Jordan Lin Operators for Qm Mechanics Dover

Ch 1 Lin Spaces & Lin Fnals

Notes J Butterfield

1.1 Vector spaces

1.2 Inner products. ; Schwarz inequality.

1.3 Hilbert space

a) converge of vectors $\psi_n \to \psi$:$= \|\psi - \psi_n\| \to 0$ as $n \to \infty$.

b) def of infinite lin combinations an analogy with $\omega = \Sigma z_k$

c) Idea of completeness: every Cauchy seq converges

Hbt space := complete I.P. space.

separable

lin manifold vs. subspace.

d) example $l^2 = \{(x_1 x_2 \ldots) \; x_c \in \mathbb{C}$ (or $\mathbb{R}$!) $\mid \Sigma^\infty (x_k)^2 < \infty\}$.

orthobasis $(1,0,0,0 \ldots), (0,1,0,0,0 \ldots) = \{\phi_k\}$

So each vector ψ is $\Sigma^\infty x_k \phi_k$

The partial sums are $(x_1 x_2 \ldots x_N, 0 0 0)$ and these converge to $(x_1 x_2 \ldots x_N, x_{N+1}, \ldots x_m \ldots)$

e) in sep Hbt space general ψ has a unique expression in terms of an orthonormal basis $\{\phi_k\}$

$\psi = \Sigma^\infty (\phi_k, \psi) \phi_k$

(Use Schwarz to show this with partial sums)

f) Any 2 Hbt space (over $\mathbb{C}$/$\mathbb{R}$) are isomorphic by basis $\to$ basis

Cor Any inf-diml sep Hbt sp can be identified with l^2

g) A space L^2 of sq-Sable functions $\psi : X \to \mathbb{C} \ni \int_X |\psi(x)|^2 d\mu < \infty$ is a Hbt space.

$L^2(0,1)$ – Fourier series give an orthonormal basis of continuous functions

h) NB Lebesgue vs Riemann integration!

i) orthocomplements & projections: $\mathcal{M}$ a subspace of $\mathcal{H}$.
(closed lin manifold)

If $\mathcal{H}$ is separable, so is $\mathcal{M}$
And $\mathcal{M}^\perp$ as usual with
$\forall \psi \in \mathcal{H}$, has ! expression $\varphi + \varphi^\perp$
$\varphi \in \mathcal{M}$ and $\varphi^\perp \in \mathcal{M}^\perp$

1.4 Lin Fnals

a) In an inner product space, each ψ defines a lin fnal F_ψ by $F_\psi(\varphi) := (\psi, \phi)$

b) For n-dimensional space: $\{\phi_k\} = \{\phi_1 \ldots \phi_n\}$ an o basis.

To each F on this space, assign the vector $\psi_F := \Sigma^n F(\phi_k)^* \phi_k$

Indeed the defn in a), applied to this ψ_F, yields F again.

That is for any vector $\phi = \Sigma(\phi_k, \phi)\phi_k$, we have

$$F(\phi) = \Sigma(\phi_k, \phi) F(\phi_k) = (\psi_F, \phi).$$

c) For ∞ diml space, use defn of continuity

d) (Riesz) every contns lin fnal on sep Hbt space F, $\exists! \psi_F \in \mathcal{H}$ with $F(\phi) = (\psi_F, \phi)$

Ch 2 Lin Operators

2.5 (1st f) Operators & matrices

For general $AB \neq BA$!

$(A\psi)(x) = \frac{d}{dx}\psi(x) \quad (A\psi)(x) = \int a(x,y)\psi(y)\,dy$

2.6 Bdd opers.

Defn of A is continuous, and of bdd with norm $\|A\|$

Thm 6.1 A lin oper is cont iff bdd

from norm for vectors.

$$\left.\begin{array}{l} \|A+B\| \leq \|A\| + \|B\|, \quad \|A\| = 0 \text{ iff } A = 0 \\ \|aA\| = |a|\,\|A\| \\ \|AB\| \leq \|A\|\|B\| \end{array}\right\} \text{so } \|\ \| \text{ is a "norm"}$$

A bdd lin oper on an ∞-dmnl but sep Hlbt sp can be represented by an inf matrix.

2.7 Inverses

A has an inverse if there is a lin oper B s.t. $AB = 1 = BA$

iff $\forall \psi \exists! \varphi$ with $\psi = A\phi$.

Thm 7.2 fin dimnl vec space. $\{\phi_i\}$ any basis NSCs for A to have inverse

(i) there is no nonzero vector ϕ s.t. $A\phi = 0$

(ii) the set $\{A\phi_1 \ldots A\phi_n\}$ is lin indpdt

(iii) there is a lin oper B s.t. $BA = 1$

(iv) the matrix corrspdg to A has non-zero detmt.

NB for inf-dimnsl space (i) (ii) (iii) are NOT sufficient.

Let $A : \ell^2 \to \ell^2$: $\phi = (x_1, x_2 \ldots) \xmapsto{A} (0, x_1, x_2 \ldots)$ right shift

Let B be "drop 1st component & left shift" $(x_1, x_2 \ldots) \xmapsto{B} (x_2, x_3, x_4 \ldots)$

Then (i) (ii) (iii) all hold

But A has no inverse. If $\psi = (x_1, x_2 \ldots)$ with $x_1 \neq 0$ then there is no vector ϕ with $\psi = A\phi$.

2.8 Unitaries.

U is unitary if it has an inverse and $\|U\psi\| = \|\psi\|$ for all ψ

every unity is bdd $\|U\| = 1$.

Thm 8.1 $(U\psi, U\varphi) = (\psi, \varphi) \; \forall \psi, \varphi$. Cor 8.2 U(o basis) is an o basis

Thm 8.3 "Conversely" to Cor: If U is bdd and U(some o basis) is an o basis, then U is unitary

2.9 Adjoints, Hermit^c Opers

Let A be bdd (and so continuous). Then for each $\psi \in \mathcal{H}$, the lin functional $F[\psi]$ defined by $F^{[\psi]}(\phi) := (\psi, A\phi)$

is continuous

So by Riesz, $\exists$ a ! vector, call it $A^+\psi$, with $F^{[\psi]}(\phi) = (A^+\psi, \phi)$

A^+ is trivially linear

Thm 9.1 A is bdd $\Rightarrow A^+$ is bdd and $\|A^+\| = \|A\|$

Usual stuff: $A^{++} = A$ $(AB)^+ = B^+A^+$ $(aA)^+ = a^*A^+$ etc.

: adjoint/Herm conjugate of the represtg matrix.

A bdd lin oper is self-adjoint or Hermitn if $A^+ = A$.

NB it is impossible to define an unbdd Herm^c oper for all vectors. This is Thm 11.1 below

i.e. $(\phi, A\psi) = (A\phi, \psi)$ i.e. $(\phi, A\psi) = (\psi, A\phi)^*$ (Herm)

& $(\psi, A\psi) \in \mathbb{R}$ $\forall \psi$

Exmple On $L^2(0,1)$ A with $(A\psi)(x) := x\psi(x)$
is bdd: $\|A\psi\|^2 \leq \|\psi\|^2$ and $\|A\| = 1$.
and (Herm). So A is Hermitian.

But "corresponding definition" $L^2(\mathbb{R})$ is NOT bdd; need to damp eg $(V\psi)(x) := e^{-|x|}\psi(x)$

Thm 9.2 If A is bdd with a bdd inverse A^{-1}, then $(A^+)^{-1}$ exists and $(A^+)^{-1} = (A^{-1})^+$

Thm 9.3 A lin oper is unitary iff $U^+U = 1 = UU^+$.

For bdd lin opers A, B, we can use the adjoints to show that the represtg matrix (c_{jk}) of the $C = AB$ is the matrix product

$$c_{jk} = \sum_{i=1}^{\infty} a_{ji} b_{ik}$$

2.10 Projectn operators

Given subspace $M < \mathcal{H}$, define projection E_M by $E_M : \mathcal{H} \to \mathcal{H}$ with $\psi \in M \Rightarrow E_M\psi = \psi$.

Then

Thm 10.1 A bdd lin op E is a projection iff $E^2 = E = E^+$

(continuity ~~for E~~)

Thm 10.2 If $E_1E_2 = E_2E_1$, then E_1E_2 is the projectn onto $\operatorname{ran}(E_1) \cap \operatorname{ran} E_2$

If $E_1 \perp E_2$ $E_1 + E_2$ projects on $\operatorname{ran} E_1 + \operatorname{ran} E_2$

If $E_1 \leq E_2$, $E_2 - E_1$ projects on rel o complement

2.11 UnBdd Operators

Example. Posn in Schrod repsn for $L^2(\mathbb{R})$

i.e. $(Q\psi)(x) := x\psi(x)$.

$\|Q\psi\|^2 \equiv \int_{-\infty}^{\infty} |x\psi(x)|^2 dx$ can be arbitly larger than $\|\psi\|^2 = \int |\psi(x)|^2 dx$

Thm 11.1 If a lin oper is defined for all vectors and if $(\phi, A\psi) = (A\phi, \psi)$ for all ψ, ϕ, then A is bdd.

a) QM Thy needs unbdd opers with the Hermitian property. So it needs dom opers with domains less than all of $\mathcal{H}$. Hence jargon of domains, dense domains, and extensions

b) If A has a dense domain, we can define $A^\dagger$ viz:-

Dom$(A^\dagger)$ = { ψ s.t. there is a vector, $\tilde{\psi}$, with: $\forall \phi \in$ dom A $(\phi, \tilde{\psi}) = (A\phi, \psi)$ }

Then define $A^\dagger$: $\psi \underset{A^\dagger}{\mapsto} \tilde{\psi}$

This defines $A^\dagger$ as linear and dom $A^\dagger$ a lin mfd. because dom A is dense.

c) An operator is symmetric if it has dense domain and $(\phi, A\psi) = (A\phi, \psi)$ $\forall \phi, \psi \in$ dom(A)

Then $A^\dagger\psi$ is defined and $A^\dagger\psi = A\psi$, for all $\psi \in$ dom A

IE: $A^\dagger$ is an extension of A.

If $A^\dagger = A$, we say A is self-adjoint or Hermitian.

d) Example. Define Q on $L^2(\mathbb{R})$ to have dom$(Q) := \{\psi \mid \int |x\psi(x)|^2 < \infty\}$

This is trivially dense. So Q is symmetric.

And so $Q^\dagger$ is defined & extends Q

One argues by "chopping" (p.31) that the extension is "vacuous"

That is:- $Q^\dagger = Q$ and Q is self-adjoint.

e) A symmetric oper that cannot be extdd to a larger dom is maximal symmetric

Thm 11.2 Every s-a operator is a maximal symmetric operator.

NB not conversely

f) Closed operators Defn of "closed" ("2nd best to continuity"

Thm 11.3 If dom(A) is dense, then $A^\dagger$ is closed

So every s-adjoint operator is closed

3.12 Eigenvalues & eigenvectors

..usual....

Let A be Hermitn or unitary
Let $a_1 \ldots a_k \ldots$ be its evalues, with corrspdg espaces M_k
Then $\oplus_k M_k$ is the subspace spanned by eigenvectors of A. "Eig(A)"

Define M reduces A if both M and $M^\perp$ are invariant under A.

Thm 12.7 M reduces A iff $EA = AE$ iff $(1-E)A = A(1-E)$

Thm 12.8 A Hermitn or unitary. Then Eig(A) reduces A.

3.13 Eigenvalue decompositn.

Thm 13.1 For fin dim, eigenvect of Hermitn span whole space.

Write , with evalues $a_1 < a_2 .. < a_k < a_{m-1} < a_m$ with esps M_k.
So m evalues in all and $k = 1, 2, \ldots m$.

$$A = \sum_{k=1}^{m} a_k I_k$$ with I_k projectors onto M_k

For each real number x, $E_x := \sum_{a_k < x} I_k$

Then $E_x = 0$ for $x < a_1$ and $E_x = 1$ for $x > a_m$.
If $x < y$, $E_x E_y = E_x = E_y E_x$. ie. $E_x \leq E_y$

Define for each x let $dE_x = E_x - E_{x-\varepsilon}$
with ε so small that there is no a_j with $x - \varepsilon \leq a_j < x$.

So dE_x is not zero only when x is an evalue a_k
— in which case $dE_x = I_k$

For $\sum_{k=1}^{m} I_k = \mathbb{1}$, we write $\int_{-\infty}^{\infty} dE_x = \mathbb{1}$

And for $A = \sum a_k I_k$ we write $A = \int_{-\infty}^{\infty} x \, dE_x$.

Now, $(\phi, E_x \psi)$ is a complex fn of x that jumps in value by $(\phi, I_k \psi)$ at $x = a_k$. So

Bvt is continuous from the right

ordinary Riemann integrals

$$(\phi, \psi) = \int d(\phi, E_x \psi)$$

$$(\phi, A\psi) = \int x \, d(\phi, E_x \psi)$$

Similarly unitaries. eigenvalues $u_k = e^{i\theta_k}$ $0 < \theta_1 < \theta_2 .. \theta_k < .. < \theta_m \leq 2\pi$

$E_x := \sum_{\theta_k < x} I_k$

$$U = \int_0^{2\pi} e^{i\theta} dE_x$$

$$(\phi, U\psi) = \int_0^{2\pi} e^{i\theta} d(\phi, E_x \psi)$$

3.14 Spectral decomposition

Define A family $\{E_x\}_{x\in\mathbb{R}}$ is spectral family iff

(i) $x \le y \Rightarrow E_x \le E_y$ i.e. $E_xE_y = E_x = E_yE_x$

(ii) If $\varepsilon > 0$, then $E_{x+\varepsilon}\psi \to E_x\psi$ as $\varepsilon \to 0$: $\forall\psi$, and $\forall x$ "continuous from right"

(iii) $E_x\psi \to 0$ as $x \to -\infty$; $E_x\psi \to \psi$ as $x \to \infty$: $\forall\psi$

Thm 14.1 For each selfadjt oper A, $\exists!$ spectral family E_x s.t.

$\forall\varphi,\psi$ $\quad (\phi, A\psi) = \int_{-\infty}^{\infty} x\, d(\phi, E_x\psi)$ $\quad$ we write $A = \int x\, dE_x$

Thm 14.2 Similarly for unitary U: $E_x = 0$ for $x \le 0$, $E_x = 1$ for $x > 2\pi$

and $(\phi, U\psi) = \int_0^{2\pi} e^{ix} d(\phi, E_x\psi)$

So we write $U = \int_0^{2\pi} e^{ix} dE_x$.

Examples "position" on $L^2(0,1)$ and $L^2(\mathbb{R})$

On $L^2(0,1)$, define E_x as "chopping" $(E_x\psi)(y) \begin{cases} = \psi(y) & \text{if } y \le x \\ = 0 & \text{if } y > x \end{cases}$

Then

$$\|E_{x+\varepsilon}\psi - E_x\psi\|^2 = \int_x^{x+\varepsilon} |\psi(y)|^2 dy \to 0$$

and $\{E_x\}$ is a spectral family.

Define A on $L^2(0,1)$ by $(A\psi)(x) = x\psi(x)$ bdd s-adjoint.

Then for any ψ, and ϕ

$$\int_{-\infty}^{\infty} x\, d(\varphi, E_x\psi) = \int_{-\infty}^{\infty} x\, d\int_0^1 \varphi(y)^*(E_x\psi)(y)dy =$$

$$= \int x\, d\int_0^x \varphi(y)^*\psi(y)\,dy = \int_0^1 \varphi(x)^* x\,\psi(x)\,dx \equiv (\phi, A\psi)$$

So we have spectral decomp of A.

$(\phi, E_x\psi)$ never jumps in value $\quad (\phi, E_x\psi) - (\phi, E_{x-\varepsilon}\psi) \equiv \int_{x-\varepsilon}^{x} \varphi(y)^*\psi(y)\,dy \to 0$ as $\varepsilon \to 0$

~~This~~ We say E_x jumps in value at x iff for some ψ $(E_x - E_{x-\varepsilon})\psi$ does not converge to 0 as $\varepsilon \to 0$ (ε positive of course). Otherwise contns at x

So in this example E_x is continuous at all x, since A has no eigenvalues/eigenvectors.

Thm 14.3 A selfadjoint with $A = \int x\, dE_x$

Then E_x jumps in value at a iff a is an evalue of A.

Let I_a be the projection onto the eigenspace for a.

Then $E_xI_a = 0$ for $x < a$ and $E_xI_a = I_a$ for $x \ge a$

And for $\varepsilon > 0$, any ψ $\quad E_a\psi - E_{a-\varepsilon}\psi \to I_a\psi$ as $\varepsilon \to 0$

Define spectrum := $\{x \in \mathbb{R} \mid E_x$ increases$\}$ i.e. x not in (a, b) on which E_x const. — Jordan p 7
point spectrum := $\{x \in \mathbb{R} \mid E_x$ jumps$\}$
conts spectrum := $\{x \in \mathbb{R} \mid E_x$ increases continuously$\}$

Thm 14.5 A self-adjoint oper is bdded iff its spectrum is bdd
A s-adjt oper is positive if $(\phi, A\phi) \geq 0$ ~~all~~ for all ϕ.
Thm 14.6 A self-adjt oper is positive iff its spectrum is non-negative.

3.15 Functions of an Operator, Stone's Theorem

Given A self-adjoint $A = \int x\,dE_x$. Let $f: \mathbb{R} \to \mathbb{C}$. We define $f(A)$ by:

$$(\varphi, f(A)\psi) = \int_{-\infty}^{\infty} f(x)\, d(\varphi, E_x\psi)$$ (for f continuous this is Riemann integral)

Clearly a) for $f(x) = x$, we have $f(A) = A$

b) for $f(x) = 1$, we have $f(A) = \mathbb{1}$, since $\int_{-\infty}^{\infty} d(\phi, E_x\psi) = (\varphi, \psi)$

c) $(f+g)(A) = f(A) + g(A)$; $(cf)(A) = c\, f(A)$

d) Also for products (fg) is defined by $(fg)(x) := f(x)g(x)$
Then $(fg)(A) = f(A)g(A)$

e) So polynomials: $f(x) = c_0 + c_1x + c_2x^2 \ldots + c_nx^n$
gives $f(A) = c_0\mathbb{1} + c_1A + c_2A^2 + \ldots + c_nA^n$

f) $[f(A)]^\dagger = (f^*)A$. If $f: \mathbb{R} \to \mathbb{R}$, then $f(A)$ is self-adjt
If $f^*f = 1$, then $[f(A)]^\dagger f(A) = \mathbb{1} = f(A)[f(A)]^\dagger$
i.e. $f(A)$ is unitary.

g) $f(A)$ is positive if $f(x) \geq 0$ on spectrum of A

h) $f(A)$ is bdd if $|f(x)|$ is bdd over spectrum of A.

Given $H = \int_{-\infty}^{\infty} x\,dE_x$, define for all t: $(\phi, U_t\psi) := \int e^{itx} d(\phi, E_x\psi)$
U_t is an operator, viz $U_t = e^{itH}$, which is unitary since $(e^{itx})^* e^{itx} = 1$.
Evidently $U_0 = \mathbb{1}$ and since $e^{itx}e^{it'x} = e^{i(t+t')x}$, we have
$U_t U_{t'} = U_{t+t'}$. The converse is

Thm 15.1 Stone's theorem

$\forall t \in \mathbb{R}$, let U_t be unitary and suppose (i) $(\phi, U_t \psi)$ is a continuous function of t, $\forall \phi$ and $\forall \psi$. (ii) $U_0 = 1$, $U_t U_{t'} = U_{t+t'}$

(a unitary representation of $(\mathbb{R}, +)$).

Then ① there is a unique s-adjoint operator H such that $U_t = \exp(itH)$ for all t.

② the domain of H is $\left\{ \psi \,\middle|\, \frac{1}{(it)}(U_t - 1)\psi \text{ converges as } t \to 0 \right\}$ and the limit vector is $H\psi$.

③ If a bdd op commutes with all U_t, it commutes w. H.

Using ②, we say:-

If $U_t \psi \in \mathrm{dom}(H)$, then $\frac{1}{i\Delta t}(U_{\Delta t} - 1) U_t \psi \to H U_t \psi$ as $\Delta t \to 0$

That is: $\frac{1}{i\Delta t}(U_{t+\Delta t} - U_t)\psi \to H U_t \psi$

So we write $-i \frac{d}{dt}(U_t \psi) = H U_t \psi$ "Schrödinger equation"