MACROECONOMICS FOR MSc

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$$V_t = \max_{\{A_t, \{w_{it}\}_{i=1}^N\}_{t=0}^\infty} E_t \sum_{\tau=0}^\infty \beta^\tau u(c_{t+\tau})$$

$$\text{s.t.} \quad c_t + A_t = W_t$$

$$W_{t+1} = \left[R^f + \sum_{i=1}^N w_{it} (R_{it+1} - R^f) \right] A_t$$

Note, can't just maximize in c_t and A_t as usual, otherwise W_{t+1} undetermined

1) $$V(W_t) = \max_{\{A_t, \{w_{it}\}_{i=1}^N\}_{t=0}^\infty} E_t \sum_\tau \beta^\tau u(c_{t+\tau}) =$$

$$= \max_{A_t, \{w_{it}\}_{i=1}^N} u(c_t) + \beta E_t V(W_{t+1})$$

$$\text{s.t.} \quad c_t + A_t = W_t$$

$$W_{t+1} = \left[R^f + \sum_{i=1}^N w_{it} (R_{it+1} - R^f) \right] A_t$$

Substitute first constraint into $u()$, solve for optimality

$$V(W_t) = \max_{A_t, \{w_{it}\}_{i=1}^N} u(W_t - A_t) + \beta E_t V(W_{t+1})$$

$$\text{s.t.} \quad W_{t+1} = \ldots$$

FOC A_t: $-u'(C_t) + \beta E_t V'(W_{t+1}) \cdot \frac{\partial W_{t+1}}{\partial A_t} = 0$

$$u'(C_t) = \beta \cdot E_t \left\{ V'(W_{t+1}) \cdot \left[R^f + \sum_{i=1}^{N} w_{it} (R_{it+1} - R_f) \right] \right\}$$

FOC w_{it}: $\beta E_t V'(W_{t+1}) \cdot (R_{it+1} - R^f) A_t = 0$

$$\cancel{\beta \cdot A_t} \; E_t \left\{ V'(W_{t+1}) (R_{it+1} - R^f) \right\} = 0$$

$$E_t \left\{ V'(W_{t+1}) (R_{it+1} - R^f) \right\} = 0 \qquad \forall i$$

~~FOC~~ ET: $V'(W_t) = u'(C_t) \cdot \frac{\partial C_t}{\partial W_t} = u'(C_t)$

Substitute, get

$$u'(C_t) = \beta \cdot E_t \left\{ \left[R^f + \sum_{i=1}^{N} w_{it} (R_{it+1} - R_f) \right] \cdot u'(C_{t+1}) \right\} \qquad *$$

$$E_t \left\{ u'(C_{t+1}) (R_{it+1} - R^f) \right\} = 0$$

$\hookrightarrow \quad E_t \left\{ u'(C_{t+1}) R_{it+1} \right\} = E_t \left\{ u'(C_{t+1}) \right\} \cdot R_f \qquad \forall i$

hence, an optimum is indifferent
and hence does not reallocate its savings
across bonds.

Plug with Euler equation (*), get

$$u'(C_t) = \beta \cdot E_t \left\{ \left[\left(1 - \sum_{i=1}^{n} \omega_{it}\right) R_f + \sum_{i=1}^{n} \omega_{it} R_{it+1} \right] u'(C_{t+1}) \right\}$$

$$= \beta \cdot \left\{ R_f E_t u'(C_{t+1}) + \sum_{i=1}^{n} \omega_{it} \underbrace{E_t \left[R_{it+1} u'(C_{t+1}) - R_f u'(C_{t+1}) \right]}_{=0,\ \forall i} \right\}$$

$$u'(C_t) = \beta \cdot E_t [u'(C_{t+1})] \cdot R_f$$

$$\frac{1}{R_f} = \beta \cdot \frac{E_t [u'(C_{t+1})]}{u'(C_t)}$$

Hence, Euler equation is

$$u'(C_t) = \beta \cdot R_f \cdot E_t [u'(C_{t+1})] = \beta E_t [u'(C_{t+1}) R_{it+1}] \quad \forall i$$

3) Given $\begin{cases} u'(C_t) = \beta E_t [u'(C_{t+1}) R_{it+1}] \\ u'(C_t) = \beta E_t [u'(C_{t+1}) R_f] \end{cases}$

Subtract, get

$$0 = \beta E_t [u'(C_{t+1}) (R_{it+1} - R_f)]$$

divide both sides by $u'(C_t)$

$$0 = \beta E_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \cdot (R_{it+1} - R_f) \right]$$

$$0 = \beta \cdot \left\{ E_t \frac{u'(C_{t+1})}{u'(C_t)} \cdot E_t (R_{it+1} - R_f) + Cov \left(\frac{u'(C_{t+1})}{u'(C_t)}, R_{it+1} - R_f \right) \right\}$$

Which gives $E[R_{i,t+1} - R_f] = -\dfrac{Cov\left(\frac{u'(C_{t+1})}{u'(C_t)}, R_{i,t+1} - R_f\right)}{E\left(\frac{u'(C_{t+1})}{u'(C_t)}\right)}$

$\dfrac{u'(C_{t+1})}{u'(C_t)}$ is inversely related to consumption growth. $Cov(\,,\,) > 0$ means that the stochastic return $R_{i,t+1}$ is negatively related to consumption growth, hence the asset will pay a lot in the states of the world where our consumption ~~is~~ is low. This bond works well as insurance, hence we will be ~~wi~~ willing to pay a higher price for this bond. Since the price of a bond is the discounted value of its returns, this bond will have a lower return. That's what the above equation prescribes.

4) $\dfrac{u'(C_{t+1})}{u'(C_t)} = \dfrac{C_t}{C_{t+1}} = e^{\log C_t / C_{t+1}} = e^{-\overbrace{\Delta \log(C_{t+1})}^{\text{consumption growth}}}$

$E_t \dfrac{u'(C_{t+1})}{u'(C_t)} = \dfrac{1}{R_f \cdot \beta}$, substitute, get

$$E[R_{i,t+1} - R_f] = -\frac{Cov(e^{-\Delta \log C_{t+1}}, R_{i,t+1}) + \overbrace{Cov(e^{-\Delta \log C_{t+1}}, R_f)}^{= 0 \text{ since } R_f \text{ constant}}}{1/R_f \beta}$$

$$= -\frac{\beta\, Cov(e^{-\Delta \log C_{t+1}}, R_{i,t+1})}{1/R_f}$$

$E[R_{i,t+1}] = R_f +$ Risk Premium

with Risk premium $= -\dfrac{\beta \, Cov\left(e^{-s \Delta \log C_{t+1}}, R_{i,t+1}\right)}{1/R_f} =$

$= f\left(\underset{-}{\beta}, \underset{-}{R_f}, \underset{+}{Cov(\Delta \log C_{t+1}, R_{i,t+1})}\right)$

$Cov(\Delta \log C_{t+1}, R_{t+1}) \uparrow \rightarrow$ a bad asset since works poorly as insurance $\rightarrow$ to have incentive to hold it need high return (or low price)

$\beta \uparrow \rightarrow$ HH cares more about the future, so firm doesn't have to offer a high premium in order to convince HH to postpone consumption

$R_f \uparrow \rightarrow E[R_{i,t+1}]$ already increases, so don't need to provide a high risk premium