Reopening the Hole Argument*

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Abstract

The aim of this expository paper is to argue that although Weatherall (2018) and Halvorson & Manchak (2022) claim to 'close the Hole Argument', its philosophical thrust may be resurrected by rephrasing the argument in terms of the theorem of Choquet-Bruhat & Geroch (1969) on the existence and uniqueness of maximally globally hyperbolic solutions to the Einstein field equations for suitably posed initial data. This not only avoids the pointwise identification of manifolds at the base of non-isometric space-times (although it remains controversial if earlier versions of the Hole Argument actually need to do this), but has the additional advantage of being based on a precise notion of determinism intrinsic to GR (as an initial value problem). We also speculate that the Hole Argument corroborates van Fraassen's empiricist structuralism.

1 Introduction

Initially, the Hole Argument (*Lochbetrachtung*) was an episode in Einstein's struggle between 1913–1915 to find the gravitational field equations of general relativity. At a time when he was already unable to find generally covariant equations for the gravitational field (i.e. the metric) that had the correct Newtonian limit and satisfied energy-momentum conservation, the Hole Argument confirmed him in at least temporarily giving up the idea of general covariance (which he later recovered without ever mentioning the Hole Argument again). More generally, Einstein's invention of the argument formed part of his analysis of the interplay between general relativity (of motion), general covariance (of equations under coordinate transformations), and determinism. The more recent emphasis on substantivalism versus relationalism (Earman & Norton, 1987) is not Einstein's, but since for him this opposition was closely related to the problem of absolute versus relative motion and hence to general relativity, (Earman, 1989) he would certainly have been interested in it.[1]

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*If anything, this paper is a tribute to the weekly Cambridge–LSE *Philosophy of Physics Bootcamp*, in which the hole argument was often discussed. I am especially indebted to the organizers of this seminar, Jeremy Butterfield and Bryan Roberts, as well as to Henrique Gomes, Hans Halvorson, and JB Manchak, for comments on earlier drafts of this paper. I also wish to thank Michel Janssen for historical comments, most of which have found a way to the Introduction.

[1]See Janssen & Renn (2022) for the final reconstruction of Einstein's struggle, with §4.1 devoted to the Hole Argument. The earliest known reference to the Hole Argument is in a memo by Einstein's friend and colleague Besso dated August 1913, provided this dating is correct (Janssen, 2007). Einstein subsequently presented his argument four times in print; I just cite Einstein (1914)

In modernized form (using a global perspective and replacing Einstein's coordinate transformations by diffeomorphisms), his reasoning was essentially as follows:[2]

- Let (M, g) be a space-time.[3] The transformation behaviour of the Einstein tensor $\mathrm{Ein}(g)$ under diffeomorphisms ψ of the underlying manifold M is

$$\psi^*(\mathrm{Ein}(g)) = \mathrm{Ein}(\psi^* g). \tag{1}$$

  Similarly, for any healthy energy-momentum tensor $T(g, F)$ constructed from the metric g and the matter fields F that matter we should have

$$\psi^*(T(g, F)) = T(\psi^* g, \psi^* F). \tag{2}$$

  Consequently, if g satisfies the Einstein equations $\mathrm{Ein}(g) = 8\pi\, T(g, \varphi)$, then $\psi^* g$ satisfies these equations for the transformed matter fields $\psi^* F$.

- Now consider an open connected vacuum region H in space-time possibly surrounded by matter (i.e. $F = 0$ in H); H is referred to as a "hole", whence the name of the argument.[4] Furthermore, find a diffeomorphism ψ that is nontrivial inside H and equals the identity outside H, so that in particular,

$$T(\psi^* g, \psi^* F) = T(\psi^* g, F) = T(g, F), \tag{3}$$

  both outside H (where ψ is the identity) and inside H (where $T(g) = 0$).

- It follows from the previous two points that if g satisfies the Einstein equations for some given energy-momentum tensor T, then so does $\psi^* g$. Hence the space-times (M, g) and $(M, \psi^* g)$ both satisfy the Einstein equations for the same matter distribution and are identical outside H, but they differ inside the hole.

Einstein saw this as a proof that the matter distribution fails to determine the metric uniquely, and regarded this as such a severe challenge to determinism that, supported by the other problems he had at the time, he retracted general covariance.

as the paper containing his final version. See Stachel (2014), Norton (2019), and Pooley (2022), and references therein for reviews of the Hole Argument in both a historical and a modern context.

[2] We write the Einstein tensor as $\mathrm{Ein}(g)$, where its dependence on the metric g is explicitly denoted; in coordinates we have $\mathrm{Ein}(g)_{\mu\nu} = G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

[3] As usual we take a space-time to be a $4d$ connected Lorentzian manifold with time orientation (this nomenclature of course hides philosophical issues to be discussed later in this paper). More generally, my notations and conventions follow Landsman (2021) and are standard.

[4] Einstein's arrangement looks unnatural compared to Hilbert's (1917) reformulation as an initial-value problem in the PDE sense, see below, but Einstein was probably inspired by Mach's principle, where "the fixed stars at infinity" determine the local inertia of matter; see Maudlin (1990), Hofer (1994), and Stachel, 2014). There is another argument that actually favours Einstein's curious setting for the Hole Argument: the smaller the hole, i.e. the larger the complement of the hole, the greater the challenge to determinism, for if even things almost everywhere except in a tiny hole fail to determine things inside that hole, then we should really worry (Butterfield, 1989). This pull admittedly gets lost in the initial-value formulation of the argument below. See Muller (1995) for the explicit construction of a hole diffeomorphism (the only one I am aware of).

From a modern point of view the energy-momentum tensor is a red herring in the argument,[5] which may just as well be carried out *in vacuo*, as will be done from now on (this also strengthens my subsequent reformulation of the argument, since the theorem on which this is based is less well developed in the presence of matter).

Earman & Norton (1987) famously revived the Hole Argument. Streamlining:

1. Although (M, g) and $(M, \psi^* g)$ are different space-times (unless of course ψ is an isometry of (M, g), i.e. $\psi^* g = g$), physicists–usually tacitly–circumvent this alleged lack of determinism of GR by simply "identifying" the two, i.e. by claiming that (M, g) and $(M, \psi^* g)$ represent "the same physical situation".

2. In this practice they are encouraged by the trivial observation that (M, g) and $(M, \psi^* g)$ are *isometric*; indeed, the pertinent isometry is none other than ψ.[6]

3. *However*–and this is their key point–this spells doom for space-time substantivalists (like Newton), who (allegedly) should be worried that if in order to save determinism, $x \in M$, carrying the metric $\psi^* g(x)$, must be identified with $\psi(x) \in M$, carrying the same metric, then points have lost their "this-ness": they cannot be identified *as such*, but only as carriers of metric information.

Apart from its intended bearing on space-time substantivalism and determinism, the Hole Argument also has implications for the general philosophy of science, but either way, there can be no implications whatsoever if the argument is void, as claimed by Weatherall (2018) and his followers.[7] The goal of this paper is not to decide if these authors are *right*, but, in the light of the indisputable Cartesian fact that their arguments can and have been *doubted*,[8] to present a version of the Hole Argument that should be uncontroversial, while leading to the same philosophical issues.

[5]Continuing footnote 4: Janssen (2007), footnote 98, notes that Einstein formulated his requirement that the matter distribution fully determines the metric only in 1917; in 1913 Einstein still thought of Mach's principle in the light of the relativity of inertia. Furthermore, Einstein (1914) explicitly introduced the final version of the hole argument in terms of a conflict between general covariance and the "law of causality" ("*Kausalgesetz*"), which was contemporary parlance for determinism. In sum, it seems safe to say, with Janssen (2007), that the 'worries about determinism and causality that are behind Einstein's hole argument have strong Machian overtones.' See Norton (1993) for Einstein's general struggle with general covariance, and its aftermath.

[6]We say that $(M', g') \xrightarrow{\psi} (M, g)$, where $\psi : M' \to M$ is a diffeomorphism, is an *isometry* iff $g' = \psi^* g$ (in particular, following Hawking & Ellis, 1973, we always take an isometry to be a diffeomorphism). Now simply take $M' = M$ and $g' = \psi^* g$. See Weatherall (2018) and Halvorson & Manchak (2022) for the meaning of this for GR and for what the alternatives could (not) be.

[7]Such as Fletcher (2020), Halvorson & Manchak (2022), and Bradley & Weatherall (2021).

[8]See e.g. Arledge & Rynasiewicz (2019), Roberts (2020), Pooley & Read (2021), and Gomes (2021) for critiques. The main issue seems to be whether it is a valid move in GR (or more generally in Lorentzian geometry) to first take a (hole) diffeomorphism ψ of M and subsequently assume a metric g and construct the pullback $\psi^* g$. This is deemed invalid since in saying that the transformed metric $\psi^* g$ at $x \in M$ is such and such and comparing it with the original metric g at x, one identifies points x across models through the identity map $\mathrm{id}_M : M \to M$ in the category **Man** of (smooth) manifolds, instead of through a morphism in the category **Man** of Lorentzian manifolds (or preferably, in view of Theorem 2 below, in the category **ST** of space-times, see footnote 3, whose isomorphism are isometries preserving time orientation), of which it is claimed (or some would say shown) that it has to be an isometry. The only hope for this to

In some sense, propagated e.g. by Stachel (2014), the version of the Hole Argument to come goes back to Hilbert (1917), who gave the first analysis of GR from a PDE point of view.[9] The initial-value problem of Einstein's equations is very involved, but due to the efforts of especially the "French school", in direct lineage of doctoral descent consisting of Darmois, Lichnerowicz, and Choquet-Bruhat (whose early papers carry the name Fourès-Bruhat), the abstract situation is well understood now, at least *in vacuo* and for initial data given on a spacelike hypersurface.[10]

The culmination of the abstract PDE theory is a theorem due to Choquet-Bruhat & Geroch (1969), which we recall in §2. I see this theorem not so much as a way to delineate possible versions of determinism that are compatible with the Hole Argument, as in e.g. Butterfield (1987, 1989), but as the least vulnerable version of the argument itself. It is not only uncontroversial, but it also yields the sharpest formulation of the Hole Argument in so far as the underlying notion of determinism is concerned; namely existence and uniqueness up to isometry of the very specific geometric intial-value problem posed by the Einstein equations (which on the one hand is unique to GR but on the other hand is close to what one expects in classical mathematical physics). In particular, it basically poses the same conceptual questions as the original version(s). These questions will briefly be touched upon in the final section, which is much more speculative than the earlier parts of the paper.

be the case in the Hole Argument is that ψ be an isometry, which possibility in the hole situation is excluded by Theorem 1 in Halvorson & Manchak (2022). This hits the final nail in the coffin for the Hole Argument built by Weatherall (2018), although it seems to me that even without a theorem like that, i.e. if there were any possibility for a hole diffeomorphism ψ to be an isometry, the Hole Argument would be empty, since in that case $\psi^* g = g$ all across M and the dilemma of having both (M, g) and $(M, \psi^* g)$ as models with the same matter distribution or other initial data simply would not arise. Even accepting the claim that the only valid comparison maps in Lorentzian geometry are isometries (although in my experience one often uses different maps from those suggested by the *a priori* categorical structure), the Hole Argument actually uses neither id_M seen as a map from $(M, \psi^* g)$ to (M, g), nor ψ seen as a map from (M, g) to (M, g), both of which indeed fail to be isometries and hence admissible maps between Lorentzian manifolds (unless of course ψ is an isometry in the usual sense). Instead, as is clear from the main text the argument relies on ψ seen as a map from $(M, \psi^* g)$ to (M, g), which is surely an isometry, cf. footnote 6. The complaint that ψ should never haven been introduced in the first place seems feeble to me, since any space-time (M', g') is an object in the category **Lor** (or even **ST**), and surely one is free, for any given metric g on $M' = M$ and (time orientation preserving) diffeomorphism ψ of M, to construct the metric $g' = \psi^* g$. It simply exists as a Lorentzian metric on M and it should not matter where it comes from: one cannot have one's cake and eat it by insisting on the use of categories like **Lor** or **ST** and then banning certain objects from them–in fact any object is of the said kind and nothing would be left. However, this entire rebuttal, even if it turns out to be invalid, is only included here to show that the nullification arguments in Weatherall (2018) and Halvorson & Manchak (2022) *can be doubted*, which seems hardly the case for the Choquet-Bruhat–Geroch theorem reviewed in §2, and which leads to similar conclusions as the original Hole Arguments. If Weatherall c.s. turn out to be correct, I would see this as a consequence of the Hole Argument.

[9]Hilbert addresses the indeterminism of Einstein's equations, and also refers to Einstein (1914) in connection with this problem, but does not explicitly relate his analysis to the *Lochbetrachtung*.

[10]See Stachel (1992) and Choquet-Bruhat (2014) for some history, summarized in Landsman (2021), §1.9. It is in fact more popular nowadays to give initial data for the Einstein equations on a *null* hypersurface (Penrose, 1963). See Landsman (2021), §7.6, for a summary of the ideas, and e.g. Christodoulou & Klainerman (1993) and Klainerman & Nicolò (2003) for full treatments.

2 The Choquet-Bruhat–Geroch theorem

The initial-value approach to GR is based on PDE-theory and the following ideology:

- *All valid* ***assumptions*** *in* GR *are assumptions about initial data* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$.

Such an initial data triple, assumed smooth, is obtained by equipping some $3d$ Riemann manifold $(\tilde{\Sigma}, \tilde{g})$ with a second symmetric tensor $\tilde{k} \in \mathfrak{X}^{(2,0)}(\tilde{\Sigma})$, i.e. of the same "kind" as the 3-metric $\tilde{g}$, such that $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ satisfies the vacuum constraints

$$\tilde{R} - \mathrm{Tr}\,(\tilde{k}^2) + \mathrm{Tr}\,(\tilde{k})^2 = 0; \qquad\qquad \tilde{\nabla}_j \tilde{k}^j_i - \tilde{\nabla}_i \mathrm{Tr}\,(\tilde{k}) = 0. \tag{4}$$

Here $\tilde{R}$ is the Ricci scalar on $\tilde{\Sigma}$ for the Riemannian metric $\tilde{g}$ and likewise $\tilde{\nabla}$ is the unique Levi-Civita (i.e. metric) connection on $\tilde{\Sigma}$ determined by $\tilde{g}$ (so that $\tilde{\nabla}\tilde{g} = 0$).

- *All valid* ***questions*** *in* GR *are questions about "the"* MGHD (M, g, ι) *thereof.*

Among these questions, the one relevant to the Hole Argument concerns the uniqueness of (M, g, ι), whence the scare quotes around 'the'. Roughly speaking, a MGHD (for *maximal globally hyperbolic development*) of $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ is a maximal space-time (M, g) "generated" by these initial data via the Einstein equations, with "time slice" $\iota : \tilde{\Sigma} \hookrightarrow M$. In more detail,[11] A *Cauchy development* or *globally hyperbolic development* of given initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ satisfying the constraints (4) is a triple (M, g, ι), where (M, g) is a space-time that solves the vacuum Einstein equations $R_{\mu\nu} = 0$ and $\iota : \tilde{\Sigma} \hookrightarrow M$ is an injection making $\iota(\tilde{\Sigma})$ a spacelike Cauchy (hyper)surface in M such that g induces these initial data on $\iota(\tilde{\Sigma}) \cong \tilde{\Sigma}$, i.e. $\tilde{g} = \iota^* g$ is the metric and $\tilde{k}$ is the extrinsic curvature of $\tilde{\Sigma}$, induced by the embedding ι and the 4-metric g.[12] It follows that (M, g) is globally hyperbolic, since it has a Cauchy surface.[13]

This formulation of the (spatial) initial-value problem for the (vacuum) Einstein equations was an achievement by itself: in particular, it cleverly circumvents the vicious circle one ends up in by trying to find initial data for an already given space-time (solving the Einstein equations), for it is part of the problem to find the latter from the given initial data.[14] However, the main achievement concerns the existence and uniqueness of (M, g, ι), which (as in the far simpler case of ODEs) depends on a suitable notion of *maximality*, which is also non-trivial and tied to GR. Namely:

[11] See also the references in footnote 19, or Landsman (2021), §7.6. Tildes adorn $3d$ objects.

[12] Let N be the unique (necessarily timelike) future-directed normal vector field on $\iota(\tilde{\Sigma})$ such that $g_x(N_x, N_x) = -1$. Then $\tilde{k}(X, Y) = -g(\nabla_X N, Y)$ defines the extrinsic curvature of $\iota(\tilde{\Sigma})$.

[13] This procedure by no means excludes the study of non-globally hyperbolic space-times in GR, which in this approach emerge as possible extensions of globally hyperbolic space-times. This is closely connected to strong cosmic censorship (Penrose, 1979), which in turn is related to a kind of indeterminism in GR that is outside the scope of the Hole Argument and may occur even if we all agree that the MGHD of given initial data is essentially unique. See e.g. Earman (1995), Doboszewski (2017, 2020), Smeenk & Wütrich (2021), or Landsman (2021), Chapter 10, and references therein.

[14] Having said this, there is a close analogy between the initial-value problem for the Einstein equations and the so-called *fundamental theorem for hypersurfaces* of nineteenth-century mathematics, see e.g. Kobayashi & Nomizu (1969), §VII.7 or Landsman (2021), §4.8, especially in so far as the role of the constraints and the Gauss–Codazzi equations are concerned.

- A *maximal Cauchy development* or *maximal globally hyperbolic development*,[15] acronym MGHD, of given smooth initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$, satisfying the constraints (4), is a smooth Cauchy development (M, g, ι) with the property that for any other Cauchy development = globally hyperbolic development (M', g', ι') of these same data there exists an embedding $\psi : M' \to M$ that preserves time orientation, metric, and Cauchy surface, i.e., one has

$$\psi^* g = g'; \qquad\qquad \psi \circ \iota' = \iota. \tag{5}$$

The Hole Argument à la Hilbert (1917) then follows from the (almost trivial) observation that if (M, g, ι) is a MGHD of the initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ and $\psi : M' \to M$ is a diffeomorphism (*pace* Weatherall c.s.!), then the triple (M', g', ι'), where g' and ι' are defined by (5), i.e. $g' = \psi^* g$ and $\iota' = \psi^{-1} \circ \iota$, with time orientation induced by ψ,[16] is a MGHD of the initial data $(\tilde{g}', \tilde{k}')$ induced on $\tilde{\Sigma}$ via ι' and g'. In particular:[17]

Proposition 1. *Given some* MGHD (M, g, ι) *of the initial data* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$*, let* U *be a neighbourhood of* $\iota(\tilde{\Sigma})$ *in* M*. Take a (time orientation preserving) diffeomorphism* ψ *of* M *that is the identity on* U*, so that in particular* $\iota' = \iota$ *and* $(\tilde{g}', \tilde{k}') = (\tilde{g}, \tilde{k})$*.*

Then the "Hilbert-triple" $(M, \psi^* g, \iota)$ *is a* MGHD *of the same initial data* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$*.*

This is a decent version of the Hole Argument,[18] but since it starts from a diffeomorphism ψ of M that only becomes an isometry from $(M, \psi^* g)$ to (M, g) with hindsight, it may be equally vulnerable to the reasoning in Weatherall (2018), Fletcher (2020), Halvorson & Manchak (2022), etc. My claim is that this is not the case for the highly nontrivial *converse* of the reasoning preceding Proposition 1, which nonetheless poses the same philosophical problems as the original Hole Argument(s). This converse is the celebrated theorem of Choquet-Bruhat & Geroch (1969):[19]

[15] It might be thought that isometries enter surreptitiously via this definition of maximality, but this is not the case. The appearance of isometries is a consequence of a local version of Theorem 2: *Any two Cauchy developments* (M, g, ι) *and* (M', g', ι') *of the same (smooth) initial data are locally isometric, in that* $\iota(\tilde{\Sigma})$ *and* $\iota'(\tilde{\Sigma})$ *have open neighbourhoods* U *and* U' *in* M *and* M'*, respectively, such that* (U, g) *and* (U', g') *are isometric through a diffeomorphism* $\psi : U' \to U$ *satisfying* (5). See Choquet-Bruhat (2009), Theorem VI.8.4, or Ringström (2009), Theorem 14.3.

[16] Defining time orientation by (the equivalence class of) a global timelike vector field T on M, so that some causal vector X is future-directed iff $g(X, T) < 0$, this means that $T' = \psi_*^{-1} T$.

[17] This construction also works if $U = J^-(\iota(\tilde{\Sigma}))$, cf. Curiel (2018) and Pooley (2022).

[18] Continuing footnote 4, it is superior to Einstein's and Earman & Norton's formulation in that it has shaken off any implicit reference to Mach's principle and is closer to the usual initial value problem for hyperbolic PDEs (with a special GR twist though). But it may be weaker as a challenge to determinism in that the open set on which initial data are given can be made arbitrarily thin.

[19] The original source is Choquet-Bruhat & Geroch (1969), who merely sketched a proof (based on Zorn's lemma, which they even had to use twice). Even the 800-page textbook by Choquet-Bruhat (2009) does not contain a proof of the theorem (which is Theorem 12.2); the treatment in Hawking & Ellis (1973), §7.6, is slightly more detailed but far from complete, too. Ringström (2009) is a book-length exposition of the theorem, but ironically its proof in §14, is wrong; it is corrected in Ringström (2013), §23. A constructive proof was given by Sbierski (2016), which is streamlined and summarized in Landsman (2021), §7.6. Though never mentioned in statements of the theorem, the isometry ψ is unique. This can be shown by the (well-known) argument in

Theorem 2. *For each initial data triple* $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ *satisfying the constraints* (4) *there exists a* MGHD (M, g, ι). *This triple is unique up to time-orientation-preserving* <u>*isometries*</u> *fixing the Cauchy surface, i.e. for any other* MGHD (M', g', ι') *there exists an* <u>*isometry*</u> $\psi : M' \to M$ *that preserves time orientation and satisfies* $\psi \circ \iota' = \iota$.

All reference to diffeomorphisms that are not (yet) isometries has gone! And yet in a sense, this theorem *is* the Hole Argument, for it forces us to choose between:

1. *Determinism*, in the precise sense that the Einstein equations for given initial data have a unique solution in the sense that we agree that triples (M, g, ι) and (M', g', ι') as in the statement of the theorem are seen as different representatives of the same physical situation (i.e., are physically "identified").

2. *Space-time substantivalism* through *denial of Leibniz equivalence*, which obtains if triples (M, g, ι) and (M', g', ι') represent "distinct states of affairs" (although they seem observationally indistinguishable). This choice saves the "this-ness" of points at the cost of accepting some invisible indeterminism.

Or, at least, this is the dilemma Earman & Norton (1987), or even Einstein (1914), left us with on the basis of their own versions of the argument. Most philosophical discussions of this dilemma, including more precise formulations thereof (e.g. Butterfield, 1989; Pooley, 2022) or even dismissals (e.g. Curiel, 2018), remain relevant if we replace controversial earlier versions of the Hole Argument by Theorem 2. This is the sense in which the Hole Argument remains alive, which is all I wish to argue.

If we go for determinism, the specific version thereof in GR that seems enforced by Theorem 2 is that we must "physically" identify all maximal globally hyperbolic space-times (M, g, ι) with Cauchy surface $\iota(\tilde{\Sigma})$ that carry fixed (and *a priori* "timeless") initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$. Theorem 2 states that all different possibilities are isometric, and hence isometries (preserving the given Cauchy surfaces) "morally" play the role of gauge symmetries.[20] This is unsurprising, since as in the Hole Argument(s) the occurrence of isometries in Theorem 2 is a shadow of the diffeomorphism invariance of the Einstein equations. But is is also surprising, since the isometries of a fixed space-time (M, g) are not given by freely specifiable functions on M, as in the case of gauge theories.[21] To assess the situation, it may help to state the analogue of Theorem 2 for special relativity, seen as a generally covariant field theory à la vacuum GR, but this time with field equation $R_{\rho\sigma\mu\nu} = 0$ instead of $R_{\mu\nu} = 0$.

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footnote 639 of Landsman (2021), to the effect that an isometry ψ is determined at least locally (i.e. in a convex nbhd of x) by its tangent map ψ'_x at some fixed $x \in M'$. Take $x \in \iota'(\tilde{\Sigma})$. Since ψ in Theorem 2 is fixed all along $\iota'(\tilde{\Sigma})$ by the second condition in (5) and since it also fixes the (future-directed) normal N_x to $\iota'(\tilde{\Sigma})$ by the first condition in (5), it is determined locally. This means that Theorem 1 in Halvorson & Manchak (2022) applies, which is a rigidity theorem for isometries going back at least to Geroch (1969), Appendix A (as they acknowledge).

[20] See Gomes (2021) for a detailed analysis of the relationship between gauge symmetries in gauge theories and diffeomorphisms in GR. What follows was inspired by correspondence with Henrique Gomes and Hans Halvorson, who proposed to look at special relativity in this context.

[21] If $\dim(M) = n$, then for any semi-Riemannian metric g the isometry group of (M, g) is at most $\frac{1}{2}n(n+1)$-dimensional. See O'Neill (1983), Lemma 9.28; Kobayashi & Nomizu (1963), Theorem VI.3.3 does the Riemannian case. Thus the Poincaré-group in $n = 4$ has maximal dimension 10.

The initial value problem may then be posed in almost the same way as in GR; the only difference is that the initial data triple $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ now satisfes the constraints

$$\tilde{R}_{ijkl} - \tilde{k}_{il}\tilde{k}_{jk} + \tilde{k}_{ik}\tilde{k}_{jl}; \qquad\qquad \tilde{\nabla}_i \tilde{k}_{jk} - \tilde{\nabla}_j \tilde{k}_{ik} = 0. \tag{6}$$

The constraints (6) are stronger than (4), which follow from (6) by contracting with $\tilde{g}^{ik}\tilde{g}^{jl}$ and $\tilde{g}^{ik}$, respectively. The reason is that in GR one merely asks for an embedding of the initial data in a Lorentzian manifold (M, g) where $R_{\mu\nu} = 0$, whereas in special relativity (M, g) is (locally) flat, i.e. $R_{\rho\sigma\mu\nu} = 0$. To avoid irrelevant global topological issues (interesting as these might be in a different context), we assume that both $\tilde{\Sigma}$ and M are connected and simply connected. In that case maximality gives $M \cong \mathbb{R}^4$ as a manifold,[22] Instead of Theorem 2, we then obtain:[23]

Theorem 3. *For each initial data triple $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ satisfying the constraints* (6) *there exists an isometric embedding $\iota : \Sigma \to \mathbb{R}^4$ (where $\mathbb{R}^4 = \mathbb{M}$ is Minkowski space-time with metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$) whose extrinsic curvature is the given $\tilde{k}$. Such an embedding is unique up to time-orientation-preserving Poincaré transformations.*

There is a complete conceptual analogy between Theorems 2 and 3, except that the former refers to the initial-value problem in general relativity, whilst the latter states the situation in special relativity (albeit in a somewhat unusual way). In particular, the role of isometries in the general theory is now played by Poincaré transformations (i.e. isometries of the Minkowski metric), as was to be expected. And yet, whereas most physicists would be happy to regard isometries in general relativity as gauge symmetries akin to coordinate transformations, few if any would regard Poincaré transformations as gauge symmetries. But on this basis, they are.

It is also interesting to compare the notion of determinism in GR provided for free by Theorem 2 to some others that have been used in the literature on the Hole Argument. To facilitate this, here is a somewhat awkward weakening of Theorem 2:

Corollary 4. *If two globally hyperbolic space-times (M, g) and (M', g') contain Cauchy surfaces $\tilde{\Sigma} \subset M$ and $\tilde{\Sigma}' \subset M'$, respectively, which carry initial data $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ and $(\tilde{\Sigma}', \tilde{g}', \tilde{k}')$ induced by the 4-metrics g and g' on M and M', respectively, where both (M, g) and (M', g') are maximal for these initial data, and there is a 3-diffeomorphism $\alpha : \tilde{\Sigma} \to \tilde{\Sigma}'$ such that $\tilde{g} = \alpha^*\tilde{g}'$ and $\tilde{k} = \alpha^*\tilde{k}'$, then there exists an isometry $\psi : M' \to M$ that preserves time orientation and reduces to α on $\tilde{\Sigma}$.*

[22] Maximality of Minkowski space-time $(\mathbb{M}, \eta)$ follows from its inextendibility; see e.g. Corollary 13.37 in O'Neill (1983) for the smooth case and Sbierski (2018ab) for inextendibility even in C^0.

[23] This is a Minkowskian version of the fundamental theorem for hypersurfaces, see e.g. Kobayashi & Nomizu (1969), §VII.7, or Landsman (2021), Theorem 4.18. The proof is the same, up to some sign changes: in the Euclidean case the first constraint in (6) is $\tilde{R}_{ijkl} + \tilde{k}_{il}\tilde{k}_{jk} - \tilde{k}_{ik}\tilde{k}_{jl}$, the sign changes going back to the different signs in the Gauss–Codazzi equations in Euclidean and Lorentzian signature, see e.g. eqs. (4.147) - (4.148) in §4.7 in Landsman (2021). These sign changes do lead to spectacularly different possibilities. For example, in one dimension lower, Hilbert (1901) proved that it is impossible to isometrically embed two-dimensional hyperbolic space (H^2, g_H) in Euclidean $\mathbb{R}^3$. But it can be isometrically embedded in $\mathbb{R}^3$ with Minkowski metric, cf. e.g. Landsman (2021), §4.4. Hence given (H^2, g_H), a symmetric tensor $\tilde{k}$ such that $(g_H, \tilde{k})$ satisfy the Euclidean constraint do not exist, but such a $\tilde{k}$ can be found satisfying the Minkowski constraints.

Recall that by convention an isometry is always a diffeomorphism. This corollary is weaker than Theorem 2, for it lacks the existence claim of (M, g) and (M', g'), which are now taken as given. We mention this corollary because it relates to an influential notion **Dm2** of determinism introduced in the context by Butterfield (1987, 1989).[24]

Indeed, if we specialize a corrected version of **Dm2** to globally hyperbolic solutions to the vacuum Einstein equations, it is clear that Butterfield takes Corollary 4 to be his *definition* of determinism in GR, allowing his claim that GR is deterministic.[25] However, apart from being amenable to correction if applied to GR, this version of determinism is unnecessarily weak since it already *assumes* the existence of the space-times in question; part of the thrust of Theorem 2 is an existence proof.

The same is true for *Property R* introduced by Halvorson & Manchak (2022);[26] This definition is *compatible* with GR because of the uniqueness of the isomorphism ψ in Theorem 2 (see footnote 19), but like Butterfield's definition **Dm2** Property R *assumes* the existence of models and then makes some uniqueness claim about it, whereas the Choquet-Bruhat–Geroch theorem *proves* their existence. In both cases the difference is between assuming that the future exists and showing it is "essentially" unique for given initial data, and also proving that it exists. In fact, none of the abstract definitions of determinism proposed in the literature does this.

[24]We quote *verbatim*: 'A theory with models $\langle M, O_i \rangle$ is **S**-deterministic, where **S** is a kind of region that occurs in manifolds of the kind occurring in the models, iff: given any two models $\langle M, O_i \rangle$ and $\langle M', O'_i \rangle$ containing regions S and S' of kind **S**, respectively, and any diffeomorphism α from S onto S': if $\alpha^*(O'_i) = \alpha(O_i)$ on $\alpha(S) = S'$, then: there is an isomorphism β from M onto M' that sends S to S', *i.e.* $\beta^* O'_i = O_i$ throughout M' and $\beta(S) = S'$.' (Butterfield, 1987, p. 29; 1989, p. 9). However, though inspired by GR this definition is too weak to cover it and needs to be amended by firstly adding the extrinsic curvature to the initial data induced on S and S' (which cannot be done by taking it to be one of the space-time tensors O_i, since in vacuum GR one just has $O_1 = g$), and secondly by adding a maximality condition on M and M', as in Corollary 4. Unless both of these are added, GR is not even deterministic in the sense expressed by **Dm2**.

[25]Butterfield (1987, 1989) contrasts **Dm2** with a Laplacian kind of definition of determinism **Dm1** he attributes to Montague and Earman: 'A theory with models $\langle M, O_i \rangle$ is **S**-deterministic, where **S** is a kind of region that occurs in manifolds of the kind occurring in the models, iff: given any two models $\langle M, O_i \rangle$ and $\langle M', O'_i \rangle$ and any diffeomorphism β from M onto M', and any region S of M of kind **S**: if $\beta(S)$ is of kind **S** and also $\beta^* O'_i = O_i$ on $\beta(S)$, then: $\beta^* O'_i = O_i$ throughout M.' If we correct this similarly to **Dm2**, Butterfield's point still stands: the Hole Argument (in any version) shows that GR violates **Dm1**. See also Pooley (2022) for a detailed analysis of similar definitions. Pooley's version of **Dm2** is a bit more general and also applies to GR: 'Theory T is deterministic just in case, for any worlds W and W' that are possible according to T, if the past of W up to some timeslice in W is qualitatively identical to the past of W up to some timeslice in W', then W and W' are qualitatively identical.' Apart from my complaint that also this definition assumes the existence of W and W' instead of proving it, a definition like this of course begs the question what is meant by 'qualitative'. This question is answered by Theorem 2; see below.

[26]Again specializing to maximal globally hyperbolic space-times solving the vacuum Einstein equations, Property R states that if (M, g) and (M', g') are two such space-times, and if two time-orientation preserving isometries $\psi : M' \to M$ and $\varphi : M' \to M$ coincide on the causal pasts $J^-(\tilde{\Sigma})$ of some Cauchy surface $\tilde{\Sigma} \subset M$, then $\psi = \varphi$ altogether. As for Butterfield, a major goal of introducing such a definition is to contrast it with some Laplacian definition of determinism (this time attributed to Montague, Lewis, and Earman), which is violated in GR because of the Hole Argument. Halvorson & Manchak (2022) take this Laplacian definition to be: if, in the situation in the main text, there is some Cauchy surface $\tilde{\Sigma} \subset M$ that also lies in M' as a Cauchy surface, such that $J^-(\tilde{\Sigma}) \subset M$ coincides with $J^-(\tilde{\Sigma}) \subset M'$, then $(M, g) = (M', g')$.

3 Resolving the Hole Argument

Despite their denial of the Hole Argument, Weatherall (2018) and Halvorson & Manchak (2022) make some of the most useful comments towards its resolution:

> Mathematical models of a physical theory are only defined up to isomorphism, where the standard of isomorphism is given by the mathematical theory of whatever mathematical objects the theory takes as its models. One consequence of this view is that isomorphic mathematical models in physics should be taken to have the same representational capacities. By this I mean that if a particular mathematical model may be used to represent a given physical situation, then any isomorphic model may be used to represent that situation equally well. Note that this does not commit me to the view that equivalence classes of isomorphic models are somehow in one-to-one correspondence with distinct physical situations. But it does imply that if two isomorphic models may be used to represent two distinct physical situations, then each of those models individually may be used to represent both situations.
>
> (Weatherall, 2018, pp. 331–332)

> Why is it, then, that there has been, and will surely continue to be, a feeling that there is some remaining open question about whether general relativity is fully deterministic? Our conjecture is that the worry here arises from the fact that general relativity, just like any other theory of contemporary mathematical physics, allows its user a degree of representational freedom, and consequently displays a kind of *trivial semantic indeterminism*: how things are represented at one time does not constrain how things must be represented at later times. (Halvorson & Manchak, 2022, p. 19)

These comments could just as well have been made about Theorem 2, which by itself makes it worth delving into the idea of "representational freedom".[27] Similarly, although it wasn't, Theorem 2 might have been invoked as an argument in favour of van Fraassen's empiricist structuralism, which he summarizes as follows:

> 1. Science represents the empirical phenomena as embeddable in certain *abstract structures* (theoretical models).
> 2. Those abstract structures are describable only up to structural isomorphism.
>
> (...) How can we answer the question of how a theory or model relates to the phenomena by pointing to a relation between theoretical and data models, both of them abstract entities? The answer has to be that the data model represent the phenomena; but why does that not just push the problem [namely: *what is the relation between the data and the phenomena it models*] one step back? The short answer is this:
>
> > construction of a data model is precisely the selective relevant depiction of the phenomena *by the user of the theory* required for the possibility of representation of the phenomenon.
> >
> > (van Fraassen, 2008, pp. 238, 253)

[27] See also Pooley (2022) and references therein, as well as Fletcher (2020) and Gomes (2021).

This last comment (which made a deep impression on me) seems decisive to me in explaining the actual practice of mathematical and theoretical physicists working in GR. Despite the unfortunate terminology, no sane person will maintain that a mathematical object like (M, g) really "is" a space-time, or that space-time (now meant in the sense of the world around us) really "is" a $4d$ Lorentzian manifold. It even seems unsound to me to say that space-time is *modeled* by some $4d$ Lorentzian manifold, since that would leave us with the Hole Argument, or even by an equivalence class thereof, which strictly speaking is the set or class of all Lorentzian manifolds isometric to some given one; this extravagant class is never used in actual physics or mathematics. What happens in practice is that some *user of the theory* chooses a member of this equivalence class, whilst some other *user* may pick another member (even the same user may prefer to work with various choices).[28]

These arguments may also work for structural realism (Ladyman, 2020), where 'modeling the phenomena' becomes 'modeling reality'. However, in structural realism the role of the user, which seems essential to me for an effective implementation of the resolution of the Hole Argument (!) proposed by Weatherall, Halvorson, and Manchak, seems weaker and less compelling than in empiricist structuralism: once we emphasize the role of the user of the theory, as van Fraassen does, we might as well take the second step of putting the empirical data, lying between some evanescent reality and this very user, as the basis for the modeling. In for a penny, in for a pound! Modeling the data instead of modeling some "real" metric structure also circumvents the problem that space-time and the metric may well be emergent.[29]

Perhaps the "user perspective" of empiricist structuralism even explains the possibility or even coherence of seemingly incompatible philosophical points of view. For example, no one can stop user Newton in thinking of elements $x \in M$ as points in space-time, which as some sort of a secondary quality carry a metric $g(x)$, but neither is there any argument against user Gelfand, who sees points of his modelling manifold as nonzero multiplicative linear functionals $C^\infty(M) \to \mathbb{R}$, whose primary quality by definition is to carry fields. More widely, 'the distinction between a Fregean, or assertoric way of understanding mathematical axioms as opposed to a Hilbertian, or algebraic one' (Gomes, 2021, p. 64) may well be in the eye of the user.

Finally, in so far as the opposition between structural realism and empiricist structuralism at least superficially reflects the famous opposition between Plato and Aristotle in the philosophy of mathematics (Bostock, 2009; Mendell, 2019), the Hole Argument again seems to me to favour the latter. Aristotle's mathematical modeling is based on *abstraction in the sense of deletion of properties*,[30] and of course it is the user rather than "reality" who does this. I plan to return to this in the future.

[28] Van Fraassen's emphasis on the user also explains why say Kerr space-time, even with fixed parameters m and a, can be used to describe different black holes, despite the mathematical identity of the two models. Indeed, one user models the phenomena produced by one black hole, whilst another user uses (!) the "space-time" in question to model the phenomena produced by another.

[29] Two examples in which the metric is at least a derived concept are two-spinor space-times (M, ε), where the metric is a quadratic expression in the antisymmetric field ε (Penrose & Rindler, 1984), and, in the Riemannian case, four-spinor space-time, where g comes from the Dirac operator via Connes's distance formula (Connes, 1994). See also Franco (2018) for Lorentzian attempts.

[30] For example, a mathematician sees a bronze sphere as a sphere, deleting its bronze-ness.

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