The Quantization of Linear Dynamical Systems I: Finite Systems

Spoken by JNB, but: taught by Caulton (adam.caulton@balliol.ox.ac.uk), Wikipedia ... ;
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This document, and its succcessor on the Quantization of Linear Dynamical Systems with Infinitely many degrees fo freedom, expound a rigorous quantization procedure developed by Irving Segal and others in the 1960s. This means we postpone to the second half of term, coverage of algebraic quantum theory; which will include e.g. inequivalent representations, 'getting out of Fock space', Haag's theorem etc. (cf. eg Emch 1972). But the present material:

(i) gives a strong grip on the first (forbiddingly concise!) third of Wald 1994, which is the basis for the rest of that book on QFT in curved spacetime and thus e.g. the Unruh effect (an essay!);

(ii) is of intrinsic interest... though please be warned that here you will find: no Lagrangian, no path integrals, no renormalization, no gauge theory, no curved spacetime, no gravitation; indeed, no interactions, and overall, not much physics ... we will focus on the harmonic oscillator (!),the free KG field and spin-chains (and without putting a Hamiltonian on the chain...). Nor will you find much straight-up philosophy ... but perhaps the light here shed on field/wave vs.particle counts as philosophy, since wave vs.particle is, like continuum vs. discrete, a perennial dichotomy of *natural philosophy*...

In this document, we consider only finitely many degrees of freedom, and lead up to the Stone-von Neumann Theorem, which essentially guarantees that the quantization of point particles in $\mathbb{R}^n$ is unique. We begin by introducing the Weyl form of the CCRs; and posing the quest for its representations (Section 1). Then we present the complexification and realification of vector spaces, complex structures etc. (Section 2); and symplectic vector spaces and manifolds (Section 3). Then we present linear systems, both classical and quantum; and thus the harmonic oscillator (Section 4). With all this in hand, we can then see the task of quantization as "unitarizing" a Hamiltonian evolution in a symplectic space so as to give an evolution in a complex Hilbert space. This gives the idea of a *one particle structure*, both in general and for the harmonic oscillator as an example (Section 5). The key to successful quantization, which see at work in the harmonic oscillator example, turns out to be the *two out of three property* of the unitary group: which concerns its relation to certain orthogonal and symplectic groups (Section 6)). Then we treat the case of finitely many harmonic oscillators, and so the occupation number representation: which can be described in a "Fock-space way" (Section 7). Finally, we state (i) the Stone-von Neumann Theorem; and (ii) an analogous theorem (the Jordan-Wigner theorem) about the uniqueness of the representation of the CARs (as against CCRs) of a *finite* system, such as a spin chain (Section 8).

Mottoes:

Let us try to introduce a quantum Poisson Bracket which shall be the analogue of the classical one....we are thus led to the following definition for the quantum Poisson Bracket of any two variables u and v: $uv - vu = i\hbar[u, v]$. Dirac (1930/1958, Section 21)

There is thus a complete harmony between the wave and light-quantum descriptions of the interaction. (Dirac, 1927, p. 245).

First quantization is a mystery, but second quantization is a functor. (E.Nelson).

Probably all these connections would have been clarified long ago, if quantum physicists had not been hampered by a prejudice in favor of complex and against real numbers. (Freeman Dyson)

The life of a theoretical physicist consists of solving harmonic oscillator at ever higher levels of abstraction. (S. Coleman)

Contents

1 Canonical quantization introduced

1.1 Commutation relations: from Heisenberg to Weyl

The idea of *canonical quantization* is familiar from elementary quantum mechanics: to "promote" the classical Poisson bracket relations

$$\{q^i, q^j\} = \{p_i, p_j\} = 0; \qquad \{q^i, p_j\} = \delta^i_j, \tag{1}$$

where $i, j \in \{1, 2, \dots n\}$, to the *Heisenberg relations* (CCRs)

$$[Q^i, Q^j] = [P_i, P_j] = 0; \qquad [Q^i, P_j] = i\hbar\delta^i_j \mathbb{1}; \tag{2}$$

(we will usually set $\hbar := 1$). This Poisson bracket-commutator correspondence originated with Dirac (cf. his *Principles of Quantum Mechanics* 1958, Section 21f.) The standard representation of eq. (2) is the familiar Schroedinger representation: namely, for n configurational degrees of freedom, e.g. a spinless particle in Euclidean n-space, or n such particles on a line:

$$Q^i\psi = q_i\psi, \qquad P_j\psi = -\frac{ih}{2\pi}\frac{\partial\psi}{\partial q_j} \qquad \text{for } \psi \in L^2(\mathbb{R}^n, d\mathbf{q}). \tag{3}$$

This prompts four main topics. They are of increasing scope, and we will consider only the first.

(a): To examine canonical quantization as just described for position and momentum in $\mathbb{R}^n$. The big positive result here is the Stone von Neumann theorem, stating (roughly) that for $\mathbb{R}^n$ as the configuration space, the Schroedinger representation of (2) is unique up to unitary equivalence. Cf Section 8. But so as to set the scene for quantum field theory, and more generally so as to get materials useful for contexts other than $\mathbb{R}^n$, we will lead up to this slowly. This will mean expounding some ideas of *Segal quantization*, which is the most straightforward generalization of the above ideas. In short: it replaces $\mathbb{R}^n$ as the classical configuration space, by an arbitrary n-dimensional manifold.

(b): To extend quantization to other quantities, in particular functions (polynomial, or even "arbitrary", functions) of position and momentum.

(c): To consider other methods of quantization.

(d) To pursue the *pure mathematical* interest of quantization. For a glimpse of this, cf. Folland (2008, p. 49; and Vogan 2005, cited there). In short: the interest lies in how it helps one find all the irreducible unitary representations of a connected Lie group G: i.e. in physical language, finding all quantum systems in which Gacts irreducibly as a symmetry group. The corresponding classical problem is to find all symplectic manifolds on which G acts transitively as a group of canonical transformations (symplectomorphisms), i.e. all symplectic homogeneous G-spaces. But this classical problem is "under good control". For the orbits of the co-adjoint action of G on $\mathfrak{g}^*$ are symplectic homogeneous G-spaces; and furthermore, all symplectic homogeneous G-spaces can be, more or less, built from orbits of such co-adjoint action. (Here, "more or less" signals issues about central extensions and covering spaces). Thus a "good" quantization procedure for such spaces is likely to be illuminating finding all the irreducible unitary representations of G.

Of course, we foreswear (d); and for the most part, we foreswear (b) and (c). For an introduction to both, and of course (a), we recommend: .

(i): N Landsman, Between Classical and quantum, especially Section 3; in J Butterfield and J Earman eds, *Handbook of Philosophy of Physics* (2006) and: quant-ph:0506082; and for more details:

(ii): S Ali and M Englis, Quantization methods: a guide for physicists and analysts, *Reviews in Mathematical Physics* 2005, math-ph: 0405065.

In particular, as to (b): Ali and Englis Section 1 review the obstructions confronting quantization of (even just a "handful" of polynomial) functions of position and momentum. These obstructions concern ambiguities of operator-ordering. That is: natural general constraints on the quantization map Q ("adding a hat") that sends a classical (real-scalar) quantity $f : \mathbb{R}^{2n} \to \mathbb{R}$ to a quantum quantity, i.e. to a self adjoint operator $Q_f : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, lead to *contradictions*. This topic originates in papers by Groenewold and van Hove. Recent developments include: Gotay et al. Obstructions in quantization theory, *Journal of Nonlinear Science*, volume 6, p. 469-498, 1996; and Gotay, On the Groenewold-van Hove Problem, *Journal of Mathematical Physics* 1999.

As to (c): Ali and Englis review (Section 3f.) geometric quantization, deformation quantization etc. But even their Section 2 gives details of e.g. the inequivalent quantizations involve din the Aharonov Bohm effect.

But the four topics are of course closely related. For example, these obstructions mean that a main motivation to pursue (c)'s other methods of quantization is to extend quantization to as many quantities as possible.

For us, concentrating on (a): the main point about (b), i.e. the obstructions, will be that (cf. Wald 1994, Section 2.2 , pp. 17-18): Segal quantization "works" for:

(i) a classical configuration space that is an arbitrary n-dimensional manifold M (so that classical quantities are real functions of the cotangent bundle T^*M); provided that

(ii) we restrict consideration to quantities that are at most linear in the momentum (i.e. the momentum canonically conjugate to an arbitrary configurational coordinates q on M).

Here, the word "works" means that the quantization map Q maps Poisson brackets into commutators, divided by $i\hbar$: (in more formal jargon: Q respects Lie algebra structure). That is: Q obeys, for classical quantities $f, g : T^*M \to \mathbb{R}$ that are appropriately restricted by (ii):

$$[Q(f), Q(g)] = i\hbar\, Q(\{f, g\}) \tag{4}$$

In this sense, Segal quantization is a good framework for the quantization of finite-dimensional systems.

And Segal quantization has other merits. We will also see that for linear classical systems, it "respects" the dynamics. That is: the Segal quantization of the classical Hamiltonian (which is essentially like that of a harmonic oscillator: "$p^2 + q^2$") is the "correct" quantum Hamiltonian. Besides, we will eventually see that it works for (some!) quantum field theory: specifically, for the quantization of the free bose field (e.g. De Faria and De Melo, Section 6.3. Furthermore, it does this in a manner that generalizes readily to constructing quantum field theories on *curved* spacetimes (Wald 1994, p. 31 and Section 3.2).

For our topic ((a)above): the first point to address is that since the classical position and momentum quantities, for a phase space $\mathbb{R}^{2n}$, are unbounded, we expect the quantum position and momentum Q^i, P_j to also be unbounded, indeed to have all of $\mathbb{R}$ as their spectra—so that, if they are to be self-adjoint, they cannot be defined on all of $L^2(\mathbb{R}^n)$.

Indeed, setting aside the physical desideratum that the spectra should be unbounded: there is a simple theorem that if two *bounded* self-adjoint operators Q, P have a commutator that is proportional to the identity, they must *commute*. That is: If $[Q, P] = \alpha I$ for some $\alpha \in \mathbb{C}$, then $\alpha = 0$. (De Faria and De Melo, Lemma 2.11; Jauch 1968, p. 205, Problem 4).

In short: we face issues of domains. We remedy this by formulating to the *Weyl form* of the CCRs. These govern unitary exponentiations of linear combinations of the position, and similarly, of the momentum operators.

Thus we define, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$U(\mathbf{a}) := e^{-i\mathbf{a}.\mathbf{P}/\hbar} \quad ; \qquad V(\mathbf{b}) := e^{-i\mathbf{b}.\mathbf{Q}/\hbar}; \tag{5}$$

Since the Us and Vs are both families of unitaries, their spectra are bounded, and are defined everywhere on $L^2(\mathbb{R}^n)$. In the Schroedinger representation, we have

$$(U(\mathbf{a})\psi)(\mathbf{x}) = \psi(\mathbf{x} - \mathbf{a}) \ ; \ (V(\mathbf{b})\psi)(\mathbf{x}) = e^{-i\mathbf{b}.\mathbf{x}/\hbar}\psi(\mathbf{x}) \tag{6}$$

so that U represents translations in space, and V represents translations in momentum-space.

We have, of course, commutation for each of position and momentum, alone:

$$U(\mathbf{a})U(\mathbf{b}) = U(\mathbf{b})U(\mathbf{a}) = U(\mathbf{a} + \mathbf{b}) \ ; \ V(\mathbf{a})V(\mathbf{b}) = V(\mathbf{b})V(\mathbf{a}) = V(\mathbf{a} + \mathbf{b}) \tag{7}$$

To deduce the commutation relations of U and V operators, we need the *Campbell-Baker-Hausdorff formula* for products of exponentials of non-commuting operators. Given a self-adjoint operator A, we say that a vector $\psi \in \mathcal{H}$ is *analytic* if for all n, $A^n(\psi)$ is defined, and so is $e^A\psi$. Then the version of the Campbell-Baker-Hausdorff formula which is appropriate here (De Faria and De Melo, Lemma 2.12) says that if:

(i) A, B and $A + B$ have a common dense domain D of analytic vectors, and

(ii) [A,B] commutes with A and with B:

then in D:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \equiv e^{A+B} e^{\frac{1}{2}[A,B]} \tag{8}$$

To apply (8) to (5), we set $A := -i\mathbf{a}.\mathbf{P}/\hbar$ and $B := -i\mathbf{b}.\mathbf{Q}/\hbar$, to deduce that

$$U(\mathbf{a})V(\mathbf{b}) = \exp(\frac{1}{2}i(\mathbf{a} \cdot \mathbf{b})/\hbar). \exp(-i(\mathbf{a}.\mathbf{P}/\hbar + \mathbf{b}.\mathbf{Q}/\hbar)) \ ; \tag{9}$$

and *mutatis mutandis*, we set $A := -i\mathbf{b}.\mathbf{Q}/\hbar$ and $B := -i\mathbf{a}.\mathbf{P}/\hbar$, to deduce that

$$V(\mathbf{b})U(\mathbf{a}) = \exp(-\frac{1}{2}i(\mathbf{a} \cdot \mathbf{b})/\hbar). \exp(-i(\mathbf{a}.\mathbf{P}/\hbar + \mathbf{b}.\mathbf{Q}/\hbar)). \tag{10}$$

Combining these immediately gives the *Weyl commutation relations*:[1]

$$U(\mathbf{a})V(\mathbf{b}) = e^{i\mathbf{a}.\mathbf{b}/\hbar} V(\mathbf{b})U(\mathbf{a}). \tag{11}$$

[1]Beware: (i) many authors 'flip' the notation of U and V, so that V represents translations in space; and (ii) some authors (even rigorous ones e.g. Prugovecki 1981, Chapter IV, Sections 6.2, 6.4!) also put the $\hbar$ in the numerator of the exponent, so that the exponent is in dire danger of having dimension action-squared! Besides, (iii): various texts also get the sign of the exponent in (11) wrong. (See later for discussion of different choices of sign in the two definitions of (5).) I am following S. Summers (2001: in *John von Neumann and the Foundations of quantum mechanics*, ed. M. Redei and M. Stoeltzner). Summers puts the $\hbar$ in the denominator of the exponent, is perfectionist about signs; and his use of U for translation in space, is like Weyl himself (1932, Chapter IV, Section 14, building on Chapter II, Section 11): this last text being no doubt correct, but—with all due respect!—incomprehensible.

1.2 The Weyl algebra

So from now on, we take as our CCRs, not the Heisenberg form (2), but (11) together with the trivial commutations of Us and Vs alone i.e. (7).

We have so far built the Us and Vs concretely from given $\mathbf{Q}, \mathbf{P}$. But in the usual tradition of physics, we can:

(i) consider an abstract algebra of Us and Vs subject to the relations (11) and (7); any such algebra is called *the Weyl algebra*; and then

(ii) try to classify the representations of this algebra, especially the unitary representations on some Hilbert space. Recall the pure mathematical topic (d) in Section 1.1.

As already announced, the main result about (ii), for finite-dimensional systems, will be the Stone-von Neumann uniqueness theorem.

Now, we first make two comments about this endeavour (in order of increasing importance for us); and then develop a more abstract formulation of the Weyl relations, which will be central in all that follows.

(1): *The relation between the Heisenberg and Weyl forms*:– The Weyl form of the CCRs implies the Heisenberg form, and so a representation of the Weyl form is also a representation of the Heisenberg form. But uniqueness (up to unitary equivalence) of a representation of the Weyl form does not imply uniqueness of the implied representation of the Heisenberg form. The reason lies in the simple theorem above, that two *bounded* self-adjoint operators Q, P cannot obey the Heisenberg form. In fact, the Heisenberg form does not imply the Weyl form, even if Q and P are essentially self-adjoint on their respective domains; though conditions can be added that make the implication go through (e.g. Dixmier's condition (1958: in French!), discussed by Jauch (1968, p. 204-205)).

(2): *Allowing for projective unitary representations*:— Of course, the quantum state is *non-redundantly* represented by a *ray* rather than a unit vector. This motivates considering *projective* representations of groups, rather than "true" representations. Such representations allow a phase to occur in equations stating the group composition law for the representing operators. Indeed, we see this even for elementary abelian groups, like the phase-space translation groups we are concerned with: cf. the phase in (11), and in in (13) below.

Equation (11) can be given a more abstract formulation, which both:

(i) brings out the role being played by the symplectic structure in the underlying framework of Hamiltonian mechanics, and

(ii) underpins how Segal quantization succeeds in quantizing linear classical systems, both finite-dimensional and infinite-dimensional.

Setting $z := (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$, we define the family of operators

$$W(z) := e^{\frac{1}{2}i\mathbf{a}.\mathbf{b}} U(\mathbf{a}) V(\mathbf{b}). \tag{12}$$

Then the Weyl form of the CCRs, i.e. (11) and and (7), are equivalent to the following, which is thus also called the *Weyl algebra*: for all $z, z_1, z_2 \in \mathbb{R}^{2n}$,

$$\begin{array}{rcl} W(z_1)W(z_2) & = & e^{\frac{1}{2}i\Omega(z_1,z_2)} W(z_1 + z_2); \\ W^\dagger(z) & = & W(-z); \end{array} \tag{13}$$

where Ω is the *symplectic product*:

$$\Omega(z_1, z_2) := \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2, \tag{14}$$

The symplectic meaning of Ω will be explained in Section 3. But as a preliminary to that, we spell out some elementary ideas and results about complexification and complex structures: which are often treated very concisely if at all, (e.g. Wald 1994, p. 190).

2 Complexification, complex structures—and all that

There is a circle of ideas which can be traversed starting from almost any point... We begin with complexification, then describe complex structures, then complex conjugation of spaces, and then the compatibility of a complex structure with a bilinear form, such as an inner product or symplectic form.

2.1 Complexification

The *complexification* $V^{\mathbb{C}}$ of a real vector space V is defined as the tensor product of V with the complex numbers $\mathbb{C}$

$$V^{\mathbb{C}} := V \otimes \mathbb{C} \, . \tag{15}$$

So far, this is just a real vector space. Every vector in $V^{\mathbb{C}}$ can be written uniquely as

$$v = v_1 \otimes 1 + v_2 \otimes i \tag{16}$$

and the (real) dimension of $V^{\mathbb{C}}$ is twice the dimension of V. But we make it into a complex vector space, by defining complex scalar multiplication by

$$\alpha(v \otimes \beta) = v \otimes (\alpha\beta) \ \text{ for all } v \in V \text{ and } \alpha, \beta \in \mathbb{C} \, ; \tag{17}$$

where we also of course require scalar multiplication to distribute over addition, i.e. we 'extend by linearity':

$$\alpha(v \otimes \beta + u \otimes \gamma) := \alpha(v \otimes \beta) + \alpha(u \otimes \gamma) \equiv v \otimes (\alpha\beta) + u \otimes (\alpha\gamma) \, . \tag{18}$$

Since every vector in $V^{\mathbb{C}}$ can be written uniquely as $v = v_1 \otimes 1 + v_2 \otimes i$, it is usual to drop the tensor product symbol and just write

$$v = v_1 + iv_2. \tag{19}$$

One then checks that the definition eq. 15 implies that the complex scalar multiplication defined by eq. 17, can be written in the usual-looking form. Namely: for a complex number $\alpha = a + ib$ with $a, b \in \mathbb{R}$

$$(a + ib)(v_1 + iv_2) = (av_1 - bv_2) + i(bv_1 + av_2). \tag{20}$$

So we regard $V^{\mathbb{C}}$ as the direct sum of two copies of V, equipped with a complex scalar multiplication defined by eq. 20 .

There is a natural embedding of V in to $V^{\mathbb{C}}$ given by

$$v \mapsto v \otimes 1 \, . \tag{21}$$

V may thus be regarded as a *real* subspace of $V^{\mathbb{C}}$. If V has a basis $\{e_i\}$ over $\mathbb{R}$ then a correponding basis for $V^{\mathbb{C}}$ is given by $\{e_i \otimes 1\}$ over $\mathbb{C}$. The *complex* dimension of $V^{\mathbb{C}}$ is therefore equal to the *real* dimension of V:

$$\dim_{\mathbb{C}} V^{\mathbb{C}} = \dim_{\mathbb{R}} V. \tag{22}$$

Alternatively: We can *define* the complexification of V as the direct sum

$$V^{\mathbb{C}} := V \oplus V \tag{23}$$

equipped with a *complex structure* (cf. below for details) given by the operator $J : V^{\mathbb{C}} \to V^{\mathbb{C}}$, where J is defined by

$$J(v, w) := (-w, v) \ . \tag{24}$$

Here J encodes multiplication by i in the sense that setting $a = 0, b = 1$ in eq. 20 yields

$$i(v_1 + iv_2) = -v_2 + iv_1 = -v_2 \otimes 1 + v_1 \otimes i \tag{25}$$

where the last expression on the right is in the notation of eq. 16.

Let $\dim_{\mathbb{R}} V = n$. Then in matrix form, J is given by a $2n \times 2n$ matrix J by

$$J := \begin{pmatrix} \mathbf{0} & -\mathbf{1_V} \\ \mathbf{1_V} & \mathbf{0} \end{pmatrix} \ . \tag{26}$$

where $-\mathbf{1_V}$ is the identity map on V. Thus $V^{\mathbb{C}}$ can be written as $V \oplus JV$ or as $V \oplus iV$, so as (i) to avoid the tensor product notation, and (ii) to signal the fact that the direct sum in eq. 23 is endowed with J. J swaps the summands in the sense that $J(v, 0) = (0, v)$.

Examples: (i) the complexification of $\mathbb{R}^n$ is $\mathbb{C}^n$; (ii) if V is the $m \times n$ matrices with real entries, then $V^{\mathbb{C}}$ is the $m \times n$ matrices with complex entries.

Again we have (cf. eq. 22): the *complex* dimension of $V^{\mathbb{C}}$ is equal to the *real* dimension of V, which is half the *real* dimension of $V \oplus V$:

$$\dim_{\mathbb{C}} V^{\mathbb{C}} = \dim_{\mathbb{R}} V = \frac{1}{2} \dim_{\mathbb{R}} (V \oplus V) \ . \tag{27}$$

2.2 Complex structures

A *complex structure* on a real vector space V is an automorphism J of V that squares to minus the identity map, - I. That is: $J^2 = -1 \equiv$ - I. Such a structure on V allows one to define multiplication by complex scalars in a canonical fashion so as to regard V as a complex vector space. Namely:

$$(x + iy)v := xv + yJ(v) \ \text{ for all } v \in V \text{ and } x, y \in \mathbb{R} \ ; \tag{28}$$

which (check!) makes V into a complex vector space, denoted V_J.

If V is any real vector space, there is a canonical complex structure J on the direct sum $V \oplus V$: namely, the complex structure on the complexification $V^{\mathbb{C}}$ of V, i.e. on the tensor product $V \otimes \mathbb{C}$, written as $V \oplus JV$ or as $V \oplus iV$. That is, J is given by $J(v, w) := (-w, v)$, i.e. by eq. 24, ; and the matrix form of J is as in eq. 26. In this notation for complexification—i.e. the notation, $V \oplus JV$ or $V \oplus iV$—we can write: $V \oplus JV = (V \oplus V)_J$ or similarly $V \oplus iV = (V \oplus V)_J$.

One can go in the other direction. Any complex vector space W is also a real vector space, with the same vector addition and real scalar multiplication. On this underlying real vector space, one defines a complex structure J by $J(w) := iw$ for all $w \in W$; where the right-hand-side is given us by W being a complex vector space. With this complex structure defined, we of course get back the original complex vector space W.

In fact, if V_J has complex dimension n, then V must have real dimension $2n$. That is, a finite-dimensional real space V admits a complex structure only if it is even-dimensional. And every even-dimensional real vector space V admits a complex structure. Indeed, many. For any basis $\{e_1, e_2, \ldots, e_{2n}\}$ of V can be divided in to n pairs, say $\{e_1, e_2\}, \ldots, \{e_{2n-1}, e_{2n}\}$, and then one can define J as the 'swap with a minus' on each such pair, i.e. $J(e_1) := e_2, J(e_2) := -e_1, \ldots, J(e_{2n-1}) := e_{2n}, J(e_{2n}) := -e_{2n-1}$, and then one extends by linearity to all of V. So $J^2 = -1$.

Suppose that we are given a real linear transformation $A : V \rightarrow V$ on a real vector space V, and that V admits a complex structure J. Then A defines a complex linear transformation of the complex space V_J if and only if A commutes with J, i.e. if and only if $AJ = JA$: (trivial check, cf. eq. 28).

Likewise, a real subspace U of V is a complex subspace of V_J (i.e. is closed under complex-linear combinations) if and only if J preserves U, i.e. if and only if $J(U) < U$; (trivial check).

Basic example:— Obviously, the main example of a complex structure is the structure on $\mathbb{R}^{2n}$ coming from the complex structure on $\mathbb{C}^n$. That is, the complex n-dimensional space $\mathbb{C}^n$ is also a real $2n$-dimensional space. Here, one uses the same vector addition and real scalar multiplication: while multiplication by the complex number i is not only a *complex* linear transform of the space, thought of as a complex vector space, but also a *real* linear transform of the space, thought of as a real vector space. This is just because scalar multiplication by i:

(a) commutes with scalar multiplication by real numbers, i.e. $i(\lambda v) = (i\lambda)v = (\lambda i)v = \lambda(iv)$, and

(b) distributes across vector addition.

As a complex $n \times n$ matrix, this complex structure is simply the scalar matrix with i on the diagonal. The corresponding real $2n \times 2n$ matrix is denoted J.

Again, there is the general equation that counts dimensions, with $V^{\mathbb{C}} = (V \oplus V)_J$ (cf. eq. 27):

$$\frac{1}{2}\dim_{\mathbb{R}}(V \oplus V)_J = \dim_{\mathbb{C}}(V \oplus V)_J = \dim_{\mathbb{R}} V = \frac{1}{2}\dim_{\mathbb{R}}(V \oplus V) \ . \tag{29}$$

And in this example, with $V = \mathbb{R}^n$: these numbers are all n.

Suppose given a complex vector space, of complex dimension n, and a basis $\{e_1, e_2, \ldots, e_n\}$. This set, together with these vectors multiplied by i, namely $\{ie_1, ie_2, \ldots, ie_n\}$, form a basis for the underlying real vector space. There are two natural ways to order this basis.

(1): If one orders the basis as $\{e_1, ie_1, e_2, ie_2, \ldots, e_n, ie_n\}$, then the matrix for J takes the following block-diagonal form, where the blocks are the 2×2 matrix $J_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. That is: J is (with subscript $2n$ added, so as to indicate dimension):

$$J_{2n} := \begin{pmatrix} J_2 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & J_2 & \ldots & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & \mathbf{0} & \ldots & J_2 \end{pmatrix} \ . \tag{30}$$

(2): If one orders the basis as $\{e_1, e_2, \ldots, e_n, ie_1, , ie_2, \ldots, ie_n\}$, then the matrix for J is block-antidiagonal:

$$J_{2n} := \begin{pmatrix} \mathbf{0} & -\mathbf{1_n} \\ \mathbf{1_n} & \mathbf{0} \end{pmatrix} \ : \tag{31}$$

This is more natural when one thinks of the real space as a direct sum of real spaces, as in the second, alternative, approach to complexification at the end of Section 2.1. Thus eq. 31 is the same as eq. 26.

2.3 Complex conjugation of spaces, and the relation of complex structures to complexifications

The complex conjugate of complex vector space W is a complex vector space $\overline{W}$ that has the same elements and additive group structure as $\overline{W}$, but whose scalar multiplication involves conjugation. That is: we define the scalar multiplication $*$ in $\overline{W}$ in terms of the scalar multiplication $\cdot$ in W by:

$$\alpha * w := \overline{\alpha} \cdot w \ , \quad \text{for all } \alpha \in \mathbb{C}, w \in \overline{W} \tag{32}$$

Properties:

(1) $\overline{\overline{W}} = W$.

(2) W and $\overline{W}$ have the same complex dimension. Indeed the identity map $id : W \rightarrow \overline{W}$ is an antilinear map, since

$$id(\alpha \cdot w) = \alpha \cdot w \equiv \overline{\alpha} * w = \overline{\alpha} * id(w) \tag{33}$$

and id maps any basis of W into a basis of $\overline{W}$. And given any two bases, $\{e_i\}$ and $\{f_i\}$, of W and $\overline{W}$ respectively, the map $f : e_i \rightarrow f_i$ can be extended by *antlinearity* to be an antilinear map, an *anti-isomorphism*, from W to $\overline{W}$. Thus there is no canonical isomorphism between W and $\overline{W}$.

(3) If W and U are complex vector spaces, an antilinear map $f : W \rightarrow U$ can be regarded as an ordinary linear map $f : \overline{W} \rightarrow U$, since:

$$f(\alpha * w) = f(\overline{\alpha} \cdot w) = \overline{\overline{\alpha}} \cdot f(w) = \alpha \cdot f(w) \ . \tag{34}$$

Conversely, any linear map g defined on $\overline{W}$, $g : \overline{W} \rightarrow U$, gives rise to an antilinear map from W to U, which again we write with a g. That is, we write: $g : W \rightarrow U$. For if we write the scalar multiplication in W as $\cdot$ (as before) and the scalar multiplication in U as $\cdot\cdot$, then the map $g : W \rightarrow U$ obeys:

$$g(\alpha \cdot w) \equiv g(\overline{\alpha} * w) = \overline{\alpha} \cdot \cdot g(w) \ , \tag{35}$$

since $g : \overline{W} \rightarrow U$ is linear. So the defined map $g : W \rightarrow U$ is antilinear.

(4) A linear map between complex vector spaces, $f : W \rightarrow U$, gives rise to a corresponding *also!* linear map $\overline{f} : \overline{W} \rightarrow \overline{U}$ which has the same action as f. For $\overline{f}$ preserves scalar multiplication, since

$$\overline{f}(\alpha * w) := f(\overline{\alpha} \cdot w) = \overline{\alpha} \cdot f(w) = \alpha * \overline{f}(w) \tag{36}$$