PROBLEM SET 9, JULLIARD MICHELE PIFFER

The central bank solves

$$\max_{\{\pi_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(y_t - \frac{a}{2} \pi_t^2 \right)$$

$$\text{s.t.} \quad y_t = \bar{y} + b(\pi_t - \pi_t^e)$$

$$\pi_t^e = \begin{cases} \hat{\pi} & \text{if } \pi_{t-1} = \hat{\pi} \text{ , } \forall s \\ \frac{b}{a} & \text{otherwise} \end{cases} \qquad \hat{\pi} \text{ a parameter}$$

Tradeoff: $\pi \uparrow \to y_t \uparrow$, unless expected
Intuition goes through firm price setting
and cost of labour

a) What is equilibrium if $\pi \neq \hat{\pi}$ in any period before time t?

$$y_t = \bar{y} + b\left(\pi_t - \frac{b}{a}\right)$$

since π^e constant, no interaction between private sector and central bank. Static problem

$$\max_{\pi_t} \sum_{t=0}^{\infty} \beta^t \left\{ \bar{y} + b\left(\pi_t - \frac{b}{a}\right) - \frac{a}{2}\pi_t^2 \right\}$$

FOC π_t: $\beta^t \{ b - a\pi_t \} \overset{!}{=} 0$

SOC π_t: $-\beta^t a < 0$, ok

$$\pi_t^* = \frac{b}{a} = \pi^e \text{, so no surprise, so}$$

$$y_t = \bar{y}$$

if they expected inflation, you have to realise it

b) if π has always been at $\hat{\pi}$ till now, does CB have incentive to actually realize $\pi_t = \hat{\pi}$ or does he give inflation surprise?

<u>if commits to $\hat{\pi}$</u>, no maximization to solve for

$$w_t = \bar{y} + b(\underbrace{\pi_t - \hat{\pi}}_{\substack{\text{no surprise}\\ \text{under}\\ \text{commitment}}}) - \frac{a}{2}\pi_t^2 = \bar{y} - \frac{a}{2}\hat{\pi}^2$$

Suppose that the choice is at $t=0$. So

$$W^c = \sum_{t=0}^{\infty} \beta^t w_t = \sum_{t=0}^{\infty} \beta^t \left(\bar{y} - \frac{a}{2}\hat{\pi}^2\right) =$$

$$= \frac{1}{1-\beta}\left(\bar{y} - \frac{a}{2}\hat{\pi}^2\right)$$

<u>if deviates from $\hat{\pi}$ and solves for optimal π_t</u>

consider that he optimizes at time zero.

$$\pi_0^e = \hat{\pi}$$
$$\pi_t^e = \frac{b}{a}, \quad t \geq 1$$

$$\max_{\{\pi_t\}_{t=0}^{\infty}} \quad \bar{y} + b(\pi_0 - \hat{\pi}) - \frac{a}{2}\pi_0^2 + $$
$$+ \sum_{t=1}^{\infty} \beta^t \left[\bar{y} + b\left(\pi_t - \frac{b}{a}\right) - \frac{a}{2}\pi_t^2\right]$$

FOC π_0: $b - a\pi_0 = 0 \Rightarrow \pi_0^* = \frac{b}{a} \neq \hat{\pi}$

FOC π_t: $\beta^t [b - a\pi_t] = 0 \Rightarrow \pi^*_t = \frac{b}{a}$

so, it is optimal to surprise with inflation and then meet inflation expectations all the time. Is it welfare-improving?

$$w_0 = \bar{y} + b(\pi_0 - \pi^e) - \frac{a}{2}\pi_0^2 =$$

$$= \bar{y} + b\left(\frac{b}{a} - \hat{\pi}\right) - \frac{a}{2}\left(\frac{b}{a}\right)^2 =$$

BENEFIT OF SURPRISE

$$= \bar{y} + \frac{b^2}{2a} - b\hat{\pi} > \bar{y} - \frac{a}{2}\hat{\pi}^2 \text{ when not surprising}$$

$$w_t = \bar{y} + b\left(\pi_t - \frac{b}{a}\right) - \frac{a}{2}\left(\frac{b}{a}\right)^2 =$$

COST OF SURPRISE

$$= \bar{y} - \frac{b^2}{2a} < \bar{y} - \frac{a}{2}\hat{\pi}^2 \text{ when not surprising (under } \tfrac{b}{a} > \hat{\pi})$$

$$W^D = \bar{y} + \frac{b^2}{2a} - b\hat{\pi} + \sum_{t=1}^{\infty} \beta^t \left(\bar{y} - \frac{b^2}{2a}\right) =$$

$$= \bar{y} + \frac{b^2}{2a} - b\hat{\pi} + \left(\bar{y} - \frac{b^2}{2a}\right) \cdot \sum_{t=1}^{\infty} \beta^t =$$

$$= \bar{y} + \frac{b^2}{2a} - b\hat{\pi} + \left(\bar{y} - \frac{b^2}{2a}\right) \cdot \beta \cdot \sum_{t=0}^{\infty} \beta^t =$$

$$= \bar{y} + \frac{b^2}{2a} - b\hat{\pi} + \left(\bar{y} - \frac{b^2}{2a}\right) \cdot \frac{\beta}{1-\beta} =$$

$$= \frac{1}{1-\beta}\bar{y} - b\hat{\pi} + \frac{1-2\beta}{1-\beta} \frac{b^2}{2a}$$

c) CB chooses $\pi = \hat{\pi}$ if

$$W^c \geq W^0$$

$$\frac{1}{1-\beta}\left(\bar{y} - \frac{a}{2}\hat{\pi}^2\right) \geq \frac{1}{1-\beta}\bar{y} - b\hat{\pi} + \frac{1-2\beta}{1-\beta}\frac{b^2}{2a}$$

$$-\frac{a}{2(1-\beta)}\hat{\pi}^2 \geq -b\hat{\pi} + \frac{1-2\beta}{1-\beta}\frac{b^2}{2a}$$

$$-a^2\hat{\pi}^2 \geq -2ab(1-\beta)\hat{\pi} + (1-2\beta)b^2$$

$$-a^2\hat{\pi}^2 + 2ab(1-\beta)\hat{\pi} - (1-2\beta)b^2 \geq 0$$

$$a^2\hat{\pi}^2 - 2ab(1-\beta)\hat{\pi} + (1-2\beta)b^2 \leq 0$$

$$\pi = \frac{ab(1-\beta) \pm \sqrt{a^2b^2(1-\beta)^2 - a^2(1-2\beta)b^2}}{a^2} =$$

$$= \cancel{\frac{ab(1-\beta) \pm \sqrt{a^2b^2 - a^2b^2\beta^2 - a^2b^2 + 2a^2b^2\beta}}{a^2}}$$

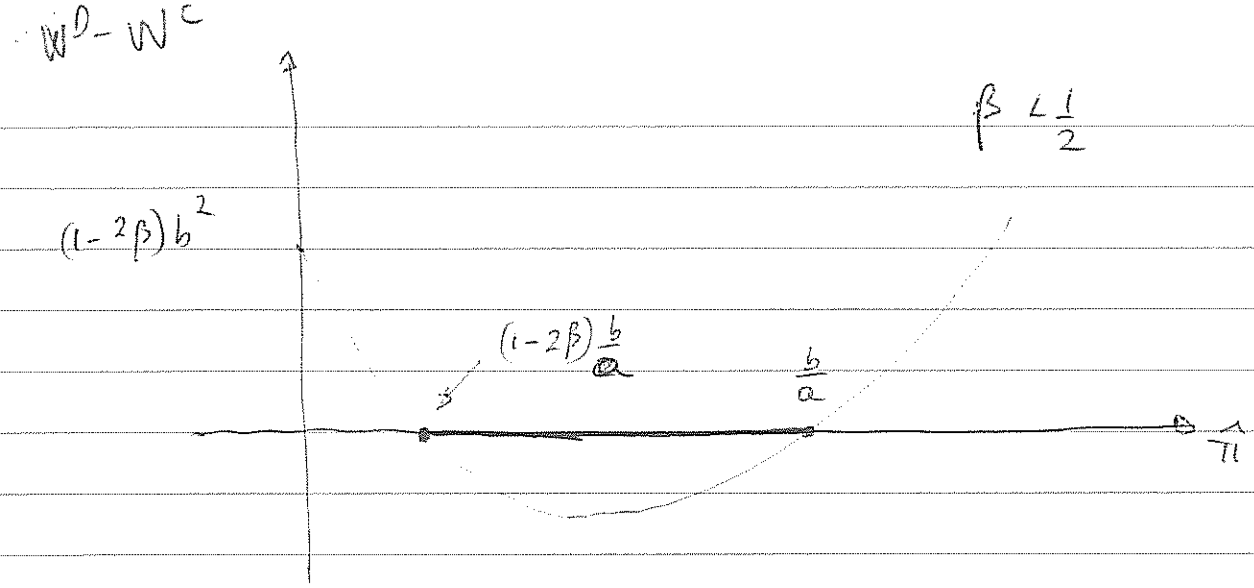
$$= \frac{ab(1-\beta) \pm \sqrt{\cancel{a^2b^2} + a^2b^2\beta^2 - \cancel{2a^2b^2\beta} - \cancel{a^2b^2} + \cancel{2a^2b^2\beta}}}{a^2}$$

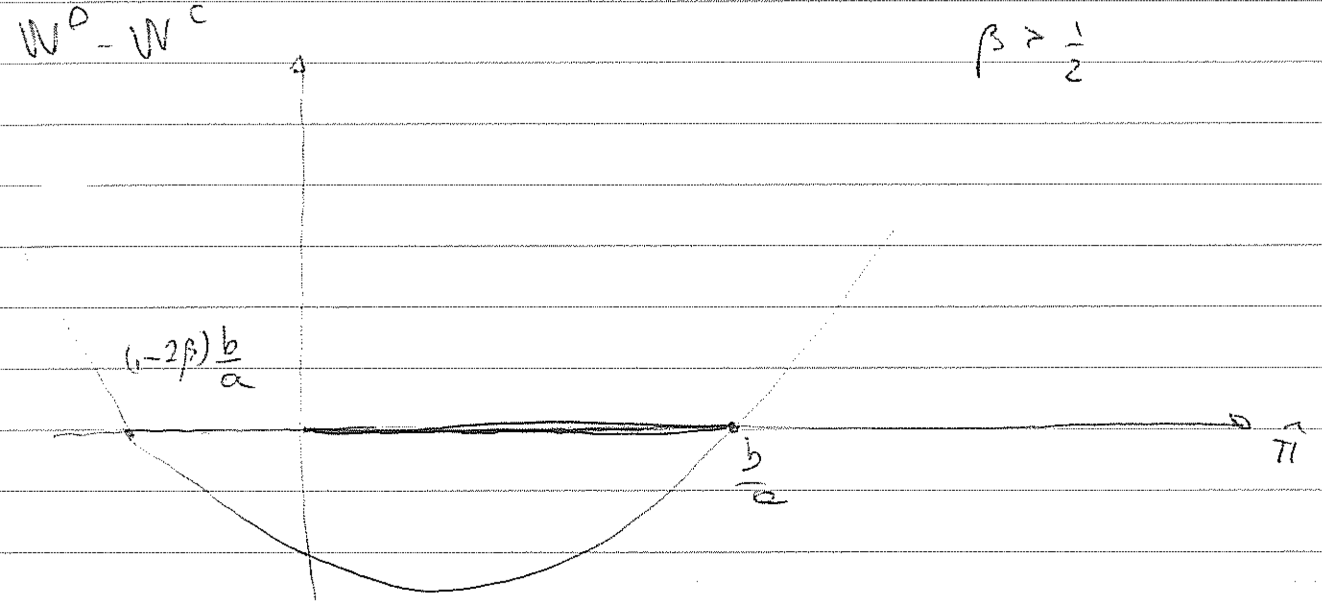
$$= \frac{ab(1-\beta) \pm ab\beta}{a^2} =$$

$$= \begin{cases} \dfrac{b}{a} \\[2ex] \dfrac{b(1-2\beta)}{a} < \dfrac{b}{a} \end{cases}$$

> 0 if $\beta < \frac{1}{2}$

< 0 if $\beta > \frac{1}{2}$





Remember, $\pi^* = \hat{\pi}$ iff $W_D - W_C(\hat{\pi}) < 0$

if $\beta > \frac{1}{2} \Rightarrow \hat{\pi} \le \frac{b}{a} \Rightarrow \pi^* = \hat{\pi}$

$\hat{\pi} > \frac{b}{a} \Rightarrow$ deviate $\pi^* = \frac{b}{a}$

if $\beta < \frac{1}{2} \Rightarrow (1-2\beta)\frac{b}{a} < \hat{\pi} < \frac{b}{a} \Rightarrow \pi^* = \hat{\pi}$

otherwise $\Rightarrow \pi^* = \frac{b}{a}$

Note, $\hat{\pi} = 0$ implies $\pi^* = \hat{\pi} = 0$ iff $(1-2\beta)\frac{b}{a} \le 0$, i.e., for

$b = 0$: [illegible] no benefit from surprise

$a \to \infty$: inflation is infinitely costly

$\beta > \frac{1}{2}$: [illegible]