Lecture 3- Concavity and convexity

Definition *A set U is a convex set if for all $x \in U$ and $y \in U$, then for all $t \in [0,1]$:*

$$t\mathbf{x} + (1-t)\mathbf{y} \in U$$

Definition *A real valued function f defined on a convex subset U of R^n is concave, if for all x, y in U and for all $t \in [0,1]$:*

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{y})$$

A real valued function g defined on a convex subset U of R^n is convex, if for all x, y in U and for all $t \in [0,1]$:

$$g(t\mathbf{x} + (1-t)\mathbf{y}) \leq tg(\mathbf{x}) + (1-t)g(\mathbf{y})$$

- Note: f is concave if and only if $-f$ is convex.
- Note: linear functions are convex and concave.
- Concave and convex functions need to have convex sets as their domain. Otherwise, we cannot use the conditions above.

Lemma *Let f be a continuous and differentiable function on a convex subset U of R^n. Then f is concave on U if and only if for all x, y in U :*

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) &\leq Df(\mathbf{x})(\mathbf{y} - \mathbf{x}) \\ &= \frac{\partial f(\mathbf{x})}{\partial x_1}(y_1 - x_1) + ... \\ &\quad + \frac{\partial f(\mathbf{x})}{\partial x_n}(y_n - x_n) \end{aligned}$$

Proof on R^1 : since f is concave, then

$$\begin{aligned} tf(y) + (1-t)f(x) &\leq f(ty + (1-t)x) \Leftrightarrow \\ t(f(y) - f(x)) + f(x) &\leq f(x + t(y-x)) \Leftrightarrow \\ f(y) - f(x) &\leq \frac{f(x + t(y-x)) - f(x)}{t} \Leftrightarrow \\ f(y) - f(x) &\leq \frac{f(x+h) - f(x)}{h}(y - x) \end{aligned}$$

for $h = t(y-x)$. Taking limits when $h \to 0$ this becomes

$$f(y) - f(x) \leq f'(x)(y-x).$$

- Why do we need concavity and convexity?
- We will make the following important assumptions, denoted by CC:
  1. The set Z is convex;
  2. The function g is concave;
  3. The function h is convex.

Recall the definition of the set B :

$$B = \{(k, v) : k \geq h(z), v \leq g(z) \text{ for some } z \in Z\}.$$

PROPOSITION *under CC, the set B is convex*

Proof: suppose that (k_1, v_1) and (k_2, v_2) are in B, so there exists z_1 and z_2 such that:

$$\begin{aligned} k_1 &\geq h(z_1)\ k_2 \geq h(z_2) \\ v_1 &\leq g(z_1)\ v_2 \leq g(z_2) \end{aligned}$$

By convexity of h :

$$\theta k_1 + (1-\theta)k_2 \geq \theta h(z_1) + (1-\theta)h(z_2) \geq h(\theta z_1 + (1-\theta)z_2)$$

and by concavity of g :

- $$\theta v_1 + (1-\theta)v_2 \leq \theta g(z_1) + (1-\theta)g(z_2) \leq g(\theta z_1 + (1-\theta)z_2)$$

thus, $(\theta k_1 + (1-\theta)k_2, \theta v_1 + (1-\theta)v_2) \in B$ for all $\theta \in [0,1]$, implying that B is convex.

PROPOSITION *assume that the maximum value exists, then under CC, the maximum value is a non decreasing, concave and continuous function of* k.

Proof: we have already shown, without assuming convexity or concavity, that the maximum value is non decreasing. Also we have shown that if the maximum value function $v(k)$ exists, it is the upper boundary of the set B. Above we proved that under CC the set B is convex. The set B can be re-written as:

$$B = \{(k,v) : v \in R, k \in K, v \leq v(k)\}.$$

But a set B is convex iff the function v is concave. Thus, v is concave, and concave functions are continuous, so v is continuous.

The Lagrangian Necessity Theorem

- Consider the COP.
- Assume CC.
- Assume that the constraint qualification holds, that is, there is a vector $z_0 \in Z$ such that $h(z_0) << k^*$.
- Finally suppose that z^* solves COP.

Then:

- (i) there is a vector $q \in R^m$ such that z^* maximizes the lagrangian $L(q, k^*, z) = g(z) + q[k^* - h(z^*)]$.

  (ii) the lagrange multiplier q is non negative for all components, $q \geq 0$.

  (iii) the vector z^* is feasible, that is $z \in Z$ and $h(z^*) \leq k^*$.

  (iv) the complementary slackness conditions are satisfied, that is: $q[k^* - h(z^*)] = 0$.