Problem of Time as a Hamiltonian shadow of the Hole Argument: Background to Gryb–Thébault

Cambridge–LSE Bootcamp

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Hole Argument: Einstein versus Hilbert

- Key point: if ψ is diffeomorphism (originally: coordinate transformation) of space-time M then $\mathrm{Ric}(\psi^* g) = \psi^* \mathrm{Ric}(g)$ so if g solves vacuum EE $\mathrm{Ric}(g) = 0$ then so does $\psi^* g$
- Einstein (1913): ψ nontrivial **inside** 4d "hole" H in M

$\Rightarrow$ boundary conditions **outside** H do not determine g **within** H

- Hilbert (1917): ψ nontrivial **outside** 4d (tubular) nbhd T of Cauchy surface $\Sigma \subset T \subset M$ (initial values EE given on Σ)

$\Rightarrow$ initial conditions **within** T (i.e. on Σ) do not determine g **outside** T, or: Cauchy problem for EE has no unique solution

- Einstein's rendition looks unnatural compared to Hilbert's but Einstein was inspired by Mach's Principle: "stars at infinity" should determine local inertia of matter (Stachel, 2014)
- Modern understanding of Cauchy problem for EE (Hilbert $\rightsquigarrow$ Darmois $\rightsquigarrow$ Lichnerowicz $\rightsquigarrow$ **Choquet-Bruhat** $\rightsquigarrow$ Geroch): EE are simultaneously **underdetermined** (Hole Argument) and **overdetermined** (initial values are constrained)

Geometric uniqueness theorem (C-B & Geroch, 1969):

- Correct initial value formulation for EE solves issue that EE $\mathrm{Ric}(g) = 0$ as PDEs for g cannot be posed on **given** $4d$ mfd M since M is typically constructed along with g, so:
- Initial data for EE are $(\Sigma, \tilde{g}, \tilde{k})$ where $(\Sigma, \tilde{g})$ is $3d$ Riemannian mfd equipped with extra covariant symmetric 2-tensor $\tilde{k}_{ij}$
- Solution of EE for such data is triple (M, g, ι), where
  (i): (M, g) is space-time whose metric g solves EE,
  (ii): map $\iota : \Sigma \hookrightarrow M$ is embedding, (iii): $\iota^* g = \tilde{g}$,
  iv): $\tilde{k}$ is extrinsic curvature of submanifold $\iota(\Sigma) \subset M$
- (M, g, ι) always globally hyperbolic with Cauchy surface $\iota(\Sigma)$ ($\Rightarrow M \cong \mathbb{R} \times \Sigma$) and M is foliated as $M = \cup_t \Sigma_t$ with $\Sigma_t \cong \Sigma$
- Maximal solution contains any other solution (up to isometry)
- Theorem: For each smooth initial data set $(\Sigma, \tilde{g}, \tilde{k})$ satisfying the constraints, EE have maximal solution (M, g, ι) which is **unique up to isometries fixing** $\iota(\Sigma) \subset M$: in other words, some solution (M', g', ι') is maximal iff there is an isometry $\psi : M \to M'$ such that $\psi^* g' = g$ and $\psi \circ \iota = \iota'$

Making the (maximal) solution unique

Goal is to single out solution (M, g, ι) within its equivalence class

- Covariant approach (C-B, 1952): give additional (covariant) equations for g like **wave gauge** $\hat{W}^\mu = g^{\rho\nu}(\hat{\Gamma}^\mu_{\rho\nu} - \Gamma^\mu_{\rho\nu}) = 0$ that uniquely fix solution g to EE (and make these hyperbolic)
- Non-covariant approach: fix (spacelike) foliation $M = \cup_t \Sigma_t$

$\Leftrightarrow$ fix lapse N and shift β ($\sim g_{00}$ and g_{0i}), write EE in $3+1$ form

$\Rightarrow$ Only (gauged) **spatial** EE $R_{ij} = 0$ (for given N and β) and initial-value constraints $G_{\mu 0} = 0$ need to be solved $\Rightarrow R_{\mu\nu} = 0$

- Fixing a foliation fixes the gauge and makes solution unique
- Connection with diffeomorphisms: foliation is $F : \mathbb{R} \times \Sigma \to M$; two such F_1, F_2 related by diffeo $\psi = F_2 \circ F_1^{-1} \Leftrightarrow F_2 = \psi \circ F_1$
- Suggestion: subjective choice of "now" ($= F$) fixes solution

  Freedom in choosing F is what makes GR truly "general"

Punch line: Hole Argument vs Problem of Time

- **Hole**: Any (covariant) gauge describes same physical situation (since different gauges give isometric solutions to EE)

$\Rightarrow$ Any two foliations F (being special cases of a gauge condition) describe same physical situation, including foliations that only differ monotonously in their labeling of t (???)

- **Time**: Moving up in time among the Σ_t is a special case of such a relabeling and hence is a gauge transformation
- Confirmed infinitesimally by Thiemann (c.s.), and globally by Fischer–Marsden (1979): "group" $\mathrm{Emb}(\Sigma, M, g)$ "acts" on initial data set $(\tilde{g}, \tilde{k})$ and pushes in gauge direction
- And yet every physicist takes "gauge" motion along the Σ_t to be real time development (FLRW, numerical relativity, ...)

Where is the mistake?

- Argument that shifts $(\Sigma_t, \tilde{g}_t, \tilde{k}_t) \mapsto (\Sigma_{t+s}, \tilde{g}_{t+s}, \tilde{k}_{t+s})$ are gauge transformations and hence are physically trivial is based on the fact that initial data $(\Sigma_t, \tilde{g}_t, \tilde{k}_t)$ and $(\Sigma_{t+s}, \tilde{g}_{t+s}, \tilde{k}_{t+s})$ produce isometric space-times (M, g) and hence define same point in reduced phase space $\{\text{solutions}\,(M,g)\,\text{of EE}\}/\text{Diff}(M)$
- So from block universe point of view these shifts are indeed physically trivial but for mortal comoving observer they are not

$\Rightarrow$ **Hole Argument** takes place entirely in block universe and seems innocent (solved by Weatherall-like manoeuvre with realization that (M, g) is not space-time but is a model of it)

- **Problem of Time** (though its Hamiltonian shadow) seems genuine issue about objective/subjective nature of time (and seems resolved *classically* by accepting the latter)