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Newton–Cartan theory and teleparallel gravity: The force of a formulation

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ABSTRACT

It is well-known that Newtonian gravity, commonly held to describe a gravitational force, can be recast in a form that incorporates gravity into the geometry of the theory: Newton–Cartan theory. It is less well-known that general relativity, a geometrical theory of gravity, can be reformulated in such a way that it resembles a force theory of gravity; teleparallel gravity does just this. This raises questions. One of these concerns theoretical underdetermination. I argue that these theories do not, in fact, represent cases of worrying underdetermination. On close examination, the alternative formulations are best interpreted as postulating the same spacetime ontology. In accepting this, we see that the ontological commitments of these theories cannot be directly deduced from their mathematical form. The spacetime geometry involved in a gravitational theory is not a straightforward consequence of anything internal to that theory as a theory of gravity. Rather, it essentially relies on the rest of nature (the non-gravitational interactions) conspiring to choose the appropriate set of inertial frames.

0. Introduction

A popular account of the development of gravitational theories might go something like this: Newtonian theory casts gravity as a force. That is, gravity causes objects to deviate from inertial trajectories. Newtonian gravity is not a consequence of the geometry of space and time; forces and fields propagate in fixed Euclidean space. With the advent of general relativity, however, we realised that gravity was best seen as a manifestation of spacetime geometry, and the force that was gravity faded from physics. Instead, it was suggested, massive bodies move towards each other because spacetime itself is curved by their presence. Thus the effects of gravity are not, in fact, the effects of a force. Bodies freely falling in a gravitational field are held to be force-free.

If the account were somewhat more sophisticated, it might mention that the conceptual move was prompted by a move to a very different mathematical form for the theory. While Newtonian gravity is written in the language of forces (or, on a more sophisticated formulation, potentials and fields), general relativity is written in the language of differential geometry.

For textbook purposes, this might be a reasonable, if simplified, summary. However, from the perspective of the philosopher of science, the situation is far more complicated. Philosophers and physicists have long known that general relativity's uniqueness does not lie in its mathematical format alone: Newtonian gravity can also be written in the language of differential geometry. Moreover, it may be reformulated in this language in such a way that Newtonian gravity, as in GR, appears to be a manifestation of geometrical spacetime structure. This account is known as Newton–Cartan theory (NCT). All this is familiar, and has been examined in depth in the literature. What is less well-known, at least among philosophers, is that general relativity has been given an analogous makeover, but in the opposite direction. *Teleparallel gravity* (TPG) reproduces the empirical content of GR, but in a format that more closely resembles the gauge theories of the standard model than gravity does. The surface form of teleparallel gravity has lead some proponents to claim that it presents a force theory of gravity.[1]

We thus find ourselves in a situation altogether less simple than a quick flick through the textbooks might suggest. If a Newtonian universe admits of geometrical gravity, and a general relativistic universe allows gravity as a force, then we face several challenges to our standard conception of physics. One obvious

[1] The suggestion that TPG is a force theory has been made explicitly by a group of researchers based in Sao Paulo (de Andrade & Pereira, 1997, 1999; de Andrade, Guillen, & Pereira, 2000; Aldrovandi, Pereira, & Vu, 2004a Aldrovandi, Pereira, & Vu, 2004b; Arcos & Pereira, 2005), but is also suggested by the common claim elsewhere that TPG is set in 'Weitzenböck spacetime'. The sense in which this latter claim implies that gravity is a force is explored in Section 1.

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worry is that we have here physical examples of the conventionalist thesis: it seems we must accept that the geometry of spacetime is underdetermined by data or else accept that it is not an objective feature of the world. I argue here that such a conclusion is not warranted; the full structure of our complete set of physical theories and the data they entail is enough to choose between geometries. This is because the concept of an inertial frame is both more central, and more robust, than the literature typically gives it credit for.

However, the existence of formulations of gravitational theories with different geometrical structure poses a challenge to popular ideas about the source of general relativity's (and Newton–Cartan theory's) geometrical nature. In light of alternative formulations, we must conclude that a theory's geometrical nature is not a straightforward consequence of any formal aspects of the theory *qua theory of gravitation*. Rather, it essentially relies on the rest of nature conspiring to choose the appropriate set of inertial frames. That is, the non-gravitational interactions must confirm that the straightest lines, the geodesics, of our gravitational theory, are indeed the straightest trajectories in spacetime itself. In GR, this amounts to the other forces obeying the strong equivalence principle, and taking their simplest form in free-fall frames. While this apparent fact is encoded by the minimal coupling principle, it is, in a sense, external to the barest mathematical form of the theory.

This paper divides naturally into two parts, the first on Newton–Cartan theory and the second on teleparallel gravity. In both the Newtonian and the relativistic cases we are presented with pairs of theories, one of which incorporates gravitational effects into the geometrical structure of the theory via the connection, and the other of which apparently postulates a division between the geometrical connection and the gravitational field. In both cases, I shall argue that there are grounds for not taking the gravity/inertia split seriously. Even the theory that appears 'non-geometrical' can be interpreted as postulating the freely falling frames as inertial frames; there is, in a sense, an effective geometry that is not reflected in the theory's explicit mathematical form. In the case of Newtonian gravity, there is also a second option: it might be the case that non-gravitational phenomena pick out inertial frames and thus impose a gravity/inertia split. However, in the teleparallel case, there is no such option: the only candidates for inertial frames within the theory are those of general relativity.

Before beginning, it is worth making a few comments about the relationship between theory equivalence and underdetermination. In a standard account, underdetermination worries come about in a relatively straightforward way. It is proposed that it is possible to have two distinct theories that make identical empirical predictions. Assuming (quite reasonably), that different theories entail different realist commitments, such a possibility is taken to undermine scientific realism. However, discussion of this kind of *theoretical* underdetermination sometimes obscures the heart of the issue, because the emphasis on the existence of distinct theories suggests the need for some criteria of theory equivalence.[2] This debate is tempting in the current context; the question of whether teleparallel gravity and general relativity constitute distinct theories, or reformulations of the same theory, seems to raise itself immediately. However, the worry for the realist is that *ontology* might be underdetermined, and it is possible to have ontological underdetermination even within a single theory; for example if the theory admits of more than one interpretation. I will focus here on the ontological worry; do these theories/formulations posit the same spacetime structure? Happily, in answering this question in the affirmative, I will also answer the question about theoretical equivalence. Given that I will argue that the mathematical accounts under consideration posit the same ontology, it is natural to see them as reformulations of a single theory.[3]

[2] See, for example, Glymour (1970) and Quine (1975).

1. Newton–Cartan theory

There is no lack of philosophical literature examining Newton–Cartan theory, and the majority of it dates back 30–50 years. Why, then, revisit such a well-worn topic? For one thing, the advent of teleparallel gravity makes possible some interesting comparisons between Newtonian theories and GR that cast the equivalence between Newton–Cartan theory and Newtonian gravity in a new light. For another, I argue that emphasising inertial structure leads to an argument against theoretical underdetermination. This argument has occasionally been hinted at, but not, to my knowledge, made explicit. I hold that, despite what the mathematical form of the theory seems to suggest, in an empirical situation that would confirm Newton–Cartan theory, Newtonian gravity should be interpreted as postulating the same set of inertial frames as its more 'geometrical' relative.[4]

In this discussion it is worth being clear on exactly which form of Newtonian gravity is under discussion. By NG here I mean the later, field theoretic formulation of the theory developed by Laplace and Poisson. Moreover, I'll assume (as a jumping off point, although I'll question this later) that this theory is set in neo-Newtonian spacetime, rather than Newton's absolute space. Although this isn't the original Newtonian force theory, I take it in this context that it is still appropriate to refer to the gravitational force, in as much as forces remain well-defined in the field theoretic version, even if they are not necessarily fundamental.

I will begin by going over the details of NCT, making explicit the precise form of the theory that is under consideration. I will then examine the case for underdetermination, looking particularly at whether NCT and NG should be regarded as giving the same empirical results, and at how NG should be interpreted. I will conclude that, if they are taken to be empirically equivalent, they postulate the same spacetime structure, and, if we insist that they represent different spacetime structures, then we should regard the two theories as empirically inequivalent. We therefore do not have a case of ontological underdetermination. Finally, I will draw some parallels between the case here and the situation in GR.

1.1. The theory

With the hindsight provided by general relativity, it is easy to ask the following question of Newton's gravitational theory: why, given that inertial and gravitational mass are equivalent in NG, just as they are in GR, should we not make the same conceptual identification in the one as we do in the other? Why not identify gravitational and inertial structure in Newtonian dynamics? The answer is, of course, that, at least when considering NG in isolation from other theories, it is perfectly possible to make this identification and thus cast NG in geometrical terms; the result of this reformulation is Newton–Cartan theory.

[3] While this answers the question of theoretical equivalence in this specific case, it's not obvious that one can simply assert that two formulations are theoretically equivalent if and only if they posit identical ontology; occasionally philosophers speak of different formulations of the same theory even when they prima facie possess different ontologies. Jones (1991) sees force and field formulations of Newtonian gravity as examples of this.

[4] The word 'geometrical' in this context, is, of course, somewhat ambiguous. Newtonian gravity, at least when presented in its neo-Newtonian form, is a geometrical theory in as much as it describes the geometry of spacetime, but is not a geometrical theory of *gravity*, because gravity is not reduced to spacetime geometry.

To convince ourselves that it is possible to introduce a connection that will 'geometrise' Newtonian gravity, simply look at the Newtonian equation of motion for a particle in free-fall in some gravitational potential ϕ

$$\frac{d^2x^j}{dt^2}+\frac{\partial\phi}{\partial x^j}=0 \quad (j=1,2,3) \tag{1}$$

We may view this path as a geodesic with affine parameter λ, which we may take to represent the time read by a clock moving along the geodesic. In our framework, time is absolute, and thus λ will be a linear function of absolute time: $\lambda = at+b$. This gives us

$$\frac{d^2t}{d\lambda^2}=0, \tag{2}$$

$$\frac{d^2x^j}{d\lambda^2}+\frac{\partial\phi}{\partial x^j}\left(\frac{dt}{d\lambda}\right)^2=0. \tag{3}$$

By comparison recall the geodesic equation

$$\frac{d^2x^\alpha}{d\lambda^2}+\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0 \tag{4}$$

this gives a connection with coefficients

$$\Gamma^j{}_{00}=\frac{\partial\phi}{\partial x^j}. \tag{5}$$

All other connection coefficients vanish. Inserting these coefficients into the Riemann tensor is straightforward; curvature is given by

$$R^j{}_{0k0}=-R^j{}_{00k}=\frac{\partial^2\phi}{\partial x^j\partial x^k} \tag{6}$$

with all other $R^\alpha_{\beta\gamma\delta}$ vanishing. It is therefore clear that this connection, unlike the usual affine connection in standard neo-Newtonian spacetime, is curved wherever the gravitational field has non-zero gradient. Moreover, by contracting to the Ricci tensor, we get a reformulation of the Poisson equation in geometrical terms

$$\nabla^2\phi=4\pi\rho \tag{7}$$

becomes

$$R_{00}=4\pi\rho, \tag{8}$$

where ρ is the usual mass density function.

The connection, Riemann and Ricci tensors above encode all the content of NG; this rather swift and easy process casts Newtonian gravity in an apparently geometrical form. However, aside from an awareness that a connection whose curvature is influenced by mass has been introduced, we know relatively little about the theory's geometrical structure. Let us recast the theory into the language of differential geometry.

Newton–Cartan theory introduces a classical spacetime $\langle M, h^{ab}, t_a, \nabla_a, \rho\rangle$, where M is a smooth four-dimensional differentiable manifold, h^{ab} is a Euclidean spatial metric (given by a tensor field of signature (0,1,1,1)), t_a is a temporal metric, with signature (1,0,0,0) and ∇_a is a derivative operator associated with the connection introduced above. ρ takes its usual significance as the mass density function. The temporal metric is stipulated to be orthogonal to the spatial metric

$$h^{ab}t_b=0 \tag{9}$$

and the connection is constrained to be compatible with both spatial and temporal metrics

$$\nabla_c h^{ab}=\nabla_a t_b=0. \tag{10}$$

In more general form, the Poisson equation (8) becomes

$$R_{ab}=4\pi\rho t_a t_b. \tag{11}$$

For a particle with some four-velocity ξ^a, the geodesic equation has the familiar form

$$\xi^a\nabla_a\xi^b=0. \tag{12}$$

Other curvature constraints may be introduced by raising indices with the spatial metric h^{ab}.[5] We now add the constraint that the connection is 'curl-free', needed to ensure that the theory provides the appropriate $c\to\infty$ limit for GR (see Malament, 1986)

$$R^{[a\ c]}{}_{[b\ d]}=0, \tag{13}$$

where [...] represents antisymmetrization. With conditions (9)–(13) in place, we have the minimal version of NCT. This posits a spacetime with a flat spatial metric and an orthogonal universal time function which will be read by any clock traversing the spacetime. As with standard NG, this gives us a spacetime foliated into 3-D Euclidean spaces coordinatised by an absolute time. However, unlike in NG, the connection here, although spatially flat, possesses curvature along timelike paths. This curvature is affected by mass distribution, and explains accelerations of bodies relative to one another in the presence of mass.

This form of NCT, which Bain (2004) calls "weak" Newton–Cartan theory, is the $c\to\infty$ limit of GR.[6] However, the constraints on the connection given by Eqs. (9)–(11) and (13) do not sufficiently restrict the class of connections to provide either an absolute standard of rotation or an absolute standard of acceleration. Given that global accelerations are unobservable in Newtonian gravity, the fact that the theory does not give an absolute standard of acceleration is, if anything, an advantage. However, the failure to provide a rotation standard is more serious, and prevents "weak" NCT from being equivalent to Newtonian gravity, and, for that matter, from being a well-defined physical theory.

In order to see that weak NCT will not be empirically equivalent to standard Newtonian gravity, consider the fact that, while absolute linear accelerations are not observable in NG, absolute rotations are. The water in Newton's bucket is predicted to be just as concave in empty space as it is in our own world. Leaving aside the substantivalist/relationist debate concerning whether this is a correct prediction of the theory,[7] let us simply note that, in order for NCT to be empirically equivalent to NG, it too must introduce an absolute standard of rotation. In the weak form of the theory developed above, there are too many degrees of freedom in the connection for it to distinguish between straight, or non-rotating, trajectories, and twisted, or rotating trajectories; the class of allowed connections is simply too large. In order to solve this problem, we may introduce the following constraint on the curvature of the Newton–Cartan connection[8]:

$$R^{ab}{}_{cd}=0. \tag{14}$$

[5] The covariant equivalent, h_{ab} does not exist in these theories, and we therefore have no means of lowering indices. The index-raising procedure given by the spatial metric is thus not quite the same as rasing indices using a full spacetime metric like g^{ab}. The above procedure is best viewed a prescription for creating tensors whose contravariant indices are purely spatial.

[6] That this is the case has been demonstrated by Malament (1986) and Ehlers (1991, 1997).

[7] Barbour–Bertotti theory (Barbour & Bertotti, 1977, 1982) predicts no concavity, and is in this sense not empirically equivalent to standard Newtonian gravity.

[8] The rotation standard is not the only way to bring NCT empirically into line with NG. The symmetry group of NCT may be radically restricted by the introduction of 'island universe' boundary conditions; that is, by the assumption that spacetime is asymptotically flat. If we impose the same boundary conditions on standard Newtonian gravity, then both theories have as their symmetry group the Galilean group; both rotations and accelerations are absolute. However, this assumption is unrealistic and introduces an unobservable absolute standard of acceleration. The variations of NCT and NG that result from this addition are therefore unattractive compared to the types considered here.

This condition was first introduced by Trautman (1965). Bain refers to it as the rotation standard, and calls the theory that results from adding it "strong" Newton–Cartan theory. A connection obeying this constraint can differentiate between 'twisted' and 'non-twisted' world lines—that is, the connection picks out some particular class of reference frames as non-rotating.

This rotation standard is *essential* if NCT is to be a well-defined physical theory. To see this, note that the restriction imposed on the connection by the rotation standard is just that, a restriction. As such, the connections allowed by strong NCT are a subset of those allowed by weak NCT. Now consider a situation in which the entire universe is put into a rotating state. Strong NCT, which, like NG, distinguishes such motion from non-accelerating motion, will predict a divergence of the inertial paths of particles. What will weak NCT predict? The connection that produces the divergence is a solution of the equations of weak NCT, but not the only one; weak NCT also allows for connections that make global rotations unobservable. As such, weak NCT does not always provide determinate solutions for a given state of motion of the universe.[9] Indeed, it is not clear that the state of motion of the universe can even be specified, given that motion is defined relative to affine structure, and this affine structure is itself underdetermined. Despite being a limit of general relativity, weak NCT is simply not a well-defined physical theory, let alone an empirically equivalent competitor to NG.

1.2. Underdetermination

In order to determine whether NCT and NG constitute a case of underdetermination, we must ask two questions. First, are they empirically equivalent? Second, do they diverge in their ontological commitments? I will argue that if the answer to the first question is yes, then the answer to the second is no, and vice versa. As a result, no underdetermination obtains. Let us see how this works.

First, we should note that weak NCT, as noted above, is not empirically equivalent to NG because it fails to make determinate predictions in cases (a rotating bucket in an empty universe), in which NG makes determinate predictions. Therefore, in evaluating empirical equivalence, we must consider strong NCT. Once we move to the case of strong NCT, we do seem to have empirical equivalence, at least insofar as gravitational phenomena are concerned. Strong NCT and NG make identical predictions for the behaviour of massive particles under gravity. To see this, we may note that not only may we derive the NCT Poisson equation (11), from the standard Newtonian Poisson equation (7), but, with the aid of the rotation standard, we may also derive (7) from (11).[10] However, it is not clear what we should say about their predictions for non-gravitational phenomena. Suppose we discover massless particles in a Newtonian universe. How do we expect these to behave? In the absence of an interpretation of NCT and NG, it is far from clear. If we insist that the connections in each theory represent inertial structure, then it seems we should expect each theory to make *different* predictions for the trajectories of free, massless particles; empirical equivalence is broken. However, I will argue in what follows that there is also another path: we can refrain from insisting that the connections in both theories must represent inertial structure.

Let us turn now to the reason that NCT and NG are generally held to postulate different spacetime structures. While NCT has a single connection, NG divides this into an inertial connection and a gravitational field

$$\Gamma'^{a}_{b\ c} = \Gamma^{\ a}_{b\ c} + h^{ad}\nabla_d\phi t_b t_c. \tag{15}$$

A standard reading holds that, from the perspective of NG, there is a unique correct way to effect the split described above; at any spacetime point, the value of the gravitational field is specified by the theory (given initial conditions). From the perspective of NCT, on the other hand, there is no preferred way to make the split. This reflects the fact, noted earlier, that NG holds to an absolute, global standard of linear acceleration, whereas NCT does not; acceleration is defined locally with respect to inertial frames. As a result, NG has a way of specifying gravitational acceleration relative to the background inertial frames. On the one hand, in NG, we have models $\langle M, h^{ab}, t_a, \nabla_a, \phi, \rho \rangle$, where ∇_a is a flat affine connection, and ϕ is the gravitational potential. On the other hand, in NCT, we have models $\langle M, h^{ab}, t_a, \nabla_a, \rho \rangle$, where ∇_a is a dynamical connection with curvature. The ontologies of the two theories appear to differ both in terms of the nature of the spacetime structure they posit (absolute and dynamical respectively) and in whether they posit the existence of the gravitational potential.

However, careful consideration of the interpretation of Newtonian gravity gives us reason to resist this conclusion. It is generally accepted that the Newtonian gravitational potential ϕ is a gauge quantity; the equations of motion are entirely unaffected by the addition of a constant component to ϕ. As a result, it is usually asserted that the real fundamental gravitational entity in NG is the field $-\nabla\phi$. However, this entity is also subject to a gauge freedom. Accelerative boosts are symmetries of the NG equations of motion, and uniform gravitational fields are unobservable; in a universe in which all bodies are subject to gravity, there *is* no unique physically motivated way to make the gravity/inertia split.[11] In such a situation, both the flat affine connection, and the gravitational field, are properly thought of as gauge quantities; their exact value makes no difference to the equations of motion; only their sum is empirically significant. It has been frequently noted (for example, by Friedman, 1983) that a great strength of NCT is that it replaces two gauge quantities with one that is non-gauge. Given that we accept the gauge argument for the potential, why not also accept it for the field? On its most perspicacious reading, NG should never have accepted that different choices of the inertia/field split corresponded to different possible worlds; the precise value of the gravitational field has no physical significance.[12]

However, it is worth pausing at this stage to consider the implications of this suggestion. When we assert that the gravitational potential is pure gauge and has no physical import, we assert that it is a piece of surplus structure, a mathematical artefact that does not reflect physical structure. If we take the same approach to the gravity/inertia split, and hence to both the gravitational field and NG's flat affine connection, then we assert the gravity/inertia division is a piece of surplus structure; only the sum of the two pieces represents real physical structure. However, because the physical structure in this case is spacetime structure, some may find this conclusion less comfortable than in the case of the gravitational potential. Some discussion of the

[9] Christian (2001) points out that weak NCT does not possess a classical Lagrangian density, a Hamiltonian density or an unambiguous phase space. The point above is connected to these facts, particularly to the ambiguity of the phase space.

[10] For a proof of this, see Bain (2004, pp. 366–372).

[11] Some might claim that this is not strictly true. If we impose island universe boundary conditions, we can impose a standard of acceleration. However, this is a strong physical assumption and cannot be applied in any realistic model of Newtonian cosmology. We could also use exactly the same conditions to impose a standard of rest, which is not an argument that most physicists would take seriously. Moreover, the fact that grounds can be found for imposing a particular gauge does not eliminate the gauge freedom of the entity in question. For a full survey of the problems caused by insisting on a unique split between gravity and inertia, see Norton (1992), Malament (1995) and Norton (1995).

[12] Malament (1995) makes the case for this.

circumstances under which mathematically geometrical structure comes to represent spacetime structure is therefore called for.

The question here concerns the claim some particular connection has to represent the geometry of spacetime; what makes it that some particular connection is the right one for some physical situation? It is well-known that, in the context of spacetime theories, a connection's link to physical structure comes about via it's capacity for representing the structure of inertial frames; this is why I refer to the 'inertial connection' above. However, standard discussions of inertial structure blur two definitions of an inertial frame in such a way as to preempt questions about the representational capacities of geometrical objects. It is common to see inertial frames *defined* as those reference frames[13] in which the components of the connection vanish, and which parallel transport their own coordinate axes according to the standard defined by the relevant connection. This is all well and good, but it presupposes that we know which connection accurately represents spacetime structure; this mathematical characterisation of an inertial frame cannot help us to determine which connection is the correct one. Happily, there is a more physical characterisation of an inertial frame available. Inertial frames are those reference frames in which force-free bodies move with constant velocities, and which are indistinguishable according to the dynamics. This second criterion is a strong one; in order for a class of reference frames to count as inertial, dynamical laws must take the same form in each frame. Moreover, inertial frames must be universal; if the theory under consideration allows for multiple types of interaction, each of these must pick out the same class of inertial frames.

If we ask what makes some connection the right one for a theory, the one that accurately represents the theory's spacetime structure, it is the second, physical, characterisation of inertial frame that is relevant. A connection only represents spacetime structure if the class of frames associated with it bears the right relations to the rest of the theory. If we rely on the mathematical characterisation of inertial frames alone, it gives the impression that we may select a connection at random, and simply assert it to represent the inertial structure of the theory. In reality, inertial structure is constrained by the rest of the theory's dynamics.

Turning back to the question of whether to extend the gauge argument to the gravity/inertia split in Newtonian gravity, we see that the question is really one of determining the correct class of inertial frames, and hence the correct connection, for the theory. However, answering this question in the context of Newtonian mechanics is not straightforward, because Newtonian physics is not fully defined: nothing internal to the Newtonian picture tells us whether all bodies have mass or how the non-gravitational interactions will transform. Let us therefore consider two Newtonian universes, and see what follows in each.

In the first, Universe A, gravity is universal. All phenomena pick out the freefall frames as the inertial frames: light rays follow geodesics of the NCT connection, and the non-gravitational laws look simplest in freely falling frames. In such a universe, it would be natural to see the Newton–Cartan connection as representing the inertial structure of spacetime; the theory would be geometrical in a sense that went beyond its mathematical form (although we might well see that form as particularly appropriate). However, such a universe is in no way at odds with Newtonian gravity. Indeed, if Newton's light corpuscles possessed mass, it may well be precisely the universe most naturally posited by the original theory.

In a second universe, Universe B, both NG and NCT give a full account of all gravitational phenomena, but the Newton–Cartan connection does not encode the behaviour of non-gravitational phenomena. In particular, light rays follow the geodesics of a flat connection, and the laws of electro-magnetism and the other fundamental forces take their simplest form in reference frames that are not the free-fall frames. In such a universe, NCT might give a useful and concise geometrical formulation of our gravitational theory, but we would not associate its content with space or time. Instead, non-gravitational phenomena would point to a division between gravity and inertia. In such a case, the gravitational field ceases to be a gauge quantity; different choices of gravity/inertia split correspond to different *non-gravitational* phenomena.

What may we conclude from our two hypothetical universes concerning underdetermination in Newtonian theories? We seem to have two choices. The first of these, and the one that I prefer, is to hold that NCT and NG need not be automatically interpreted as postulating different spacetime structure. Rather, the bare theories simply fail to return a single interpretation when taken in isolation. In order to determine whether their connections in fact represent spacetime structure, we must look outside Newtonian gravity, to our whole physical theory. If gravity in fact turns out to be universal, and the non-gravitational laws take their simplest form with respect to the freely falling frames, then inertial structure is well-represented by the Newton–Cartan connection. However, this is exactly the kind of situation in Newtonian gravity in which the inertia/gravity split is not well-defined, and should not be taken seriously. As a result, a correct reading of Newtonian gravity, even without the insight given by Newton–Cartan theory, should result in the conclusion that the inertial structure of the theory is represented not by the connection alone, but by the right hand side of Eq. (15). As a result, there is no ontological divergence, and no underdetermination. Likewise, if non-gravitational phenomena *did* specify a gravity/inertia split, NCT would remain a correct theory of gravity, but lose its claim to a spatiotemporal interpretation. If we accept this option, we see that the spatiotemporal status of the objects of a theory is not merely a matter of the theory's mathematical form, but rather a subtle matter of the interplay of those objects with our total physical theory.

The second, to my mind less attractive, option is to insist that both theories can be interpreted in isolation from a complete account of physics. If we take this option, we insist that it is part and parcel of Newtonian gravity that it postulates flat inertial structure, and that Newton–Cartan theory postulates curved structure. However, if one is genuinely committed to the idea that these connections must represent a piece of spacetime structure, then one should expect observable manifestations of the resulting inertial structure.[14] In this case, we should not see the two theories as empirically equivalent: they give different predictions for the behaviour of massless bodies, and different predictions for the correct form for the non-gravitational interactions. Even if there turn out to be no massless bodies, the constraints placed by a given inertial structure on the form of non-gravitational theories is far from trivial. Thus, even on this reading, we do not have a case of underdetermination. However, it should be noted that the second reading might quite well commit us to the idea that Newtonian gravity is disproved by a universe in which all bodies have mass, assuming that the NCT inertial structure is the most natural in such a case. Given that this may well have been exactly the universe that Newton himself envisioned, this makes the position decidedly odd.

1.3. Comparing NCT and GR

Of course, we do not live in a Newtonian universe, geometrical gravitation or no. The above debate is primarily interesting in the light of the kind of universe we *do* think we live in, namely, one

[13] The formal definition of a frame of reference requires some subtlety, but in this context it is helpful to think of a reference frame as a class of coordinate systems related by rotations and translations.

[14] Earman & Friedman (1973) also make this point.

described by general relativity. We might therefore wish to ask how a fully geometrical Newton–Cartan theory (in our second kind of universe above) compares to general relativity; does the latter still provide a more geometrical account of gravity?

There is certainly a sense in which GR is the more *naturally* geometrical theory, in as much as a GR universe is more simply described in geometrical terms than a Newtonian one. Where NCT must postulate a time metric, a spatial metric, and a connection as basic geometrical structures, general relativity is so formulated that all relevant structure follows directly from the spacetime metric. NCT spacetime is a less coherent and cohesive entity than that of GR. However, it is possible to distinguish between the broad geometric content of a theory and the extent to which it geometrizes gravity; in terms of the extent to which each theory describes gravity as a manifestation of spacetime geometry, there does not appear to be any deep difference between GR and NCT. In each case, gravitational phenomena arise as a result of the structure of spacetime, albeit a somewhat impoverished structure in the NCT case.

On the other hand, if we compare general relativity to NCT in the absence of any assumptions about the non-gravitational forces, there is a clear difference. General relativity comes with a spatiotemporal interpretation as part of the package; the strong equivalence principle is generally presented as part and parcel of the theory. But we might formulate NCT in just such a way, if we wished; simply exchange the Levi-Civita connection for the Newton–Cartan connection in our minimal coupling rule. In a universe where such a rule applied, one would be hard-pushed to cite a deep sense in which gravity was less a matter of spacetime geometry than it is in GR.

Before closing this issue, one more point deserves consideration. In a 1973 paper on Newton–Cartan theory John Earman and Michael Friedman argue that the geometrization of gravity achieved by GR is both 'more effective' and 'a geometrization in a very different sense' from that of NCT. Their basis for this is that

> If we demand that the affine connection of a relativistic spacetime be symmetric and compatible with the spacetime metric, there is only one such connection ∇^R, and therefore, no possibility of splitting ∇^R into an inertial and a gravitational part as there is with the Newtonian connection ∇^1 (Earman & Friedman, 1973, p. 355).

As a result of this, they claim, the very notion of a gravitational force is incoherent in GR; there is no method, even an arbitrary one, to divide the gravitational force from inertial effects. We shall see in the next section that technically, the above is correct. If the affine connection is *symmetric*, the split is impossible. However, if we allow a non-symmetric spacetime connection, we can indeed effect a division of the Levi-Civita connection in a way analogous to standard Newtonian gravity.

2. Teleparallel gravity

The elegance of the reduction of gravity to geometry effected by general relativity (GR) is deeply seductive. But aesthetics, though important, are not everything in physics. Theoretical physicists have long been aware that it is possible to create theories similar to GR in their empirical consequences, but which replace the simple beauty of Einstein's original theory with an often messier, but perhaps more useful, gauge theory. Of interest here is teleparallel gravity (TPG), a particular variant of gravitational gauge theories. In this theory, the Levi-Civita connection of GR, which has curvature but no torsion, is replaced with a Weitzenböck connection, with torsion but no curvature.

Although 'teleparallel theory' can be used to refer to a family of theories using the Weitzenböck connection, the variant of interest here recreates the results of general relativity exactly.[15] However, it models gravity in a way that at first glance resembles a theory like electromagnetism much more closely than GR does. This resemblance, and the fact that the formal geometrical structure of the theory is such that gravity does not lead to motion along geodesics of the connection, makes it tempting to think of TPG as a 'force' theory of gravity. This claim is made explicitly by a group working on teleparallel gravity in Sao Paulo (de Andrade & Pereira, 1997, 1999; de Andrade et al., 2000; Aldrovandi et al., 2004a, 2004b; Arcos & Pereira, 2005), but is also implicit in the common claim that teleparallel gravity involves a 'Weitzenböck spacetime'.[16] Doubtless this phrase is not intended, by most of its users, to carry a heavy interpretational burden, but it is suggestive; it implies that the spacetime, and hence inertial structure, of the theory is not that of general relativity, and that particles moving as predicted by GR are thus accelerating. There will be more discussion of the inertial structure of teleparallel gravity in Section 1.2.

This half of the paper will question this interpretation of TPG. In fact, TPG is to GR what Newtonian gravity is to Newton–Cartan theory, and my suggestions here will proceed along the same lines as those in the previous section. Therefore, I will argue that TPG and GR do not constitute a case of underdetermination; they in fact postulate the same spacetime structure. If anything, the case for this is more clear cut, because the geometrical structure of these theories, and its interplay with the rest of physics, is better defined than it was in the Newtonian case.

Teleparallel gravity has not been much discussed in the philosophical literature,[17] and considerably more stage setting will therefore be necessary than was the case in the Newtonian discussion. I will start by introducing the formalism of the theory and examining its motivation. I will then argue that, on close examination, ontological differences between the two theories are illusory. In fact, this new pair of formulations turns out to have much in common with our two formulations of Newtonian gravity, not least in that an emphasis on inertial structure, and not mathematical form, seems crucial.

2.1. The theory

Our discussion here will be concerned with a particular, modern variant of a group of theories that have taken advantage of the possibility of a connection with torsion. As a result, it's worth some brief mention of the history of the subject, and the contrast between the theory under discussion here, and historical theories of the same name. Teleparallel gravity has its roots in work simultaneously developed by Einstein and Cartan (1979), concerning the possibility of establishing a theory similar to general relativity, but with the feature of absolute parallelism—the possibility determining the angles between distant vectors. This *Fernparallelismus* theory introduced a curvature free connection with torsion, and an associated tetrad field in the hope that the extra degrees of freedom associated with these might be used to encode electromagnetism, and hence produce a unified field theory. The details of this project, and the reasons for its abandonment, are detailed in an excellent article by Sauer (2006).

The idea of using a connection with torsion in a gravitational theory did not die with Einstein–Cartan theory. Following the model of Yang–Mills gauge theories, Poincaré gauge theory was developed in early 1960s in the hope of providing a gauge theory

[15] In this paper, the abbreviation TPG always refers to this empirically equivalent theory, rather than any other.

[16] See, for example, Blagojevic (2001, p. 69) or Ortin (2004, p. 11).

[17] Lyre & Eynck (2003) is an exception.

of both gravitation and other fundamental forces.[18] Poincaré gauge theories allow connections with both torsion and curvature on the tangent bundle in order to produce gauge theories of the Poincaré group. Their aim is to incorporate the spin of matter fields into gravitational dynamics; torsion is associated with spin much as curvature is associated with energy-momentum in GR. General relativity may be thought of as that variant of Poincaré gauge theory in which the torsion is set to zero. Likewise, various theories that go under the name 'teleparallel gravity' may be thought of as variants in which curvature is set to zero.[19]

In this context, the theory under discussion here has a rather odd fit. As mentioned, TPG is a version of teleparallel theory that is engineered to be empirically equivalent to general relativity. As such, spin plays no role in the theory; torsion cannot be interpreted as it is in Poincaré gauge theory. Moreover, the extra degrees of freedom conferred by the introduction of the tetrad field are not used to represent additional forces, as in Einstein–Cartan theory, but are gauge freedoms. These disanalogies mean that it will be most instructive to interpret TPG primarily in isolation, rather than in comparison to these related theories.

2.1.1. Curvature, torsion and geodesics

Teleparallel theory tears apart several geometrical notions that we have, in GR, become used to associating. It also introduces the relatively unfamiliar notion of torsion. It will therefore be useful, before looking at teleparallel theory, to go over some GR territory and reexamine some geometrical notions.

The most fundamental field in GR is the metric field, which encodes spatiotemporal distances. However, much of the formal apparatus of GR uses the Levi-Civita connection, which can be derived from the metric only if one stipulates that the connection must be metric compatible; i.e. that the metric be covariantly constant with respect to the connection

$$\nabla_\rho g_{\mu\nu} = 0. \qquad (16)$$

This ensures that the angle between two vectors remains the same under parallel transport. However, this is not enough to specify the connection uniquely. In addition, it is necessary to specify that the connection must be symmetric in its lower indices

$$\Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}. \qquad (17)$$

It is this second condition that will be replaced in teleparallel theory. However, before moving on to the possibility of dropping the condition, it is worthwhile to look at its original motivation. After all, on the surface, the symmetry condition appears to have nothing to do with the metric.

In order to see why the symmetry condition is introduced, we may turn to the concept of a geodesic. Clearly, this idea is fundamental to general relativity and its geometrical nature; the idea that freely falling particles follow geodesics of the metric is an essential component of GR. However, it is important to bear in mind that there are *two* notions of geodesic at play in differential geometry. The first is a metrical notion: geodesics are paths of extremal length; they represent local maxima and minima of the interval, *ds*. There is no need to have defined a connection in order to introduce this notion of geodesic. The second notion of geodesic centres around the connection: a geodesic is a path that parallel transports its own tangent vector. This notion is entirely affine; it would allow us to define geodesics in a space with no metric. In fact, it is the notion of geodesic we use in Newton–Cartan theory, where no spacetime metric is defined. The motivation for the symmetry condition is this: *The symmetry condition ensures that the two notions of geodesic coincide*.[20] In general, even for a metric compatible connection, there will be two sets of geodesics, one associated with the metric, and another associated with the connection.[21] This will prove important when looking at teleparallel gravity.

In addition to the notion of geodesic, the concepts of torsion and curvature will play an important part in what follows, and are worth a closer look. As a result of its central position in general relativity, most of us are familiar with the ideas underlying curvature. Because it can be conveniently realised on two-dimensional surfaces, curvature appears intuitively familiar. This can sometimes obscure the fact that the connection between pictorial geometry and more formal geometry can be subtle, because more than one notion of curvature is at play. Nonetheless, two-dimensional surfaces are helpful in seeing that curvature is a measure of the failure of a vector to return to itself when parallel transported around a closed loop. Put like this, curvature is obviously a property of the connection, and we can write the Riemann curvature tensor in terms of the connection

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}. \qquad (18)$$

Obviously, curvature is not the only formal property a connection may possess. Another is torsion. Torsion is less easy to visualize; it corresponds, not to the failure of a single vector to come back to itself when transported around a loop, but to the failure of two vectors to form a parallelogram when parallel transported along one another. Taking two infinitesimal vectors in the tangent space, χ^α and ζ^α, first parallel transport χ^α along ζ^α, and then transport ζ^α along χ^α. In space with no torsion, the result of these two processes will be the same; a parallelogram is formed. However, if the connection has torsion, the parallelogram will not close, as shown in Fig. 1. The non-closure is proportional to torsion.

A more formal definition of torsion can be given in terms of in terms of the Lie bracket of two vector fields and the covariant derivative of one vector field along another. If *X* and *Y* are vector fields, torsion is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \qquad (19)$$

In co-ordinate notation, the torsion tensor is given by

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}. \qquad (20)$$

As such, torsion is a measure of the antisymmetric part of a connection. Thus, general relativity's demand for a symmetric connection amounts to a demand that the torsion tensor vanish. Teleparallel gravity, on the other hand, postulates a metric compatible connection with zero curvature, but non-zero torsion.

The above account makes curvature and torsion, and the connections postulated by GR and TPG, sound pleasantly symmetrical. However, there are some important asymmetries. As was

[18] Poincaré gauge theory was first proposed by Kibble (1961) and Sciama (1962). For a good textbook overview of Poincaré gauge theories, see Blagojevic (2001, Chap. 3). For some discussions that focus on the role of torsion in these theories, see Hehl (1984) or Hammond (2002).

[19] It is natural to think of these as 'limiting cases' of Poincaré gauge theories, but it is worth noting that, unlike some other 'limiting cases' of theories, GR and TPG are not derived by considering the approach to a particular limit, but rather by simply setting curvature/torsion to zero.

[20] The symmetry condition is sufficient, but not necessary, for the two classes of geodesic to coincide; the geodesics associated with a given connection depend only on its symmetric part. Where torsion is not completely antisymmetric, its symmetric part will contribute to geodesic structure and cause deviation from metric geodesics. However, the geodesics of a connection with torsion will coincide with metric geodesics if the torsion is completely antisymmetric.

[21] In the literature where torsion is considered, the straight lines associated with the connection are often known as *autoparallels*, while the term geodesic is reserved for the extremal paths determined by the metric. However, to retain continuity with more familiar GR literature, I'll refer to both as geodesics here.

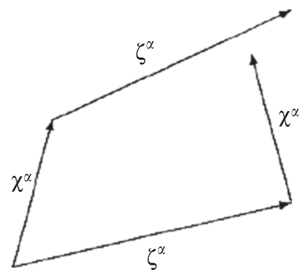

Fig. 1. Torsion.

the case for geodesics, there is more than one concept of curvature. This comes about because curvature may be defined without making use of the connection. Curvature may be defined as the relative acceleration of neighbouring geodesics. If we use the metrical notion of geodesic, this does not rely on the connection. Torsion, on the other hand, has no metrical interpretation; it is purely a property of the connection.

Another important asymmetry concerns the uniqueness of the connection. While there is a unique symmetric metric-compatible connection for a given metric, demanding metric compatibility and zero curvature fails to uniquely specify a connection. Teleparallel gravity therefore involves an equivalence class of connections.

2.1.2. *Summary of the formalism*

Armed with the above geometrical concepts, we are ready to examine teleparallel theory. As noted, TPG introduces a connection with torsion, but zero curvature.[22] This is called the Weitzenböck connection. Of course, it is hardly surprising that such a connection can be defined. What is far more surprising is that the introduction of the Weitzenböck connection can lead to a theory of gravity that reproduces the results of general relativity. In order to understand how this comes about, it is necessary to move into tetrad notation, which requires further explanation.[23]

In standard GR, we generally work within the constraints of a coordinate basis for our tangent space. However, in curved spaces, these coordinates won't generally be orthogonal. Nonetheless, locally, we can always assign an orthonormal basis for the tangent space. A tetrad field is a set of four vector fields which at each point in the space provide an orthonormal basis for the tangent space at that point. Writing the components of a tetrad field in a coordinate basis gives us a means of transforming between orthonormal and coordinate bases: if **V** is a vector in the tangent space, and roman and greek indices represent components of vectors in orthonormal and coordinate bases respectively, then the tetrad h^a_μ gives us

$$V^a = h_\mu{}^a V^\mu. \tag{21}$$

Given a non-trivial tetrad field, the Weitzenböck connection[24] is given by

$$\Gamma^\rho{}_{\mu\nu} = h_a{}^\rho \partial_\nu h^a{}_\mu. \tag{22}$$

This means that the Weitzenböck covariant derivative of the tetrad field vanishes identically, which is the feature that ensures absolute parallelism

$$\nabla_\nu h^a{}_\mu = \partial_\nu h^a{}_\mu - \Gamma^\theta{}_{\mu\nu} h^a{}_\theta = 0. \tag{23}$$

Teleparallel gravity is an Abelian gauge theory of the translation group. The gauge potential $B^a{}_\mu$ is the non-trivial part of the tetrad field[25]

$$h^a{}_\mu = \partial_\mu x^a + B^a{}_\mu. \tag{24}$$

The field strength may be derived from the potential in the usual way, and turns out to be simply the torsion of the connection written in the tetrad basis

$$F^a{}_{\mu\nu} = \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu = h^a{}_\rho T^\rho{}_{\mu\nu}. \tag{25}$$

The action is given by the following integral:

$$S = \int_a^b [-m\sqrt{-u^2} + mB^\alpha{}_\mu u_\alpha u^\mu]\, ds, \tag{26}$$

where $ds = (\eta_{ab} dx^a dx^b)^{1/2}$ is the Minkowski tangent space invariant interval, and u^α is the particle four-velocity. Use of the Euler–Lagrange equations leads to a force equation analogous to the Lorentz force law

$$(\partial_m ux^\alpha + B^a{}_{mu})\frac{du_\alpha}{ds} = F^a{}_{\mu\rho} u_a u^\rho. \tag{27}$$

This can be reexpressed via Eqs. (24) and (25) to give

$$\frac{du_\mu}{ds} - \Gamma_{\theta\mu\nu} u^\theta u^\nu = T_{\theta\mu\nu} u^\theta u^\nu, \tag{28}$$

which is the equation of motion for teleparallel gravity. This predicts deviation away from the geodesics of the Weitzenböck connection, and these deviations depend on torsion, and thus on the field strength of the theory.

2.1.3. *Motivation*

It is worth pausing here to ask why one might prefer the messier teleparallel theory to the more elegant general relativistic formulation. TPG aims to reproduce the results of GR exactly, and thus manifestly fails to have any empirical benefits. The perceived benefits then, must be theoretical, and I shall try to outline some possibilities here.

A first possibility concerns the motivation for all gauge theories of gravity, including that first suggested by Einstein and Cartan: unification. The overwhelming success of the gauge heuristic gives good reason to believe that the way forwards in the unification programme lies in a fully fledged gauge theory of gravity. However, unlike Einstein's version of a teleparallel theory, which was intended to unify gravity and electromagnetism, modern teleparallel theory simply reproduces the results of general relativity, and does not pretend to posit any new results connecting gravity to other forces. In this case, then, the unificationist motivation boils down to a vague conviction that nature has written all her laws in the language of gauge theories; it is a methodological unification and not an ontological one.

A second possible motivation concerns the energy of the gravitational field. In standard GR, this is represented by the energy–momentum pseudo-tensor of the gravitational field: $t_\lambda{}^\rho$. In teleparallel gravity, it is now more naturally represented by the gauge current, $j_a{}^\rho$, which is given in Eq. (36). de Andrade et al. (2000) consider this a major advantage of the theory. They make much of the fact that this quantity is invariant under local translations of the tangent space coordinates, and transforms covariantly under global tangent space Lorentz transformations. This means it is both a well-behaved gauge invariant quantity of the theory (remembering that the gauge group of teleparallel

[22] This connection is, in fact, metric compatible. However, the metric is not explicitly introduced in the TPG formalism.

[23] The account below will be, of necessity, only a sketch. For a fuller account of the precise theory described here, look at e.g. de Andrade et al. (2000). For an introduction to the tetrad formalism, see Carroll (2003, Chap. 10) or Wald (1984, pp. 50–53). Jensen (2005) provides a helpful introduction to differential geometry with torsion. Blagojevic (2001, Chap. 3) gives an introductory overview of Poincaré gauge theories, including teleparallel theories.

[24] Henceforth the Weitzenböck connection will be represented by Γ, and the Levi-Civita connection by $\mathring{\Gamma}$. I will also work in units where $c=1$.

[25] Note that this division depends on our choice of coordinates.

theory is the translation group), and what they call a "*spacetime tensor*".[26] This contrasts with the pseudo-tensorial nature of $t_\lambda{}^\rho$.[27] Moreover, although both the gauge current and the energy–momentum pseudo-tensor are conserved quantities, this fact can be expressed with a covariant derivative for the gauge current, but cannot be expressed covariantly for the pseudo-tensor. This is perhaps an advantage, but hardly reason enough to abandon GR in favour of teleparallel theory, especially when we note that $j_a{}^\rho$ is only covariant under global, and not local, Lorentz transformations.

A final motivation for the theory is a claim, put forward in two recent papers by Aldrovandi et al. (2004a, 2004b) that teleparallel gravity may be formulated in such a way that it remains valid in the face of violation of the weak equivalence principle (WEP). That is, teleparallel theory could accommodate the discovery that, for some matter, gravitational and inertial mass are not identical, and that all bodies do not therefore fall at the same rate regardless of their constitution. Of course, we have no current reason to think that WEP is violated, and one might well think that, if TPG loses GR's ability to explain the validity of the WEP, this is a disadvantage of the theory. We will return to this issue, and its consequences, in the final section of this paper.

2.2. Underdetermination

Teleparallel gravity and general relativity appear to be candidates for underdetermination. Is this really the case? As in the Newtonian discussion, this rests on whether they are really empirically equivalent, and whether they propose different ontologies. However, the issues here are slightly more complicated than in the Newtonian case, and, while I'll argue that we don't in fact have a case of underdetermination, I will also argue that one of the options available in the Newtonian case, insisting that the two theories represented different inertial structures and provided different predictions, is not available here.

First, we need to understand the results that lead to the claim that TPG is empirically indistinguishable from GR. The Lagrangian for a pure gravitational field may be written in the following notation[28]:

$$\mathcal{L}_G = \frac{h}{16\pi G} S^{\rho\mu\nu} T_{\rho\mu\nu}, \tag{29}$$

where $h = det(h^a{}_\mu)$, and

$$S^{\rho\mu\nu} = \frac{1}{2}[K^{\mu\nu\rho} - g^{\rho\nu}T^{\sigma\mu}{}_\sigma + g^{\rho\mu}T^{\sigma\nu}{}_\sigma] \tag{30}$$

with $K^{\mu\nu\rho}$ being the contortion tensor

$$K^\rho{}_{\mu\nu} = \frac{1}{2}[T_\mu{}^\rho{}_\nu + T_\nu{}^\rho{}_\mu - T_\mu{}^\rho{}_\nu]. \tag{31}$$

How does this lead to empirical equivalence? It turns out that the contortion tensor defined above is simply the difference between the Weitzenböck connection and the Levi Civita connection

$$\mathring{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - K^\rho{}_{\mu\nu}. \tag{32}$$

Given that $K^\rho{}_{\mu\nu}$ is built up from the torsion, which is the field strength of the theory, the parallel with Eq. (15), which gave the relationship between the Newton–Cartan and Newtonian connections, is obvious. In fact, the above seems to represent a gravity/inertia split in much the same way that Eq. (15) does. This relation is also essential in establishing the empirical equivalence of TPG and GR, because it acts as a translation dictionary between the equations off teleparallel gravity and those of standard general relativity. Given this translation, the Lagrangian above turns out to be identical, up to a divergence, to the Einstein–Hilbert Lagrangian in standard GR.[29]

$$\mathcal{L} = \frac{h}{16\pi G}\sqrt{-g}\,\mathring{\mathcal{R}}. \tag{33}$$

The vacuum field equation can likewise either be translated into the language of teleparallel gravity or alternatively obtained by performing variations with respect to the gauge potential. It turns out to be

$$\partial_\sigma(hS_a{}^{\sigma\rho}) - 4\pi G(hj_a{}^\rho) = 0, \tag{34}$$

where j_a^ρ is the gravitational gauge current[30]

$$hj_a{}^\rho \equiv \frac{\partial\mathcal{L}}{\partial h^a{}_\rho} = -\frac{c^4}{4\pi G} hh_a{}^\lambda S_\mu{}^{\nu\rho} T^\mu{}_{\nu\lambda} + h_a{}^\rho \mathcal{L}. \tag{36}$$

This quantity is conserved as a result of the field equations

$$D_\rho j_a{}^\rho = 0. \tag{37}$$

The equivalence of the Lagrangians is enough to establish empirical equivalence, but we may, at this stage, begin to have some doubts about TPG's status as an original and coherent theory. In the above equation, D_ρ is the so-called *teleparallel* covariant derivative, *not* the Weitzenböck covariant derivative. The teleparallel covariant derivative is simply the Levi-Civita covariant derivative reexpressed in terms of the Weitzenböck connection via Eq. (32)

$$D_\rho j_a{}^\rho \equiv \partial_\rho j_a{}^\rho + (\Gamma^\rho{}_{\lambda\rho} - K^\mu{}_{\lambda\rho}) j_a{}^\lambda. \tag{38}$$

Thus the gravitational gauge current is not conserved with respect to the connection most natural to the theory, but rather with respect to the Levi-Civita connection. The importance of the *teleparallel* covariant derivative does not stop there. The minimal coupling principle applies in TPG just as it does in GR. In order to maintain empirical equivalence, however, ordinary derivatives are converted not into the Weitzenböck covariant derivative, but into the teleparallel covariant derivative.[31] We shall see shortly that this has important consequences for the inertial frames picked out by TPG.

So far we have not mentioned the metric. In teleparallel gravity, this is quite possible; it does not appear in the formalism of the theory. Nonetheless, it is worth noticing that it has been hiding in the shadows all along, closely tied to the tetrad field

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^a{}_\nu, \tag{39}$$

[26] This term is just another way of expressing the covariance under global Lorentz transformations.

[27] This quantity does not transform like a tensor, and vanishes in freely falling frames.

[28] I use the vacuum Lagrangian and field equations here for ease of exposition. The presence of a source field adds an extra term to the Lagrangian, and results in the presence of the energy–momentum tensor on the right-hand side of the field equations.

[29] Although this certainly guarantees the empirical equivalence of the two theories locally, we might have some global worries. Certain topologies do not admit of a global tetrad field. Nonetheless, the Einstein equations as standardly written are solvable for these manifolds. As a result, it appears that there must be some GR solutions that cannot be instantiated in TPG. Of course, this also applies to *any* formulation of GR using tetrads, not just TPG. Furthermore, Bob Geroch (1970) has argued that it is reasonable to restrict the GR space of solutions to those that admit of spinor fields. Many thanks to an anonymous referee for bringing this paper to my attention.

[30] $j_a{}^\rho$ may be compared to the standard GR energy–momentum pseudo-tensor of the gravitational field, which can be expressed in our notation as:

$$ht_\lambda{}^\rho = \frac{c^4 h}{16\pi G}\Gamma^\mu{}_{\nu\lambda} S_\mu{}^{\nu\rho} + \delta^\rho_\lambda \mathcal{L}_G. \tag{35}$$

[31] See de Andrade & Pereira (1999).

where η_{ab} is the Minkowski metric. In fact, $g_{\mu\nu}$ is still used to raise and lower indices, just as it is in GR.

One might therefore have doubts that teleparallel gravity really postulates a different ontology; the old entities from GR appear to be waiting in the wings. I suggested in the Newtonian case that, under circumstances in which non-gravitational phenomena failed to determine the gravity/inertia split, we should regard Newtonian mechanics as postulating the same spacetime structure as Newton–Cartan theory. Eq. (33), like Eq. (15), postulates a split between gravity and inertia that constitutes the purported ontological difference between TPG and GR. Might we have reason here, as in the Newtonian case, not to take this split seriously? Might TPG not *really* be a force theory?

Let us briefly review some reasons one might view TPG as a force theory, and thus as ontologically divergent from GR. A reasonable place to begin is with the common claim that TPG involves a Weitzenböck spacetime. Unlike the straightforward mathematical fact that the theory involves a Weitzenböck *connection*, this claim carries some metaphysical weight; it makes a statement about the nature of the spacetime in which the theory is set. It implies that the Weitzenböck connection is a 'spacetime' connection, in contrast to, say, the connections of electromagnetic fibre bundles. In Newtonian gravity, special relativity, and general relativity, spacetime connections are those that determine geodesics and the structure of inertial frames. Indeed, this seems constitutive of what it is to be a spacetime connection. So the claim that TPG involves a Weitzenböck spacetime implies that the Weitzenböck connection represents the inertial structure of the theory. This in turn implies that particles moving in a gravitational field are subject to accelerations; the geodesics of the Weitzenböck connection differ from those of the Levi-Civita connection, and, in a theory empirically equivalent to GR, particles moving under gravity alone follow Levi-Civita geodesics. This grounds the claim that gravity is a force in TPG: if the Weitzenböck connection represents geodesic and inertial structure, then gravity causes non-inertial motion.

This view appears to be supported by the form of the equations listed above. Eq. (27) resembles the Lorentz force law, while Eq. (28) appears to be an equation of motion involving a force. Some further support might be garnered from the gauge form of the theory; Poincaré gauge theories are modelled on Yang–Mills gauge theories, which describe forces.

However, when we dig a little deeper, there is reason to question this force interpretation of TPG. The analogy with Yang–Mills gauge theories is somewhat misleading; TPG proposes a Minkowskian frame bundle on the base space $\mathbf{R}^4$. The gauge group of the bundle is the translation group, and the gauge potential is the non-trivial part of the tetrad field. This bundle has markedly different features from the fibre bundles of Yang-Mills theories. For one this bundle possesses a soldering form relating the fibre spaces to the base manifold. For another, the algebra of constraints on the TPG bundle fails to form a Lie algebra. One should therefore be wary of simplistic comparisons with standard model force theories.

But the problems for the force interpretation of TPG really arise when we consider the inertial structure of the theory in more detail. It is essential to ask relative to which inertial frames TPG represents gravity as a force. The answer might seem simple: relative to those frames defined with respect to the Weitzenböck connection. But which frames are these? In general relativity, the inertial frames are those with respect to which the Levi-Civita connection coefficients vanish. In these frames, the laws of physics are returned to their special relativistic form because of minimal coupling: non-gravitational interactions couple to gravity only via the connection and when the connection coefficients vanish, local gravitational effects vanish as well. However, no such frames exist for a non-symmetric connection; the components of the Weitzenböck connection cannot be made to vanish by an appropriate choice of reference frame.[32] It seems that the geodesics of the Weitzenböck connection simply don't have a full inertial structure associated with them; although we can define a class of frames that parallel transport their own axes relative to the Weitzenböck connection, these frames can't be the ones in which the laws of physics take a uniform form by virtue of the connection vanishing.

Is TPG therefore a theory without inertial structure, and therefore without full spacetime structure? Perhaps not. As noted in Section 1.1, the minimal coupling prescription in TPG uses the teleparallel covariant derivative (which is simply a recasting of the Levi-Civita covariant derivative), and not the Weitzenböck covariant derivative. As a result, all other fundamental forces take their simplest form relative to the inertial frames picked out by the Levi-Civita connection. Moreover, freely falling bodies still follow geodesics of the metric, which, along with the Levi-Civita connection, is still present in the theory, albeit in disguised form. It therefore seems most sensible to conclude that the spacetime of TPG is in fact precisely the same as the one in GR; only the mathematical form is different.

This conclusion becomes still more compelling when we note that neither the Weitzenböck connection nor the tetrad field are uniquely determined by the theory. The tetrad field is defined only up to a local gauge transformation,[33] and this gauge freedom passes on to the connection defined in terms of the tetrads. As a result, the gravity/inertia split expressed in Eq. (32) is just as much a gauge matter as it was in the Newtonian case. In fact, the situation is worse. Because the tetrad field is subject to a *local* gauge freedom, no amount of information about the tetrad field on, say, a spacelike hypersurface will determine the value of the tetrad field in other regions of spacetime. As a result, it is tempting to think that the metric, and the Levi-Civita connection, should be taken as ontologically prior to the tetrad field and Weitzenböck connection.

Both TPG and GR appear to take the metric to be fundamental. Looked at another way, we can note that both theories admit of both the tetrad field and the metric; tetrad formulations of GR have various uses. Moreover, rods and clocks survey the self-same metric in both theories. The only difference is the way in which this comes about—either via the Weitzenböck connection or the Levi-Civita connection. It seems that both theories posit the 'same' spacetime; if the connections in the two theories are thought of as modelling properties of this spacetime, they should perhaps be seen as alternative representations of the same properties. The closer we look, the less there appears to be any underdetermination at all.

However, this does present us with a puzzle. Torsion and curvature represent very different geometrical properties. How then, can TPG and GR be interpreted identically? This query becomes more pressing when we note that in Poincaré gauge theories, the torsion and curvature represent translational and Lorentz gauge field strengths respectively. These then couple to different fields, and have markedly different effects. While the curvature of the connection plays its traditional role in General Relativity; coupling to all matter fields, torsion, at least at the macroscopic level, couples only to fields with spin. Prima facie, then,

[32] The standard way of seeing this is to note that the antisymmetric part of the connection forms a tensor, and therefore its components cannot be made to vanish by a coordinate transformation. In fact, this story turns out to be something of a simplification; it is now recognised that one can define 'normal frames' for a non-symmetric connection, but these frames don't correspond to coordinates in the right way to represent extended inertial structure. See Hartley (1995) and Iliev (2006) for details.

[33] Note that this gauge freedom exists in addition to the translational gauge freedom that standardly defines the gauge group of the theory.

we would expect very different theories to be produced by postulating connections with only torsion and curvature respectively.

However, perhaps the analogy with Poincaré gauge theory is misleading. The equations of TPG are set up in such a way that the torsion of the connection plays a very different role from that in Poincaré gauge theories. In particular, despite being broadly associated with spin in the non-teleparallel literature,[34] torsion has no such association in teleparallel theory; given the equations, the effects of the Weitzenböck connection on matter must in every way match the effects of the Levi-Civita connection. That identical physical effects can be modelled either by a connection with curvature or one with torsion is truly remarkable, but not because there is any necessary link between torsion and particular particles or effects. The fact that the role of torsion in similar theories can be so different teaches us that interpreting a theory involves more than just looking at its mathematical form. In this case, we can only understand the theory by thinking hard about the inertial structure it posits. Despite appearing to possess very different mathematically geometrical meanings, given the right laws, two theories involving connections with torsion and curvature can, under very special circumstances, have the same physical geometry.

2.3. *Alternative inertial structure?*

The above discussion closes with a stronger conclusion than in the Newtonian case. What has happened to our two universes and our second interpretative option? If Eq. (32) represents a gravity/inertia split, can't we imagine a situation in which non-gravitational phenomena suggested that teleparallel gravity represented the more natural spacetime theory? Is there no GR analog of the Universe B mentioned in Section 1.2?

The difference between Universe A and Universe B lays in the satisfaction of the equivalence principle. Recall that in GR the strong equivalence principle has two components: minimal coupling and universality of gravitation. Minimal coupling is maintained in TPG with respect to the *Levi-Civita* connection, so there is no reason to think that TPG might correspond to a universe in which minimal coupling is incorrect. In particular, we can't even consider modifying the theory to a form with minimal coupling with respect to the Weitzenböck connection, because there are no inertial frames in which this connection vanishes, and therefore applying the comma goes to semi-colon rule with respect to the Weitzenböck connection would not give back the pre-GR forms of non-gravitational interactions in any frame whatsoever. However, there has been a suggestion that teleparallel gravity might be compatible with a universe in which gravity is not universal, namely one in which gravitational and inertial mass were not equivalent.

Aldrovandi et al. (2004a, 2004b), derive an equation analogous to the Lorentz force equation which can accommodate different values for gravitational charge m_g, and inertial mass m_i. They reproduce the action for TPG, but without assuming the weak equivalence principle

$$S=\int_a^b[-m_i d\sigma - m_g B^a{}_\mu u_a\, dx^\mu]. \tag{40}$$

The force equation is now

$$\left(\partial_\mu+\frac{m_g}{m_i}B^a{}_\mu\right)\frac{du_a}{ds}=\frac{m_g}{m_i}F^a{}_{\mu\rho}u_a u^\rho, \tag{41}$$

where $F^a_{\mu\rho}$ is the field strength defined in Eq. (7). When $m_g=m_i$, this equation reduces to our original equation of motion, and, if we substitute appropriately, can be shown to be identical to the geodesic equation of GR. However, the motions of particles are still well-defined even for $m_g \neq m_i$.

What are we to make of this? First we should note that what is being proposed is quite different from my suggestion for Universe B, and much stronger. I proposed violation of a Newtonian version of the strong equivalence principle; massless phenomena were to select the geodesics of a flat spacetime. However, in this case, we have violation of the weak equivalence principle. It is not exactly clear what massless bodies do according to the above prescription, but it seems plausible to hold that these bodies, at least, still follow geodesics of the Levi-Civita connection. It is the behaviour of *massive* bodies that changes. However, the change does not help us interpret TPG as a force theory on flat spacetime. While WEP violation certainly threatens the claim of the Levi-Civita connection to represent inertial structure, it does nothing to award the Weitzenböck connection inertial significance. A theory like the above appears simply to be a theory without inertial structure at all, and hence without any proper notion of force. It may be that such notions must go by the wayside in some eventual theory, but it scarcely seems that TPG is doing any work in showing us this. Given our ability to translate teleparallel quantities to GR ones, we can always back-engineer the above equations into GR form.

Thus, when it comes to postulating the a flat spacetime, the analogy between the general relativistic case and the Newtonian one breaks down. We don't have underdetermination in the TPG/GR case, but the sheer complexity of the geometrical form of GR, and the subtlety of the notion of inertial frame within it, restrict our options. The only coherent spacetime to be found in these theories is the curved spacetime of GR.

3. Conclusions

Two morals may be drawn from this discussion. The first is that examples of theoretical underdetermination are harder to come by than one might think, especially if one takes a relatively liberal attitude to the metaphysical commitments of a theory. Genuine potential examples of underdetermination are scarce; I have examined two rare examples and found them lacking.

Significantly, the illusion of underdetermination came about as a result of what John Stachel has called "the fetishism of mathematics".[35] It is only when we take the mathematical form of the theories too seriously that they appear to diverge. Careful consideration of inertial structure revealed that geometrical form does not always determine a theory's spatiotemporal commitments. This is the second moral. We see that geometrical form is not a sufficient condition for representing spacetime structure. Inertial considerations play an important role in the process by which mathematical structure comes by its spatiotemporal credentials. Moreover, the requirements on our total theory for a particular inertial structure to be represented are stringent.

The consequences of ignoring the above are apparent in the teleparallel literature. GR's geometrization of gravity is a very deep and subtle matter; we should be wary of claims that a theory either reverses or extends that geometrization.

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[34] Hehl (1984) justifies this association by suggesting that the spins of particles act like tiny gyroscopes which detect the translational field.

[35] Stachel (2005, p. 17).

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