Uncertainty Principles, Time—and Montevideo?

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The discussion of time in quantum theory is liable to turn into a survey of the *pros* and *cons* of the various contending interpretations. Agreed: they say very different things about time ...

Collapse: does the irreducible indeterminism need branching, or merely divergence, of histories?

The pilot-wave: most versions return us to an absolute time.

Everett: as Deutsch laments, we await a mathematically exact formulation of the multiverse.

But such interpretations aside: quantum physics seems to treat time as the classical cousin theories do.

I will address two novelties: uncertainty principles (especially time-energy), and the Montevideo interpretation.

In Sections 1 to 3, I report the work by P. Busch et al. in sorting out long-standing confusions, especially about the time-energy uncertainty principle, including such topics as time operators, i.e. Pauli's 'proof'. I begin with uncertainty principles in general (Section 1), and then, following Busch, distinguish three roles for time (Section 2).

I will set aside many details. And I will wholly exclude: (i) historical aspects, e.g. the legacy of Pauli's proof (for which cf. Busch 1990, 2007, Hilgevoord 2005); (ii) treating time as a canonical coordinate.

Overall, we will see that there are very different versions of the time-energy uncertainty principle, i.e. a relation like

$$\Delta T \Delta E \geq \frac{1}{2}\hbar \tag{0.1}$$

that are valid in different contexts.

Some will use, not the variance, but rather some other measure of spread; some will not involve a time operator (which frequently cannot exist, for the problem/Hamiltonian in question) ... But so far, there is no general theory of time measurements...

Besides: Some of these novelties, e.g. measures of spread other than variance, are also important for quantities other than time and energy.

Plan

1. Uncertainty principles in general
2. Three roles for time
3. Time-energy uncertainty principles with intrinsic times
4. The Montevideo interpretation

We recall the ‘Heisenberg-Robertson uncertainty principle’

$$\Delta_\rho A \Delta_\rho B \geq \frac{1}{2} |\langle [A, B] \rangle_\rho| \tag{1.1}$$

with the special case, from $[Q, P] = i\hbar I$:

$$\Delta_\rho Q \Delta_\rho P \geq \frac{1}{2}\hbar \tag{1.2}$$

But the standard deviation has defects.
E.g.: it can be very large for states that ‘look like’ a delta-function.
So the traditional eq. 1.1 and 1.2 allows very narrow distributions, e.g. in Q and P.
So we should consider other measures of spread: some of these will not require a corresponding operator ...

1.1: An uncertainty principle for Width of the Bulk

Often, a better measure of the spread of a distribution is the length W_α of the smallest interval on which a sizeable fraction α of distribution is supported; (where 'sizeable' can be taken to mean $\alpha \geq \frac{1}{2}$).
That is: we represent the spread of a distribution in terms of the smallest interval on which the bulk of the distribution is found. We call this a *width*.
Then we have (Uffink 1990), for position and momentum:

$$W_\alpha(Q)W_\alpha(P) \geq c_\alpha \hbar \text{ if } \alpha \geq \frac{1}{2}; \text{with } c_\alpha \approx 1 \tag{1.3}$$

A bit more generally and explicitly: For any normalized L^2 function i.e. $\int |f|^2 dx = 1$:—
we define $W_\alpha(|f|^2)$ to be the width of the smallest interval J such that $\int_J |f|^2 dx = \alpha$;
and similarly for the Fourier transform $\hat{f}$.

Then we have

$$W_\alpha(|f|^2) W_\alpha(|\hat{f}|^2) \geq c_\alpha \hbar \text{ if } \alpha \geq \frac{1}{2}; \text{with } c_\alpha \approx 1. \tag{1.4}$$

1.2: An uncertainty principle for Translation Width, and Width of the Bulk

Another measure of uncertainty reflects an uncertainty about the *state*, rather than the value of a quantity: and has the merit of combining W_α as in eq. 1.3, to give uncertainty principles—with an exactly similar treatment of space and time. But I postpone the case of time till Section 3.

Given $|\langle\phi|\psi\rangle| = 1 - r$, with $0 \leq r \leq 1$, we call r the *reliability* with which $|\phi\rangle$ and $|\psi\rangle$ can be distinguished. So if $|\phi\rangle = |\psi\rangle$, then $r = 0$; while if $|\phi\rangle$ and $|\psi\rangle$ are orthogonal, r attains its maximum value, 1.

We apply reliability to the translation of a quantum state $|\psi\rangle$ in space (in one spatial dimension, labelled x).

Translation is effected by the exponentiation of the total momentum, i.e. by the unitary operators (with $\hbar$ set equal to 1):

$$U_x(\xi) = \exp(-iP_x\xi) \; . \tag{1.5}$$

Then for given $r \in [0, 1]$, we define ξ_r as the smallest distance for which

$$|\langle\psi|U_x(\xi_r)|\psi\rangle| = 1 - r \; . \tag{1.6}$$

ξ_r may be called the *spatial translation width* of the state $|\psi\rangle$.

NB: ξ_r should be distinguished from the spatial width $W_\alpha(q)$, i.e. the width of the probability distribution $|\langle q|\psi\rangle|^2$ of the position operator $\hat{Q}$ in a given (pure) state ψ.

Agreed: if $|\langle q|\psi\rangle|^2$ has a single peak, and the bulk of the distribution is on an interval of length d, then W_α, for α close to 1, is close to d. And ξ_r, for r close to 1, will also be close to d.

But if $|\langle q|\psi\rangle|^2$ has many narrow peaks of a small width e, while the entire distribution is spread over an interval d (example: interference), then ξ_r will be of the order e, while $W_\alpha(q)$ will be of order d.

This translation width combines with the width W introduced previously (cf. eq. 1.3) in uncertainty relations. To define W for momentum, let $|p_x\rangle$ denote a complete set of eigenstates of P_x. We set aside degeneracy, to simplify notation. So, with integration perhaps including a sum over discrete eigenstates

$$\int |p_x\rangle\langle p_x| dp_x = \mathbf{I} . \tag{1.7}$$

Then we define the width $W_\alpha(P_x)$ of the momentum distribution as the smallest interval such that

$$\int_{W_\alpha(P_x)} |\langle p_x|\psi\rangle|^2 dp_x = \alpha . \tag{1.8}$$

Then it can be shown that for $r \geq 2(1-\alpha)$

$$\xi_r W_\alpha(P_x) \geq C(\alpha, r)\hbar \tag{1.9}$$

where for sensible values of the parameters, say $\alpha = 0.9$ or 0.8, and $0.5 \leq r \leq 1$, the constant $C(\alpha, r)$ is of order 1.

1.3: Comments

Formal comments:
(1): Completely general: eq. 1.9 depends only on the existence of translation operators and completeness relations, eq. 1.7.
(2): Also, it is relativistically valid.

Comments about physical significance:
(1): We will see an exactly parallel treatment for translation in time.
(2): Broadly speaking: eq. 1.9 gives more information than the traditional Heisenberg-Robertson UP, eq. 1.2. It is not just that as we said, the standard deviation has some defects. Also...

(2a): Recall that ξ_r should be distinguished from the spatial width $W_\alpha(q)$: in an interference pattern we can have $\xi_r << W_\alpha(q)$. In such a case, eq 1.9 is stronger than eq. 1.2.

(2b): eq. 1.9 shows that in a many-particle system the width of the total momentum can control whether the positions of the component particles can be sharply determined.
For if the spread of just one position variable is small, then ξ_r for the entire system's state is small.
(For 'distinguishing under translation requires distinguishing only one component'; cf. orthogonality in one factor of a tensor product implies orthogonality in the whole.)
And so, by eq. 1.9, the width in the total momentum must be large.
Conversely, if $W_\alpha(P_x)$ is small, ξ_r must be large; and so the spread of all position variables must be large.

2.1: External time

(a) Measured by clocks not coupled to the objects studied in the experiment.
(b): Used to specify a parameter, e.g. instant or duration of preparation or measurement, or of the time-interval between them. So no scope for uncertainty...

Yet there is tradition (Landau, Peierls ...) of an uncertainty principle between (i) the duration of an energy measurement, and (ii) *either* the range of an uncontrollable change of the measured system's energy *or* the resolution of the measurement.

But Aharonov and Bohm (1961) describe an arbitrarily accurate and arbitrarily rapid energy measurement. Two particles are confined to a line and are both free except for an impulsive measurement of the momentum and so energy of the first by the second, with the momentum of the second being the pointer-quantity.

2.2: Intrinsic times

A dynamical variable of the studied system, that functions to measure the time.
For example: The position (motion) of a clock dial, or of a free particle!
In principle: every non-stationary quantity A defines for any state ρ a characteristic time $\tau_\rho(A)$ in which $\langle A \rangle$ changes 'significantly'.

For example: if $A = Q$, and ρ is a wave packet, say $\rho = |\psi\rangle\langle\psi|$, then $\tau_\rho(A)$ could be defined as the time for the bulk of the wave packet to shift by its width ...

So we expect various uncertainty principles for various definitions of intrinsic times. Details in Section 3. Some of the scenarios suggest time of arrival, or time of decay, or time of flight; so they raise the quest for time operators...

2.3: Time operators: a glimpse of 'Pauli's legacy'

A self-adjoint operator T generating translations in energy according to

$$\exp(i\tau T/\hbar)H\exp(-i\tau T/\hbar) = H + \tau\mathbf{I}\ , \quad \forall\tau \in \mathbb{R} \tag{2.1}$$

would imply that the spectrum of H is $\mathbb{R}$.

Informal reason as follows:
$UHU^\dagger = H + \tau\mathbf{I} \Rightarrow [U, H]U^\dagger = \tau\mathbf{I}$, i.e. $[U, H] = \tau U$.
Applying this to an eigenvector ψ_E of H with eigenvalue E, we get:
$UE\psi_E - HU\psi_E = \tau U\psi_E$,
i.e. $U\psi_E$ is an eigenvector of H with eigenvalue $E - \tau$.

Eq. 2.1 would imply that in a dense domain:

$$[H, T] = i\hbar\mathbf{I}\ ; \tag{2.2}$$

which would imply (cf. eq 1.1) our opening 'prototype form' of the time-energy uncertainty principle for any state ρ

$$\Delta_\rho T \Delta_\rho H \geq \frac{1}{2}\hbar \, . \tag{2.3}$$

But eq 2.2 does *not* imply eq. 2.1. So the door is open for some realistic Hamiltonians (i.e. bounded below and-or with discrete spectrum; without the whole of $\mathbb{R}$ as spectrum), to have a T satisfying eq 2.2—which Busch calls a *canonical time operator*.

Examples include the harmonic oscillator ... But there is little general theory of which Hamiltonians have such a T (even when we generalize to POVMs).

3.1: Mandelstam-Tamm principle

This is one of many such principles, all with time as a parameter: cf. Busch 1990, 2007.

We combine the Heisenberg equation of motion of an arbitrary quantity A

$$i\hbar \frac{dA}{dt} = [A, H] \tag{3.1}$$

with the Heisenberg-Robertson uncertainty principle, eq. 1.1, and the definition of a characteristic time

$$\tau_\rho(A) \; := \; \Delta_\rho A \; / \; | \; d\langle A \rangle_\rho \; / dt \; | \tag{3.2}$$

and deduce

$$\tau_\rho(A)\Delta_\rho(H) \geq \frac{1}{2}\hbar. \tag{3.3}$$

Example: A free particle in a pure state ψ with a sharp momentum i.e. $\Delta_\psi P << |\langle P \rangle_\psi|$.

We can deduce that $\Delta_\psi Q(t) \approx \Delta_\psi Q(t_0)$, i.e. slow spreading of the wave-packet; so that the Mandelstam-Tamm time $\tau_\rho(Q)$, as defined by eq 3.2, is indeed the time it takes for the packet to propagate a distance equal to its standard-deviation.

Example: Lifetime of a property $\hat{P}$:
Define $p(t) := \langle\psi_0|U_t^{-1}\hat{P}U_t\psi_0\rangle$, with of course $U_t := \exp(-itH/\hbar)$.
Then the Mandelstam-Tamm uncertainty relation gives

$$|\frac{dp}{dt}| \leq \frac{2}{\hbar}(\Delta_{\psi_0}H)[p(1-p)]^{\frac{1}{2}} \ . \qquad (3.4)$$

Integration with initial condition $p(0) \equiv 1$ (i.e. $\hat{P}$ actual at $t = 0$) yields

$$p(t) \geq \cos^2(t(\Delta_{\psi_0}H) \ / \ \hbar) \ \text{ for } \ 0 < t < \frac{\pi}{2}\frac{\hbar}{(\Delta_{\psi_0}H)} \qquad (3.5)$$

so that if we define the lifetime $\tau_{\hat{P}}$ of the property $\hat{P}$ by $p(\tau_{\hat{P}}) := \frac{1}{2}$, we deduce

$$\tau_{\hat{P}}.\Delta_{\psi_0}H \geq \frac{\pi\hbar}{4}. \qquad (3.6)$$

The general point is that the rate of change of a property of the system decreases with increasingly sharp energy: and for an energy eigenstate, of course, all quantities have stationary distributions.

3.2: Hilgevoord-Uffink 'Translation-Bulk' uncertainty principle

We report how to combine:

(i) W_α as in Section 1.1, cf. eq. 1.3;
(ii) translations-widths as in Section 1.2;
(iii) equal treatment of space and time.

We apply reliability to the translation of a quantum state $|\psi\rangle$ in time *instead of* space. Translation is effected by the exponentiation of the energy, i.e. by the unitary operators (with $\hbar$ set equal to 1):

$$U_t(\tau) = \exp(-iH\tau) \quad . \tag{3.7}$$

Then for given $r \in [0, 1]$, we define τ_r as the smallest time for which

$$|\langle\psi|U_t(\tau_r)|\psi\rangle| = 1 - r \; . \tag{3.8}$$

τ_r may be called the *temporal translation width* of the state $|\psi\rangle$. If we choose r such that $(1 - r)^2 = \frac{1}{2}$, i.e. $r = 1 - \sqrt{\frac{1}{2}}$, then τ_r is the *half-life* of the state $|\psi\rangle$.

The translation widths τ_r, and ξ_r from Section 1.2, combine with the widths W introduced previously (cf. eq. 1.3) in uncertainty relations.

To avoid referring back, and yet bring out the analogy between position-momentum and time-energy, I now repeat the case of position-momentum.

In order to define the widths W for energy and momentum, let $|E\rangle$ and $|p_x\rangle$ denote complete sets of eigenstates of H and P_x respectively. I again set aside degeneracy, to simplify notation. So, with integration perhaps including a sum over discrete eigenstates

$$\int |E\rangle\langle E| dE = \mathbf{I} \quad \text{and} \quad \int |p_x\rangle\langle p_x| dp_x \; = \mathbf{I} \, . \tag{3.9}$$

We define the widths $W_\alpha(E)$ and $W_\alpha(P_x)$ of the energy and momentum distributions as the smallest intervals such that

$$\int_{W_\alpha(E)} |\langle E|\psi\rangle|^2 dE = \alpha; \int_{W_\alpha(P_x)} |\langle p_x|\psi\rangle|^2 dp_x = \alpha. \tag{3.10}$$

Then it can be shown (Uffink and Hilgevoord 1985, Uffink 1993) that for $r \geq 2(1-\alpha)$

$$\tau_r W_\alpha(E) \geq C(\alpha, r)\hbar \; ; \; \xi_r W_\alpha(P_x) \geq C(\alpha, r)\hbar \tag{3.11}$$

where for sensible values of the parameters, say $\alpha = 09$ or 0.8, and $0.5 \leq r \leq 1$, the constant $C(\alpha, r)$ is of order 1.

The comments from Section 1.3, both formal and as regard physical significance, carry over.

We can add:

Broadly speaking: the second equation of eq. 3.11 gives more information than the traditional Heisenberg-Walker uncertainty principle, eq. 1.2. The extra information is not merely what we said in 1.3. Also eq. 3.11 shows that in a many-particle system it is the width in the total energy that determines whether the time variables can be sharply determined.

3.3: Quantum clocks

The general idea of such a clock is a sequence $\psi_1, \psi_2, ...$ of orthogonal clock-pointer states that are occupied at equi-spaced times $t_1, t_2, ...$; so that the clock resolution is $\delta t := t_{k+1} - t_k$. We apply (i) the Mandelstam-Tamm inequality eq. 3.6, or (ii) the Hilgevoord-Uffink inequality eq. 3.11, to get uncertainty relations.

(i): If the clock-pointer is taken to be the mean position of a wave packet, then eq. 3.6 and the constraint $\delta t \geq \tau_\psi(Q)$ implies

$$\delta t \geq \tau_\psi(Q) \geq \frac{\pi\hbar}{4}\frac{1}{\Delta_\psi H}. \tag{3.12}$$

(ii): The temporal translation width must be less than the time resolution, so that eq. 3.11.1 implies:

$$\delta t \geq \tau_r \geq \frac{C(\alpha, r)\hbar}{W_\alpha(E)}. \tag{3.13}$$

4.1: Evolution with respect to a real clock

We envisage measuring $\hat{O}$ on a quantum system, when $\hat{T}$ on another quantum system ('the clock') has value 'T'.
In the Heisenberg picture (with respect to the background time t), the projector for the clock system for the value T lying in the interval $[T_0 - \Delta T, T_0 + \Delta T]$ is

$$P_{T_0}(t) := \int_{T_0-\Delta T}^{T_0+\Delta T} dT \; \Sigma_k \; |T, k, t\rangle\langle T, k, t| \quad . \qquad (4.1)$$

Similarly the projector for the measured system for the value O lying in the interval $[O_0 - \Delta O, O_0 + \Delta O]$ is

$$P_{O_0}(t) := \int_{O_0-\Delta O}^{O_0+\Delta O} dO \; \Sigma_j \; |O, j, t\rangle\langle O, j, t| \; . \qquad (4.2)$$

Then the (orthodox Born-rule) conditional probability that $\hat{O}$ takes value O_0 given that the clock indicates time T_0

$$\mathcal{P}(O \in [O_0 - \Delta O, O_0 + \Delta O] \text{ given that } T \in [T_0 - \Delta T, T_0 + \Delta T]) \tag{4.3}$$

is

$$\lim_{\tau \to \infty} \frac{\int_{-\tau}^{\tau} dt \, \mathrm{Tr}(P_{O_0}(t) P_{T_0}(t) \rho P_{T_0}(t))}{\int_{-\tau}^{\tau} dt \, \mathrm{Tr}(P_{T_0}(t) \rho)}$$

where the integrals over all t reflect our ignorance when in the background time t the clock takes the value T_0.

Assume $\rho = \rho_{\rm sys} \otimes \rho_{\rm cl}$ and $U = U_{\rm sys} \otimes U_{\rm cl}$. And define (i) a probability density of t at given T:

$$\mathcal{P}_t(T) := \frac{\mathrm{Tr}(P_{T_0}(0) U_{\rm cl}(t) \rho_{\rm cl} U_{\rm cl}(t)^\dagger)}{\int_{-\infty}^{\infty} dt \, \mathrm{Tr}(P_{T_0}(t) \rho_{\rm cl})} \; ; \tag{4.4}$$

and (ii) an "effective" density matrix of the observed system as a function of T:

$$\rho_{\rm eff}(T) := \int_{-\infty}^{\infty} dt \; U_{\rm sys}(t) \rho_{\rm sys} U_{\rm sys}(t)^\dagger \mathcal{P}_t(T) \, . \tag{4.5}$$

It is easy to verify, that the conditional probability eq. 4.4 is equal to

$$\mathcal{P}(O_0|T) := \frac{\mathrm{Tr}(P_{O_0}(0)\rho_{\mathrm{eff}}(T))}{\mathrm{Tr}(\rho_{\mathrm{eff}}(T))} \quad : \tag{4.6}$$

which is of the familiar form except with the effective density matrix $\rho_{\mathrm{eff}}(T)$ substituted for the background-time density matrix $\rho(t)$.

"I measure O when the clock reads 'T'. But the statistics of O that I gather are as if each measurement draws the system from an urn, in the state $U_{\mathrm{sys}}(t)\rho_{\mathrm{sys}}U_{\mathrm{sys}}(t)^{\dagger}$ with a probability $\mathcal{P}_t(T)dt$."

In terms of T, evolution is not unitary: for $\rho_{\rm eff}(T)$ is a convex combination of density matrices, each of which is unitarily evolving with respect to t, but which are associated with *different* values of t.

Under some assumptions, the equation of motion for $\rho_{\rm eff}(T)$ in terms of T has the exact solution

$$\rho_{\rm eff}(T)_{nm} = [\rho_{\rm eff}(0)_{nm}] \times \exp(-i\omega_{nm}T)\exp(-\sigma\omega_{nm}^2 T) \tag{4.7}$$

where: (i) $\omega_{nm} := \omega_n - \omega_m$ is the difference of the frequencies for levels n and m; and (ii) σ is the (constant) rate of change, with respect to T, of the width of $\mathcal{P}_t$. Thus the off-diagonal go to zero exponentially.

This *temporal decoherence* is at the centre of the Montevideo interpretation.

4.2: The meaning of a mixture

The two usual roles for a mixture:

(Basis): To define a preferred quantity, or basis, or set of nearly orthogonal states: that intuitively is definite at the end of measurement.

(Outcome): To secure that in each specific measurement trial, there is an outcome (represented mathematically by one component of the mixture).

Various aspects for various approaches to the measurement problem, such as:

(Ein): Decoherence by the environment.

(DRP): Dynamical reduction programme.

In short: (Ein) fulfills (Basis), but not (Outcome). The *total* system being in a pure state can in principle be revealed by measuring a 'global quantity'.

(DRP) fulfills both (Basis) and (Outcome).

Agreed, there are various debates:

(1): *Can we ignore small amplitudes?* 'The problem of revivals'

(2): *Is vagueness a virtue or a vice?*

As to the Montevideans:

(Basis): They appeal to a combination of (Ein) and temporal decoherence, eq. 4.8.

Temporal decoherence solves the problem of revivals that besets (Ein)'s fulfillment of (Basis), by making revivals unobservable in principle.

(Outcome): The Montevideans succeed in justifying the ascription of a proper (i.e. ignorance-interpretable) mixture.

Temporal decoherence, eq. 4.8, answers the 'global quantity' objection against (Ein): the global quantity is unmeasurable in principle.

But: what about securing an individual outcome in each specific measurement trial?

I think the interpretation fits best in an Everettian picture, for reasons to do with the vagueness debate, (2) above.

4.3: Throwing away the ladder

(i) Some heuristic arguments about black holes suggest fundamental limits to how accurate a quantum clock can be.

Measuring a time interval T has a minimum uncertainty δT proportional to $T_{\mathrm{Planck}}^{2/3} T^{1/3}$, where $T_{\mathrm{Planck}} = 10^{-44}$ seconds is the Planck time.

(ii) The equation of motion in terms of an optimal clock-quantity T should be taken as fundamental: 'throw away of the ladder' of background time t.

The exact solution, for a system with discrete energy levels now has width $\sigma(T) = \left(\frac{T_{\mathrm{Planck}}}{T}\right)^{1/3} . T_{\mathrm{Planck}}$

$$\rho_{\mathrm{eff}}(T)_{nm} = [\rho_{\mathrm{eff}}(0)_{nm}] \times \exp(-\omega_{nm} T) \exp(-\omega_{nm}^2 T_{\mathrm{Planck}}^{4/3} T^{2/3}). \tag{4.8}$$

(iii) This fundamental temporal decoherence, together with environmental decoherence, solves:

(a) The problem of revivals, i.e. the fact that in some models *without* temporal decoherence, the off-diagonal terms can in the very long run, become large.

Detail: the off-diagonal terms get multiplied by an N-fold product of exponentials whose exponents are proportional to $T^{2/3}$. This gives unobservably small tails, and so no revivals even in very long times, even for $N \sim 100$.

(b) The problem of global quantities, i.e. the fact that in some models *without* temporal decoherence, one in principle measure a quantity revealing the total system to be in a pure (entangled) state..

(iv) Although the arguments in (i) vary in the accuracy limits they suggest, the ensuing solution to the measurement problem is robust.

BUT: There are still 'tails'. So we have not secured that in each specific measurement trial, there is an individual outcome. We have not yet passed from Bell's 'and' to his 'or'.

The Montevideans reply:
"There is an individual outcome, a transition to 'or', exactly when it becomes in principle undecidable (thanks to temporal decoherence) whether or not the evolution is unitary."

But they have not yet given us a rigorous statement of what is their proposed restriction on quantities.

So I suggest that the Montevideo interpretation fits best in an Everettian approach, in which a 'branching' is effective and approximate: an approximation whose value does not depend on its being precisely defined—indeed, the value depends on its *not* being precisely defined.

5: References

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