The Quantization of Linear Dynamical Systems I: Finite Systems

Spoken by JNB, but: taught by Caulton (adam.caulton@balliol.ox.ac.uk), Wikipedia ... ;
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This document, and its succcessor on the Quantization of Linear Dynamical Systems with Infinitely many degrees fo freedom, expound a rigorous quantization procedure developed by Irving Segal and others in the 1960s. This means we postpone to the second half of term, coverage of algebraic quantum theory; which will include e.g. inequivalent representations, 'getting out of Fock space', Haag's theorem etc. (cf. eg Emch 1972). But the present material:

(i) gives a strong grip on the first (forbiddingly concise!) third of Wald 1994, which is the basis for the rest of that book on QFT in curved spacetime and thus e.g. the Unruh effect (an essay!);

(ii) is of intrinsic interest... though please be warned that here you will find: no Lagrangian, no path integrals, no renormalization, no gauge theory, no curved spacetime, no gravitation; indeed, no interactions, and overall, not much physics ... we will focus on the harmonic oscillator (!),the free KG field and spin-chains (and without putting a Hamiltonian on the chain...). Nor will you find much straight-up philosophy ... but perhaps the light here shed on field/wave vs.particle counts as philosophy, since wave vs.particle is, like continuum vs. discrete, a perennial dichotomy of *natural philosophy*...

In this document, we consider only finitely many degrees of freedom, and lead up to the Stone-von Neumann Theorem, which essentially guarantees that the quantization of point particles in $\mathbb{R}^n$ is unique. We begin by introducing the Weyl form of the CCRs; and posing the quest for its representations (Section 1). Then we present the complexification and realification of vector spaces, complex structures etc. (Section 2); and symplectic vector spaces and manifolds (Section 3). Then we present linear systems, both classical and quantum; and thus the harmonic oscillator (Section **??**). With all this in hand, we can then see the task of quantization as "unitarizing" a Hamiltonian evolution in a symplectic space so as to give an evolution in a complex Hilbert space. This gives the idea of a *one particle structure*, both in general and for the harmonic oscillator as an example (Section 5). The key to successful quantization, which see at work in the harmonic oscillator example, turns out to be the *two out of three property* of the unitary group: which concerns its relation to certain orthogonal and symplectic groups (Section 6). Then we treat the case of finitely many harmonic oscillators, and so the occupation number representation: which can be described in a "Fock-space way" (Section 7). Finally, we state (i) the Stone-von Neumann Theorem; and (ii) an analogous theorem (the Jordan-Wigner theorem) about the uniqueness of the representation of the CARs (as against CCRs) of a *finite* system, such as a spin chain (Section 8).

Mottoes:

Let us try to introduce a quantum Poisson Bracket which shall be the analogue of the classical one....we are thus led to the following definition for the quantum Poisson Bracket of any two variables u and v: $uv - vu = i\hbar[u, v]$. Dirac (1930/1958, Section 21)

There is thus a complete harmony between the wave and light-quantum descriptions of the interaction. (Dirac, 1927, p. 245).

First quantization is a mystery, but second quantization is a functor. (E. Nelson).

Probably all these connections would have been clarified long ago, if quantum physicists had not been hampered by a prejudice in favor of complex and against real numbers. (Freeman Dyson)

The life of a theoretical physicist consists of solving harmonic oscillator at ever higher levels of abstraction. (S. Coleman)

Contents

1 Canonical quantization introduced

1.1 Commutation relations: from Heisenberg to Weyl

The idea of *canonical quantization* is familiar from elementary quantum mechanics: to "promote" the classical Poisson bracket relations

$$\{q^i, q^j\} = \{p_i, p_j\} = 0; \qquad \{q^i, p_j\} = \delta^i_j, \tag{1}$$

where $i, j \in \{1, 2, \ldots n\}$, to the *Heisenberg canonical commutation relations* (CCRs)

$$[Q^i, Q^j] = [P_i, P_j] = 0; \qquad [Q^i, P_j] = i\hbar\delta^i_j \mathbb{1}; \tag{2}$$

(we will usually set $\hbar := 1$). This Poisson bracket-commutator correspondence originated with Dirac (cf. his *Principles of Quantum Mechanics* 1958, Section 21f.) The standard representation of eq. (2) is the familiar Schroedinger representation: namely, for n configurational degrees of freedom, e.g. a spinless particle in Euclidean n-space, or n such particles on a line:

$$Q^i\psi = q_i\psi, \qquad P_j\psi = -\frac{ih}{2\pi}\frac{\partial\psi}{\partial q_j} \qquad \text{for } \psi \in L^2(\mathbb{R}^n, d\mathbf{q}). \tag{3}$$

This prompts four main topics. They are of increasing scope, and we will consider only the first.

(a): To examine canonical quantization as just described for position and momentum in $\mathbb{R}^n$. The big positive result here is the Stone von Neumann theorem, stating (roughly) that for $\mathbb{R}^n$ as the configuration space, the Schroedinger representation of (2) is unique up to unitary equivalence. Cf Section 8. But so as to set the scene for quantum field theory, and more generally so as to get materials useful for contexts other than $\mathbb{R}^n$, we will lead up to this slowly. This will mean expounding some ideas of *Segal quantization*, which is the most straightforward generalization of the above ideas. In short: it replaces $\mathbb{R}^n$ as the classical configuration space, by an arbitrary n-dimensional manifold.

(b): To extend quantization to other quantities, in particular functions (polynomial, or even "arbitrary", functions) of position and momentum.

(c): To consider other methods of quantization.

(d) To pursue the *pure mathematical* interest of quantization. For a glimpse of this, cf. Folland (2008, p. 49; and Vogan 2005, cited there). In short: the interest lies in how it helps one find all the irreducible unitary representations of a connected Lie group G: i.e. in physical language, finding all quantum systems in which G acts irreducibly as a symmetry group. The corresponding classical problem is to find all symplectic manifolds on which G acts transitively as a group of canonical transformations (symplectomorphisms), i.e. all symplectic homogeneous G-spaces. But this classical problem is "under good control". For the orbits of the co-adjoint action of G on $\mathfrak{g}^*$ are symplectic homogeneous G-spaces; and furthermore, all symplectic homogeneous G-spaces can be, more or less, built from orbits of such co-adjoint action. (Here, "more or less" signals issues about central extensions and covering spaces). Thus a "good" quantization procedure for such spaces is likely to be illuminating for the task of finding all the irreducible unitary representations of G.

Of course, we foreswear (d); and for the most part, we foreswear (b) and (c). For an introduction to both, and of course (a), we recommend:

(i): N Landsman, Between Classical and quantum, especially Section 3; in J Butterfield and J Earman eds, *Handbook of Philosophy of Physics* (2006) and: quant-ph:0506082; and for

more details:

(ii): S Ali and M Englis, Quantization methods: a guide for physicists and analysts, *Reviews in Mathematical Physics* 2005, math-ph: 0405065.

In particular, as to (b): Ali and Englis Section 1 review the obstructions confronting quantization of (even just a "handful" of polynomial) functions of position and momentum. These obstructions concern ambiguities of operator-ordering. That is: natural general constraints on the quantization map Q ("adding a hat") that sends a classical (real-scalar) quantity $f : \mathbb{R}^{2n} \to \mathbb{R}$ to a quantum quantity, i.e. to a self adjoint operator $Q_f : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, lead to *contradictions*. This topic originates in papers by Groenewold and van Hove. Recent developments include: Gotay et al. Obstructions in quantization theory, *Journal of Nonlinear Science*, volume 6, p. 469-498, 1996; and Gotay, On the Groenewold-van Hove Problem, *Journal of Mathematical Physics* 1999.

As to (c): Ali and Englis review (Section 3f.) geometric quantization, deformation quantization etc. But even their Section 2 gives details of e.g. the inequivalent quantizations involved in the Aharonov Bohm effect.

But the four topics are of course closely related. For example, these obstructions mean that a main motivation to pursue (c)'s other methods of quantization is to extend quantization to as many quantities as possible.

For us, concentrating on (a): the main point about (b), i.e. the obstructions, will be that (cf. Wald 1994, Section 2.2 , pp. 17-18): Segal quantization "works" for:

(i) a classical configuration space that is an arbitrary n-dimensional manifold M (so that classical quantities are real functions of the cotangent bundle T^*M); *provided that*

(ii) we restrict consideration to quantities that are at most linear in the momenta (i.e. the momenta canonically conjugate to arbitrary configurational coordinates q on M).

Here, the word "works" means that the quantization map Q maps Poisson brackets into commutators, divided by $i\hbar$. (In more formal jargon: "Q respects Lie algebra structure"). That is: Q obeys, for classical quantities $f, g : T^*M \to \mathbb{R}$ that are appropriately restricted by condition (ii) above:

$$[Q(f), Q(g)] = i\hbar\, Q(\{f, g\}) \tag{4}$$

In this sense, Segal quantization is a good framework for the quantization of finite-dimensional systems.

And Segal quantization has other merits. We will also see that for linear classical systems, it "respects" the dynamics. That is: the Segal quantization of the classical Hamiltonian (which is essentially like that of a harmonic oscillator: "$p^2 + q^2$") is the "correct" quantum Hamiltonian. Besides, we will eventually see that it works for (some!) quantum field theories. Specifically, it works for the quantization of the free bose field (e.g. De Faria and De Melo, Section 6.3). Furthermore, it does this in a manner that generalizes readily to constructing quantum field theories on *curved* spacetimes (Wald 1994, p. 31 and Section 3.2).

So much by way of preamble. For our main topic, i.e. (a) above, the first job is to pass from the Heisenberg CCRs to the *Weyl form of the CCRs*. The point here is that since the classical position and momentum quantities, for a phase space $\mathbb{R}^{2n}$, are unbounded, we expect the quantum position and momentum Q^i, P_j to also be unbounded, indeed to have all of $\mathbb{R}$ as their spectra—so that, if they are to be self-adjoint, they cannot be defined on all of $L^2(\mathbb{R}^n)$.

Indeed, setting aside the physical desideratum that the spectra should be unbounded: there

is a simple theorem that if two *bounded* self-adjoint operators Q, P have a commutator that is proportional to the identity, they must *commute*. That is: If $[Q, P] = \alpha I$ for some $\alpha \in \mathbb{C}$, then $\alpha = 0$. (De Faria and De Melo, Lemma 2.11; Jauch 1968, p. 205, Problem 4).

In short: we face issues of domains. We remedy this by formulating to the *Weyl form* of the CCRs. These govern unitary exponentiations of linear combinations of the position, and similarly, of the momentum operators.

Thus we define, for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

$$U(\mathbf{a}) := e^{-i\mathbf{a}.\mathbf{P}/\hbar} \ ; \qquad V(\mathbf{b}) := e^{-i\mathbf{b}.\mathbf{Q}/\hbar}; \tag{5}$$

Since the Us and Vs are both families of unitaries, their spectra are bounded, and are defined everywhere on $L^2(\mathbb{R}^n)$. In the Schroedinger representation, we have

$$(U(\mathbf{a})\psi)(\mathbf{x}) = \psi(\mathbf{x} - \mathbf{a}) \ ; \ (V(\mathbf{b})\psi)(\mathbf{x}) = e^{-i\mathbf{b}.\mathbf{x}/\hbar}\psi(\mathbf{x}) \tag{6}$$

so that U represents translations in space, and V represents translations in momentum-space.

We have, of course, commutation for each of position and momentum, alone:

$$U(\mathbf{a})U(\mathbf{b}) = U(\mathbf{b})U(\mathbf{a}) = U(\mathbf{a} + \mathbf{b}) \ ; \ V(\mathbf{a})V(\mathbf{b}) = V(\mathbf{b})V(\mathbf{a}) = V(\mathbf{a} + \mathbf{b}) \tag{7}$$

To deduce the commutation relations of U and V operators, we need the *Campbell-Baker-Hausdorff formula* for products of exponentials of non-commuting operators. Given a self-adjoint operator A, we say that a vector $\psi \in \mathcal{H}$ is *analytic* if for all n, $A^n(\psi)$ is defined, and so is $e^A\psi$. Then the version of the Campbell-Baker-Hausdorff formula which is appropriate here (De Faria and De Melo, Lemma 2.12) says that if:

(i) A, B and $A + B$ have a common dense domain D of analytic vectors, and

(ii) [A,B] commutes with A and with B:

then in D:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \equiv e^{A+B} e^{\frac{1}{2}[A,B]} \tag{8}$$

To apply (8) to (5), we set $A := -i\mathbf{a}.\mathbf{P}/\hbar$ and $B := -i\mathbf{b}.\mathbf{Q}/\hbar$, to deduce that

$$U(\mathbf{a})V(\mathbf{b}) = \exp(\frac{1}{2}i(\mathbf{a} \cdot \mathbf{b})/\hbar).\exp(-i(\mathbf{a}.\mathbf{P}/\hbar + \mathbf{b}.\mathbf{Q}/\hbar)) \ ; \tag{9}$$

and *mutatis mutandis*, we set $A := -i\mathbf{b}.\mathbf{Q}/\hbar$ and $B := -i\mathbf{a}.\mathbf{P}/\hbar$, to deduce that

$$V(\mathbf{b})U(\mathbf{a}) = \exp(-\frac{1}{2}i(\mathbf{a} \cdot \mathbf{b})/\hbar).\exp(-i(\mathbf{a}.\mathbf{P}/\hbar + \mathbf{b}.\mathbf{Q}/\hbar)). \tag{10}$$

Combining these immediately gives the *Weyl commutation relations*:[1]

$$U(\mathbf{a})V(\mathbf{b}) = e^{i\mathbf{a}.\mathbf{b}/\hbar}V(\mathbf{b})U(\mathbf{a}). \tag{11}$$

[1]Beware: (i) many authors 'flip' the notation of U and V, so that V represents translations in space; and (ii) some authors (even rigorous ones e.g. Prugovecki 1981, Chapter IV, Sections 6.2, 6.4!) also put the $\hbar$ in the numerator of the exponent, so that the exponent is in dire danger of having dimension action-squared! Besides, (iii): various texts also get the sign of the exponent in (11) wrong. (See later for discussion of different choices of sign in the two definitions of (5).) I am following S. Summers (2001: in *John von Neumann and the Foundations of quantum mechanics*, ed. M. Redei and M. Stoeltzner). Summers puts the $\hbar$ in the denominator of the exponent, is perfectionist about signs; and his use of U for translation in space, is like Weyl himself (1932, Chapter IV, Section 14, building on Chapter II, Section 11): this last text being no doubt correct, but—with all due respect!—incomprehensible.

1.2 The Weyl algebra

So from now on, we take as our CCRs, not the Heisenberg form (2), but (11) together with the trivial commutations of Us and Vs alone i.e. (7).

We have so far built the Us and Vs concretely from given $\mathbf{Q}, \mathbf{P}$. But in the usual tradition of physics, we can:

(i) consider an abstract algebra of Us and Vs subject to the relations (11) and (7); any such algebra is called *the Weyl algebra*; and then

(ii) try to classify the representations of this algebra, especially the unitary representations on some Hilbert space.

As already announced at the start of Section 1.1, the main result about (ii), for finite-dimensional systems, will be the Stone-von Neumann uniqueness theorem. But as that discussion also suggested: the Weyl algebra, and Segal quantization, will also be centre-stage for quantizing fileds (including on curved spacetime) and for the pure mathematical topic (d) of Section 1.1.

Now, we first make two comments about this endeavour (in order of increasing importance for us); and then develop a more abstract formulation of the Weyl relations, which will be central in all that follows.

(1): *The relation between the Heisenberg and Weyl forms*:– The Weyl form of the CCRs implies the Heisenberg form, and so a representation of the Weyl form is also a representation of the Heisenberg form. But uniqueness (up to unitary equivalence) of a representation of the Weyl form does not imply uniqueness of the implied representation of the Heisenberg form. The reason lies in the simple theorem above, that two *bounded* self-adjoint operators Q, P cannot obey the Heisenberg form. In fact, the Heisenberg form does not imply the Weyl form, even if Q and P are essentially self-adjoint on their respective domains; though conditions can be added that make the implication go through (e.g. Dixmier's condition (1958: in French!), discussed by Jauch (1968, p. 204-205)).

(2): *Allowing for projective unitary representations*:— Of course, the quantum state is *non-redundantly* represented by a *ray* rather than a unit vector. This motivates considering *projective* representations of groups, rather than "true" representations. Such representations allow a phase to occur in equations stating the group composition law for the representing operators. Indeed, we see this even for elementary abelian groups, like the phase-space translation groups we are concerned with: cf. the phase in (11), and in (63) below.

Equation (11) can be given a more abstract formulation, which both:

(i) brings out the role being played by the symplectic structure in the underlying framework of Hamiltonian mechanics, and

(ii) underpins how Segal quantization succeeds in quantizing linear classical systems, both finite-dimensional and infinite-dimensional.

Setting $z := (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$, we define the family of operators

$$W(z) := e^{\frac{1}{2}i\mathbf{a}.\mathbf{b}} U(\mathbf{a}) V(\mathbf{b}). \tag{12}$$

Then the Weyl form of the CCRs, i.e. (11) and and (7), are equivalent to the following, which is thus also called the *Weyl algebra*: for all $z, z_1, z_2 \in \mathbb{R}^{2n}$,

$$\begin{array}{rcl} W(z_1)W(z_2) & = & e^{\frac{1}{2}i\Omega(z_1,z_2)} W(z_1 + z_2); \\ W^\dagger(z) & = & W(-z); \end{array} \tag{13}$$

where Ω is the *symplectic product*:

$$\Omega(z_1, z_2) := \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2, \tag{14}$$

The symplectic meaning of Ω will be explained in Section 3. But as a preliminary to that, we spell out some elementary ideas and results about complexification and complex structures: which are often treated very concisely if at all, (e.g. Wald 1994, p. 190).

2 Complexification, complex structures—and all that

There is a circle of ideas which can be traversed starting from almost any point... We begin with complexification, then describe complex structures, then the compatibility of a complex structure with a bilinear form, such as an inner product or symplectic form. This will give us a glimpse of how we can "go back and forth" between certain classical phase spaces (viz. symplectic vector spaces) and Hilbert spaces. It will also give us a glimpse of (i) Kahler manifolds, and (ii) how in a quantum theory different choices of a complex structure are associated with different splittings of positive and negative frequencies, and thereby (iii) the Unruh effect.The Section ends with discussion of the complex conjugation of spaces.

2.1 Complexification

2.1.A Complexification as tensor product:— The *complexification* $V^{\mathbb{C}}$ of a real vector space V is defined as the tensor product of V with the complex numbers $\mathbb{C}$

$$V^{\mathbb{C}} := V \otimes \mathbb{C} \, . \tag{15}$$

Here we think of $\mathbb{C}$ as a copy of $\mathbb{R}^2$, with a basis $\{(1,0),(0,i)\}$. So far, this is just a real vector space. Every vector in $V^{\mathbb{C}}$ can be written uniquely as

$$v = v_1 \otimes 1 + v_2 \otimes i \tag{16}$$

and the (real) dimension of $V^{\mathbb{C}}$ is twice the dimension of V. But we make it into a complex vector space, by defining complex scalar multiplication by

$$\alpha(v \otimes \beta) = v \otimes (\alpha\beta) \;\text{ for all } v \in V \text{ and } \alpha, \beta \in \mathbb{C} \, ; \tag{17}$$

where we also of course require scalar multiplication to distribute over addition, i.e. we 'extend by linearity':

$$\alpha(v \otimes \beta + u \otimes \gamma) := \alpha(v \otimes \beta) + \alpha(u \otimes \gamma) \equiv v \otimes (\alpha\beta) + u \otimes (\alpha\gamma) \, . \tag{18}$$

Since every vector in $V^{\mathbb{C}}$ can be written uniquely as $v = v_1 \otimes 1 + v_2 \otimes i$, it is usual to drop the tensor product symbol and just write

$$v = v_1 + iv_2. \tag{19}$$

One then checks that the definition eq. 15, equivalently eq. 16, implies that the complex scalar multiplication defined by eq. 17, can be written in the usual-looking form. Namely: for a complex number $\alpha = a + ib$ with $a, b \in \mathbb{R}$

$$(a + ib)(v_1 + iv_2) = (av_1 - bv_2) + i(bv_1 + av_2). \tag{20}$$

So we regard $V^{\mathbb{C}}$ as the direct sum of two copies of V, equipped with a complex scalar multiplication defined by eq. 20 .

There is a natural embedding of V in to $V^{\mathbb{C}}$ given by

$$v \mapsto v \otimes 1 \ . \tag{21}$$

V may thus be regarded as a *real* subspace of $V^{\mathbb{C}}$. If V has a basis $\{e_i\}$ over $\mathbb{R}$ then a correponding basis for $V^{\mathbb{C}}$ is given by $\{e_i \otimes 1\}$ over $\mathbb{C}$. The *complex* dimension of $V^{\mathbb{C}}$ is therefore equal to the *real* dimension of V:

$$\dim_{\mathbb{C}} V^{\mathbb{C}} = \dim_{\mathbb{R}} V. \tag{22}$$

2.1.B Complexification as direct sum:— Alternatively, we can *define* the complexification of V as the direct sum

$$V^{\mathbb{C}} := V \oplus V \tag{23}$$

equipped with a *complex structure* (cf. below for details) given by the operator $J : V^{\mathbb{C}} \to V^{\mathbb{C}}$, where J is defined by

$$J(v, w) := (-w, v) \ . \tag{24}$$

Here J encodes multiplication by i in the sense that setting $a = 0, b = 1$ in eq. 20 yields

$$i(v_1 + iv_2) = -v_2 + iv_1 = -v_2 \otimes 1 + v_1 \otimes i \tag{25}$$

where the last expression on the right is in the notation of eq. 16.

Let $\dim_{\mathbb{R}} V = n$. Then in matrix form, J is given by a $2n \times 2n$ matrix J, viz.

$$J = \begin{pmatrix} \mathbf{0} & -\mathbf{1_V} \\ \mathbf{1_V} & \mathbf{0} \end{pmatrix} \ . \tag{26}$$

where $-\mathbf{1_V}$ is the identity map on V. Thus $V^{\mathbb{C}}$ can be written as $V \oplus JV$ or as $V \oplus iV$, so as (i) to avoid the tensor product notation, and (ii) to signal the fact that the direct sum in eq. 23 is endowed with J. J swaps the summands in the sense that $J(v, 0) = (0, v)$.

Examples: (i) the complexification of $\mathbb{R}^n$ is $\mathbb{C}^n$; (ii) if V is the $m \times n$ matrices with real entries, then $V^{\mathbb{C}}$ is the $m \times n$ matrices with complex entries.

Again we have (cf. eq. 22): the *complex* dimension of $V^{\mathbb{C}}$ is equal to the *real* dimension of V, which is half the *real* dimension of $V \oplus V$:

$$\dim_{\mathbb{C}} V^{\mathbb{C}} = \dim_{\mathbb{R}} V = \frac{1}{2} \dim_{\mathbb{R}} (V \oplus V) \ . \tag{27}$$

2.1.C A matter of convention:— The above discussion (in 2.1.A and 2.1.B) has an obviously conventional aspect. Suppose that in 2.1.A, we had taken the basis of $\mathbb{C}$ as a copy of $\mathbb{R}^2$, to be in the opposite order, i.e. $\{(0, i), (1, 0)\}$. Then eq. 16 would become

$$v = v_1 \otimes i + v_2 \otimes 1 \tag{28}$$

Then the definition of complex scalar multiplication, eq. 17 and 18, remain as they are. But the notation that drops the tensor product, i.e. eq. 19, becomes

$$v = iv_1 + v_2 \ ; \tag{29}$$

and the usual-looking form of the complex scalar multiplication that we now deduce is the following analogue of eq. 20: for a complex number $\alpha = a + ib$ with $a, b \in \mathbb{R}$

$$(a + ib)(iv_1 + v_2) = (av_2 - bv_1) + i(av_1 + bv_2). \tag{30}$$

Similarly, for the alternative direct sum approach of 2.1.B. Instead of eq. 24, we define the complex structure J on the direct sum $V \oplus V$ by

$$J(v, w) := (w, -v) \ . \tag{31}$$

Then, setting $a = 0, b = 1$ in eq. 30 yields

$$i(iv_1 + v_2) = -v_1 + iv_2 = iv_2 - v_1 = v_2 \otimes i - v_1 \otimes 1 \tag{32}$$

where the last expression on the right is in the notation of eq. 28. This J as defined by eq. 31 is of course just minus the J defined by eq. 24. The matrix form of J as defined by eq. 31 is thus the negative of eq. 26. That is:

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{1_V} \\ \mathbf{-1_V} & \mathbf{0} \end{pmatrix} \ . \tag{33}$$

This last equation will give us, shortly, an obvious comparison with the matrix expression of a symplectic form.

2.2 Complex structures

2.2.A Basics:— A *complex structure* on a real vector space V is an automorphism J of V that squares to minus the identity map, - I. That is: $J^2 = -1 \equiv$ - I. Such a structure on V allows one to define multiplication by complex scalars in a canonical fashion so as to regard V as a complex vector space. Namely:

$$(x + iy)v := xv + yJ(v) \ \text{ for all } v \in V \text{ and } x, y \in \mathbb{R} \ ; \tag{34}$$

which (check!) makes V into a complex vector space, denoted V_J.

If V is any real vector space, there is a canonical complex structure J on the direct sum $V \oplus V$: namely, the complex structure on the complexification $V^{\mathbb{C}}$ of V, i.e. on the tensor product $V \otimes \mathbb{C}$, written as $V \oplus JV$ or as $V \oplus iV$. That is, J is given by $J(v, w) := (-w, v)$, i.e. by eq. 24, ; and the matrix form of J is as in eq. 26. In this notation for complexification—i.e. the notation, $V \oplus JV$ or $V \oplus iV$—we can write: $V \oplus JV = (V \oplus V)_J$ or similarly $V \oplus iV = (V \oplus V)_J$.

One can go in the other direction. Any complex vector space W is also a real vector space, with the same vector addition and real scalar multiplication. On this underlying real vector space, one defines a complex structure J by $J(w) := iw$ for all $w \in W$; where the right-hand-side is given us by W being a complex vector space. With this complex structure defined, we of course get back the original complex vector space W.

In fact, if V_J has complex dimension n, then V must have real dimension $2n$. That is, a finite-dimensional real space V admits a complex structure only if it is even-dimensional. If

$\{v_1, ..., v_m\}$ is a basis of the complex vector space V_J, then $\{v_1, J(v_1)..., v_m, J(v_m)\}$ is a basis of the underlying real vector space V.

Every even-dimensional real vector space V admits a complex structure. Indeed, many. For any basis $\{e_1, e_2, \ldots, e_{2n}\}$ of V can be divided in to n pairs, say $\{e_1, e_2\}, \ldots, \{e_{2n-1}, e_{2n}\}$, and then one can define J as the 'swap with a minus' on each such pair, i.e. $J(e_1) := e_2, J(e_2) := -e_1, \ldots, J(e_{2n-1}) := e_{2n}, J(e_{2n}) := -e_{2n-1}$, and then one extends by linearity to all of V. So $J^2 = -1$.

Suppose that we are given a real linear transformation $A : V \to V$ on a real vector space V, and that V admits a complex structure J. Then A defines a complex linear transformation of the complex space V_J if and only if A commutes with J, i.e. if and only if $AJ = JA$: (trivial check, cf. eq. 34).

Likewise, a real subspace U of V is a complex subspace of V_J (i.e. is closed under complex-linear combinations) if and only if J preserves U, i.e. if and only if $J(U) < U$; (trivial check).

2.2.B: Basic example:— Obviously, the main example of a complex structure is the structure on $\mathbb{R}^{2n}$ coming from the complex structure on $\mathbb{C}^n$. That is, the complex n-dimensional space $\mathbb{C}^n$ is also a real $2n$-dimensional space. Here, one uses the same vector addition and real scalar multiplication: while multiplication by the complex number i is not only a *complex* linear transform of the space, thought of as a complex vector space, but also a *real* linear transform of the space, thought of as a real vector space. This is just because scalar multiplication by i:

(a) commutes with scalar multiplication by real numbers, i.e. $i(\lambda v) = (i\lambda)v = (\lambda i)v = \lambda(iv)$, and

(b) distributes across vector addition.

As a complex $n \times n$ matrix, this complex structure is simply the diagonal matrix with i on the diagonal. The corresponding real $2n \times 2n$ matrix is denoted J. What this matrix J looks like will depend on how we order the basis: cf. eq. 36 and 37 in (1) and (2) below.

Again, there is the general equation that counts dimensions, with $V^{\mathbb{C}} = (V \oplus V)_J$ (cf. eq. 27):

$$\frac{1}{2}\dim_{\mathbb{R}}(V \oplus V)_J = \dim_{\mathbb{C}}(V \oplus V)_J = \dim_{\mathbb{R}} V = \frac{1}{2}\dim_{\mathbb{R}}(V \oplus V) \ . \tag{35}$$

And in this example, with $V = \mathbb{R}^n$: these numbers are all n.

2.2.C: The "look" of J:— Suppose given a complex vector space, of complex dimension n, and a basis $\{e_1, e_2, \ldots, e_n\}$. This set, together with these vectors multiplied by i, namely $\{ie_1, ie_2, \ldots, ie_n\}$, form a basis for the underlying real vector space. (Cf. 2.2.A, paragraph 4, above.) There are two natural ways to order this basis.

(1): If one orders the basis as $\{e_1, ie_1, e_2, ie_2, \ldots, e_n, ie_n\}$, then the matrix for J takes the following block-diagonal form, where the blocks are the 2×2 matrix $J_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. That is: J is (with subscript $2n$ added, so as to indicate dimension):

$$J_{2n} := \begin{pmatrix} J_2 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & J_2 & \ldots & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & \mathbf{0} & \ldots & J_2 \end{pmatrix} \ . \tag{36}$$

(2): If one orders the basis as $\{e_1, e_2, \ldots, e_n, ie_1, , ie_2, \ldots, ie_n\}$, then the matrix for J is

block-antidiagonal:

$$J_{2n} := \begin{pmatrix} \mathbf{0} & -\mathbf{1_n} \\ \mathbf{1_n} & \mathbf{0} \end{pmatrix} \ : \tag{37}$$

This is more natural when one thinks of the real space as a direct sum of real spaces, as in the second, alternative, approach to complexification at the end of Section 2.1. Thus eq. 37 is the same as eq. 26.

2.3 Compatibility of a complex structure with bilinear forms

2.3.A: Basics:— Later we will be much concerned with vector spaces that have: either an inner product (like a Hilbert space) or a symplectic product (as in Hamiltonian mechanics; cf. Section 3). So we here consider, in general, the "meshing" of a complex structure with bilinear forms. This will lead, in 2.3.B and 2.3.C, to "building a Hilbert space", and to the construction in the reverse direction, from a Hilbert space to a symplectic space.

If B is a bilinear form on a real vector space V, i.e. $B : V \times V \to \mathbb{R}$, then we say that J *preserves* B if for all $u, v \in V$

$$B(Ju, Jv) = B(u, v) \ . \tag{38}$$

Recall that since J is an automorphism with $J^2 = -1$, we have $J^{-1} = -J$. This implies that eq. 38 is equivalent to J being skew-adjoint with respect to B. That is:

$$B(Ju, v) = -B(u, Jv) \ . \tag{39}$$

Examples of bilinear forms are inner products and symplectic products. If g is an inner product on V then J preserves g if and only if J is an orthogonal transformation. Likewise, J preserves a non-degenerate, skew-symmetric form ω, i.e. a symplectic product, if and only if J is a symplectic transformation, i.e. $\omega(Ju, Jv) = \omega(u, v)$. If ω and J obey, for all non-zero $u \in V$, $\omega(u, Ju) > 0$, we say that J *tames* ω.

2.3.B: From symplectic form and compatible J to real-valued inner product:— A symplectic form ω on a real vector space V, together with a complex structure J that preserves ω, define: a symmetric bilinear form g_J on the complex vector space V_J. Namely, by:

$$g_J(u, v) := \omega(u, Jv) \ . \tag{40}$$

This is called *the Kähler condition*. We note that g_J is symmetric because J being skew-adjoint with respect to ω, i.e. eq. 39, implies that the rhs of eq. 40, i.e. $\omega(u, Jv) = -\omega(Ju, v) \equiv \omega(v, Ju) =: g_J(v, u)$. One similarly checks trivially that: (i) J preserves g_J; (ii) if J *tames* ω, then g_J is positive-definite, i.e. an inner product.

One also checks trivially that on the complex vector space V_J: g_j is *complex-linear*, even though g_J is *real-valued*. Thus, applying the initial definition of complex scalar multiplication for V_J, eq. 34, we write:

$$\begin{aligned} g_J((x+iy)u, v) &:= \omega((x+iy)u, J(v)) \equiv \omega((xu + yJ(u)), J(v)) \\ \equiv \omega(xu, J(v)) + \omega(yJ(u), J(v)) &\equiv x\omega((u, J(v)) + y\omega(J(u), J(v)) \\ &\equiv x g_J(u, v) + y g_J(Ju, v) \ . \end{aligned} \tag{41}$$

2.3.C: Defining a complex-valued inner product:— From 2.3.B, we assume we are given: (i) a real vector space V with (ii) a symplectic form ω, and (iii) a complex structure J that preserves and tames ω; and thereby (iv), on the complex vector space V_J, a positive-definite real-valued inner product g_J: namely as defined by the Kähler condition, eq. 40.

Now let us define a *complex-valued* function on $V \times V$ in terms of g_J and ω by

$$\langle u, v\rangle \equiv \langle u, v\rangle_{\omega,J} := g_J(u, v) + i\omega(u, v) \tag{42}$$

where the subscript shows the dependence on the given ω and J. It is trivial that this function is additive in each argument, i.e. $\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$ and similarly for additivity of the second argument. One checks (exercise!) that it is *sesquilinear*. That is: it is complex-linear in the second argument, but antiinear in the first argument. That is, with $x, y \in \mathbb{R}$:

$$\langle (x + iy)u, v\rangle = x\langle u, v\rangle - iy\langle u, v\rangle \;\text{ and }\; \langle u, (x + iy)v\rangle = x\langle u, v\rangle + iy\langle u, v\rangle \; . \tag{43}$$

The check of eq. 43 uses most of the properties we have postulated. Namely: the definition eq. 40 of g_J in terms of ω and J; the antisymmetry of ω and the symmetry of g_J; and the fact that J preserves ω.

Besides, recall that we assumed that J tames ω, so that g_J is positive-definite, i.e. a real-valued inner product (cf. 2.3.B). Then since ω is also non-degenerate, one checks (exercise!) that $\langle \cdot, \cdot\rangle$ is positive-definite. To conclude: $\langle \cdot, \cdot\rangle$ is a complex inner product in the usual sense: sesquilinear and positive-definite.

We recall that a (complex) *Hilbert space* is a a complex inner product space, that is complete in the norm induced by the inner-product. That is: Cauchy sequences, in the norm, converge to a vector in the space. This completeness does not follow from the above assumptions, unless the given real vector space V is finite-dimensional. (Thus a complex inner product space is often called a *pre-HIlbert space*.) But even if V is infinite-dimensional, and not complete in the norm, there is a canonical construction of a Hilbert space from it. This is like the canonical construction, for an arbitrary metric space (X, d), of a complete metric space $(\bar{X}, \bar{d})$, into which (X, d) can be isometrically embedded. Namely, the points of $\bar{X}$ are appropriately defined equivalence classes of Cauchy sequences in X. For details, cf. e.g. Prugovecki (REF).

2.4 A compatible J is not unique—and encodes some dynamics

There are three remarks to make at this point, about this development from 2.2.A to 2.3.C. They all concern the non-uniqueness of J, and they give a glimpse of further, more physical, developments, including *the Unruh effect*—glimpsed in 2.4.C below.

2.4.A: On the non-uniqueness of J:— We stressed in 2.2.A that an even-dimensional real vector space V admits many complex structures. For any basis $\{e_1, e_2, ..., e_{2n}\}$ can be divided into n pairs (in many ways), with J can then defined as the 'swap with a minus' on each such pair, extended by linearity.

But in this Subsection, since 2.3.B, we have assumed that a symplectic form ω is given, and that J is compatible with it. So does fixing ω and requiring compatibility still leave freedom in the definition of J? In fact, it does.

Fixing ω defines (by an analogue of the Gram-Schmidt diagonalization of a bilinear form) bases such that ω's matrix form is that of J in eq. 26 (cf. Section 2.1, and eq. 58 in Section

3.1 below, about symplectic structure). This is best understood in terms of how Hamiltonian mechanics defines a symplectic form on the phase space "of qs and ps", i..e. of positions and momenta. This naturally associates each q one-to-one with a p, and so the basis of $2n$ vectors breaks down in to n pairs. We might write the basis as $\{q_1, q_2, ..., q_n, p_1, p_2, ...p_n\}$, with each (q_i, p_i) forming a pair that J is to "swap with a minus". (Cf. the discussions above about the direct-sum way of thinking about complexification and complex structure.) The J thus defined will be, by construction, compatible with the given ω. So does compatibility with this fixed ω also fix, i.e. determine, J?

No. For we must remember that our vector space has no concept of *length* of vectors: it has only a concept of area given by the symplectic form (cf. the discussion in Section 3.1). So for each $i = 1, ..., n$, and each q_i in the basis yielding the matrix form in eq. 26, there is a positive-real-parameter family of vectors p_i, any one of which can be chosen while preserving ω's form in eq. 26. So with $\dim(V) = 2n$, there is an entire $(\mathbb{R}^+)^n$ 'hyperquadrant' in $\mathbb{R}^n$ of choices of the n vectors p_i. (Note that this freedom in J is not just a choice of sign, as discussed for complexifications in Section 2.1.C.)

We will see later a physical rationale for this: elegant and helpful, since it concerns the simple harmonic oscillator (SHO). In one spatial dimension, the SHO has a phase space $\mathbb{R}^2 \ni (q_1, p_1)$, with the system's possible trajectories (histories) being ellipses. But this copy of $\mathbb{R}^2$ has no concept of length, but only of area. A choice of J will thus encode facts about the eccentricity of the ellipses, and thus about the dynamics (the Hamiltonian). (The image $J((q_1, p_1))$ of a point (q_1, p_1) under the action of J will lie on the same ellipse as (q_1, p_1).)

The idea that J—and a closely associated map K that "maps from the (complex!) classical solution space to the quantum Hilbert space"—encode facts about the dynamics will be important in the sequel: also for understanding the *Unruh effect*. Cf. 2.4.C below.

2.4.B: From vector space to manifold:— In Hamiltonian mechanics, the phase space is in general a manifold, not a vector space. Namely, a symplectic manifold. Usually, this is the cotangent bundle of the configuration space. But if it is not, Darboux' theorem secures that locally it can be written as a cotangent bundle, and so has a canonical decomposition in to qs and ps, that associates each q one-to-one with a p.

However, in the sequel, we will be mostly concerned with the "happy" case of a phase space that is a vector space. It may be infinite-dimensional, as for classical fields; or it may be finite-dimensional, as for n uncoupled SHOs. In either case, a linear combination of solutions is itself a solution. For classical fields on a spatial manifold, e.g. $\mathbb{R}^3$, we add—or more generally, linearly combine—the field configurations and the momenta pointwise. For n uncoupled SHOs, we add (linearly combine) for each SHO independently. If we are given two solutions for the ith SHO (with a frequency ω_i say), labelled by their amplitude and phase (i.e. amplitude at time $t = 0$), we just add the two amplitudes and the two phases.

For any symplectic manifold M, we can of course rehearse for the tangent space T_pM at each point $p \in M$, and for its dual space T_p^*M, the development above from 2.2.A to 2.3.C. This means that given a symplectic form ω that smoothly varies across a local neighbourhood $U \subset M$, the bases it defines as in (1) above, i.e. the bases of T_pM at each point $p \in U$ such that ω's matrix is as in eq. 26 (cf. Section 2.1), also vary smoothly. And so the expression of J varies smoothly. In short, the local constructions presented above, from 2.2.A to 2.3.C, can be smoothly meshed with each other at the points in a local neighbourhood $U \subset M$.

But this still leaves open the question of *global existence* of a smooth J compatible with the global smooth ω. There can be obstructions to global existence. (Wald's exposition (1994) assumes there are none.) So when we do the local construction of J at each point $p \in M$, as above, we say there is an *almost complex structure*. For details of this, cf. e.g. Ana da Silva, *Lectures on Symplectic Geometry*.

2.4.C: Complexifying the classical solution space; and then splitting the frequencies in different ways:— When we study linear systems (Section **??**), we will see that a complex structure J corresponds to a splitting of the frequencies of complex classical solutions into positive and negative frequencies; and we will later see that having more than one complex structure J underlies the *Unruh effect*. The idea will be that in the Unruh effect, there are two different notions of time-evolution (two different Killing fields, two different Hamiltonians), that determine different *one particle structures* (cf. Section 5), and so different complex structures J. The general ideas are as follows; (cf. Wald 1994, (i) p. 24-29, for finite systems; and (ii) pp. 35-43, especially 39-41, for infinite systems, i.e. the Klein-Gordon field).

We first take the complexification of the solution space of the classical linear system. Here, we identify the solutions with the initial states, thanks to the determinism of the classical equations of motion. So writing S for the real symplectic vector space of solutions, the complexification is $S^{\mathbb{C}}$ (cf. Section 2.1).

We then define a 'positive frequency'/'positive energy' Hilbert space $\mathcal{H}$ by its being spanned by (as the span of) the complex classical solutions that oscillate with purely positive frequency (NB: also written ω!). For the simple harmonic oscillator, this means the complex classical solutions: $q(t) = \alpha \exp(-i\omega t)$, α a constant in $\mathbb{C}$. (Think of the momentum information being in the imaginary part.) For n uncoupled simple harmonic oscillators with frequencies $\omega_1, ..., \omega_n$, this means: $q_j(t) = \alpha_j \exp(-i\omega_j t)$ with $j = 1, ...n$. So for the latter case, $\mathcal{H}$ has complex dimension n.

Then the 'negative frequency'/'negative energy' Hilbert space $\bar{\mathcal{H}}$ is the span of the complex classical solutions that oscillate with purely negative frequency. In Section 2.5, just below, we will see that $\bar{\mathcal{H}}$ can be taken as the *complex conjugate of* $\mathcal{H}$, as defined there.

$S^{\mathbb{C}}$ is then the direct sum of the positive and negative frequency Hilbert spaces: $S^{\mathbb{C}} = \mathcal{H} \oplus \bar{\mathcal{H}}$. This direct sum structure means that there is a real-linear one-to-one onto "projection map" $K : S \to \mathcal{H}$ that extracts the positive frequency part of any real classical solution. This map K "maps from the (complex!) classical solution space to the quantum Hilbert space". It is the (main part of the definition of) *one particle structure*, which wil be central in the sequel, both for quantization in general (obviously!) and for e.g. the Unruh effect. Cf. Section 5.

The Unruh effect then arises in a scenario (defined on Minkowski spacetime!) in which two different notions of time-evolution (two different Killing fields, two different Hamiltonians) yield: two different frequency-splittings in (two different direct sum decompositions of) $S^{\mathbb{C}}$, and so two different Js; and so two different maps K; and thus two different vacua (ground states), and two different Fock spaces built from these vacua.

Besides: the failure of the Stone von Neumann theorem for infinite systems, means that here, 'different' means 'unitarily inequivalent'. That is: the two different Fock spaces built from the two vacua give unitarily inequivalent representations of the Weyl algebra.

Incidentally, Wald points out (p. 29 paragraph 2) that also for finite systems, e.g. n uncoupled time-independent simple harmonic oscillators, one can choose a different frequency-splitting

than the usual one, and so define a different vacuum (ground) state, which is usually called a *squeezed vacuum*. But here, there *is* unitary equivalence of representations.

There is a general philosophico-mathematical theme hereabouts: *singular limits*. That is: for every finite n, we have unitary equivalence; but for $n = \infty$, there is unitary inequivalence. We will see exactly the same for *spin-chains*. There, the canonical *anti*-commutation relations (CARs)—rather than CCRs—have for finite spin chains a unique representation upto unitary equivalence (the Jordan-Wigner theorem). But for infinite spin chains there are countless unitarily inequivalent representations.

2.5 Complex conjugation of spaces

2.5.A: Basics:— The *complex conjugate of complex vector space* W is the complex vector space $\overline{W}$ that has the same elements and additive group structure as $\overline{W}$, but whose scalar multiplication involves conjugation. That is: we define the scalar multiplication $*$ in $\overline{W}$ in terms of the scalar multiplication $\cdot$ in W by:

$$\alpha * w := \overline{\alpha} \cdot w \ , \quad \text{for all } \alpha \in \mathbb{C}, w \in \overline{W} \tag{44}$$

Various properties and results ensue!

(1) $\overline{\overline{W}} = W$.

(2) W and $\overline{W}$ have the same complex dimension. Note that the identity map $id : W \to \overline{W}$ is an antilinear map, since

$$id(\alpha \cdot w) = \alpha \cdot w \equiv \overline{\alpha} * w = \overline{\alpha} * id(w) \tag{45}$$

and id maps any basis of W into a basis of $\overline{W}$. So id is an *anti-isomorphism* from W to $\overline{W}$. It is a "canonical" one in the sense that its definition needs no choice of basis. That is: it is defined in terms of the underlying identity of vectors.

But of course, there are countless anti-isomorphisms defined in terms of such bases (just like there are countless isomorphisms!). For given any two bases, $\{e_i\}$ and $\{f_i\}$, of W and $\overline{W}$ respectively, the map $\Theta : e_i \to f_i$ can be extended by *antilinearity* to be an antilinear map, an *anti-isomorphism*, from W to $\overline{W}$.

(3) If W and U are complex vector spaces, an antilinear map $f : W \to U$ can be regarded as an ordinary linear map $f : \overline{W} \to U$, since:

$$f(\alpha * w) = f(\overline{\alpha} \cdot w) = \overline{\overline{\alpha}} \cdot f(w) = \alpha \cdot f(w) \ ; \tag{46}$$

where in the last two expressions, $\overline{\overline{\alpha}} \cdot f(w)$ and $\alpha \cdot f(w)$, the $\cdot$ is of course scalar multiplication in the codomain space U.

Conversely, any linear map g defined on $\overline{W}$, $g : \overline{W} \to U$, gives rise to an antilinear map from W to U, which again we write with a g. That is, we write: $g : W \to U$. For if we write the scalar multiplication in W as $\cdot$ (as before) and the scalar multiplication in U as $\cdot\cdot$, then the map $g : W \to U$ obeys:

$$g(\alpha \cdot w) \equiv g(\overline{\alpha} * w) = \overline{\alpha} \cdot \cdot g(w) \ , \tag{47}$$

since $g : \overline{W} \to U$ is linear. So the defined map $g : W \to U$ is antilinear.

(4) A linear map between complex vector spaces, $f : W \to U$, gives rise to a corresponding *also!* linear map $\overline{f} : \overline{W} \to \overline{U}$ which has the same action as f. For $\overline{f}$ preserves scalar multiplication, since

$$\overline{f}(\alpha * w) := f(\overline{\alpha} \cdot w) = \overline{\alpha} \cdot f(w) = \alpha * \overline{f}(w) \ . \tag{48}$$

If W, U are finite-dimensional, and the matrix of f with respect to bases $\{e_i\}$ of W and $\{g_j\}$ of U is (c_{ij}), i.e. $f(e_i) = c_{ij} g_j$, then the matrix of the linear map $\overline{f} : \overline{W} \to \overline{U}$ with respect to the *same* (as regards the underlying identity of vectors!) bases, i.e. $\{e_i\}$ of $\overline{W}$ and $\{g_j\}$ of $\overline{U}$, is the matrix whose entries are the complex conjugates of the c_{ij}. For in U, $c_{ij} g_j$ is short for $c_{ij} \cdot g_j$. But $c_{ij} \cdot g_j = \overline{c_{ij}} * g_j$. In short: to get the matrix of $\overline{f}$ from the matrix of f, we take complex conjugates of entries—but we do not transpose!

(5) *The complex conjugate of a Hilbert space.* That a Hilbert space $\mathcal{H}$ has extra structure additional to being a vector space, viz. the inner product, implies that there *is* a canonical aka natural, i..e basis-independent, isomorphism between $\mathcal{H}$ and $\overline{\mathcal{H}}$.

Indeed, recall *Riesz' theorem*: for a separable Hilbert space $\mathcal{H}$, every continuous linear functional $F : \mathcal{H} \to \mathbb{C}$ is given by taking the inner product with a unique vector $\psi_F \in \mathcal{H}$. That is: $F(\cdot) = (\psi_F, \cdot)$. Since this inner product is *sesquilinear*, i.e. $(\alpha\psi, \beta\phi) = \overline{\alpha}\beta(\psi, \phi)$, there is natural *antilinear* bijection between continuous linear functionals and vectors in $\mathcal{H}$: $F \mapsto \psi_F$. This is antilinear because $(\alpha F) \mapsto \psi_{(\alpha F)} \equiv \overline{\alpha}.\psi_F$. (Here, the . is good old scalar multiplication in $\mathcal{H}$!).

So there is natural *linear* bijection—i.e. an isomorphism!—between continuous linear functionals and vectors in the complex conjugate Hilbert space $\overline{\mathcal{H}}$. That is the dual space of linear functionals, $\mathcal{H}^*$ can be identified with $\overline{\mathcal{H}}$. It then follows that if we identify $\mathcal{H}^{**}$ with $\mathcal{H}$, there is natural isomorphism between $\mathcal{H}^{**} \equiv \mathcal{H}$ and $(\overline{\mathcal{H}})^*$.

Exercise! : Is there a natural isomorphism between $(\overline{\mathcal{H}})^*$ and $\overline{\mathcal{H}^*}$?

(6) *The relation of complexifications to complex structures.*

YET TO DO

(a) general ideas then (b) physics, i.e. about the complexification of classical solutions as direct sum of positive-frequency and negative frequency subspaces, with the J thus encoding a choice of positive-frequency.

3 Symplectic structure

We first recall elements of the symplectic structure underlying Hamiltonian mechanics (Section 3.1). This will show us how to write the classical Poisson brackets in terms of the symplectic product (Section 3.2). Thus we will return to the ideas of the Weyl algebra (cf. Section 1.2), in the form using operators W—which combine the translations in position and in momentum that were given separately by the operators U and V. Then we generalize to symplectic manifolds (Section 3.3).

3.1 Symplectic vector spaces

We will rewrite the classical Poisson brackets, eq. 1, but repeated here:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \ ; \ \{q_i, p_j\} = \delta_{ij} \tag{49}$$

in terms of a symplectic product on a vector space.

We begin with *Hamilton's equations*

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \quad ; \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \, . \tag{50}$$

Defining

$$\xi^\alpha = q^\alpha, \;\; \alpha = 1, ..., n \quad ; \quad \xi^\alpha = p_{\alpha-n}, \;\; \alpha = n+1, ..., 2n \tag{51}$$

Hamilton's equations become

$$\dot{\xi}^\alpha = \frac{\partial H}{\partial \xi^{\alpha+n}}, \;\; \alpha = 1, ..., n \quad ; \quad \dot{\xi}^\alpha = -\frac{\partial H}{\partial \xi^{\alpha-n}}, \;\; \alpha = n+1, ..., 2n \;\; . \tag{52}$$

Writing $\mathbf{1}$ and $\mathbf{0}$ for the $n \times n$ identity and zero matrices respectively, we define the $2n \times 2n$ *symplectic matrix* ω by

$$\omega := \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \;\; . \tag{53}$$

ω is antisymmetric, and has the properties, writing $\tilde{}$ for the transpose of a matrix, that

$$\tilde{\omega} = -\omega = \omega^{-1} \quad \text{so that} \quad \omega^2 = -\mathbf{1} \quad ; \quad \text{also} \quad \det \omega = 1. \tag{54}$$

Using ω, Hamilton's equations eq. 52 get the more symmetric form, in matrix notation

$$\dot{\xi} = \omega \frac{\partial H}{\partial \xi} \;\; . \tag{55}$$

In terms of components, writing $\omega^{\alpha\beta}$ for the matrix elements of ω, and $\partial_\alpha := \partial \, / \partial \xi^\alpha$, eq. 52 become

$$\dot{\xi}^\alpha = \omega^{\alpha\beta} \partial_\beta H. \tag{56}$$

Eq. 55 and 56 show how ω forms, from the naive gradient (column vector) ∇H of H on the phase space Γ of qs and ps, the vector field on Γ that gives the system's evolution: the *Hamiltonian vector field*, often written X_H. At a point $z = (q, p) \in \Gamma$, eq. 55 can be written

$$X_H(z) = \omega \nabla H(z). \tag{57}$$

Interpretation in terms of areas: Let us begin with the simplest possible case: $\mathbb{R}^2 \ni (q, p)$, representing the phase space of a particle constrained to one spatial dimension. Here, the 2×2 matrix

$$\omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{58}$$

defines the antisymmetric bilinear form on $\mathbb{R}^2$:

$$A : ((q^1, p_1), (q^2, p_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto q^1 p_2 - q^2 p_1 \in \mathbb{R} \tag{59}$$

since

$$q^1 p_2 - q^2 p_1 = \begin{pmatrix} q^1 & p_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q^2 \\ p_2 \end{pmatrix} = \det \begin{pmatrix} q^1 & q^2 \\ p_1 & p_2 \end{pmatrix} \;\; . \tag{60}$$

It is easy to prove that $A((q^1, p_1), (q^2, p_2)) \equiv q^1 p_2 - q^2 p_1$ is the signed area of the parallelogram spanned by $(q^1, p_1), (q^2, p_2)$, where the sign is positive (negative) if the shortest rotation from (q^1, p_1) to (q^2, p_2) is anti-clockwise (clockwise).

Similarly in $\mathbb{R}^{2n}$: the matrix ω of eq. 53 defines an antisymmetric bilinear form on $\mathbb{R}^{2n}$ whose value on a pair $(q,p) \equiv (q^1,...q^n;p_1,...,p_n),(q',p') \equiv (q'^1,...q'^n;p'_1,...,p'_n)$ is the sum of the signed areas of the n parallelograms formed by the projections of the vectors $(q,p),(q',p')$ onto the n pairs of coordinate planes labelled $1,...,n$. That is to say, the value is:

$$\Sigma_{i=1}^{n}\, q^i p'_i - q'^i p_i \quad . \tag{61}$$

3.2 Returning to the Weyl algebra

If we are lucky enough for our classical phase space to be vector space (as when $S = \mathbb{R}^{2n}$), then we can make it a *symplectic vector space*, which is a pair (S,Ω), where S is a phase space—also a vector space—and Ω is a symplectic product. The symplectic product $\Omega : S \times S \to \mathbb{R}$ is, by definition, anti-symmetric, linear and non-degenerate (i.e. if $\Omega(z_1,z_2) = 0$ for all z_2, then $z_1 = \mathbf{0}$).

We define *the symplectic product* Ω on $S = \mathbb{R}^{2n} \ni z_1, z_2$ as in (14): which we repeat here:

$$\Omega(z_1,z_2) := \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2, \tag{62}$$

Then as we saw in Section 1.2, the Weyl form of the CCRs, i.e. (11) and (7), are equivalent to the following: for all $z, z_1, z_2 \in \mathbb{R}^{2n}$,

$$\begin{array}{rcl} W(z_1)W(z_2) & = & e^{\frac{1}{2}i\Omega(z_1,z_2)}W(z_1+z_2); \\ W^\dagger(z) & = & W(-z)\ . \end{array} \tag{63}$$

Note that $\Omega(z,\cdot) : S \to \mathbb{R}$ is a real-valued function on S, and so a classical observable. In particular, $\Omega(z,\cdot) = q^i$ iff z has $(n+i)$th component $b_i = 1$ and the rest 0, and $\Omega(z,\cdot) = p_i$ iff z has ith component $a^i = -1$ and the rest 0. In general, $\Omega(z,\cdot)$ is some linear combination of p_is and q^is.

In this formulation, the classical Poisson bracket relations (1: repeated as 49) may be written

$$\{\Omega(z_1,\cdot),\Omega(z_2,\cdot)\} = -\Omega(z_1,z_2)\ . \tag{64}$$

So the corresponding Heisenberg form of the CCRs are

$$[\hat{\Omega}(z_1,\cdot),\hat{\Omega}(z_2,\cdot)] = -i\Omega(z_1,z_2)\mathbb{1}\ . \tag{65}$$

Thus we seek a representation in which the map $z \mapsto \hat{\Omega}(z,\cdot)$ takes elements of S to self-adjoint operators, and in which the Weyl unitaries defined by

$$W(z) := e^{i\hat{\Omega}(z,\cdot)}. \tag{66}$$

obey the Weyl algebra, eq. 63.

This is Wald's presentation: see Wald (1994, Ch. 2). Later we will use field operators Φ, for which $\Phi(Jz) = \hat{\Omega}(z,\cdot)$, or $\Phi(z) = -\hat{\Omega}(Jz,\cdot) = \hat{\Omega}(\cdot,Jz)$.

3.3 Symplectic manifolds, more generally

In the case where the classical phase space S is not a vector space, we must resort to a longer route. In this case, we seek a group whose action on S is *transitive* and preserves the symplectic

form $\omega := \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i$. (In the case that S is a vector space, this group is just the (abelian) additive group of translations in S, which is isomorphic to S. That is what allowed us to treat S as a symplectic vector space above.) For illustration, taking the case $S = \mathbb{R}^{2n}$, the group action is a $2n$-parameter family of diffeomorphisms associated with the vector fields (with constant coefficients)

$$X_z = \sum_{i=1}^{n} b_i \frac{\partial}{\partial q^i} - a^i \frac{\partial}{\partial p_i}, \tag{67}$$

for any $z := (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{2n}$. We may now act on any two such vector fields with the *symplectic form* ω with which S, being a classical phase space, is equipped. This yields

$$\omega(X_{z_1}, X_{z_2}) = \mathbf{a}_2.\mathbf{b}_1 - \mathbf{a}_1.\mathbf{b}_2. \tag{68}$$

Our quantization problem then becomes the search for continuous families of unitaries $z \mapsto W(z)$ which respect this symplectic structure, as expressed in the Weyl algebra (63), setting $e^{\frac{1}{2}i\Omega(z_1,z_2)} = e^{\frac{1}{2}i\omega(X_{z_1},X_{z_2})}$. Since the Weyl algebra (63) is unitary up to the phase factor $e^{\frac{1}{2}i\omega(X_{z_1},X_{z_2})}$, it is a *projective unitary representation* of the group of symplectomorphisms on S.

4 Linear systems

THIS Section to be expanded! The Section 4.2 is half-written as for classical field theory

4.1 Classical linear dynamics

The dynamics are linear if they obey the superposition principle. We use lowercase Fraktur letters $\mathfrak{z}$ to denote point sin the phase space Γ. The Hamiltonian is given by

$$H(\mathfrak{z}) = \frac{1}{2}(\mathfrak{z}^T, \boldsymbol{H}\mathfrak{z}) \tag{69}$$

4.2 The simple harmonic oscillator

The Hamiltonian is given by

$$H(\mathfrak{z}) = \frac{1}{2}(\mathfrak{z}^T, \boldsymbol{H}\mathfrak{z}) \tag{70}$$

We will specifically focus on systems for which

$$H(q,p) = \frac{1}{2}\left[(q, \omega^2 q) + (p,p)\right] \ , \tag{71}$$

and so

$$\mathbf{H} = \begin{pmatrix} \omega^2 & \\ & \mathbb{1} \end{pmatrix} \ , \tag{72}$$

where ω^2 is some strictly positive and self-adjoint operator on the configuration space $\mathcal{L}^2(M, V, \mu) \ni q$. Since it is strictly positive, we can express it as the square of a self-adjoint operator ω with dimensions of angular frequency. This Hamiltonian is associated with the following action:

$$S = \int \mathrm{d}^4x \ \frac{1}{2}\left(\dot{q}^2(x) - q(x)(\omega^2 q)(x)\right) \ , \tag{73}$$

which gives the field dimensions of $[action]^{\frac{1}{2}}[length]^{-\frac{3}{2}}[time]^{\frac{1}{2}}$, and the field momentum dimensions of $[action]^{\frac{1}{2}}[length]^{-\frac{3}{2}}[time]^{-\frac{1}{2}}$. In natural units, these yield mass dimensions 1 and 2, respectively.

Write the symplectic product as

$$\Omega(\mathfrak{z}_1, \mathfrak{z}_2) =: (\mathfrak{z}_1^T, \mathbf{\Omega}\mathfrak{z}_2) \tag{74}$$

where we introduce the symplectic matrix

$$\mathbf{\Omega} = \begin{pmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix} . \tag{75}$$

The equations of motion are then given by

$$\begin{aligned} \dot{\mathfrak{z}} &= \mathbf{\Omega}\boldsymbol{H}_{\mathfrak{z}} \\ (\dot{q}, \dot{p}) &= \mathbf{\Omega}(\omega^2 q, p) \\ &= (p, -\omega^2 q) \end{aligned} \tag{76}$$

(a linear equation). It follows that the field configuration q obeys the second-order differential equation

$$\ddot{q} + \omega^2 q = 0 . \tag{77}$$

5 One-particle structures

5.1 The general idea

We begin with a symplectic vector space (S, Ω, Φ_t), where Φ_t is the one-parameter family of symplectomorphisms generated by the Hamiltonian vector field D_t. *Note now the presence of dynamics, indicated by* D_t. We find the single-particle Hilbert space by "Hilbertizing" S and "unitarizing" D_t to obtain a *one-particle structure* $(\mathcal{H}, \langle\cdot,\cdot\rangle, U(t))$, i.e. a Hilbert space equipped with an inner product and a one-parameter family of unitaries, representing the quantum dynamical evolution.

The one-particle structure is related to our original symplectic vector space by the map $K : S \to \mathcal{H}$ such that:

(i) $\mathrm{ran}(K)$ is dense in $\mathcal{H}$;

(ii) $2\Im m\langle Kz_1, Kz_2\rangle = \Omega(z_1, z_2)$;

(iii) $U(t)K = K\Phi_t$, where $U(t) = e^{-itA}$ and A is a positive operator.

(See Weinless 1969; and Halvorson 2001 for an application.) If a one-particle structure exists, it is unique; see Kay (1979).

S is a vector space over the reals. Key to defining the map K is finding some way to make sense of complex multiplication of the elements of S: i.e. we seek a *complex structure* $J : S \to S$. Recall Section 2.2! Then for any $a, b \in \mathbb{R}$ we may define, for any $z \in S$:

$$(a + ib)K(z) = K(az) + K(bJz). \tag{78}$$

We may then write (iii) in infinitesimal form to derive a "Schrödinger equation"

$$AK(z) = iKD_t(z) = KJD_t(z). \tag{79}$$

(In some cases K will be the embedding map, which simplifies the equation above to $Az = JD_tz$.)

Recall from Section 2.2 that J must satisfy three conditions:

(a) J is a symplectomorphism; i.e. $\Omega(Jz_1, Jz_2) = \Omega(z_1, z_2)$; (compatibility with Ω: cf. Section 2.3.A);

(b) $J^2 = -\mathbb{1}$; (cf. Section 2.2.A);

(c) $\Omega(z, Jz) > 0$, for all $z \neq 0$ ('taming': cf. Section 2.3.A).

It follows from (iii) and (a) that $[J, \Phi_t] = 0$, i.e. J is equivariant under the classical dynamics.

Given J, we may define a complex inner product on (S, Ω, Φ_t, J):

$$\langle z_1, z_2\rangle_S = \frac{1}{2}\Omega(z_1, Jz_2) + \frac{1}{2}i\Omega(z_1, z_2), \tag{80}$$

Here we recall: from 2.3.B: the definition of the *complex-linear* but *real-valued* symmetric bilinear form g_J on the complex vector space $V_J g_J$ by eq. 40. It was

$$g_J(u, v) := \omega(u, Jv)\ . \tag{81}$$

and from 2.3.C: the definition of the *sesquilinear*, *complex-valued* function on $V \times V$, i.e. complex inner product, in terms of g_J and ω, by eq. 42. It was

$$\langle u, v\rangle \equiv \langle u, v\rangle_{\omega,J} := g_J(u, v) + i\omega(u, v)\ . \tag{82}$$

The definition eq. 80 guarantees (ii).

We can then demand that $\langle Kz_1, Kz_2\rangle = \langle z_1, z_2\rangle_S$ on $K[S]$. By completing $K[S]$ in the norm induced by this inner product (if required), we obtain $\mathcal{H}$.

5.2 Example: the simple harmonic oscillator

Perhaps surprisingly, the above strategy works where (S, Ω, Φ_t) is the theory of the classical simple harmonic oscillator. We will see that, bizarrely enough, a single-particle Hilbert space with unitary dynamics corresponds to this theory.

Recall that for the simple harmonic oscillator, $S = \mathbb{R}^2 \ni (q, p)$, Ω is defined as usual, i.e.

$$\Omega((q_1, p_1), (q_2, p_2)) = q_1p_2 - q_2p_1, \tag{83}$$

and Φ_t is generated by the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2q^2. \tag{84}$$

Hamilton's equations yield

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}; \qquad \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q; \tag{85}$$

and so $\ddot{q} + \omega^2 q = 0$. Solutions take the form

$$q = \frac{1}{\sqrt{2m\omega}} \left(a e^{-i\omega t} + a^* e^{i\omega t} \right), \qquad p = -i\sqrt{\frac{m\omega}{2}} \left(a e^{-i\omega t} - a^* e^{i\omega t} \right), \tag{86}$$

which defines trajectories in the phase space which are ellipses centred at $(0, 0)$. (The reason for our choice of constants will be clear soon.) These trajectories already look *almost* like unitary evolution in $\mathbb{C}$; i.e. circular motion centred at $(0, 0)$. Heuristically, we need to rescale q and p to send these ellipses to circles. *One* way of doing so is to define the map $K : \mathbb{R}^2 \to \mathbb{C}$ by

$$K(q, p) = \sqrt{\frac{m\omega}{2}} q + \frac{i}{\sqrt{2m\omega}} p \;=:\; \xi_{(q,p)}. \tag{87}$$

Solutions in $\mathbb{C}$ are then defined by $K(q(t), p(t)) = a e^{-i\omega t}$. This suggests that the quantum Hamiltonian $A = \omega$, which determines the complex structure

$$J(q, p) = \left(-\frac{p}{m\omega}, m\omega q \right); \tag{88}$$

for then we have

$$JD_t(q, p) = J(\dot{q}, \dot{p}) = J\left(\frac{p}{m}, -m\omega^2 q \right) = (\omega q, \omega p) = \omega(q, p), \tag{89}$$

which is the classical counterpart of our "Schrödinger equation" in S.

It is interesting to note that if we define the "positive frequency" component of $(q(t), p(t))$:

$$q^{(+)}(t) \;:=\; \frac{1}{\sqrt{2}} \left(q(t) + \frac{i}{m\omega} p(t) \right) = \frac{1}{\sqrt{m\omega}} a e^{-i\omega t}; \tag{90}$$

$$p^{(+)}(t) \;:=\; \frac{1}{\sqrt{2}} \left(p(t) - im\omega q(t) \right) = -i\sqrt{m\omega}\, a e^{-i\omega t} = -im\omega q^{(+)}(t); \tag{91}$$

then

$$J\left(q^{(+)}(t), p^{(+)}(t) \right) = J\left(1, -im\omega \right) q^{(+)}(t) = \left(i, m\omega \right) q^{(+)}(t) = i\left(q^{(+)}(t), p^{(+)}(t) \right); \tag{92}$$

i.e., J acts as multiplication by i; while for the "negative frequency" component:

$$q^{(-)}(t) \;:=\; \frac{1}{\sqrt{2}} \left(q(t) - \frac{i}{m\omega} p(t) \right) = \frac{1}{\sqrt{m\omega}} a^* e^{i\omega t}; \tag{93}$$

$$p^{(-)}(t) \;:=\; \frac{1}{\sqrt{2}} \left(p(t) + im\omega q(t) \right) = i\sqrt{m\omega}\, a^* e^{i\omega t} = im\omega q^{(-)}(t); \tag{94}$$

we have

$$J\left(q^{(-)}(t), p^{(-)}(t) \right) = J\left(1, im\omega \right) q^{(-)}(t) = \left(-i, m\omega \right) q^{(-)}(t) = -i\left(q^{(-)}(t), p^{(-)}(t) \right); \tag{95}$$

i.e., J acts as multiplication by $-i$. This a toy analogue of the solution to the "negative energy problem" in quantum field theory: for negative-frequency solutions, the complex structure has the opposite sign, allowing for positive-energy solutions in all cases.

Our inner product in S is given by

$$\begin{aligned}
\langle (q_1,p_1),(q_2,p_2)\rangle_S &= \frac{1}{2}\Omega\left((q_1,p_1),J(q_2,p_2)\right)+\frac{1}{2}i\Omega\left((q_1,p_1),(q_2,p_2)\right) && (96)\\
&= \frac{1}{2}\Omega\left((q_1,p_1),\left(-\frac{p_2}{m\omega},m\omega q_2\right)\right)+\frac{1}{2}i\Omega\left((q_1,p_1),(q_2,p_2)\right) && (97)\\
&= \frac{p_1p_2}{2m\omega}+\frac{1}{2}m\omega q_1q_2+\frac{i}{2}\left(q_1p_2-q_2p_1\right) && (98)\\
&= \left(\sqrt{\frac{m\omega}{2}}q_1-\frac{i}{\sqrt{2m\omega}}p_1\right)\left(\sqrt{\frac{m\omega}{2}}q_2+\frac{i}{\sqrt{2m\omega}}p_2\right) && (99)\\
&= K(q_1,p_1)^*K(q_2,p_2), && (100)
\end{aligned}$$

so our inner product in $\mathbb{C}$ is just

$$\langle \xi_1,\xi_2\rangle = \xi_1^*\xi_2. \tag{101}$$

This elegant form for the inner product is a consequence of our particular choice of the map K.

6 "Unitarization": complex structures, metrics, and the 2-out-of-3 property

"Unitarization" of linear Hamiltonian systems appeals to the "2-out-of-3" property of the group $U(n)$ of unitary transformations in n complex dimensions:

$$U(n) = Sp(2n,\mathbb{R})\cap GL(n,\mathbb{C})\cap O(2n) \tag{102}$$

The identity above continues to hold for any two of the three groups on the righthand side: i.e. $Sp(2n,\mathbb{R})\cap GL(n,\mathbb{C}) = GL(n,\mathbb{C})\cap O(2n) = O(2n)\cap Sp(2n,\mathbb{R}) = U(n)$.

- $Sp(2n,\mathbb{R})$ is the *symplectic group in* $2n$ *dimensions*, the group of automorphisms of the symplectic vector space $\langle\mathbb{R}^{2n},\Omega\rangle$, the *linear symplectic transformations*, and so is the most general group of symmetries pertaining to a linear Hamiltonian system. It is the group of linear transformations $A:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ which preserve some symplectic product Ω, according to $\Omega(Au,Av)=\Omega(u,v)$ for all $u,v\in\mathbb{R}^{2n}$. The symplectic product Ω is any bilinear, alternating and non-degenerate map $\Omega:\mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}$. If Ω is represented as a matrix $\mathbf{\Omega}$, so that $\Omega(u,v)\equiv u^T\mathbf{\Omega}v$, then this condition is equivalent to $A^T\mathbf{\Omega}A=\mathbf{\Omega}$. (The bilinearity of $\mathbf{\Omega}$ is baked in, since it is a matrix; the non-degeneracy of Ω means that $\mathbf{\Omega}^{-1}$ exists; Ω's being alternating means that $u^T\mathbf{\Omega}v=-v^T\mathbf{\Omega}u$ for all $u,v\in\mathbb{R}^n$, which is equivalent to $\mathbf{\Omega}^T=-\mathbf{\Omega}$.) By means of some similarity transformation the matrix $\mathbf{\Omega}$ may be brought to the form

  $$\mathbf{\Omega} = \begin{pmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix}. \tag{103}$$

  All symplectic matrices $A\in Sp(2n,\mathbb{R})$ have unit determinant, which means that they preserve area in the symplectic vector space—the specialisation of preserving the symplectic form to vector spaces. In the case $n=1$ the elements of $Sp(2,\mathbb{R})$ just are all and only the area-preserving 2×2 matrices.

- $GL(n,\mathbb{C})$ is the group of general linear transformations on $\mathbb{C}^n$, i.e. the group of all invertible $n\times n$ matrices with complex entries. This group is isomorphic to the group of invertible $2n\times 2n$ matrices A with *real* entries which commute with some *complex structure* $J:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$, so that $AJ=JA$. The complex structure J here is some $2n\times 2n$ matrix such that $J^2=-\mathbb{1}$, so that $J^{-1}=-J$. J allows us to pass from the real vector space $\mathbb{R}^{2n}$ to the complex vector space $\mathbb{C}^n$, where J takes the role of multiplication by i.

- $O(2n)$ is the group of orthogonal transformations on $\mathbb{R}^{2n}$, which can be thought of as the group of real $2n \times 2n$ matrices A such that $A^{-1} = A^T$. More generally, given some positive-definite, symmetric bilinear form $M : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$, $O(2n)$ is the group of linear transformations $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $M(Au, Av) = M(u, v)$ for all $u, v \in \mathbb{R}^{2n}$. If M is represented as a matrix $\mathbf{M}$, so that $M(u, v) \equiv u^T \mathbf{M} v$, then this condition is equivalent to $A^T \mathbf{M} A = \mathbf{M}$. (The bilinearity of $\mathbf{M}$ is baked in, since it is a matrix; the positive-definiteness of M means that $\mathbf{M}$ has only positive eigenvalues; M's being symmetric means that $u^T \mathbf{M} v = v^T \mathbf{M} u$ for all $u, v \in \mathbb{R}^n$, which is equivalent to $\mathbf{M}^T = \mathbf{M}$.) By means of some similarity transformation the matrix $\mathbf{M}$ may be brought to the form $\mathbb{1}$; then we recover the condition $AA^T = A^T A = \mathbb{1}$, i.e. $A^{-1} = A^T$. The crucial idea is that the elements of $O(2n)$ preserve the lengths of all vectors in $\mathbb{R}^{2n}$ according to *some* metric, given by M.

The intersection of any two of these groups generates a subgroup of the third by means of the identification $M(\cdot, \cdot) = \Omega(\cdot, J\cdot)$, or $\mathbf{M} = \mathbf{\Omega} J$, with the crucial compatibility requirement that $\Omega(J\cdot, J\cdot) = \Omega(\cdot, \cdot)$, or $J^T \mathbf{\Omega} J = \mathbf{\Omega}$; equivalent to $J^T \mathbf{M} J = \mathbf{M}$ and $\mathbf{M}^{-1} \mathbf{\Omega} = -\mathbf{\Omega}^{-1} \mathbf{M}$. It may then be checked that any two of: (i) $\mathbf{\Omega}^T = -\mathbf{\Omega}$; (ii) $J^{-1} = -J$; and (iii) $\mathbf{M}^T = \mathbf{M}$ implies the third. For example: $\mathbf{M}^T = (\mathbf{\Omega} J)^T = J^T \mathbf{\Omega}^T = -J^T \mathbf{\Omega} = J^T \mathbf{\Omega} J^2 = \mathbf{\Omega} J = \mathbf{M}$, where we use the identification $\mathbf{M} = \mathbf{\Omega} J$, the property $\mathbf{\Omega}^T = -\mathbf{\Omega}$, the property $J^2 = -\mathbb{1}$ and the compatibility condition $J^T \mathbf{\Omega} J = \mathbf{\Omega}$, respectively.

Starting with any linear Hamiltonian system, we are given $Sp(2n, \mathbb{R})$ and $\mathbf{\Omega}$ at the outset. Our goal is $U(n)$. The "2-out-of-3" identity above entails that we can get there so long as we can find some metric $\mathbf{M}$ or some complex structure J obeying the compatibility conditions (and, if we can find one of these, then we automatically have the other). In other words, if we can find some natural way to give the symplectic vector space $\langle \mathbb{R}^{2n}, \Omega \rangle$ the structure of a real inner-product space $\langle \mathbb{R}^{2n}, M \rangle$ or the structure of the complex vector space $\mathbb{C}^n \cong \langle \mathbb{R}^{2n}, J \rangle$, then we will end up with the Hilbert space $\langle \mathbb{C}^n, \langle \cdot, \cdot \rangle \rangle$ and its associated group of automorphisms $U(n)$.

It turns out that the metric $\mathbf{M}$ is determined by the classical Hamiltonian *so long as* the associated Hamiltonian matrix $\mathbf{H}$ is positive-definite (it is already symmetric and bilinear). This corresponds to the existence of a unique global vacuum at $\mathbf{0}$ in the symplectic vector space. The compatibility requirement relies on the dynamics decomposing into normal modes. For then we can find some invertible symmetric matrix A, diagonalised in the normal-mode basis, such that $[A, \mathbf{H}] = [A, \mathbf{\Omega}] = 0$ and such that we can set $\mathbf{M} = \mathbf{H} A^{-1}$. It follows that $J = \mathbf{\Omega}^{-1} \mathbf{H} A^{-1}$ and $[A, J] = 0$. It turns out that A is proportional to the *quantum* Hamiltonian of the resulting quantum theory.

These conditions on A give us a recipe for finding it. For in coordinates such that $\mathbf{\Omega}^{-1} = \mathbf{\Omega}^T = -\mathbf{\Omega}$ (which we can always find) we have $(\mathbf{\Omega H})^2 = \mathbf{\Omega H \Omega H} = \mathbf{\Omega} A \mathbf{M \Omega} A \mathbf{M} = A \mathbf{\Omega M \Omega M} A = A \mathbf{\Omega}^2 A = -A^2$. We may therefore set $A := (-(\mathbf{\Omega H})^2)^{\frac{1}{2}}$. Crucial here is that $\mathbf{H}$ is at least positive-semi-definite; otherwise A is not a real matrix. We now rely on the fact that we can find coordinates in which not only $\mathbf{\Omega}$ takes the standard form in which $\mathbf{\Omega}^{-1} = \mathbf{\Omega}^T = -\mathbf{\Omega}$, but also $\mathbf{H}$ takes the form

$$\mathbf{H} = \begin{pmatrix} B & \\ & C \end{pmatrix} \tag{104}$$

(so that there are no q-p cross terms), and where $[B, C] = 0$. (Usually we expect $C = \mathbb{1}$.) In this case we find that

$$A = \begin{pmatrix} (BC)^{\frac{1}{2}} & \\ & (BC)^{\frac{1}{2}} \end{pmatrix} . \tag{105}$$

It may now be verified that $[A, \mathbf{H}] = [A, \mathbf{\Omega}] = 0$. The metric $\mathbf{M}$ then takes the form

$$\mathbf{M} = \begin{pmatrix} B^{\frac{1}{2}}C^{-\frac{1}{2}} & \\ & B^{-\frac{1}{2}}C^{\frac{1}{2}} \end{pmatrix} , \tag{106}$$

and it may be checked that the compatibility requirement $\mathbf{M}^{-1}\mathbf{\Omega} = -\mathbf{\Omega}^{-1}\mathbf{M} = \mathbf{\Omega}\mathbf{M}$ is satisfied. Crucial here is that $\mathbf{H}$ is not only positive-semi-definite but positive-definite: for, both $B^{\frac{1}{2}}$ and $C^{\frac{1}{2}}$ must exist and be invertible; otherwise $\mathbf{M}$ is not well defined. The complex structure J then takes the form

$$J = \begin{pmatrix} & -B^{-\frac{1}{2}}C^{\frac{1}{2}} \\ B^{\frac{1}{2}}C^{-\frac{1}{2}} & \end{pmatrix} . \tag{107}$$

In the case $B = \omega^2, C = \mathbb{1}$, we have

$$\mathbf{H} = \begin{pmatrix} \omega^2 & \\ & \mathbb{1} \end{pmatrix} ; \quad A = \begin{pmatrix} \omega & \\ & \omega \end{pmatrix} ; \quad \mathbf{M} = \begin{pmatrix} \omega & \\ & \omega^{-1} \end{pmatrix} ; \quad J = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} . \tag{108}$$

We may then define the inner product

$$\langle \cdot, \cdot \rangle = \lambda \left(M(\cdot, \cdot) + i\Omega(\cdot, \cdot) \right) , \tag{109}$$

where λ is some constant. Since $\Omega(\mathfrak{z}_1, \mathfrak{z}_2)$ has units of action, it seems sensible to set $\lambda \sim \frac{1}{\hbar}$. The standard choice is $\lambda = \frac{1}{2\hbar}$ (this will end up yielding the usual ladder operators in the case of of the s.h.o.).

7 Many simple harmonic oscillators: the occupation number representation

THIS Section to be expanded!

Many SHOs: even with time dependent Hamiltonians, and interactions between the SHOs. (Cf. Wald 1994 SEction 2.3, and e.g. Coleman *Lectures on Quantum Field Theory*, Lecture 2, pp. 17-30)

Occupation number representation for fixed number N of SHOs. Fock space way of saying it: ie the number operator counts excitations and so has all positive integers as spectrum even for N particles /oscillators (This is in Wald of course, though he does not call it Occupation number representation! Cf. his p. 25, with the naive N-fold tensor product construction on p. 24.)

8 The Stone-von Neumann uniqueness theorem

8.1 Weak continuity

The Stone-von Neumann uniqueness theorem applies to *weakly continuous* representations of the Weyl algebra. A representation on some Hilbert space $\mathcal{H}$, with inner product $\langle \cdot, \cdot \rangle$, is weakly continuous iff it is continuous in the weak topology provided by $\langle \cdot, \cdot \rangle$. I.e. for all $\psi \in \mathcal{H}$, $z_1, z_2 \in S$ and $\epsilon \in \mathbb{R}^+$,

$$\langle \psi, W(z_1 + \epsilon z_2)\psi \rangle \to \langle \psi, W(z_1)\psi \rangle \quad \text{as} \quad \epsilon \to 0. \tag{110}$$

8.2 The theorem

The following theorem guarantees that, in certain cases, the representation of the Weyl algebra is effectively unique (i.e., unique up to unitary equivalence). Therefore there is a reasonable sense in which, in these cases, there is only one quantization of a classical system.

Theorem 8.1 (Stone-von Neumann Uniqueness Theorem). (Prugovecki, p. 371 ; Wald p. 20) Let (S, Ω) be a symplectic vector space, with $S = \mathbb{R}^{2n}$. Every weakly continuous irreducible representation of the Weyl algebra over (S, Ω) is unitarily equivalent to the Schrödinger representation, in which, for all $\psi(\mathbf{x}) \in L^2(\mathbb{R}^n)$,

$$(W(\mathbf{a}, \mathbf{b})\psi)(\mathbf{x}) := e^{-i\mathbf{a}.(\mathbf{x}-\frac{1}{2}\mathbf{b})}\psi(\mathbf{x} - \mathbf{b}). \tag{111}$$

Note as special cases that $(W(\mathbf{a}, \mathbf{0})\psi)(\mathbf{x}) \equiv (U(\mathbf{a})\psi)(\mathbf{x}) = e^{-i\mathbf{a}.\mathbf{x}}\psi(\mathbf{x})$ and $(W(\mathbf{0}, \mathbf{b})\psi)(\mathbf{x}) \equiv (V(\mathbf{b})\psi)(\mathbf{x}) = \psi(\mathbf{x} - \mathbf{b})$. In fact, the Schrödinger representation is strongly continuous, so by Stone's Theorem there are $2n$ self-adjoint operators, Q^i and P_i, such that $U(\mathbf{a}) = e^{-i\mathbf{a}.\mathbf{Q}}$, $V(\mathbf{b}) = e^{-i\mathbf{b}.\mathbf{P}}$ and for all $\psi(\mathbf{x}) \in L^2(\mathbb{R}^n)$ in suitable domains,

$$(\mathbf{Q}\psi)(\mathbf{x}) = \mathbf{x}\psi(\mathbf{x}); \qquad (\mathbf{P}\psi)(\mathbf{x}) = -i\nabla\psi(\mathbf{x}). \tag{112}$$

The Stone-von Neumann theorem fails to apply if either of its antecedent conditions fail; i.e. if either the classical phase space is not $\mathbb{R}^{2n}$, or else the representation of the Weyl algebra is not weakly continuous. Following Ruestche (2011, Ch. 3), it is helpful to break the various possible failures into three cases:

(i) weak continuity fails;

(ii) classical phase space is finite-dimensional, but not $\mathbb{R}^{2n}$;

(iii) classical phase space is infinite-dimensional.

In each of these cases, we have no guarantee that the quantization of our classical system is unique. In fact, for each of these cases we know that the quantization is not unique. This is considered in Part II.

8.3 The CARs; the Jordan-Wigner theorem

Now consider first a sequence of quantum theories, each corresponding to a chain of spin-$\frac{1}{2}$ systems. The first theory describes a single spin-$\frac{1}{2}$ system, with observables $\{\sigma(x), \sigma(y), \sigma(z)\}$, which satisfy the Pauli relations

$$[\sigma(x), \sigma(y)] = 2i\sigma(z) \quad \text{and cyclic perms}; \qquad \sigma^2 := \sigma(x)^2 + \sigma(y)^2 + \sigma(z)^2 = 3\mathbb{1}. \tag{113}$$

This is equivalent to satisfying the canonical *anti*-commutation relations (CARs; see Ruetsche (2011, pp. 60-62))

$$d^2 = \left(d^\dagger\right)^2 = 0; \qquad [d, d^\dagger]_+ = 1; \tag{114}$$

where

$$\sigma(x) = d + d^\dagger; \quad \sigma(y) = -i\left(d - d^\dagger\right); \quad \sigma(z) = dd^\dagger - d^\dagger d. \tag{115}$$

We now consider a theory describing a linear chains of n spin-$\frac{1}{2}$ systems, with observables $\{\sigma_k(x), \sigma_k(y), \sigma_k(z) \mid k \in \{1, 2, \ldots n\}\}$, satisfying

$$[\sigma_j(x), \sigma_k(y)] = 2i\delta_{jk}\sigma_k(z) \quad \text{and cyclic perms;} \qquad \sigma_k^2 := \sigma_k(x)^2 + \sigma_k(y)^2 + \sigma_k(z)^2 = 3\mathbb{1}. \tag{116}$$

Now, our theory falls outside the scope of the Stone-von Neumann theorem, because it is characterized by CARs, rather than CCRs. However, there is an analogous uniqueness theorem:

Theorem 8.2 (Jordan-Wigner Uniqueness Theorem). For each finite n, every irreducible representation of the CARs (equivalently, the Pauli relations) is unitarily equivalent to the Pauli representation, in which

$$\begin{aligned}
\sigma_k^P(x) &= \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{k-1} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{n-k}; \\
\sigma_k^P(y) &= \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{k-1} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{n-k}; \\
\sigma_k^P(z) &= \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{k-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \underbrace{\mathbb{1} \otimes \ldots \otimes \mathbb{1}}_{n-k}.
\end{aligned} \tag{117}$$

An alternative—though by the Jordan-Wigner theorem: unitarily equivalent—representation S (for 'switch') defines the spin matrices according to

$$\sigma_k^S(x) = \sigma_k^P(y); \qquad \sigma_k^S(y) = \sigma_k^P(z); \qquad \sigma_k^S(z) = \sigma_k^P(x); \tag{118}$$

i.e. the switch representation of $\sigma_k(x)$ in $\mathcal{H}_S$ has the same matrix elements as the Pauli representation of $\sigma_k(x)$ in $\mathcal{H}_P$, etc. Now let $U : \mathbb{C}_P^2 \to \mathbb{C}_S^2$ be the unitary such that $U\sigma_k^P(x)U^\dagger = \sigma_k^S(x)$, etc. Then the unitary $\otimes^n U : \mathcal{H}_P \to \mathcal{H}_S$ establishes the unitary equivalence between the switch and Pauli representations.

This equivalence extends to all operators in $\mathcal{B}(\mathcal{H}_S)$ and $\mathcal{B}(\mathcal{H}_P)$. In particular, let $\{f_i(\{\sigma_k^P(i)\})\}$ be a sequence of linear functions of the $\{\sigma_k^P(i)\}$ which converges in $\mathcal{H}_P$'s weak topology to the operator F_P. Each $f_i(\{\sigma_k^P(i)\}) \in \mathcal{B}(\mathcal{H}_P)$ and $\mathcal{B}(\mathcal{H}_P)$ is closed under weak convergence; so $F_P \in \mathcal{B}(\mathcal{H}_P)$. Similarly, let $\{f_i(\{\sigma_k^S(i)\})\}$ be a sequence of linear functions of the $\{\sigma_k^S(i)\}$, where

$$f_i(\{\sigma_k^S(i)\}) = U f_i(\{\sigma_k^P(i)\})U^\dagger. \tag{119}$$

Weak convergence is preserved under unitary transformations, so the $\{f_i(\{\sigma_k^S(i)\})\}$ converge in $\mathcal{H}_S$'s weak topology to some operator $F_S \in \mathcal{B}(\mathcal{H}_S)$, and $F_S = UF_PU^\dagger$.

In the Pauli representation $\mathcal{H}_P \cong \mathbb{C}^{2n}$, we may define the *polarization* observable $\hat{\mathbf{m}}^P := (m_x^P, m_y^P, m_z^P)$, where

$$m_x^P := \frac{1}{n}\sum_{k=1}^{n} \sigma_k^P(x), \quad \text{etc.} \tag{120}$$

Clearly, $\hat{\mathbf{m}}^P \in \mathcal{B}(\mathcal{H}_P)$, and the spectrum of $\hat{\mathbf{m}}^P$ is parametrized by points on the unit sphere. From the above considerations, we know that the similarly defined polarization observable $\hat{\mathbf{m}}^S := (m_x^S, m_y^S, m_z^S)$ in the switch representation satisfies

$$\hat{\mathbf{m}}^S = U\hat{\mathbf{m}}^P U^\dagger, \tag{121}$$

and so expectation values in S are identical to corresponding (given U) expectation values in P.

In Part II, we consider the theory of the *infinite spin-chain*, in which we have a spin-$\frac{1}{2}$ system for every integer in $\mathbb{Z}$. This theory has observables satisfying the Pauli relations (116). Representations of the Pauli relations in such a theory will be carried by a separable Hilbert space only if we make some hard choices about which of the uncountably many *prima facie* possible states are to be excluded.

We will see that with the obvious ferromagnetic ("neighbours like to be aligned") Hamiltonian, we get a ground state for each spatial direction (unit vector) $\mathbf{u}$ in $\mathbb{R}^3$. Namely: the state in which all the spins are aligned in the direction $\mathbf{u}$. On each such ground state, we build a representation. But they are *unitarily inequivalent*. Thus we return to general philosophico-mathematical theme mentioned at the end of Section 2.4 (end of 2.4.C): *singular limits*. That is: for every finite n (the length of the spin-chain), we have unitary equivalence; but for $n = \infty$, there is unitary inequivalence.

9 References

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