MACROECONOMICS FOR MSc

PROBLEM SET 7, JULLIARD MICHELE PIFFER

$$c_t = A \cdot e^{\mu \cdot t} \cdot e^{-\frac{1}{2}\sigma^2} \cdot \varepsilon_t$$

grows as t increases but is subject to shock ε_t:

$\log(\varepsilon_t) \sim N(0, \sigma^2)$, so $\varepsilon_t \sim$ log Normal

In general, $\log A \sim N(B, C)$ $E(A) = e^{B + \frac{C}{2}}$, so here

$E(\varepsilon_t) = e^{0 + \frac{\sigma^2}{2}} = e^{\frac{\sigma^2}{2}}$. Hence

$$E(c_t) = E\left[A \cdot e^{\mu \cdot t} e^{-\frac{1}{2}\sigma^2} \cdot \varepsilon_t\right] =$$

$$= A \cdot e^{\mu \cdot t} e^{-\frac{1}{2}\sigma^2} \cdot E(\varepsilon_t) = A e^{\mu t} e^{-\frac{1}{2}\sigma^2} e^{\frac{1}{2}\sigma^2} =$$

$$= A \cdot e^{\mu \cdot t}$$

Given lifetime utility $E_0\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \frac{c_t^{1-\gamma}}{1-\gamma}\right]$, compute λ st.

$$E_0\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \frac{[(1+\lambda)c_t]^{1-\gamma}}{1-\gamma}\right] = \sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t \frac{\left[A e^{\mu \cdot t}\right]^{1-\gamma}}{1-\gamma}$$

that is, compute λ that gives certainty equivalence

Multiply both sides by $1-\gamma$, get rid of the denominator

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RHS is $\sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \; A^{1-\gamma} \cdot e^{(1-\gamma)\mu \cdot t} =$

$= A^{1-\gamma} \cdot \sum_{t=0}^{\infty} \left(\frac{e^{(1-\gamma)\mu}}{1+\rho}\right)^t = A^{1-\gamma} \frac{1}{1-\frac{e^{(1-\gamma)\mu}}{1+\rho}} =$

$= A^{1-\gamma} \frac{1+\rho}{1+\rho - e^{(1-\gamma)\mu}}$

RHS is $\sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t (1+\lambda)^{1-\gamma} \cdot E(c_t^{1-\gamma}) = \ldots$

of ~~course~~ course, given Jensen inequality
$E(c_t^{1-\gamma}) \neq [E(c_t)]^{1-\gamma}$.

$= (1+\lambda)^{1-\gamma} \cdot \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \cdot E(c_t^{1-\gamma})$

Note, $c_t = A \cdot e^{\mu \cdot t} e^{-\frac{1}{2}\sigma^2} \cdot \varepsilon_t$; ~~so~~ ε_t log normal,
so c_t log normal

$c_t^{1-\gamma} = \left[A e^{\mu t} e^{-\frac{1}{2}\sigma^2}\right]^{1-\gamma} \cdot \varepsilon_t^{1-\gamma}$

given $\log(\varepsilon_t)$ normal, $\log(\varepsilon_t^{1-\gamma}) = (1-\gamma)\log \varepsilon_t$
still normal, with mean 0 and variance $(1-\gamma)^2\sigma^2$.
Hence

$E(\varepsilon_t^{1-\gamma}) = e^{\frac{(1-\gamma)^2\sigma^2}{2}}$ since will be log normal

Substitute in, get

$$E(c_t^{1-\gamma}) = \left[A e^{-\frac{1}{2}\sigma^2}\right]^{1-\gamma} \cdot e^{(1-\gamma)\mu t} \cdot E(\varepsilon_t^{1-\gamma}) =$$

$$= \left[A e^{-\frac{1}{2}\sigma^2}\right]^{1-\gamma} \cdot e^{(1-\gamma)\mu t} \cdot e^{\frac{(1-\gamma)^2\sigma^2}{2}} =$$

$$= A^{1-\gamma} e^{(1-\gamma)\mu t} \cdot e^{\frac{(1-\gamma)^2\sigma^2}{2} - \frac{(1-\gamma)\sigma^2}{2}} =$$

$$= A^{1-\gamma} \cdot e^{(1-\gamma)\mu \cdot t} \cdot e^{-\frac{\gamma(1-\gamma)\sigma^2}{2}}$$

Hence, LHS is

$$(1+\lambda)^{1-\gamma} \cdot \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t A^{1-\gamma} e^{(1-\gamma)\mu \cdot t} e^{-\frac{\gamma(1-\gamma)\sigma^2}{2}} =$$

$$= \left[(1+\lambda) A \cdot e^{-\frac{\gamma\sigma^2}{2}}\right]^{1-\gamma} \cdot \sum_{t=0}^{\infty} \left(\frac{e^{(1-\gamma)\mu}}{1+\rho}\right)^t =$$

$$= \left[(1+\lambda) A e^{-\frac{\gamma\sigma^2}{2}}\right]^{1-\gamma} \cdot \frac{1+\rho}{1+\rho - e^{(1-\gamma)\mu}} =$$

$$= \left[(1+\lambda) e^{-\frac{\gamma\sigma^2}{2}}\right]^{1-\gamma} \cdot A^{1-\gamma} \cdot \frac{1+\rho}{1+\rho - e^{(1-\gamma)\mu}}$$

From LHS = RHS, get

$$\left[(1+\lambda) e^{-\frac{\gamma\sigma^2}{2}}\right]^{1-\gamma} = 1$$

$$(1+\lambda) e^{-\frac{\gamma\sigma^2}{2}} = 1$$

$$(1+\lambda) = e^{\frac{\gamma\sigma^2}{2}}$$

True result: $\lambda = e^{\frac{\gamma \sigma^2}{2}} - 1$

Approximated result: take log

$$\log(1+\lambda) = \frac{\gamma \sigma^2}{2} ;$$

but $\log(1+\lambda) \simeq \lambda$ if λ close to zero, so

$$\lambda \simeq \frac{\gamma \sigma^2}{2}$$

if $\gamma \in (\frac{1}{2}, 10)$ and $\sigma = 0.025$,

λ can be very small, so no big cost of business cycle fluctuations