Galilei invariance and the form of the classical Hamiltonian

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Abstract. We show how invariance under spatial translations and Galilei boosts constrains the classical Hamiltonian to have the form, $h = (\mu/2)v^2 + f(q)$.

1. Introduction

It is possible to view certain symmetries of phase space as underlying and justifying a common form taken by the classical Hamiltonian, $h = (\mu/2)v^2 + f(q)$. This was first pointed out in the context of quantum mechanics by Jauch [3, 4]. Lévy-Leblond later suggested that a similar argument should be available in the context of classical Hamiltonian mechanics. Here we give a rigorous derivation of Jauch's theorem in the context of classical symplectic mechanics.

2. Notation

2.1. **Phase space.** My notation for classical mechanics will roughly follow that of Geroch [2, §1-2]. Let $\mathcal{P}$ (for "$\mathcal{P}$hase space") be a smooth, connected, $2n$-dimensional manifold. Each point $x \in \mathcal{P}$ will be interpreted as a "possible state" of a classical system. A function $f : \mathcal{P} \to \mathbb{R}$ will be interpreted as an "observable." Observables assign real values to each possible state of our system, and can represent physical quantities such as the energy or position of that state.

We will adopt the "abstract index" notation of Penrose, and accordingly denote a vector v^a with an index upstairs, and a covector w_a with an index downstairs. The operation of contraction (sometimes called "interior multiplication" or "index summation") between tensors will be indicated by a common index in both upper and lower positions, such as $w^a v_a$. The unique exterior derivative on k-forms of a manifold will be denoted d_a.

2.2. **Symplectic structure.** The central features of Hamiltonian mechanics are captured by a *symplectic form* on $\mathcal{P}$. Mathematically, a symplectic form is a 2-form on $\mathcal{P}$, denoted Ω_{ab}; that is, Ω_{ab} is a skew-symmetric ($\Omega_{ab} = -\Omega_{ba}$), bilinear mapping from pairs of vectors in $T\mathcal{P}$ to the reals, $\Omega_{ab} : v^a w^b \mapsto r \in \mathbb{R}$. It is also closed ($d_a\Omega_{bc} = \mathbf{0}$) and non-degenerate ($\Omega_{ab}v^a = \mathbf{0} \Rightarrow v^a = \mathbf{0}$). This implies that Ω_{ab} is a bijection from vectors to covectors, and thus has an inverse; we denote its inverse by Ω^{ab}.

The interpretive significance of the symplectic form is that it allows us to input an observable h, and output a unique smooth vector field $H^a := \Omega^{ba}d_bh$, such

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that the value of h is conserved along the trajectories that thread the vector field H^a. This generalizes the traditional role that Hamilton's equations play, in providing a space of deterministic trajectories along which energy is conserved. There do exist classical descriptions that fail to satisfy these conditions, and thus that fail to admit a symplectic form. However, the scope of our discussion will be restricted to the broad class of classical descriptions that do.

Given a manifold and a symplectic form $(\mathcal{P}, \Omega_{ab})$, it will be convenient to define the *Poisson bracket* $\{\cdot\,,\cdot\}$ on smooth functions $f, h : \mathcal{P} \to \mathbb{R}$, given by

$$\{f, h\} := \Omega^{ab}(d_a h)(d_b f).$$

The right hand side is itself a smooth function on $\mathcal{P}$. So, the Poisson bracket takes a pair of scalar fields to a scalar field. From Ω_{ab} and d_a, the Poisson bracket inherits the properties of being antisymmetric, linear in both terms, satisfying the Leibniz rule in both terms, and vanishing for constant functions. If f, h generate vector fields F^a and H^a by the prescription above, let φ^f_α and φ^h_β denote the diffeomorphism flows with tangent fields F^a and H^a, respectively. It will be useful in what follows to observe that, by our definitions,

$$\{f, h\} := \Omega^{ab}(d_a h)(d_b f) = H^b d_b f = \frac{d}{d\beta}\left(f \circ \varphi^h_\beta\right)\Big|_{\beta=0} \tag{1}$$

where the last equality is an expression of the chain rule. In other words, the Poisson bracket $\{f, h\}$ at a point p is equal to the directional derivative of the scalar field f at p, in the direction of the vector field H^a determined by h.

2.3. **Classical systems.** We will take a classical system to consist of a $2n$-dimensional symplectic manifold $(\mathcal{P}, \Omega_{ab})$, together with a smooth function $h : \mathcal{P} \to \mathbb{R}$ that we refer to as the "Hamiltonian." The interpretive significance of h will be (1) that we take the quantity it assigns to states in $\mathcal{P}$ to be their *energy*, and (2) that the trajectories h generates (the integral curves that thread H^a) are the possible motions of the classical system in time.

3. Position and velocity in Hamiltonian mechanics

We will now impose some additional structure on a classical system $(\mathcal{P}, \Omega_{ab}, h)$. Our classical systems will be taken to have have a certain property that can be thought of as "position," and will satisfy certain symmetries with respect to that property.

3.1. **Defining position and velocity.** The "position in space" of a classical system will be defined in terms of what is sometimes called a "maximal orthogonal set" or a "real polarization" on $\mathcal{P}$.

Definition 1. A *maximal orthogonal set* for a $2n$-dimensional symplectic manifold $(\mathcal{P}, \Omega^{ab})$ is a set $\{\overset{1}{q}, \overset{2}{q}, \ldots, \overset{n}{q}\}$ of n smooth functions $\overset{i}{q} : \mathcal{P} \to \mathbb{R}$ such that (i) $\{\overset{i}{q}, \overset{j}{q}\} = 0$ for each $i, j = 1, \ldots, n$, and (ii) if f is another smooth function satisfying $\{f, \overset{i}{q}\} = 0$ for all i, then $f = f(\overset{1}{q}, \ldots, \overset{n}{q})$ is a function of the $\overset{i}{q}$.

It makes sense to think of position as forming such a set, for example, if we represent possible positions as points in $\mathbb{R}^n$, and represent phase space by the cotangent bundle $\mathcal{P} = T^*\mathbb{R}^n$. Then, for any Cartesian coordinate chart $\{\overset{1}{q}, \overset{2}{q}, \ldots, \overset{n}{q}\}$ on $\mathbb{R}^n$, the set $\{\overset{1}{q} \circ \pi, \overset{2}{q} \circ \pi, \ldots, \overset{n}{q} \circ \pi\}$ is a maximal orthogonal set for $\mathcal{P}$ (where π is the canonical projection, $\pi : (q, p) \mapsto q$). This maximal orthogonal set is one typical way of representing position in classical mechanics[1]. However, our more abstract formulation has the advantage of allowing us to speak more generally about the spatial position of a classical system. Indeed, we follow Woodhouse [6] in observing that a maximal orthogonal set is the natural classical analogue of a complete set of a commuting observables in quantum mechanics. In this sense, the assumption that "classical position" is a maximal orthogonal set is analogous to the assumption that "quantum position" as a complete set of commuting observables, and hence, that there are no internal degrees of freedom like spin or charge.

Given a classical system $(\mathcal{P}, \Omega_{ab}, h)$ with a maximal orthogonal set $\{\overset{1}{q}, \overset{2}{q}, \ldots, \overset{n}{q}\}$, we can define the "velocity" or instantaneous change in this set over time. Since change over time is given by the phase flow φ^h_t generated by h, the velocity of a function q is given by

$$v(t) := \frac{d}{dt}(q \circ \varphi^h_t),$$

In what follows, we will make use in particular of the *initial velocity* v of a classical system, defined by

$$v := v(0) = \left.\frac{d}{dt}(q \circ \varphi^h_t)\right|_{t=0} = \{q, h\}, \tag{2}$$

where the last equality follows from our observation in Equation (1).

4. Defining translations and boosts

In Galilean physics, spatial translations and Galilei boosts are transformations that involve the simple "linear addition" of a vector to the value of position and velocity, respectively.

Definition 2 (Translations and Boosts)**.** We take a *spatial translation and Galilei boost group* for a classical system $(\mathcal{P}, \Omega_{ab}, h)$ to be a $2n$-parameter group of diffeomorphisms $\Phi(\sigma, \rho) : \mathcal{P} \to \mathcal{P}$, which forms a representation of $\mathbb{R}^{2n}$, and such that

(i) $q \circ \Phi(\sigma, \rho) = q + \sigma$
(ii) $v \circ \Phi(\sigma, \rho) = v + \rho$

where $q = \{\overset{1}{q}, \ldots, \overset{n}{q}\}$ is a maximal set of orthogonal functions, and v is the corresponding initial velocity. We define two associated diffeomorphism groups $\varphi^s_\sigma := \Phi(\sigma, 0)$ and $\varphi^r_\rho := \Phi(0, \rho)$, and refer to them as the *translation group* and the *boost group*, respectively. When these groups have a generator, we denote those generators by $s : \mathcal{P} \to \mathbb{R}$ and $r : \mathcal{P} \to \mathbb{R}$, respectively. To ensure that these generators

[1] This particular set is sometimes called the *vertical polarization* over $\mathbb{R}^n$. The "polarization" language comes from the fact that a maximal orthogonal set induces a foliation on $\mathcal{P}$, consisting of n-dimensional surfaces on which the values of the functions in $\{\overset{1}{q}, \overset{2}{q}, \ldots, \overset{n}{q}\}$ are constant. In the vertical polarization, each of these surfaces corresponds to the cotangent space at a point in $\mathbb{R}^n$.

correspond to n independent directions of space, we assume that $\{\overset{i}{s}, \overset{j}{s}\} = \{\overset{i}{r}, \overset{j}{r}\} = 0$ and $\{\overset{i}{s}, \overset{j}{r}\} = 0$ if and only if $i \neq j$.

Note that in adopting the shorthand $q = (\overset{1}{q}, \ldots, \overset{n}{q})$, Definition 2 really says that when $\overset{i}{q}$ and $\overset{j}{s}$ have the same index $i = j$, then $q \circ \varphi^s_\sigma = q + \sigma$, but $q \circ \varphi^s_\sigma = q$ when $i \neq j$. On the other hand, no matter what the values of i and j for q_i and r_j, we always have that $q \circ \varphi^r_\rho = q$. Similarly for initial velocity: when $\overset{i}{v}$ and $\overset{j}{r}$ have the same index $i = j$, then $v \circ \varphi^r_\rho = v + \rho$, and otherwise $v \circ \varphi^r_\rho = v$. Moreover, for all indices i, j of $\overset{i}{v}$ and $\overset{j}{s}$, we have that $v \circ \varphi^s_\sigma = v$. In what follows, we will sometimes spare ourselves having to write out all these indices by simply adopting the shorthand $q \circ \varphi^s_\sigma = q + \sigma$ and $v \circ \varphi^r_\rho = v + \rho$.

We now turn to "invariance" of the classical laws under a transformation $\Phi : \mathcal{P} \to \mathcal{P}$, by which we mean that a set of dynamical trajectories $H^a := \Omega^{ba} d_b h$ corresponding to the Hamiltonian h is possible only if the transformed set of trajectories $\tilde{H}^a := \Phi^* H^a$ is also possible with respect to some Hamiltonian $\tilde{h}$; that is, $\tilde{H}^a = \Omega^{ba} d_b \tilde{h}$ for some smooth $\tilde{h} : \mathcal{P} \to \mathbb{R}$. It is well known that this notion of invariance is equivalent to the statement that Φ preserves the symplectic form Ω_{ab} [5, Proposition 2.6.1]. So, the following definition of invariance makes sense.

Definition 3 (Translation and Boost invariance)**.** A classical system $(\mathcal{P}, \Omega_{ab}, h)$ is *invariant under spatial translations and Galilei boosts* if there exists a translation and boost group $\Phi(\sigma, \rho)$ on $\mathcal{P}$ such that each element of the group is symplectic, in that $\Phi^*(\sigma, \rho)\Omega_{ab} = \Omega_{ab}$ for all σ, ρ.

5. Classical Analogue of the Jauch Theorem

Theorem. *If a classical system $(\mathcal{P}, \Omega_{ab}, h)$ is invariant under spatial translations and Galilei boosts with respect to a maximal orthogonal set $\{\overset{1}{q}, \ldots, \overset{n}{q}\}$, then $\{\overset{i}{q}, \mu\overset{j}{v}\} = \delta_{ij}$ (i.e., 1 if $i = j$ and 0 otherwise) for some non-zero $\mu \in \mathbb{R}$, and $h = (\mu/2)v^2 + v(q)$ for some function v of q alone.*

It is convenient to build the proof of the theorem using three lemmas.

Lemma 1. *Let $(\overset{1}{r}, \ldots, \overset{n}{r})$ and $(\overset{1}{s}, \ldots, \overset{n}{s})$ be sets of functions on a symplectic manifold $(\mathcal{P}, \Omega_{ab})$, such that $\{\overset{i}{r}, \overset{j}{r}\} = \{\overset{i}{s}, \overset{j}{s}\} = 0$ and $\{\overset{i}{r}, \overset{j}{s}\} = \delta_{ij}$ (where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise). If f is any function such that $\{f, \overset{i}{r}\} = \{f, \overset{i}{s}\} = 0$ for each $i = 1, \ldots, n$, then f is a constant function.*

Proof. The $2n$ pairs $(\overset{i}{r}, \overset{j}{s})$ provide a local coordinate chart $\phi : U \to \mathbb{R}^{2n}$ for $\mathcal{P}$, defined by $\phi(x) := (r(x), s(x)) = \big(\overset{1}{r}(x), \ldots, \overset{n}{r}(x); \overset{1}{s}(x), \ldots, \overset{n}{s}(x)\big)$. So, any smooth function $f : \mathcal{P} \to \mathbb{R}$ may be written $f = f(r, s)$. Let $\overset{i}{R}{}^b = \Omega^{ab} d_a \overset{i}{r}$ and $\overset{i}{S}{}^b = \Omega^{ab} d_a \overset{i}{s}$ be the vector fields generated by $\overset{i}{r}$ and $\overset{i}{s}$, respectively. Then by assumption,

$$0 = \{f, \overset{i}{r}\} := \Omega^{ab}(d_a \overset{i}{r})(d_b f) = \overset{i}{R}{}^b d_b f,$$

$$0 = \{f, \overset{i}{s}\} := \Omega^{ab}(d_a \overset{i}{s})(d_b f) = \overset{i}{S}{}^b d_b f$$

for each $i = 1, \ldots, n$. This says that each of the $2n$ distinct directional derivatives of $f(r, s)$ vanish. Therefore, f is a constant function. $\square$

Lemma 2. *If $(\mathcal{P}, \Omega_{ab}, h)$ is invariant under spatial translations and Galilei boosts, with s generating the translation group and r generating the boost group, then* $\{\overset{i}{q}, \overset{j}{r}\} = \{\overset{i}{v}, \overset{j}{s}\} = 0$ *and* $\{\overset{i}{q}, \overset{j}{s}\} = \{\overset{i}{v}, \overset{j}{r}\} = \delta_{ij}$ *for all* i, j.

Proof. From the definition of spatial translations and Galilei boosts introduced above, we find by direct calculation that,

$$\{\overset{i}{q}, \overset{j}{s}\} = \frac{d}{d\sigma}\left(q \circ \varphi^s_\sigma\right) = \frac{d}{d\sigma}(q + \sigma\delta_{ij}) = \delta_{ij}, \quad \{\overset{i}{q}, \overset{j}{r}\} = \frac{d}{d\rho}\left(q \circ \varphi^r_\rho\right) = \frac{d}{d\rho}(q) = 0,$$
$$\{\overset{i}{v}, \overset{j}{r}\} = \frac{d}{d\rho}\left(v \circ \varphi^r_\rho\right) = \frac{d}{d\rho}(v + \rho\delta_{ij}) = \delta_{ij}, \quad \{\overset{i}{v}, \overset{j}{s}\} = \frac{d}{d\sigma}\left(v \circ \varphi^s_\sigma\right) = \frac{d}{d\sigma}(v) = 0.$$

□

Lemma 3. *If $(\mathcal{P}, \Omega_{ab}, h)$ is translation and Galilei boost invariant, with s generating the translation group and r generating the boost group, then* $\{\overset{i}{s}, \overset{j}{r}\} = \mu \in \mathbb{R}$, *where* $\mu = 0$ *if and only if* $i \neq j$.

Proof. We have assumed in the definition of Galilei invariance that $\{\overset{i}{s}, \overset{j}{r}\} = 0$ if and only if $i \neq j$, so it only remains to show that $\{\overset{i}{s}, \overset{j}{r}\}$ is a constant when $i = j$. This follows from the fact that the translation and boost group $\Phi(\sigma, \rho)$ is defined to be a representation of the additive group of real vectors. Since the latter is abelian, $\Phi(\sigma, \rho) = \Phi(\sigma, 0)\Phi(0, \rho) = \Phi(0, \rho)\Phi(\sigma, 0)$. Thus, the translation group $\varphi^s_\sigma := \Phi(\sigma, 0)$ and the boost group $\varphi^r_\rho := \Phi(0, \rho)$ are commuting diffeomorphism flows, in that $\varphi^s_\sigma\varphi^r_\rho = \varphi^r_\rho\varphi^s_\sigma$. Moreover, the invariance assumption entails that these flows are symplectic. But the symplectic flows generated by s and r commute if and only if $\{s, r\}$ is a constant function [1, p.218 Cor.9]. Therefore, $\{\overset{i}{s}, \overset{j}{r}\} = \mu \in \mathbb{R}$. □

The theorem is now established by the following two propositions.

Proposition 1. *If $(\mathcal{P}, \Omega_{ab}, h)$ is translation and Galilei boost invariant with respect to a maximal orthogonal set* $\{\overset{1}{q}, \ldots, \overset{n}{q}\}$, *then* $\{\overset{i}{q}, \mu\overset{j}{v}\} = \delta_{ij}$, *where* $\delta_{ij} = 1$ *if* $i = j$ *and* 0 *otherwise, and* $\mu = 0$ *if and only if* $i \neq j$.

Proof. By our invariance assumption, the translation and Galilei boost groups are symplectic. This is a necessary and sufficient condition for each to have a generator [5, Proposition 2.6.1], which we denote by s and r, respectively. We have assumed that $\{\overset{i}{s}, \overset{j}{s}\} = \{\overset{i}{r}, \overset{j}{r}\} = 0$ in the definition of these transformations. Moreover, we know by Lemma 3 that

(i) $\{\overset{i}{s}, \overset{j}{r}\} = \mu \in \mathbb{R}$,

where $\mu = 0$ if and only if $i \neq j$. This implies that the set of functions s and r provide a local orthonormal coordinate chart for $\mathcal{P}$. So, whenever a function Poisson commutes with both s and r, we may conclude from Lemma 1 that it is a constant function. We will now use this fact to show that the function $\overset{i}{r} + \mu\overset{i}{q}$ is a constant function each $i = 1, \ldots, n$.

Since the Poisson bracket is skew-symmetric, $\{\overset{i}{s}, \overset{j}{r}\} = \mu$ is equivalent to $\{\overset{i}{r}, \overset{j}{s}\} = -\mu$. From Lemma 2, we also have the relations,

(ii) $\{\overset{i}{q}, \overset{j}{s}\} = \{\overset{i}{v}, \overset{j}{r}\} = \delta_{ij}$,

(iii) $\{\overset{i}{q}, \overset{j}{r}\} = \{\overset{i}{v}, \overset{j}{s}\} = 0$.

Multiplying both sides of $\{\overset{i}{q}, \overset{j}{s}\} = \delta_{ij}$ by μ, we get $\{\mu\overset{i}{q}, \overset{j}{s}\} = \mu$. Using the linearity of the Poisson bracket, we thus find that for all j,

$$\{\overset{i}{r} + \mu\overset{i}{q}, \overset{j}{s}\} = \{\overset{i}{r}, \overset{j}{s}\} + \{\mu\overset{i}{q}, \overset{j}{s}\} = -\mu + \mu = 0.$$

But the function $(\overset{i}{r} + \mu\overset{i}{q})$ also Poisson commutes with $\overset{j}{r}$ for all j, since $\{\overset{i}{r}, \overset{j}{r}\} = \{\overset{i}{r}, \overset{j}{q}\} = 0$. So, $\overset{i}{r} + \mu\overset{i}{q}$ Poisson commutes with both $\overset{j}{r}$ and $\overset{j}{s}$ for all j, and we may conclude that $\overset{i}{r} + \mu\overset{i}{q} = k$ for some constant k, or equivalently, $\overset{j}{r} = -\mu\overset{j}{q} + \mu k$. Substituting this into $\{\overset{i}{v}, \overset{j}{r}\} = \delta_{ij}$, we have

$$\delta_{ij} = \{\overset{i}{v}, \overset{j}{r}\} = \{\overset{i}{v}, (-\mu\overset{j}{q} + \mu k)\} = \{\overset{i}{v}, (-\mu\overset{j}{q})\} = \{\overset{j}{q}, \mu\overset{i}{v}\},$$

where the penultimate equality follows from the fact that the Poisson bracket is linear and vanishes for constants in either term, and the last equality is an application of skew symmetry. $\square$

Proposition 2. *If $(\mathcal{P}, \Omega_{ab}, h)$ is a classical system with $\{q, \mu v\} = 1$ for some $\mu \in \mathbb{R}$, then $h = \frac{\mu}{2}v^2 + v(q)$ for some function v of q alone.*

Proof. From the fact that the Poisson bracket satisfies the Leibniz rule,

$$\left\{q, \frac{\mu}{2}v^2\right\} = \frac{1}{2}v\,\{q, \mu v\} + \frac{1}{2}\,\{q, \mu v\}\,v = \frac{1}{2}v + \frac{1}{2}v = v,$$

where the penultimate equality follows from our hypothesis that $\{q, \mu v\} = 1$. But by definition, $v = \{q, h\}$ (see Equation (2)). Subtracting the expression for v just calculated from this definition, we see that $\{q, h - \frac{\mu}{2}v^2\} = 0$. But q is a maximal orthogonal set (Definition 1), and so by definition, $h - \frac{\mu}{2}v^2 = v(q)$ for some function v of q alone. $\square$

Corollary 1. *On the same assumptions, $\mu v = s + k$, where s is the generator of the spatial translation group φ^s_σ and k is a constant.*

Proof. We have assumed that $\{\overset{i}{q}, \overset{j}{q}\} = \{\overset{i}{s}, \overset{j}{s}\} = 0$, and Lemma 2 implies that $\{\overset{i}{q}, \overset{j}{s}\} = \delta_{ij}$. Therefore, q and s provide a local orthonormal coordinate chart. Lemma 1 thus implies that the only functions that Poisson commutes with both q and s are the constant functions. We now show that $\mu\overset{i}{v} - \overset{j}{s}$ is one of them.

We know that $\{\overset{i}{q}, \overset{j}{s}\} = \delta_{ij}$ by Lemma 2, and $\{\overset{i}{q}, \mu\overset{j}{v}\} = \delta_{ij}$ by Proposition 1. Therefore, applying the linearity of the Poisson bracket, we find that

$$\{\overset{i}{q}, (\overset{j}{s} - \mu\overset{j}{v})\} = \{\overset{i}{q}, \overset{j}{s}\} - \{\overset{i}{q}, \mu\overset{j}{v}\} = 0$$

for all $j = 1, \ldots, n$. Moreover, we have assumed that $\{\overset{i}{s}, \overset{j}{s}\}$ in the definition of Galilei invariance, and found that $\{\overset{i}{s}, \overset{\}}{v} = 0$ in Lemma 2. Therefore,

$$\{\overset{i}{s}, (\mu\overset{j}{v} - \overset{j}{s})\} = \{\overset{i}{s}, \mu\overset{j}{v}\} - \{\overset{i}{s}, \overset{j}{s}\} = 0.$$

We conclude that $\mu\overset{j}{v} - \overset{j}{s}$ must be a constant function for all j, and so we may write $\mu v = s + k$ for some constant $k \in \mathbb{R}^n$. $\square$

Since the generator s of a group of symplectic transformations φ^s_σ is only defined up to an additive constant, this implies that we can choose a generator s of the same group of spatial translations such that $s = \mu v$. Taking this generator to be the definition of "momentum," we may interpret this result as the well-known relation that the momentum of a classical system is proportional to its velocity. We thus have the following.

Corollary 2. *On the same assumptions, q and μv form an orthonormal coordinate chart.*

Proof. Combing the result of Corollary 1 with our assumption that $\{\overset{i}{s}, \overset{j}{s}\} = 0$ for all i, j, it follows immediately that $\{\mu\overset{i}{v}, \mu\overset{j}{v}\} = 0$ for all i, j. Moreover, $\{\overset{i}{q}, \overset{j}{q}\} = 0$ as a consequence of our assumption that q forms a maximal orthogonal set. Our conclusion thus follows from the result of Proposition 1 that $\{\overset{i}{q}, \mu\overset{j}{v}\} = \delta_{ij}$. □

As a final observation for this section, we note that systems satisfying the assumptions of our theorem are guaranteed to be time reversal invariant. A *time reversal transformation* is an antisymplectic mapping (i.e., one that reverses the sign of the symplectic form Ω_{ab}). For a local coordinate system (q, p), a time reversal transformation has the form $\tau : (q, p) \mapsto (q, -p)$. Setting $p = \mu v$, we can thus immediately see that Galilei invariant classical systems are time reversal invariant, since the Hamiltonian $h(q, p) = (1/2\mu)p^2 + f(q)$ remains unchanged under the transformation τ. In summary, we have the following.

Corollary 3. *On the same assumptions, the system is time reversal invariant, in that the transformation $\tau : (q, \mu v) \mapsto (q, -\mu v)$ is antisymplectic and leaves the Hamiltonian unchanged.*

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