Lecture 3 Handout: Time Observables, and Classical Mechanics III

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1. Time Observables, Introduced

Is time a real-number parameter that serves to index physical quantities? Or, is it a **physical quantity that can be measured?** We seem to have measuring devices to measure quantities like duration, time of arrival, or the expected time of a particle jet. And yet, time does not appear to be an observable in quantum physics.

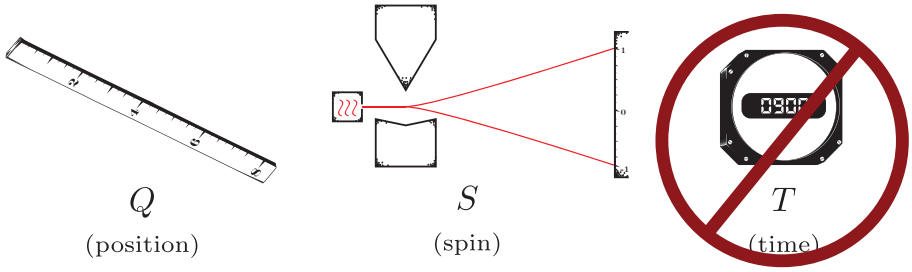

Figure 1. Why no time observables?

Pauli claimed that this was actually a novel requirement of the new quantum theory, in his influential quantum mechanics textbook:

> "In the older literature on quantum mechanics, we often find the operator equation
>
> $$Ht - tH = \frac{\hbar}{i} I$$
>
> ... It is generally not possible, however, to construct a Hermitian operator [t] (e.g. as function of P and Q) which satisfies this equation." (Pauli; 1980, p.63)

Are the technical details of this statement correct? Do we really need quantum mechanics in order to discuss the possibility of time observables? The answers are **no and no**. A little philosophical clarity is needed in order to see this. But, before we get to that, we should first motivate the classical framework in which we can discussion time observables, which is classical Hamiltonian mechanics.

2. Hamiltonian Mechanics

When you first learn Hamiltonian mechanics, it's often from an **extremely practical physics perspective:** when you're analysing a tricky system like the motion of a double-pendulum, it's just easier to model with a different formalism. You begin by describing the kinematics of a system in terms of position $\mathbf{q} = (q_1, q_2, \ldots, q_n)$

and momentum $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then you write down a system of differential equations called *Hamilton's equations*, which describe how those quantities change over time:

$$\frac{d}{dt}q_i(t) = \frac{\partial h(\mathbf{q},\mathbf{p})}{\partial p_i}, \qquad \frac{d}{dt}p_i(t) = \frac{\partial h(\mathbf{q},\mathbf{p})}{\partial p_i},$$

where h is a real-valued function called the *Hamiltonian*, usually of the form,

$$h(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n} \tfrac{1}{2m_i}p_i^2 + V(q_1, \ldots, q_n),$$

for some set of real numbers $m_i > 0$ and some real-valued function V of the q_i variables, which is related to the forces in the system by the relation, $\mathbf{F} = \nabla \cdot V$.

Work by 20th century mathematicians brought a significantly richer perspective on **what this framework is, why it works, and what it says about the world**.

2.1. **Phase space.** Hamiltonian mechanics begins by writing down a space representing **all possible states** that a physical system can occupy in a particular structured way. We assume this space of states has the structure of a smooth manifold M, with each point $p \in M$ representing one possible state of the world. That we assume this space is a manifold encodes the facts that,

- *(coordinates)* The state of the system can be described using a (possibly infinite-dimensional) local coordinate system; and
- *(smoothness)* It is possible to pass smoothly from one state to another.

We next assume that the space of physical states has a **symmetry structure** that is captured by a mathematical object called a 'symplectic form'. A *symplectic form* is a 2-form on M, denoted ω; that is, ω is a bilinear mapping from pairs of vectors on M to $\mathbb{R}$ which skew-symmetric (i.e. $\omega(X,Y) = -\omega(Y,X)$ for vector fields X and Y). It is also closed ($d\omega = \mathbf{0}$) and non-degenerate (i.e. $\omega(X,Y) = 0$ for all Y only if $X = \mathbf{0}$). The pair (M,ω) is called a *symplectic manifold*. When it used to represent mechanical systems, it is often also called a *phase space*.

Why do we introduce some tricky gadget as a symplectic form for representing physical states? One reason is that it allows us to **build in an assumption about symmetries** that is deep in the practice of physics. That assumption, in rough form, is the following:

> (symmetry-observable correspondence) *Every observable admits a continuous group of symmetries along which it is conserved, and vice versa: every continuous group of symmetries generates an observable that it conserves.*

This statement expresses a Noether-like idea. It is **built into the fabric of Hamiltonian mechanics** through the use of the symplectic form.

To make the statement above meaningful, we need to make precise **what we mean by observables and by symmetries**. Suppose we identify observables with a smooth assignment of a real-number values to each state of the world. That is, an observable is a smooth function $f : M \to \mathbb{R}$. This is standard practice: such a function can be used in an obvious way to represent a physical quantity (like total energy, angular momentum, etc.) associated with each possible state of the system.

We take a continuous "symmetry" to be represented by a diffeomorphism $t \mapsto \varphi$ that preserves the symplectic form, in that $\varphi_t^* \omega = \varphi$ for all $t \in \mathbb{R}$, where φ^* is the pushforward of φ. Such a map is called a *symplectomorphism*, and is the basic concept of isomorphism for a symplectic manifold. This allows us to express our Noether-like assumption above in precise form.

Fact 1. *Let (M, ω) be a symplectic manifold. Then for every smooth function h, there exists a smooth vector field X saisfying,*[1]

$$dh = \iota_X \Omega. \tag{1}$$

When this is true, the family of integral curves $t \mapsto \varphi_t$ are a group of symplectomorphisms (symmetries) that preserve h, i.e. $h \circ \varphi_t(p) = h(p)$ for all t. Conversely, for every smooth 1-parameter group of symplectomorphisms $t \mapsto \varphi_t$ there exists a smooth function h (unique up to addition by a constant) for which Equation 1 is true.

The vector field X associated with a smooth function in this way is sometimes called the *symplectic flow* associated with h, and h is called *Hamiltonian generator* of the vector field X. Note that this symmetry is **not necessarily associated with a dynamics**; it could be interpreted as spatial translation, momentum boosts, rotations, etc. Nevertheless, his is one of the driving ideas in Hamiltonian mechanics: **functions generate flows, and flows are generated by functions**, via Equation (1).

Like in quantum theory, it is often of interest to examine how an observable $f : M \to \mathbb{R}$ changes along the flow φ_t generated by h. This is given by, $f(t) f \circ \varphi_t$. To express this as a local differential equation, we can thus write,

$$\tfrac{d}{dt} f(t) = \tfrac{d}{dt} f \circ \varphi_t = \{f, h\},$$

where the bracket $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is by definition the derivative of f in the direction of the flow generated by h, and is known as the *Poisson bracket*. In geometric terms, it can also be written $\{f, h\} = X_h df$, where X_h is the vector field generated by h. When put in terms of standard coordinate form (discused below), it

[1]Here, d is the exterior derivative for M, and $\iota_X \Omega$ denotes the contraction of the vector field X with the first index of Ω. This latter statement in Penrose notation is expressed $d_b h = X^a \Omega_{ab}$, or equivalently, $X^a := \Omega^{ab} d_b h$, where Ω^{ab} is the inverse of Ω_{ab} (i.e. $\Omega^{ab}\Omega_{bc} = \delta^a_c$). These definitions are discussed in more detail in any differential geometry textbook, although the notation (sadly!) varies dramatically from book to book.

takes the form,

$$\{f, h\} = \frac{\partial f}{\partial q}\frac{\partial h}{\partial p} - \frac{\partial h}{\partial q}\frac{\partial f}{\partial p}.$$

2.2. **Hamiltonian dynamics.** Although it might not look like it, the statement of Equation (1) above is actually the familiar Hamilton's equations in disguise. Seeing this requires two observations.

2.2.1. *Time translation symmetry.* First, note that most fundamental physical systems are assumed to admit a continuous **time-translation symmetry**. This captures the assumption that time is continuous, and that experiments can be repeated day to day: apply the same conditions tomorrow, and you will get the same experimental results. In the context of Hamiltonian mechanics, this means that we can view **the passage of time as a 1-parameter group of symplectomorphisms**. By the discussion above, this 1-parameter group has a generator; when the group is interpreted as time-translation, the generator is called the *Hamiltonian* or the *Hamiltonian function* $h : M \to \mathbb{R}$ for the physical system.

In practice, the vector field associated with time translation is usually assumed to be **complete**, meaning that its integral curves can be simultaneously parametrised by the entire real line $\mathbb{R}$. This is in fact an empirical assumption, which encodes the statement that time is not finite to the past or future on any dynamical trajectory. We will adopt this assumption for most purposes here.

Thus, for the purposes of modelling dynamical systems, we will take a *Hamiltonian system* is a triple (M, ω, h), where the symplectic manifold (M, ω) is interpreted as representing the possible states of a system, and $h : M \to \mathbb{R}$ generates a complete vector field interpreted as representing time translation.

2.2.2. *Darboux's theorem.* Second, it turns out that if (M, ω) is a symplectic manifold, then M has **even dimension** $2n$. And, a basic theorem of symplectic mechanics makes it is possible to define a special local coordinate system around every point $p \in M$, using a binary operator $\wedge$ on forms called the *wedge product*.[2]

Fact 2 (Darboux's Theorem)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Every point $x \in M$ admits a neighbourhood U with a local coordinate chart $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ centred at x, called a* local Darboux coordinate system*, such that,*

$$\omega = (dq_1 \wedge dp_1) + (dq_2 \wedge dp_2) + \cdots + (dq_n \wedge dp_n).$$

In this local Darboux coordinates, the statement of Equation (1) about time translation symmetry is then **equivalent to the usual expression of Hamilton's equations given at the outset of this section**, with t interpreted as time.

[2]For simplicity, I'll define the wedge product here only for the relevant case of two 1-forms *alpha* and β: then their *wedge product* is the 2-form defined by $(\alpha \wedge \beta)(X, Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$. For the general definition, see Marsden and Ratiu (2010, §4.2).

2.2.3. *Systems of standard form.* Like Lagrangian and Force mechanics, Hamiltonian mechanics has a standard form associated with the Hamiltonian h. In coordinates interpreted as canonical position q and momentum p, that form is,

$$h(q,p) = \tfrac{1}{2m}p^2 + v(q)$$

for some real-valued function v. There are various ways to motivate this. For example, given a Hamiltonian system (M, ω, h) and a local coordinate system (q, p), suppose it is the case that,

$$\tfrac{1}{m}p = \dot{q},$$

where $m > 0$ and $\dot{q} := (d/dt)q(t)$ along integral curves. From Hamilton's equations we know that $\dot{q} = \partial h/\partial p$. Combining these two we get $\frac{1}{m}p\partial p = \partial h$. Partial integration thus gives us,

$$h(q,p) = \tfrac{1}{m}\int p\partial p = \tfrac{1}{2m}p^2 + v(q)$$

for some function $v(q,p) = v(q)$ of the q variable alone. So, an **assumption about velocity** along integral curves here gives rise to a **constraint on the Hamiltonian** that generates those curves, which in this case yields its standard form. This kind of thinking can be used to get an even more general result due to the Swiss mathematician Josef Jauch: that an assumption of symmetry under the action of the *Galilei group* of non-relativistic boosts, translations and rotations can be used to constrain velocity in a similar way, and derive a similarly general form of the Hamiltonian. This is known as **Jauch's theorem**. Although Jauch originally proved this theorem for quantum mechanics, a Hamiltonian mechanical generalisation exists as well; see the "Jauch's theorem" note posted on the website for this course.[3]

There is also a "standard form" of the manifold itself in Hamiltonian mechanics, which is the case that there exists a manifold Q such that $M = T^*Q$, where T^*Q is the *cotangent bundle* over Q. Note that this is *not necessarily the case.* It is only true given an empirical assumption, that it is possible to globally separate the canonical position and momentum variables associated with M so that it can be written as a cotangent bundle. However, it is quite standard to use Hamiltonian mechanics in situations where this is the case. And, when it is the case, there exists an isomorphism (called the *Legendre transformation*) which transforms between Lagrangian and Hamiltonian mechanics. It is thus thus exactly in this case that the two are empirically equivalent. For a discussion of the details of this equivalence, see Marsden and Ratiu (2010).

2.2.4. *Noether's theorem.* Although our thinking above about conserved quantities and symmetries is reminiscent of Noether's theorem, it is only "Noether-like", and not quite Noether's theorem. There is a more complete statement of **Noether's theorem in Hamiltonian mechanics**, which is the following.

[3] `http://personal.lse.ac.uk/robert49/teaching/partiii/`

Fact 3 (Noether's theorem)**.** *Let (M, ω, h) be a Hamiltonian system. If the vector field X is a symmetry, i.e., its integral curves are h-preserving symplectomorphisms, then X admits a generator $f : M \to \mathbb{R}$ satisfying $df = \iota_X \omega$ and which is preserved under the flow generated by h. And conversely: if $f : M \to \mathbb{R}$ is preserved under the flow generated by h, then the vector field X that it generates is a symmetry.*

Further discussion and interpretation of Noether's theorem in this context can be found in Butterfield (2007).

2.3. **Time observables, Again.** Let (M, ω, h) be a Hamiltonian system. We defined an *observable* above to be a smooth function $f : M \to \mathbb{R}$. So, let us simply suppose that **time is represented by one such smooth function** $\tau : M \to \mathbb{R}$. We interpret this as the assignment of real-number values to each state in M by an appropriate measuring device, like a clock. What would the properties of such an assignment be like?

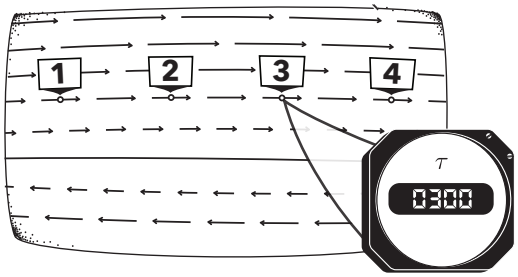

FIGURE 2. A time observable for a free particle

One seemingly necessary assumption is to suppose that, as we move along a trajectory $p(t) = \big(q(t), p(t)\big)$, the value assigned by the time observable τ changes in a way that is compatible with the parameter t associated with the dynamics. One might take this to be the defining property of a time observable: for any integral curve $p(t)$ of the dynamics generated by h and for all $t \in \mathbb{R}$,

$$\tau\big(p(t)\big) = \tau\big(p(0)\big) + t. \tag{2}$$

This property, which is called "timeliness" by Roberts (2014), captures the idea that the measuring device associated with τ **appropriately "tracks" the time parameter** associated with the dynamics of the physical system: whatever the assignment $\tau(p(0)$ is given to the initial state, a general state $p(t)$ on the trajectory will be assigned that value plus t.

Note that the time observable statement of Equation (2) implies[4] a local statement in terms of the Poisson bracket, which is that,

$$\{h, \tau\} = 1.$$

[4]Proof: by definition of the bracket, $\{h, \tau\} = \frac{d}{dt}\tau(p(t)) = \frac{d}{dt}(p(0) + t) = 1$.

at every point $p \in M$.

Do time observables satisfying Equation (2) exist? They obviously cannot at any stationary point $p(t) = p$. However, they are known to *locally* exist around every non-stationary point (Roberts; 2014, Prop. 1); there is also a uniqueness result about them (Prop. 2 of the same paper):

Fact 4 (Time observable existence)**.** *If (M, ω, h) is a Hamiltonian system, then there exists a neighbourhood U around every non-stationary point p that admits a time observable, i.e. a smooth function τ satisfying Equation* (2)*.*

Do time observables exist *globally?* Here the answer is more subtle. First, we assume (as we did in Lagrangian mechanics) that energy is bounded from below. Then it turns out that, if a time observable for this Hamiltonian exists, then the vector field it generates **is an incomplete vector field**. This is just a formal property in Hamiltonian mechanics; but, it has "no-go" implications for time observables in quantum mechanics. In Hamiltonian mechanics, the statement is the following (Roberts; 2014, Prop. 3):

Fact 5 (Hamiltonian Pauli theorem)**.** *Let (M, ω, h) be a Hamiltonian system, with the range of h bounded from below. If $\tau : M \to \mathbb{R}$ is a smooth function that satisfies $\{h, \tau\} = 1$, then the vector field generated by τ is incomplete.*

Proof. Let $\{h, \tau\} = 1$. Then, by the skew-symmetry of the bracket, $\{\tau, h\} = -1$. Let c_s be an integral curve of the Hamiltonian vector field generated by τ, and define $h(s) := h(c_s)$. Then $-1 = \{\tau, h\} = \frac{d}{ds}h(s)$, and so,

$$h(s) - h(0) = \int_0^s \frac{d}{ds}h(s)ds = -\int_0^s ds = -s.$$

If τ were to generate a complete Hamiltonian vector field, then s could take any real value, and we would have $h(s) = h(0) - s$ for all $s \in \mathbb{R}$. This would imply that the range of h is the entire real line, contradicting the assumption that h is half-bounded. Therefore, the vector field generated by τ is incomplete. □

The possibility of generating an incomplete vector field, it turns out, is **impossible in quantum mechanics** when observables are restricted to self-adjoint operators. For any self-adjoint Hilbert space operator T, Stone's theorem implies that for every state ψ and for all $s \in \mathbb{R}$, there exists $\phi := e^{-isT}\psi$. The "integral curve" through Hilbert space $\phi(s)$ is therefore defined for all real numbers s, no matter what. It cannot arise that s is only defined for some part of the real line if T is self-adjoint.

This suggests that, in the context of quantum mechanics, the result above has the character of a "no-go theorem" precluding the existence of time observables. There is a precise sense in which this is indeed the case; see Roberts (2014) for details.[5]

[5]Quantum enthusiasts are also encouraged to try to prove for themselves this fact: if $\mathcal{H}$ is a Hilbert space, and H is a self-adjoint operator on $\mathcal{H}$ whose spectrum is bounded from below, then the "time

However, there are also senses in which time observables *are* possible in quantum mechanics, which simply requires extending the orthodox conception of an observable. We will discuss this more when we turn to the foundations of quantum theory.

3. Some philosophical questions

- Compare how the frameworks of Force Mechanics, Lagrangian Mechanics, and Hamiltonian mechanics expand or restrict what is physically possible.
- How can the standard form of a Hamiltonian system be motivated?
- What is the appropriate definition of a time observable? In what sense do time observables exist?

References

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observable" equation,

$$e^{itH}Te^{-itH} = T + tI \quad \text{for all } t \in \mathbb{R}$$

cannot be satisfied by any self-adjoint operator T on $\mathcal{H}$. (Hint: This equation is equivalent to the Weyl form of the CCRs.)