Lecture 1 Handout: Classical Mechanics

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1. The Classical World

"Theoretical physicists live in a classical world, looking out into a quantum-mechanical world. The latter we describe only subjectively, in terms of procedures and results in our classical domain." (Bell; 2004, p.29)

How we understand classical mechanics (or the "classical domain") depends on how we answer a number of subtle questions. Some of them worth thinking about as we proceed are the following.

- What kinds of things make up the "classical domain"?
- If it contains forces, can those forces depend on higher derivatives than velocity? Can fundamental forces be non-conservative?
- Is the "classical world" deterministic?
- Must the things that make up the classical world be in some sense "locally" defined?
- Must the things that make up the classical world be "definite" in some meaning of the word?

In physics, classical mechanics consists in a **collection of mathematical tools, laws, and interpretation schemes** that allow one to represent and make predictions about the world. Beginning its life around the early 17th century with the work of Galileo, and continuing through the seminal works Newton and Leibniz in the late 17th century, and developed Lavoisier, Lagrange, Legendre, Hamilton, Jacobi, and many others all the way through to the present day. Much of our deepest knowledge about classical mechanics was developed in fact developed in the second half of the 20th century, and indeed our understanding of the theory is still developing.

Viewed in this way, contrary to how Kuhn (1962) and other "revolutionary" philosophers of science narrate theory change, **classical mechanics is not an archaic or "false" theory that was overthrown in light of the modern paradigm**. There are indeed some senses in which some of the original uses of the theory, such as Newton's description of gravitation, are now known to be mere approximations of what actually occurs in nature. However, in many other senses, the techniques of classical mechanics are still thought to be correct today, having simply been absorbed into modern physical theories like general relativity and quantum mechanics. For example:

- $F = ma$ exists in general relativity, It describing the force needed for a test particle to deviate from a geodesic trajectory: $F^a = m\,(\xi^n \nabla_n \xi^a)$.
- Action principles and the Euler-Lagrange equations are used in a variety of modern extremisation problems: to calculate geodesics in general relativity, dynamical evolution in quantum field theory, and many other things.
- Hamiltonian mechanics contains the essential structure of quantum mechanics, through appropriate quantisation, and other ways.

Classical mechanics remains alive and well in our most modern approaches to fundamental physics. It is sometimes even treated as the conceptually clear touchstone against which we can interpret the conceptually messy quantum mechanics. So, we had better do our best to make it so!

2. Force mechanics

2.1. **Particles in space.** When Newtonian mechanics was first formulated, it was a theory about **forces pushing particles around in space**[1]. It is helpful to begin by thinking in this way. Take a particle's position in space to be described as a point in a smooth n-dimensional manifold[2]. Often we take that manifold to be $\mathbb{R}^3$, when we imagine that the particle could occupy any point in 3-dimensional space. But really this space could be **any smooth manifold** M. For example, if we are considering the position of an object constrained to the surface of the earth, then we could take $M = S^2$ to be a 2-sphere. Multiple particles in space will then be expressed in terms of multiple points in M.

For the sake of simplicity, our discussion will involve just a **single particle**. And, we will take our manifold to be the familiar $M = \mathbb{R}^3$, in which the position of the particle can be described by a vector $x = (x_1, x_2, x_3)$ in Euclidean coordinates, keeping in mind that more general manifolds are possible as well.

A *curve* in a manifold M is a smooth function $\gamma : I \to M$ of some open interval $I \subseteq \mathbb{R}$. A curve can be used to describe how the position of a particle changes over time, by associating each time $t \in I$ with a position $\gamma(t)$. In this case it is often called a *trajectory*. Since our manifold in this section is $M = \mathbb{R}^3$, we will denote curves in this section as $\mathbf{x}(t)$. We use the "$\mathbf{x}$-dot" notation to describe the velocity of a curve, writing $\dot{\mathbf{x}}(t) := \frac{d}{dt}\mathbf{x}(t)$ as shorthand for the triple, $(\frac{d}{dt}x_1(t), \frac{d}{dt}x_2(t), \frac{d}{dt}x_3(t)$. We similarly write the acceleration "$\mathbf{x}$-double-dot" as, $\ddot{\mathbf{x}}(t) := \frac{d^2}{dt^2}\mathbf{x}(t)$.

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[1] We will soon see some problems with this picture; for more, see Butterfield (2004).

[2] Manifolds are a central concept of physics and of differential geometry. We first define an *n-chart* on a point set M to be an injective mapping φ from a subset $U \subseteq M$ to an open subset of $\mathbb{R}^n$. This allows one to assign "coordinates" to subsets of the set M. Two n-charts φ_1 and φ_2 are called *compatible* if either their intersection $U = U_1 \cap U_2$ is empty, or else it is an open subset of $\mathbb{R}^n$ such that $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U) \to \mathbb{R}^n$ and $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U) \to \mathbb{R}^n$ are both smooth (infinitely differentiable). A smooth, n-dimensional *manifold* M is then a point set M together with a set of n-charts C such that, (1) any two n-charts are compatible; (2) the domains of the n-charts cover M; (3) (Hausdorff) distinct points p_1 and p_2 admit charts containing them such that $U_1 \cap U_2$ is non-empty; and (4) (maximal) every n-chart that is compatible with all the charts in C is an element of C.

2.2. **Forces and potentials.** To introduce forces, a little more mathematical structure is needed. The bare manifold M does not have a notion of length associated with it; but we represent the strength of a force using the length of a vector. So, a manifold with a metric is needed. We will adopt the manifold $\mathbb{R}^3$ with the Euclidean metric.

As a simple example of forces in action: a particle above the earth is postulated to be "pulled" towards the centre of the earth by the force of gravity $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ with magnitutde $-1/\mathbf{x}^2$, where $\mathbf{x}^2 := \mathbf{x} \cdot \mathbf{x}$ is the square of the radial vector $\mathbf{x}$ from the centre of the earth in the Euclidean metric. **A force at a point is represented by a vector:** its strength is characterised by the vector's norm, and its direction is characterised by the vector's angle.

A few comments on subtleties: first, what would the gravitational force on a particle as it approaches the centre of the earth, $\mathbf{x} = \mathbf{0}$? By the definition above, it would diverge to infinity. This is not allowed by our convention that $\mathbf{F}$ take finite values in $\mathbb{R}^3 \times \mathbb{R}^3$. Thus, in practice, **infinite forces are often removed** by removing the point-particle idealisation, and instead representing the earth and particle as hard spheres. This also has the advantage of capturing the fact that electromagnetic forces that eventually overcome gravity at short distances and "push" the two particles apart. That said, it may still of some philosophical interest to see what happens when we allow a force $\mathbf{F}$ to be infinite. One effect, it turns out, is that it allows for the appearance of rampant indeterminism through 'space invaders'; but, there are ways for indeterminism to occur with finite forces too.[3]

Second, the postulate that **a force depends only on position and velocity** is an empirical postulate, sometimes taken to be a basic presumption of classical mechanics. However, it too can be relaxed: nothing prevents us from considering forces that depend on higher derivatives and on time as well, and indeed this is sometimes done. However, it is also possible to motivate the standard convention that forces depend on velocity from below: once we come to define energy, it is possible to show that it is required by the assumption that energy be non-negative, or at least bounded from below.[4]

There is one very **important class of force** that we must introduce in order to understand how forces relate to other concepts in classical mechanics. A force $\mathbf{F}$ is called *conservative* if there exists a function $U : \mathbb{R}^3 \to \mathbb{R}$ (or more generally from an arbitrary manifold M to $\mathbb{R}$) such that,

$$\mathbf{F} = -\nabla \cdot U$$

[3]See Earman (1986) for an overview of some ways that indeterminism can happen in classical force mechanics, including space invaders; it is freely available here: `http://pitt.edu/~jearman/Earman_1986PrimerOnDeterminism.pdf`. See Norton (2008) for another (particularly simple) of determinism's failure on a dome in a gravitational potential. A quick overview of this example is here: `http://www.pitt.edu/~jdnorton/Goodies/Dome/`.

[4]This is a consequence of what is known as "Jauch's theorem". See Jauch (1968, §12.5 and §13.5).

The function U is then called the (scalar) *potential* associated with $\mathbf{F}$.

Not all forces are conservative. For example, notice that any force that depends on velocity or time cannot be conservative. However, many Newtonian systems do have this property. A conservative force turns out to imply that no "free work" can be extracted from a system.[5] But more importantly for our purposes, we will soon see that a conservative system in Force Mechanics is one that can be expressed in Lagrangian or Hamiltonian form too.

3. Some philosophical questions

- How can an infinite force lead to indeterminism? What (if anything) does this say about the possibility of infinite forces?
- How can one motivate the assumption that forces depend only on position and velocity?
- What reasons can (and cannot) be given to motivate the assumption that a force is conservative?

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[5] The *work* along a curve γ in $\mathbb{R}^3$ is the sum of the forces along that curve, given by the path integral $W(\gamma) = \oint_\gamma \mathbf{F} \cdot d\mathbf{x}$. If a system is conservative, then one can show that for a closed curve C (i.e. a smooth map from the circle to $\mathbb{R}^3$), the work $W(C)$ vanishes whenever C is a closed curve. This says that if you complete a circuit that brings you right back to where you started, you will not have extracted any work in the end. In this sense, being conservative means that there is "no free work" available. The converse fails: note that a force given by $\mathbf{F} = \dot{\mathbf{x}} \times \mathbf{x}$) leads to vanishing work on closed paths, but is not conservative. See Roberts (2013) discussion.