+r)^j} + \sum_{j=0}^{\infty} \frac{c_{2,t+j+1}}{(1+r)^{j+1}} + \underbrace{\lim_{N\to\infty} \frac{B_{t+N+1}}{(1+r)^{N+1}}}_{=0 \text{ under No Ponzi}} =$$

$$= \frac{1+r}{r} \cdot y + B_t$$

Hence, the old maximization problem is equivalent to

$$\max_{\{c_{1,t+j},\, c_{2,t+j+1}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \theta^j u(c_{1,t+j}, c_{2,t+j+1})$$

s.t. $$\sum_{j=0}^{\infty} \frac{c_{1,t+j}}{(1+r)^j} + \sum_{j=0}^{\infty} \frac{c_{2,t+j+1}}{(1+r)^{j+1}} = \frac{1+r}{r} y + B_t$$

c) Assume $u() = \frac{c_{1t}^{1-\sigma}}{1-\sigma} + \beta \cdot \frac{c_{2t+1}^{1-\sigma}}{1-\sigma}$ and solve for optimality

$$\mathcal{L} = \sum_{j=0}^{\infty} \theta^j \left[\frac{c_{1,t+j}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{2,t+j+1}^{1-\sigma}}{1-\sigma} \right] +$$

$$+ \lambda \left\{ \frac{1+r}{r} y + B_t - \sum_{j=0}^{\infty} \frac{c_{1,t+j}}{(1+r)^j} - \sum_{j=0}^{\infty} \frac{c_{2,t+j+1}}{(1+r)^{j+1}} \right\}$$

FOC $c_{1,t+j}$: $\theta^j c_{1,t+j}^{-\sigma} = \frac{\lambda}{(1+r)^j}$

will need

$j=0$: $c_{1,t}$ $\quad c_{1,t}^{-\sigma} = \lambda$

$j=1$: $c_{1,t+1}$ $\quad \theta c_{1,t+1}^{-\sigma} = \frac{\lambda}{1+r}$

FOC $c_{2,t+j+1}$: $\theta^j \beta c_{2,t+j+1}^{-\sigma} = \frac{\lambda}{(1+r)^{j+1}}$

will need:

$j=-1$: $c_{2,t}$ $\frac{\beta}{\theta} c_{2,t}^{-\sigma} = \lambda$

$j=0$: $c_{2,t+1}$ $\beta c_{2,t+1}^{-\sigma} = \frac{\lambda}{(1+r)}$

To sum up first order conditions

$$\begin{cases} \text{FOC } c_{1,t}: & c_{1,t}^{-\sigma} = \lambda \\ \text{FOC } c_{1,t+1}: & \theta c_{1,t+1}^{-\sigma} = \frac{\lambda}{1+r} \\ \text{FOC } c_{2,t}: & \frac{\beta}{\theta} c_{2,t}^{-\sigma} = \lambda \\ \text{FOC } c_{2,t+1}: & \beta c_{2,t+1}^{-\sigma} = \frac{\lambda}{1+r} \end{cases}$$

i) Combine FOC $c_{1,t}$ & FOC $c_{2,t+1}$

$$c_{1,t}^{-\sigma} = \beta(1+r)\, c_{2,t+1}^{-\sigma}$$

$$\frac{c_{2,t+1}}{c_{1,t}} = [\beta(1+r)]^{\frac{1}{\sigma}}$$

which is our classic Euler Equation: the optimal intertemporal allocation by the same agent out of his 2 periods of living is pinned down by β, r, σ. Smooth consumption

ii) Combine FOC $c_{1,t}$ & FOC $c_{2,t}$

$$c_{1,t}^{-\sigma} = \frac{\beta}{\theta} c_{2,t}^{-\sigma}$$

$$\frac{c_{2,t}}{c_{1,t}} = \left(\frac{\beta}{\theta}\right)^{\frac{1}{\sigma}}$$

This condition pins down optimal relation between consumption of young agent and old agent within period.

$\theta \uparrow \rightarrow$ the more old cares about future generation $\rightarrow c_{2,t} \downarrow \rightarrow$ consumes less and leaves higher bequests

$\beta \uparrow \rightarrow$ the more the old cares about him self in its second period of life (here t) ~~so~~ consumes more for himself and leaves less for bequests

$\sigma \uparrow \rightarrow$ EIS $\downarrow \rightarrow$ old one wants to postpone consumption less $\rightarrow$ consume more when young $(c_{1,t-1}) \rightarrow c_{2,t} \downarrow$

iii) Combine FOC $c_{1,t}$ & FOC $c_{1,t+1}$

$$c_{1,t}^{-\sigma} = \theta(1+r)\, c_{1,t+1}^{-\sigma}$$

$$\frac{c_{1,t+1}}{c_{1,t}} = \left[\theta(1+r)\right]^{\frac{1}{\sigma}}$$

Interpretation: do consumption smoothing across generations, not only along your life period.

links consumption when young ($c_{1,t}$) to consumption of next generation when young ($c_{1,t+1}$)

$r \uparrow \Rightarrow$ young at t wants to take advantage of higher interest rate in the interest of future generation in order to leave higher bequests $\Rightarrow c_{1,t} \downarrow$

$\theta \uparrow \Rightarrow$ care more about future generations $\Rightarrow c_{1,t} \downarrow$

$\sigma \Rightarrow$ (same as $r \uparrow$)

d) Define Steady State as a situation where young consume the same independently on when they leave, and old as well:

$$c_{1,t} = \bar{c}_1 \ , \forall t \qquad c_{2,t} = \bar{c}_2 \ , \forall t$$

From $\frac{c_{1,t+1}}{c_{1,t}} = [\theta(1+r)]^{\frac{1}{\sigma}}$ get

$$1+r = \theta^{-1}$$

that is, r is pinned down by the

parameter detecting your interest in future generations

(Note, in the classic case of no OLG you get

$$u'(c_t) = \beta(1+r)\, u'(c_{t+1}) \Big|_{SS}$$

$$\cancel{u'(c_{SS})} = \beta(1+r)\, \cancel{u'(c_{SS})}$$

$$(1+r) = \beta^{-1}$$

which is the condition required for keeping consumption constant.
)

In SS:

$$\bar{c}_1 \cdot \frac{1+r}{r} + \bar{c}_2 \cdot \underbrace{\frac{1+r}{r}} = \frac{1+r}{r}\, y + B_t$$

$$\sum_{j=0}^{\infty} \frac{1}{(1+r)^{j+1}} = \frac{1}{1+r} \cdot \sum_{j=0} \frac{1}{(1+r)^j} =$$

$$= \frac{1}{1+r} \cdot \frac{1+r}{r} = \frac{1}{r}$$

$$\bar{c}_1 \cdot \frac{1+r}{r} + \frac{\bar{c}_2}{r} = \frac{1+r}{r}\, y + B_t$$

$$(1+r)\,\bar{c}_1 + \bar{c}_2 = (1+r)\, y + r B_t$$

The optimality conditions become

$$\frac{c_{2,t+1}}{c_{1,t}} = [\beta(1+r)]^{\frac{1}{\sigma}} \Rightarrow \frac{\bar{c}_2}{\bar{c}_1} = [\beta(1+r)]^{\frac{1}{\sigma}}$$

$$\frac{c_{2t}}{c_{1t}} = \left(\frac{\beta}{\theta}\right)^{\frac{1}{\sigma}} \Rightarrow \frac{\bar{c}_2}{c_1} = \left(\frac{\beta}{\theta}\right)^{\frac{1}{\sigma}}$$

$$\frac{c_{1,t+1}}{c_{2,t}} = [\theta(1+r)]^{\frac{1}{\sigma}} \Rightarrow (1+r) = \theta^{-1}$$

Combine:

$$\frac{\bar{c}_2}{c_1} = \left(\frac{\beta}{\theta}\right)^{\frac{1}{\sigma}}$$ Euler eq. in Steady State

Combine, can get reduced-form solutions

e) From IBC^{HH}, lifetime income is $W = \frac{1+r}{r} y + B_t$

If taxation is not distortionary it doesn't enter the optimality conditions. If $\{g_t\}_{t=0}^{\infty}$ is constant, changing the timing for taxes won't affect levels of consumption. R.E. holds

AN EXTRA

How do we know for sure that the optimization from each individual gives the same solution from the intertemporal optimization? How do you know that $\{c_{1,t+j}, c_{2,t+j+1}\}_{j=0}^{\infty}$ that solves

$$\max_{\{c_{1,t+j}, c_{2,t+j+1}, B_{t+1+j}\}} u(c_{1,t+j}, c_{2,t+j+1}) + \theta V_{t+1}(B_{t+j+1})$$

$$\text{s.t.} \quad c_{1,t+j} + \frac{c_{2,t+j+1}}{1+r} + \frac{B_{t+j+1}}{1+r} = y + B_{t+j}$$

$$\underline{\forall j}$$

is the same as $\{c_{1,t+j}, c_{2,t+j+1}\}$ from

$$\max_{\{\}} \sum_{j=0}^{\infty} \theta^{j} u(c_{1,t+j}, c_{2,t+j+1})$$

$$\text{s.t.} \quad IBC \qquad ?$$

Solve the individual maximization problem and show that optimality conditions are the same:

$$\max_{\{c_{1,t}, c_{2,t+1}, B_{t+1}\}} \frac{c_{1,t}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{2,t+1}^{1-\sigma}}{1-\sigma} + \theta V_{t+1}(B_{t+1})$$

$$\text{s.t.} \quad c_{1,t} + \frac{c_{2,t+1}}{1+r} + \frac{B_{t+1}}{1+r} = y + B_t$$

$\mathcal{L} = \quad \ldots$

FOC $c_{1,t}$: $\quad c_{1,t}^{-\sigma} = \lambda$

FOC $c_{2,t+1}$: $\quad \beta c_{2,t+1}^{-\sigma} = \frac{\lambda}{1+r}$

FOC B_{t+1} $\quad \underbrace{\theta \cdot V'_{t+1}(B_{t+1})}_{\text{marginal benefit of leaving more bequests}} = \underbrace{\frac{\lambda}{1+r}}_{\text{marginal cost}}$

Note, $V'_{t+1}(B_{t+1})$ coincides with the marginal benefit of relaxing the constraint on the next generation, i.e, the multiplier of the maximization at $t+1$;

$\tilde{\lambda}$, which from FOC will be

$$\tilde{\lambda} = c_{1,t+1}^{-\sigma}$$

$$\tilde{\lambda} = (1+r)\, c_{2,t+2}^{-\sigma}$$

Combine those and get

$$c_{1,t}^{-\sigma} = (1+r) \cdot c_{2,t+1}^{-\sigma}$$

$$\Rightarrow \frac{c_{2,t+1}}{c_{1,t}} = \left[\beta(1+r)\right]^{\frac{1}{\sigma}}$$

which is the same condition as derived from i)

$\theta \cdot \tilde{\lambda} = \frac{\lambda}{1+r}$

$\theta \, c_{1,t+1}^{-\sigma} = \frac{c_{1,t}^{-\sigma}}{1+r}$

$$\frac{c_{1,t+1}}{c_{1,t}} = [\theta (1+r)]^{\frac{1}{\sigma}}$$

same as from iii)

Combine, get

$$\frac{c_{2,t+1}}{[\beta(1+r)]^{\frac{1}{\sigma}}} = \frac{c_{1,t+1}}{[\theta(1+r)]^{\frac{1}{\sigma}}}$$

$$\frac{c_{2,t+1}}{c_{1,t+1}} = \left(\frac{\beta}{\theta}\right)^{\frac{1}{\sigma}} = \frac{c_{2,t}}{c_{1,t}}$$

Same as iii)

Hence, it's the same solution