3.16 Functions of Commuting Operators

Thm 16.1 Let A be self-adjoint with spectral decomposition $A = \int x\, dE_x$. If B is bdd, s-adjt, and $AB = BA$ then $E_x B = B E_x$

Proof for A has pure point spectrum, $A = \sum a_k I_k$, the proof is trivial for unbdd A, a more careful statement...

Indeed for A_1 and A_2, both unbdd, we take $E_x^{(1)} E_y^{(2)} = E_y^{(2)} E_{(x)}^{(1)}$ $\forall x, y \in \mathbb{R}$ as the definition of $[A_1, A_2] = 0$.

Given $f: \mathbb{R}^2 \to \mathbb{C}$ $(x,y) \mapsto f(x,y) \in \mathbb{C}$, we define $f(A_1, A_2)$ for A_1, A_2 commuting s-adjoint operators by:-

$$(\phi, f(A_1, A_2)\psi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, d_x d_y\, (\phi, E_x^{(1)} E_y^{(2)} \psi) \quad \forall \phi, \psi \in \mathcal{H}.$$

Sums, scalar multiples, products of functions, and adjoints.

$$[f(A_1, A_2)]^{+} = (f^{*})(A_1, A_2) \quad \text{where} \quad (f^{*})(x,y) := f(x,y)^{*}$$

Like f) g) h) on p.7 lam.

3.17 Complete sets of Commuting Operators, Spectral Representation

Let $A_1, A_2 \ldots A_N$ be mutually commuting self-adjoint with pure point spectra. For $r = 1, 2, \ldots N$

$$A_r = \sum_k a_k^{(r)} I_k^{(r)}$$

spectral decomposition
$a_k^{(r)}$ evalues of A_r
$I_k^{(r)}$ eigen projectors of A_r

Then for all r, s and $j k$

$$I_j^{(r)} I_k^{(s)} = I_k^{(s)} I_j^{(r)}$$

and for any $j k \ldots l$, the product $I_j^{(1)} I_k^{(2)} \ldots I_l^{(N)}$ is a projector onto simultaneous eigenvectors with corrpdg eigenvalues, i.e. ψ with $A_1\psi = a_j^{(1)}\psi$, $A_2\psi = a_k^{(2)}\psi$, ... $A_N\psi = a_l^{(N)}\psi$.

If none of these projects onto a subspace of dimension larger than one, then we say $\{A_1, A_2 \ldots A_N\}$ is complete set of commuting operators

So $I_j^{(1)} I_k^{(2)} \ldots I_l^{(N)} = |a_j^{(1)}, a_k^{(2)} \ldots a_l^{(N)}\rangle\langle a_j^{(1)}, a_k^{(2)}, \ldots a_l^{(N)}|$

Thm 17.1 Let $\{A_1 \dots A_N\}$ be mutually commuting s-adjoint operators, with pure point spectra. This is a complete set iff: every bdd oper that commutes with all $A_1 \dots A_N$ is a function of the $A_1 \dots A_N$.

The orthonormal basis $|a_j^{(1)}, a_k^{(2)} \dots a_\ell^{(N)}\rangle$ gives a spectral representation of the $A_1 \dots A_N$, and their functions $f(A_1 \dots A_N)$, as diagonal matrices:

$$\langle a_j^{(1)}, a_k^{(2)} \dots a_\ell^{(N)} | f(A_1 A_2 \dots A_N)\psi\rangle = f(a_j^{(1)}, a_j^{(2)} \dots a_\ell^{(N)}) \langle a_j^{(1)}, a_k^{(2)} \dots a_\ell^{(N)} | \psi\rangle.$$

For operators with continuous spectra, a complete set of commuting operators is defined by the condition (rhs) in Thm 17.1

Thm 17.2 Every bdd oper which commutes with Q (beware: care about domains!) is a function of Q

(a) For continuous quantities like Q, we write the spectral representation as

$$\langle x|\psi\rangle = \psi(x) \qquad \psi : \mathbb{R} \to \mathbb{C}$$

and $\langle x|Q\psi\rangle = x\langle x|\psi\rangle$

and $\langle x|f(Q)\psi\rangle = f(x)\langle x|\psi\rangle$

(b) Q has no eigenvectors (if $x\psi(x) = a\psi(x)$, then $\psi(x) = 0$ for $x \neq a$ and $\|\psi\|^2 = 0$)
But we use delta-functions, so that

$$x\,\delta(x-a) = a\,\delta(x-a)$$

"justifies"

$$Q|a\rangle = a|a\rangle$$

$$\langle a|\psi\rangle = \psi(a) = \int_{-\infty}^{\infty} \delta(x-a)\psi(x)\,dx$$

and $\psi(x) = \int \psi(a)\delta(x-a)\,da$ "justifies" $|\psi\rangle = \int \langle a|\psi\rangle |a\rangle\,da$

i.e. any vec "thought of" as lin comb delta functions

(c) For each $a \in \mathbb{R}$, define $|a\rangle\langle a|$ by $(|a\rangle\langle a|\psi\rangle)(x) = \psi(a)\delta(x-a)$

i.e. $|a\rangle\langle a|\psi = \langle a|\psi\rangle|a\rangle$

Then since $(E_x\psi)(y) = \int_{-\infty}^{x} \psi(a)\delta(y-a)\,da = \int^{x} (|a\rangle\langle a|\psi)(y)\,da \quad \forall\psi$

E_x spectral family of Q

we write $E_x = \int_{-\infty}^{x} |a\rangle\langle a|\,da$

and $Q = \int_{-\infty}^{\infty} x\,dE_x$ is written also as $Q = \int_{-\infty}^{\infty} x\,|x\rangle\langle x|\,dx$

(d) Similarly for 3 dimensions $\vec{Q} = (Q_1, Q_2 Q_3)$ is complete, $\vec{Q} = \int \vec{x}\,|\vec{x}\rangle\langle\vec{x}|\,d\vec{x}$

3.18 Fourier Transforms, Spectral Representation of $-i\vec{\nabla}$

p.11 Jordan

On $L^2(\mathbb{R}^3)$, we define
$$(P_r\psi)(\vec{x}) = i\frac{\partial}{\partial x_r}\psi(\vec{x}) \qquad r=1,2,3$$
These are self-adjoint i.e. the symmetric property $(\psi, P_r\phi) = (P_r\psi,\phi)$ and the domain is dense so that $P_r^\dagger$ is defined, and indeed $P_r^\dagger = P_r$. ⊛ (cf item c) on p.4)

We write $\vec{P} = (P_1, P_2, P_3)$ $(\vec{P}\psi)(\vec{x}) = -i\vec{\nabla}\psi(\vec{x})$

Thm 18.1 For any $\psi \in L^2(\mathbb{R}^3)$
$$\chi_n(\vec{k}) := (2\pi)^{-3/2}\int_{-n}^{n}dx_1\int_{-n}^{n}dx_2\int_{-n}^{n}dx_3\, \exp(-i\vec{k}\cdot\vec{x})\,\psi(\vec{x})$$
defines a sequence $\{\chi_n\}$ in $L^2(\mathbb{R}^3)$, that converges to a limit $F\psi$ which obeys:—
$$\|F\psi\|^2 = \|\psi\|^2 \qquad F\psi \text{ is fourier transform of } \psi$$
$\psi \in \mathrm{dom}(P_r)$ iff $k_r(F\psi)(\vec{k})$ is square-integrable.

in which case: $(FP_r\psi)(\vec{k}) = k_r(F\psi)(\vec{k})$. $\qquad (\ddagger) \qquad \square$

a) We write $(F\psi)(\vec{k}) = (2\pi)^{-3/2}\int e^{-i\vec{k}\cdot\vec{x}}\psi(\vec{x})\,d\vec{x}$

and $\psi(\vec{x}) = (2\pi)^{-3/2}\int (F\psi)(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\,d\vec{k} \qquad (\S)$

Since F ~~...~~ preserves norm and has an inverse, it is unitary (§2.8, p2 low), and preserves inner products

b) We write the inverse as: $(F^{-1}\phi)(\vec{x}) = (2\pi)^{-3/2}\int\phi(\vec{k})e^{i\vec{k}\cdot\vec{x}}\,d\vec{k}$

or $(F^{-1}\phi)(\vec{x}) = (F\phi)(-\vec{x})$

and $\int (F\phi)(\vec{k})^*(F\psi)(\vec{k})\,d\vec{k} = \int\phi(\vec{x})^*\psi(\vec{x})\,d\vec{x}$

c) (§) implies $(P_r\psi)(\vec{x}) = (2\pi)^{-3/2}\int (F\psi)(\vec{k})\,k_r\exp(i\vec{k}\cdot\vec{x})\,d\vec{k}$

or for 3 dims $\qquad -i\vec{\nabla}\psi(\vec{x}) = (2\pi)^{-3/2}\int (F\psi)(\vec{k})\,\vec{k}\,\exp(i\vec{k}\cdot\vec{x})\,d\vec{k}$

Writing $(Q_r\psi)(\vec{x}) = x_r\psi(\vec{x})$, $(\ddagger)$ then implies
$$FP_r = Q_rF \quad \text{i.e. } \vec{P} = F^{-1}\vec{Q}F.$$

(d) This now implies the spectral decomposition of P_r. One checks that with $Q = \int x \, dE_x^{(r)}$, the set $\{F^{-1} E_x^{(r)} F\}$ is a spectral family (definition in § 3.14 p.6). Then with $F^{-1} = F^{\dagger}$,

$$(\phi, P_r \psi) = (\phi, F^{\dagger} Q_r F\psi) = (F\phi, Q_r F\psi)$$
$$= \int_{-\infty}^{\infty} x \, d(F\phi, E_x^{(r)} F\psi) = \int_{-\infty}^{\infty} x \, d(\phi, F^{-1} E_x^{(r)} F\psi)$$

So the spectral decomposition for P_r is

$$P_r = \int_{-\infty}^{\infty} x \; d F^{-1} E_x^{(r)} F \, .$$

(e) We now do for momentum, $\vec{P} = (P_1, P_2, P_3)$, the discussion of complete commuting operators, and their functions, that we did for position $\vec{Q} = (Q_1, Q_2, Q_3)$ in Sections 3.16 & 3.17, (p.9 and p.10, ending in (d) p.10 low)

(e1) Recall Section 3.16's definition of a function $f(A_1, A_2)$ of two commuting operators A_1, A_2 in terms of inner products $(\phi, E_x^{(1)} E_y^{(2)} \psi)$ (p.9)

Then functions of P_1, P_2, P_3 are defined by integrals wrt inner products

$$(\phi, F^{-1} E_x^{(1)} F . F^{-1} E_y^{(2)} F . F^{-1} E_z^{(3)} F\psi) = (F\phi, E_x^{(1)} E_y^{(2)} E_z^{(3)} F\psi)$$

So for any $f : \mathbb{R}^3 \to \mathbb{C}$, the operator $f(\vec{P}) = f(P_1, P_2, P_3)$ is determined by inner products:

$$(\phi, f(\vec{P})\psi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z) \, d_x d_y d_z (F\phi, E_x^{(1)} E_y^{(2)} E_z^{(3)} F\psi)$$
$$\equiv (F\phi, f(Q_1, Q_2, Q_3) F\psi) = (\phi, F^{-1} f(\vec{Q}) F\psi)$$

So $f(\vec{P}) = F^{-1} f(\vec{Q}) F$. We can write:

(cf. (‡) p.11) $\quad (F f(\vec{P}) \psi)(\vec{k}) = f(\vec{k}) (F\psi)(\vec{k})$

(cf. (§) p.11) $\quad (f(\vec{P}) \psi)(\vec{x}) = (2\pi)^{-3/2} \int (F\psi)(\vec{k}) \, f(\vec{k}) \, e^{i\vec{k}.\vec{x}} \, d\vec{k}$

p.13

e2 [i] The operators $\vec{P}$ have no eigenvectors, but they have eigenfunctions (unnormalized, not in L^2!) which are like Jordan eigenvectors.

Write $|\vec{k}\rangle$ for $(2\pi)^{-3/2}\exp(i\vec{k}\cdot\vec{x})$

Then $\vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$ for $-i\vec{\nabla}(2\pi)^{-3/2}e^{i\vec{k}\cdot\vec{x}} = \vec{k}(2\pi)^{-3/2}e^{i\vec{k}\cdot\vec{x}}$

[ii] For the components of $\psi \in \mathcal{H}$, in the spectral representation of $\vec{P}$, we have

$$\langle\vec{k}|\psi\rangle = \int [(2\pi)^{-3/2}e^{i\vec{k}\cdot\vec{x}}]^* \psi(\vec{x})\, d\vec{x}$$

And repeating (§) p.11, i.e. $\psi(\vec{x}) = (2\pi)^{-3/2}\int (F\psi)(\vec{k})e^{i\vec{k}\cdot\vec{x}}d\vec{k}$,

it is now:

$$|\psi\rangle = \int \langle\vec{k}|\psi\rangle |\vec{k}\rangle d\vec{k}$$

[iii] The eigenfunctions are the inverse Fourier trsfm of a delta-function.

$$(2\pi)^{-3/2}e^{i\vec{k}\cdot\vec{x}} = (2\pi)^{-3/2}\int \delta^3(\vec{k}' - \vec{k})e^{i\vec{k}'\cdot\vec{x}}d\vec{k}'$$

[iv] The expression for the spectral projectors of $\vec{P}$ in terms of improper spectral projectors $|\vec{k}\rangle\langle\vec{k}|$:-

$$((F^{-1}E_x^{(r)}F)\psi)(\vec{y}) = \int_{k_r \leq x} (|\vec{k}\rangle\langle\vec{k}|\psi)(\vec{y})\, d\vec{k}$$

That is, $$F^{-1}E_x^{(r)}F = \int_{k_r \leq x} |\vec{k}\rangle\langle\vec{k}|\, d\vec{k}$$

and so the spectral decomposition (d, top of p.12) is written

$$\vec{P} = \int \vec{k}\, |\vec{k}\rangle\langle\vec{k}|\, d\vec{k}$$

on analogy with $\vec{Q}$, and with spectral decomposition with evalues and eigenvectors.

Ch 4 Operator Algebras

Recall Thm 12.7 (p.5) on M reduces A iff $E_M A = A E_M$

4.19 Irreducible Operators, Schur's Lemma

Definitions of: a set of operators is reducible
a set of operators is irreducible
a subspace M is invariant under a set of operators

So M reduces a set $\{A_i\}$ of operators iff M and $M^\perp$ are invariant under A_i

a set of operators is symmetric (viz. closed under adjoint)

Thm 19.1 If M is invariant under a symmetric set of operators then M reduces the set. That is: $M^\perp$ is also invariant.

Thm 19.2 (Schur's Lemma) A symmetric set of bdd or Hermitian operators is irreducible iff: multiples of the identity are the only bdd operators which commute with ~~all~~ operators in the set.

Proof The leftward implication is almost trivial.
The rightward implication uses previous work straightforwardly – as follows, considering the cases:–

(1) E a projection operator commuting with every operator in the set, assumed irreducible
So $E = \mathbb{1}$ or $E = 0$

(2) B a bdd Hermitian operator commuting with....
then (*) every spectral projection E_x of B commutes....
So $E_x = \mathbb{1}$ or $E_x = 0$. So $\exists b \in \mathbb{R}$ with $E_x = 0$ for $x < b$
$E_x = \mathbb{1}$ for $x \geq b$ etc

(3) B a bdd non-Hermitian operator commuting with...
Take Adjoints (since set symmetric) and Re and Im parts.

(*) This uses the set containing only bdd or Hermitian operators: fn. 3, p 68.

Example the set $\{\vec{Q}, \vec{P}\} = \{Q_1, Q_2, Q_3, P_1, P_2, P_3\}$ is irreducible on $L^2(\mathbb{R}^3)$.

4.20 Functions of Non-Commuting Operators, von Neumann algebras

Governing question: Given a set of non-commuting Hermitian operators, which bdd operators should be considered functions of them?

① A set of bdd opers is a symmetric ring / symmetric algebra / *algebra iff it is closed under: scalar mult, addit, product (composition, even non-commuting) and taking adjoints.

Example: all bdd functions of a set of commuting Hermitian operators.

② For the question, we certainly want: all operators we can get by scalar multiple, sum, product, starting from bdd functions of each Hermitian.

This is a symmetric ring of bdd opers, each a polynomial of bdd functions of the individual Hermitian opers.

③ But it is natural to include some sort of limit. It turns out that weak limit is the right notion.

A bdd oper B is the weak limit of a set of bdd operators iff $\forall \varepsilon > 0\ \forall n \in \mathbb{N}\ \forall \psi_1 \dots \psi_n\ \forall \phi_1, \phi_2 \dots \phi_n$, there is A in the set s.t.

$$\forall k = 1, 2 \dots n \quad |(\phi_k, A\psi_k) - (\phi_k, B\psi_k)| < \varepsilon$$

Hence weak closure of a set of bdd operators;
a set being weakly closed
a weakly closed symmetric ring of bdd opers is a von Neumann algebra or W^* algebra

④ For functions of commuting operators, we have

Thm 20.1 A bdd oper B is a function of commuting Hermitian opers $A_1 \dots A_N$ iff: B commutes with every bdd oper that commutes with $A_1, A_2, \dots A_N$.

The commutant S' of a set S of operators is the set of bdd operators that commute with all of S.

So Thm 20.1 says "B is a (bdd) function of $\{A_1 \dots A_N\}$ iff $B \in \{A_1 \dots A_N\}''$".

⑤ von Neumann's double commutant theorem generalizes Thm 20.1 to non-commuting case, using weak limits ("combine ③ & ④")

Thm 20.2 Given a set S of Hermitian operators, S'' is a von Neumann algebra. It is weak closure of symmetric ring of polynomials generated by bdd functions of each element of S.

S'' is the von Neumann algebra generated by S: the smallest such algebra.

If S is commuting, Thm 20.2 reduces to Thm 20.1

Corollary 20.3 (Combine Schur's Lemma & Thm 20.2)
If S is an irreducible set of Hermitian ~~operators~~ operators, then $S' = \{c\mathbb{1} \mid c \in \mathbb{C}\}$. So $S'' = \mathcal{B}(\mathcal{H})$:= set of all bdd operators on $\mathcal{H}$

Example Since $\{\vec{Q}, \vec{P}\}$ is irreducible on $L^2(\mathbb{R}^3)$

$$\{\vec{Q}, \vec{P}\}'' = \mathcal{B}(L^2(\mathbb{R}^3)).$$

Addendum by JNB on Operator topologies

① A sequence $\{A_n\}$ of bdd ops on $\mathcal{H}$ converges to A in the uniform aka norm topology iff $\|A_n - A\| \to 0$ as $n \to \infty$

uniform

② ... in the strong topology iff $\forall \psi \in \mathcal{H}$, $(A_n - A)\psi \to 0$ as $n \to \infty$

pointwise, "vector by vector"

③ ... in the weak topology iff $\forall \psi, \varphi \in \mathcal{H}$, $|(\psi, (A_n - A)\varphi)| \to 0$, $n \to \infty$

again, pointwise

ⓐ Converge uniformly $\Rightarrow$ converge strongly $\Rightarrow$ converge weakly
$\not\Leftarrow$ $\not\Leftarrow$

In finite dims, all equal/equivalent

ⓑ So an order of strength: in stronger topologies, fewer sequences converge

ⓒ Consider $\{\varphi_n\}$ be an onbasis, and E_n the projector onto the subspace spanned by $\{\varphi_1, \dots \varphi_n\}$.
The sequence $\{E_n\}$ converges to identity, strongly but not uniformly.
Thus $\{E_n\}$ is a Cauchy sequence in strong topology:

$$\forall \psi, \forall \varepsilon > 0 \; \exists N \; \forall n > m > N \quad \|(E_n - E_m)\psi\| < \varepsilon$$

i.e. the component of each fixed ψ in $(\mathrm{ran}(E_n))^\perp$ ~~tends~~ tends to zero.
But not uniformly! $\|E_n - E_m\| = 1$ for $n \neq m$.

ⓓ For "weak but not strong", consider $A_n : \mathcal{H} \to \mathcal{H}$ defined by
$A_n(\varphi_k) := \varphi_{n+k}$
"n-shift"
$\{A_n\}$ converges weakly to zero, not strongly

5.21 Measurable quantities

Generalities about the classical & quantum cases eg a non-commutative algebra, instead of commutative.

5.22 Density matrices and traces

A density matrix W is a positive self-adjoint operator, such that if $\{\psi_k\}$ is an o.n. basis then $\sum_k (\psi_k, W\psi_k) = 1$

Thm 22.1 A density matrix has pure point spectrum so that

$$W = \sum_k w_k |\phi_k\rangle\langle\phi_k| \quad \text{with } \phi_k \text{ orthonormal}, w_k \geq 0 \text{ and } \sum w_k = 1.$$

Thm 22.2 If W is a density matrix, B is bounded and $\{\phi_k\}$ $\{\psi_j\}$ are both o.n. bases then

$$\sum_k (\phi_k, WB\phi_k) \quad \sum (\psi_j, WB\psi_j) \quad \sum_k (\phi_k, BW\phi_k) \quad \sum_j (\psi_j, BW\psi_j)$$

are all absolutely convergent and equal.

Hence the definition of trace and from Thm 22.1, we have

(§) $\mathrm{Tr}(WB) = \sum_k w_k (\phi_k, B\phi_k)$ with $\{\phi_k\}$ an eigenbasis of W

If B is self-adjoint, $\mathrm{Tr}(WB)$ is real.

" " " " and positive, $\mathrm{Tr}(WB) \geq 0$.

A "(analog of) total probability": If E_k are mutually orthogonal, so that $\sum_k E_k$ is a projector

$$\mathrm{Tr}(W \sum_k E_k) = \sum_k \mathrm{Tr}(WE_k)$$

The linearity of trace implies:

$$\forall c \in \mathbb{C}, \quad \mathrm{Tr}(WcB) = c\,\mathrm{Tr}(WB)$$

$\forall$ bounded operators A, B $\quad \mathrm{Tr}\, W(A+B) = \mathrm{tr}\, WA + \mathrm{tr}\, WB$

Beware of this last in J.S. Bell's critique of von Neumann's "no-hidden-variable" proof cf. (Bell 1966).

(b) the "proper"/"ignorance-interpretation" mixture way of thinking of (§) fails for density matrices obtained by partial tracing of an entangled composite state.

5.23 Representation of States

von Neumann's "no-hidden-variables" proof (1932, Chap IV §2)

Assume a state assigns to every bounded operator B a complex number ("expectation value") $\langle B \rangle$, subject to

(i) If B is self-adjoint, $\langle B \rangle \in \mathbb{R}$

(ii) If B is self-adjoint and positive, $\langle B \rangle \geq 0$

(iii) $\forall c \in \mathbb{C}$ $\quad \langle cB \rangle = c \langle B \rangle$

(iv) $\quad \langle A + B \rangle = \langle A \rangle + \langle B \rangle \quad \leftarrow$ beware!

(v) $\quad \langle 1 \rangle = 1$

(vi) countable additivity: $\{E_k\}$ mutually orthogonal

$$\left\langle \sum_k E_k \right\rangle = \sum_k \langle E_k \rangle .$$

Thm 23.1 Any state in the above sense, for all bdd operators B, is represented by a unique density matrix W, i.e. there is a unique W such that

$$\forall B, \quad \langle B \rangle = \mathrm{Tr}(WB)$$

Proof Ideas (a) uniqueness is straightforward algebra to get ~~the ma~~ arbitrary matrix element $(\varphi, W\chi)$ of W fixed by letting each one dimensional projector $|\phi\rangle\langle\phi|$ substitute for B

(b) Straightforward that any such W is Hermitian, positive and trace 1. So is a density matrix

(c) For $\mathcal{H}$ fin-dimensional, the space $\mathrm{Lin}(\mathcal{H}) \equiv \mathrm{End}(\mathcal{H}, \mathcal{H})$ of linear operators on $\mathcal{H}$ has an inner product

$$(A, B) := \mathrm{Tr}(A^\dagger B)$$

Properties (iii) and (iv) imply that $\langle\ \rangle$ is a linear functional on $\mathrm{Lin}(\mathcal{H})$, and so there is an operator ("Riesz!") W such that

$$\langle A \rangle = (W, A) \equiv \mathrm{Tr}(W^\dagger A) \ \ldots$$ but W is unique and a density matrix

(d) For $\mathcal{H}$ infinite-dimensional, $\mathrm{Tr}(A^\dagger B)$ does not converge for all bdd operators A, B.

5.24 Probabilities

a) Discussion of probability distributions, probability density functions, and characteristic functions for quantities with continuous spectra, for vector states.

b) For example joint distribution for two quantities A_1 A_2 that commute with each other: —

Given $A_r = \int_{-\infty}^{\infty} x \, dE_x^{(r)}$ $\quad r = 1, 2$

Then the expectation value in state ψ, $\|\psi\| = 1$, of bounded function $f(A_1, A_2)$ is

$$\langle f(A_1, A_2) \rangle_\psi \equiv (\psi, f(A_1, A_2)\psi) = \iint f(x,y) \, d_x d_y (\psi, E_x^{(1)} E_y^{(2)} \psi)$$

$$= \iint f(x,y) \, d_x d_y \| E_x^{(1)} E_y^{(2)} \psi \|^2$$

And so the probability that the value of $A_1 \leq x$, and the value of $A_2 \leq y$, is:

$$\| E_x^{(1)} E_y^{(2)} \psi \|^2 = (\psi, E_x^{(1)} E_y^{(2)} \psi) = \langle E_x^{(1)} E_y^{(2)} \rangle_\psi$$

c) Probabilities of projectors as sufficient to determine a state, as in

Gleason's Theorem ~~If~~ $\mathcal{H}$ is separable and $\dim(\mathcal{H}) > 2$, ~~then~~ and to each projector E is assigned $\langle E \rangle \in \mathbb{R}$, $\langle E \rangle \geq 0$, such that $\langle 1 \rangle = 1$

and (countable additivity) $\quad \langle \sum_k E_k \rangle = \sum_k \langle E_k \rangle$ for any mutually orthogonal family $\{E_k\}$

then there is a unique density matrix W such that $\langle E \rangle = \mathrm{Tr}\, WE$ for all E.

Beware (Bell) that countable additivity implies constraints on the assignments to non-commuting probabilities, even though its explicit topic is the mutually orthogonal (and so mutually commuting) E_k.

5.25 Probabilities for complete sets of commuting operators

5.26 Uncertainty Principle

5.27 Simultaneous Measurability

An argument that if A, B are simultaneously measurable with unlimited precision, they must commute.

5.28 Superselection Rules

a) Idea: a restriction on the set of operators that represent measurable quantities.
A bounded operator that commutes with every operator representing a measurable quantity is a superselection operator

b) Superselection means that the density operator representing a state need not be unique.

Thm 2.8.| A cousin of von Neumann's Thm 23.1, but without the uniqueness claim. (Dixmier)

c) Suppose there is a complete set $\{A_\ell\}$ of commuting Hermitian operators representing measurable quantities.
Then every superselection operator S commutes with all A_ℓ
So S is a function of the A_ℓ
So all such superselection operators S, S', S'' commute with each other, ~~&~~ and so can be simultaneously diagonalized
Assume this gives a discrete decomposition, ~~e~~
That is: there is an orthogonal family $\{I_k\}$, with:
$\sum_k I_k = \mathbb{1}$; and every S has the form $\sum_k c_k I_k$;
and if $k \neq k'$, then there is a superselection operator $S = \sum_j c_j I_j$ such that $c_k \neq c_{k'}$. This is the usual structure of superselection.

d) Let $\mathcal{M}$ be the von Neumann algebra generated by the Hermitian operators representing measurable quantities
Proposition A bounded operator B commutes with all I_k iff $B \in \mathcal{M}$

If W is a density matrix, so also is $\sum_k I_k W I_k$
If B commutes with all I_k, then since
$$\mathrm{Tr}\, WB = \mathrm{Tr} \sum_k I_k W B I_k = \mathrm{Tr} \sum_k I_k W I_k B$$
and so W and $\sum I_k W I_k$ represent the same ~~state~~
So a state is represented equally well by vector ψ, $\|\psi\| = 1$
~~and~~ the density matrix $\sum_k I_k |\psi\rangle\langle\psi| I_k$