Lecture 1- The constrained optimization problem

- The role of optimization in economic theory is important because we assume that individuals are rational.
- Why constrained optimization? the problem of scarcity.

Example:

ABC is a perfectly competitive, profit maximizing firm, producing y from input z according to $y = z^{.5}$. The price of output is 2, and of input is 1. Negative levels of z are impossible. Also, the firm cannot buy more than k units of input.

- Consider the following general problem:

$$\text{(COP)} \qquad \max \ \mathbf{g}(\mathbf{z}) \quad \textbf{s.t.} \ \mathbf{h}(\mathbf{z}) \leq k, \quad \mathbf{z} \in Z$$

where $g : Z \to R$, Z is a subset of R^n, $h : Z \to R^m$ and k is a fixed vector in R^m.

Let us denote this problem by COP.

DEFINITION: *z^* solves COP if $z^* \in Z$, $h(z^*) \leq k$, and for any other $z \in Z$ satisfying $h(z) \leq k$, we have that $g(z^*) \geq g(z)$.*

DEFINITION: *The feasible set is the set of vectors in R^n satisfying $z \in Z$ and $h(z) \leq k$.*

Some constraint maximization problems may have no solution. For example, consider:

maximize $\ln z$ subject to $z \leq 1$ and $z \geq 2$.
maximize $\ln z$ subject to $z \geq 1$.
maximize $\ln z$ subject to $1 \leq z < 2$.

- THEOREM *If the feasible set is non empty, closed and bounded (compact), and the objective function g is continuous on the feasible set then the COP has a solution.*

- Continuous functions map compact sets onto compact sets.
- If Z is closed and the constraint function $h(z)$ is continuous, the feasible set is closed. This is because it is an intersection of two closed sets.
- The conditions are sufficient and not necessary. For example x^2 has a minimum on R^2 even if it is not bounded.

Maximum value functions

Consider the COP, and then consider the set C :

$$C = \{(k, v) : k = h(z), v = g(z) \text{ for some } z \in Z\}.$$

- What is the interpretation of the set C?

- A graphical approach, using the example:

The formal COP is:

$$\text{(COP)} \qquad \max\ \mathbf{g}(\mathbf{z}) = \mathbf{2}z^{.5} - z$$
$$\textbf{s.t.}\ \ \mathbf{h}(\mathbf{z}) = \mathbf{z} \leq k$$
$$\mathbf{z} \in Z$$

where $Z = \{z : z \in R,\ z \geq 0\}$ is a Subset of R and k is a fixed scalar.

The set C is therefore

$$C = \{(k, v) : k = z, v = 2z^{.5} - z \text{ for some } z \geq 0\}.$$

This set if actually the graph of the function g, for non negative z. It is strictly increasing on $z < 1$ and reaches a maximum at $z = 1$ and then decreases.

If the constraint k varies, we can use it to realize how the maximum profits change. If $k = .25$, $z \leq .25$,and in this interval the maximum is achieved by .25. The point $(.25, .75)$ is indeed in the set C. In fact, for all $k < 1$, the problem is solved by setting $k = z$, giving rise to points $(k, 2k^{.5} - k)$. All these points are in C, so maybe C can tell us the maximum value v attainable with constraint k.

Now suppose that $k = 4$, so z is constrained by $z \leq 4$. From the figure, profits are maximized using $z = 1$. Thus, with $k = 4$,we have $(k, v) = (k, 1)$. But a part of $(1, 1)$, none of these points is in C. This implies that the set C may sometimes not show the maximum value v attainable with k.

Let us try a different approach. Define:

$$B = \{(k, v) : k \geq h(z), v \leq g(z) \text{ for some } z \in Z\}.$$

Why do we need the set B?

- Define the possible constraint set as the subset K of R^m for which the feasible set is non empty.

Let

$$s(k) = \sup\{g(z); z \in Z, h(z) \leq k\}$$

LEMMA *For any $k \in K$, (i) for all $v < s(k)$, $(k, v) \in B$ (ii) for all $v > s(k)$, $(k, v) \notin B$.*

Proof: If $v < s(k)$ he definition of sup implies that there is a $z \in Z$ with $v > g(z)$ and $h(z) \leq k$ which implies that $(k, v) \in B$. If $v > s(k)$ then for all $z \in Z$ with $h(z) \leq k$, $g(z) < s(k) < v$, which implies that $(k, v) \notin B$.

- Thus, the set B has an upper boundary. We do not know if the boundary itself is in B, this depends on whether the objective function does or does not attain a maximum.
- Note that $s(k)$ is a well defined function on the possible constraint set K, although it may take infinite values.

LEMMA *If $k_1 \in K$ and $k_1 \leq k_2$ then $k_2 \in K$ and $s(k_1) \leq s(k_2)$.*

Proof: If $k_1 \in K$ there exists a $z \in Z$ such that $h(z) \leq k_1 \leq k_2$ so $k_2 \leq K$. Now consider any $v < s(k_1)$. From the definition there exists a $z \in Z$ such that $v < g(z)$ and $h(z) \leq k_1 \leq k_2$, which implies that $s(k_2) = \sup\{g(z); z \in Z, h(z) \leq k_2\} > v$. Since this is true for all $v < s(k_1)$ it implies that $s(k_2) \geq s(k_1)$.

- The boundary defines a non decreasing function, and if the set B is closed, that is, it includes its boundary, this is the maximum value function, that is, for each value of k it shows the maximum attainable value of the objective function.

- We can confine ourselves to maximum rather than supremum. This is possible if the COP has a solution. To ensure a solution, we can either find it, or show that the objective function is continuous and the feasible set is compact.

DEFINITION (the maximum value function) *If* $z(k)$ *solves the COP with constraint* k, *the maximum value function is* $v(k) = g(z(k))$.

- The maximum value function, if exists, has all the properties of $s(k)$. In particular, it is non decreasing.
- So we have reached the conclusion that z^* is a solution to COP if and only if $(k, g(z^*))$ lies on the upper boundary of B.