Lecture 6 Handout: Quantum Cornerstones for POVMs

Bryan W. Roberts

1. Overview

One can view the cornerstones of orthodox quantum mechanics as consisting in four crucial theorems:

- Gleason's theorem (statistical origin of Hilbert space)
- The spectral theorem (observables and statistics)
- Wigner's theorem (symmetries)
- Stone's theorem (Noether correspondence)

When evaluating an extension of the quantum framework, it is good philosophical practice to evaluate the fate of these cornerstones. Here we consider the extension of quantum theory to include Positive Operator-Valued Measures (POVMs), often associated with an operational and information-theoretic approaches to quantum theory (c.f. Busch et al.; 1995; Nielsen and Chuang; 2000).

2. POVMs and Effect Spaces

The set of projection operators on a Hilbert space is not a Boolean logic; however, it is a generalisation of this logic known as an orthomodular lattice, on which it is possible to define a generalised probability measure. However, it has been argued that even these logics are overly restrictive. Might there be even more general logics of interest?

For example, an idealised experiment is often described as consisting in a sequence of definite outcomes. This is the case, for example, when we say that the Stern-Gerlach outcomes are described by two projections $P_\uparrow$ and $P_\downarrow$ such that $P_\uparrow + P_\downarrow = I$ and $P_\uparrow P_\downarrow = \varnothing$, associated with two "spots" of silver atoms on a screen. However, as Busch et al. (1995, §1) point out, this is an idealised description of the experiment Stern and Gerlach produced, which actually consists in two "lips" of silver atoms due to the magnet failing to be perfectly homogeneous, and so failing to produce a perfectly pure beam (Figure 1).

This can be corrected inn the following way. Suppose the localisation of silver atoms on the upper and lower halves of the screen is described in terms of projections on a new "position on screen" degree of freedom:

- $\mathcal{H}_s$ is a Hilbert space representing "position on the screen" states;
- P_+ (upper) and P_- (lower) are projections representing deflection onto the upper and lower halves of the screen;

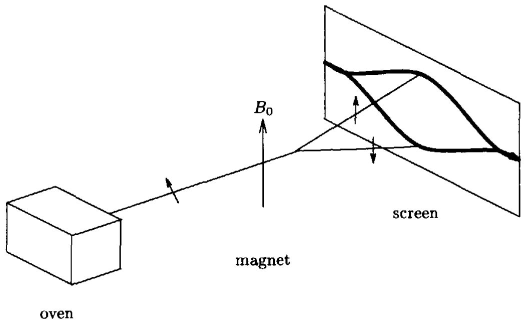

FIGURE 1. The deflection pattern observed by Stern and Gerlach (Busch et al.; 1995, p.8).

- $P_+ P_- = \varnothing$ and $P_+ + P_- = I$.

When a given silver atom is measured spin-up, this corresponds to the measurement of a "deflection up" state ϕ_+, and similar for spin-down and a "deflection down" state ϕ_-. However, since the silver atom's centre of mass is not perfectly localised on one or the other half, these states are not necessarily eigenstates of $P_\pm$.

These tools allow us to redescribe the Stern-Gerlach experiment in more detail. For a given silver atom, the two general "effects" of motion into the upper and lower halves of the screen can be described by operators of the form,

$$F_+ := \langle \phi_+, P_+\phi_+\rangle P_\uparrow + \langle \phi_-, P_-\phi_-\rangle P_\downarrow$$
$$F_- := \langle \phi_+, P_-\phi_+\rangle P_\uparrow + \langle \phi_-, P_-\phi_-\rangle P_\downarrow$$

These clearly satisfy $F_+ + F_- = P_\uparrow + P_\downarrow = I$. But, they are not projections unless $\langle \phi_+, P_\pm\phi_+\rangle$ and $\langle \phi_-, P_\pm\phi_-\rangle$ are 0 or 1. Each operator F_+ and F_- rather has two positive eigenvalues, respectively $\langle \phi_\pm, P_+\phi_\pm\rangle$ and $\langle \phi_\pm, P_-\phi_\pm\rangle$. Operators with a positive spectrum of this kind are called *positive operators*. That is, we have replaced projections with positive operators.

Now let $\varphi = c_\uparrow\varphi_\uparrow + c_\downarrow\varphi_\downarrow$ be the spin-state of the silver atom, with eigenstates $\varphi_\uparrow$ and $\varphi_\downarrow$. The more detailed, combined description of a given silver atom in the experiment can then be given in terms of a mixture of the form,

$$\psi_f = c_\uparrow(\phi_+ \otimes \varphi_\uparrow) + c_\downarrow(\phi_- \otimes \varphi_\downarrow).$$

The expected deflection for the silver atom can then be expressed in terms of effects as,

$$\begin{aligned}\langle \psi_f, P_\pm\psi_f\rangle &= |c_\uparrow|^2\langle \phi_+, P\phi_+\rangle + |c_\downarrow|^2\langle \phi_-, P_\pm\phi_-\rangle \\ &= \langle \varphi, F_\pm\varphi\rangle.\end{aligned}$$

Just as the Hilbert space projections have a logic generalising classical logic, so do the positive operators. This is known as an *effect algebra*. In a Hilbert space, this is found by extending the set of projection operators to the set of operators with spectrum in the interval $[0, 1]$, known in the literature as 'effects'. It is easy to check that it forms an algebra; and, one might try to view this as a logic, on the following analogy:

- A *projection* $P \in \mathcal{P}(\mathcal{H})$ has eigenvalues $0, 1$, which can be viewed as representing "true" or "false" outcomes; probabilities on projections correspond to the probability of the proposition being true.
- An *effect* $E \in \mathcal{E}(\mathcal{H})$ has eigenvalues in the interval $[0, 1]$. These are sometimes said to represent "unsharp measurement" outcomes of the kind described above.

Just as the Hilbert space projections can be described axiomatically as a logic in terms of an orthomodular lattice, so the effects can be described in terms of a logic as well. This approach was developed by Ludwig (1983, 1985); see Busch et al. (1995) for a textbook treatment.

3. Cornerstone 1: Gleason's theorem

Gleason's theorem shows that the probability measures on projections are completely characterised by the Born rule, so long as $\dim \mathcal{H} \geq 3$. But by expanding attention from projections to the larger space of effects $\mathcal{E}(\mathcal{H})$ on a Hilbert space, Busch (2003) found a similar (and much simpler) result. Remarkably, the result also holds for Hilbert spaces of arbitrary dimension.

We first define a *positive operator valued measure* to be an association $\Delta \mapsto F_\Delta$ of elements of a σ-algebra (usually Borel sets of reals) with positive self-adjoint operators that sum to the identity. In the discrete case, this is written, $\sum_i F_i = I$ for $i \in \mathbb{Z}^+$. In the case of continuous spectrum operators, it is written, $\int_{\mathbb{R}} dF_\lambda = I$. We can now state the Busch-Gleason theorem:

Generalised Gleason Theorem (Busch)**.** *Given a generalised probability measure*[1] $p : \mathcal{E}(\mathcal{H}) \to [0, 1]$, *there exists a density operator* ρ *on* $\mathcal{H}$ *such that* $p(E) = \mathrm{Tr}\rho E$ *for all* $E \in \mathcal{E}(\mathcal{H})$.

So, analogous to our interpretation of Gleason's theorem, we may conceive of a general statistical experiment in terms of generalised probabilities on an effect algebra, and conclude that statistical experiments can always be described in terms of Hilbert spaces and the Born rule. Moreover, unlike Gleason's theorem, the proof of the Busch-Gleason theorem is surprisingly elementary, and can be understood in just a couple of pages of reading and elementary methods.

4. Cornerstone 2: The spectral theorem

Elements of a projection-valued measure (PVM) $\Delta \mapsto E_\Delta$ sum to give a self-adjoint operator, in that $A := \sum_i \lambda_i E_i =$ is self-adjoint (discrete case), or $A :=$

[1]A *generalised probability measure* on a set of effects $\mathcal{E}(\mathcal{H})$ is a function $p : \mathcal{E}(\mathcal{H}) \to [0, 1]$ such that $p(I) = 1$ and $p(\sum_i E_i) \leq \sum_i p(E_i)$ whenever the countable sequence E_i satisfies $\sum_i E_i \leq I$.

$\int_\lambda \in \mathbb{R}\lambda dE_\lambda$ is self-adjoint (continuous case). The spectral theorem guarantees that the converse is true: every self-adjoint operator admits a PVM decomposition of the kind written above. Is there a spectral theorem analogue for positive operator-valued measures or POVMs?

The answer is "yes". Let $\mathcal{H}$ be a (possibly infinite dimensional) Hilbert space. An operator A is *symmetric* if,

$$A^*\psi = A\psi$$

for all ψ in the common domain $D_A \cap D_{A^*}$ of A and A^*. It is *self-adjoint* iff $D_A = D_{A^*}$.

It is easy to check that the weighted sum of elements of a POVM is symmetric, though not necessarily self-adjoint. It turns out that the converse is also true:

Naimark Spectral Theorem. *Let A be a closed, densely defined symmetric operator. Then there exists a POVM $\Delta \mapsto F_\Delta$ such that $A = \int_{\mathbb{R}} \lambda dF_\lambda$, which is unique (up to unitary equivalence) if and only if A is maximal symmetric, and which is a Projection Valued Measure if and only if A is self-adjoint. (Dubin and Hennings; 1990, Thm. 5.16, pg.135)*

5. Cornerstone 3: Wigner's theorem and Stone's theorem

5.1. **Wigner's theorem.** In our analysis of Wigner's theorem, we proposed to view symmetries as the bijections preserving the 1-dimensional subspaces (the rays), or equivalently the one-dimensional projections, of a Hilbert space $\mathcal{H}$. By shifting attention from projections to positive operators, we now have a new definition of symmetries:

> An *effect symmetry* is a bijection U on the effect algebra $\mathcal{E}(\mathcal{H})$ associated with a Hilbert space.

One might think that an entirely new analysis of Wigner's theorem would be needed in light of this change. In fact, no such change is needed: this symmetry group turns out to be equivalent to the one on rays.

Fact 1 (Cassinelli et al. 1997)**.** *Every effect symmetry preserves ray-space orthogonality; conversely, every ray-space orthogonality-preserving transformation can be implemented by a unique effect symmetry.*

See also Chevalier (2007). The analysis of symmetry for this generalisation of quantum theory is therefore essentially the same: every symmetry can be implemented by either a unitary or antiunitary operator, as in ordinary quantum theory.

5.2. **Generalised Stone Theorem.** The assumptions of Stone's theorem fail for maximal symmetric operators, so they do not generate a unitary group in the usual sense. However, they do satisfy a closely related result. Stating this result uses the concept of an *isometry*, that is, a linear Hilbert space operator U for which $U^*U = E$ is a projection operator (a unitary operator is thus a particular isometry for which

$U^*U = UU^* = I$). An isometry is a symmetry transformation in much the same sense as a unitary operator, but in a more restricted domain, in that $|\langle U\psi, U\phi\rangle| = |\langle \psi, \phi\rangle|$ for all $\psi, \phi \in E\mathcal{H}$. Isometries also allow one to state the following generalisation of Stone's theorem.[2]

Generalised Stone Theorem. *If $s \mapsto U_s$ is a strongly continuous, one-parameter set of isometries satisfying $U_r U_s = U_{r+s}$ for all $r, s \geq 0$ (or for all $r, s \leq 0$, but not both), then there exists a unique maximal symmetric operator A such that $U_s = e^{isA}$. Conversely, every maximal symmetric operator A generates a strongly continuous one parameter set of isometries $s \mapsto U_s = e^{isA}$ satisfying $U_r U_s = U_{r+s}$, for all $r, s \geq 0$ (or for all $r, s \leq 0$, but not both). (Cooper; 1947, 1948)*

This means that maximal symmetric operators are associated with a set of symmetries after all, in much the same way as self-adjoint operators. These symmetries are simply limited to a restricted subspace, in addition to being limited by the parameter values of the set.

When a maximal symmetric observable is a Hamiltonian, the Generalised Stone Theorem says that a unique solution to the Schrödinger equation exists, although it is only defined for non-negative times or non-positive times (but not both). As far as determinism is concerned, this situation is an improvement on the failure of essential self-adjointness considered by Earman (2009). The generalised Stone theorem says that the dynamical evolution generated by a maximal symmetric Hamiltonian is unique, much like the dynamics of an essentially self-adjoint Hamiltonian. The dynamics is time-translation invariant, in the restricted sense of an isometry. The limitation is merely that this dynamics is not defined for all times $t \in \mathbb{R}$. But as discussed above, having a dynamics for all times is a very strong requirement, which we may have good reason to relax.

References

Busch, P. (2003). Quantum states and generalized observables: A simple proof of Gleason's theorem, *Physical Review Letters* **91**(12): 120403. `https://arxiv.org/abs/quant-ph/9909073`.

Busch, P., Grabowski, M. and Lahti, P. J. (1995). *Operational Quantum Physics*, Berlin and Heidelberg: Springer-Verlag.

Cassinelli, G., de Vito, E., Lahti, P. J. and Levrero, A. (1997). Symmetry groups in quantum mechanics and the theorem of wigner on the symmetry transformations, *Reviews in Mathematical Physics* **9**(08): 921–941.

[2]This result follows naturally from the work of Naimark (1940, 1968) on the theory of self-adjoint extensions, although it was proved independently by Cooper (1947, 1948). The same technique turns out to allow for a notion of 'weak localizability' for relativistic photons (Jauch and Piron; 1976).

Chevalier, G. (2007). Wigner's theorem and its generalizations, *in* D. L. Kurt Engesser, Dov M. Gabbay (ed.), *Handbook of Quantum Logic and Quantum Structures*, pp. 429–475.

Cooper, J. L. B. (1947). One-Parameter Semigroups of Isometric Operators in Hilbert Space, *Annals of Mathematics* **48**(4): 827–842.

Cooper, J. L. B. (1948). Symmetric operators in Hilbert space, *Proceedings of the London Mathematical Society* **2**(1): 11–55.

Dubin, D. and Hennings, M. (1990). *Quantum mechanics, algebras and distributions*, Essex, UK: Longman Group UK Limited.

Earman, J. (2009). Essential self-adjointness: implications for determinism and the classical–quantum correspondence, *Synthese* **169**(1): 27–50.

Jauch, J. M. and Piron, C. (1976). Generalized localizability, *Helvetica Physica Acta* **40**.

Ludwig, G. (1983). *Foundations of Quantum Mechanics*, Vol. I, Springer-Verlag New York, Inc. Translated by Carl A. Hein from the 1954 German edition.

Ludwig, G. (1985). *Foundations of Quantum Mechanics*, Vol. II, Springer-Verlag New York, Inc. Translated by Carl A. Hein from the 1954 German edition.

Naimark, M. A. (1940). Self-adjoint extensions of the second kind of a symmetric operator, *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya* **4**(1): 53–104. Originally published as 'O samosoprjažennyh rasširenijah vtorogo roda simmetričeskogo operatora' by the Steklov Mathematical Institute, Russian Academy of Sciences.

Naimark, M. A. (1968). *Linear differential operators*, London:Frederick Ungar Publishing Co., Inc. Parts I and II, Translated by E. R. Dawson and edited by W. N. Everitt.

Nielsen, M. A. and Chuang, I. L. (2000). *Quantum Computation and Quantum Information*, Cambridge: Cambridge University Press.

Email address: b.w.roberts@lse.ac.uk