The GNS theorem for Pauli operators

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The GNS representation theorem is a result due to Gel'fand, Naimark and Segal, and is one of the core results of the representation theory lying at the heart of quantum mechanics. The result is that one can begin with the algebraic facts describing an arbitrary quantum system, and use them to construct a representation of those facts in a familiar form as operators on an ordinary Hilbert space. A general statement and proof of the GNS theorem can be found in Araki (1999), or any standard textbook on algebraic quantum theory. However, to get a concrete sense of how it works, it is a good exercise to see how the construction looks in the concrete case of the Pauli spin algebra. In this note we carry out that exercise.

Consider the Pauli spin operators, σ_x, σ_y, σ_z. But suppose that we don't know that these operators can be represented as the usual matrices on a 2-dimensional complex vector space. Suppose we can only conceive of them abstractly as self-adjoint operators ($\sigma_i = \sigma_i^*$) in an algebra with a $*$-operation and a norm $|\cdot|$, and that these operators satisfy,

$$\sigma_i^2 = I \qquad \sigma_i\sigma_j = -\sigma_j\sigma_i$$
$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k \qquad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

where $\epsilon_{ijk} = 1$ for cyclic permutations of ijk and is -1 otherwise. We denote the resulting algebra by $\mathcal{A}$. It consists in the norm-closure of these operators.

Let $\varphi : \mathcal{A} \to \mathbb{C}$ be a pure state[1]. Secretly we might think of this state as assigning an expectation value $\langle\psi, \sigma\psi\rangle$ to each element of the algebra $\sigma \in \mathcal{A}$. But the point of this construction is that we're pretending we don't yet know how to represent the elements of $\mathcal{A}$ as Hilbert space operators. So, we don't yet have an inner product $\langle,\rangle$ operating on vectors ψ. All we can currently do is specify abstract properties that φ has in virtue of its being a state, such as that $\varphi(A)$ is a real number when A is self-adjoint (e.g. when $A = \sigma_z$), is complex when A is not self-adjoint (e.g. when $A = \sigma_x\sigma_y$), and so on.

Let's now be quite specific and choose a particular state φ. Let $\varphi(\sigma_z) = 1$ and $\varphi(\sigma_y) = \varphi(\sigma_x) = 0$. Since the entire algebra $\mathcal{A}$ is generated by algebraic

[1] A *state* $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear function that satisfies $\varphi(I) = 1$ and $\varphi(A^*A) \geq 0$ for all $A \in \mathcal{A}$. A state is *not pure* or *mixed* if it can be written as convex combination of states $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$ with $\lambda \neq 0$ and $\varphi_1 \neq \varphi_2$. Otherwise it is *pure*.

combinations of these operators, this defines φ on the entire algebra. Thus, we begin with an algebra-state pair $(\mathcal{A}, \varphi)$.

We're now going to do something remarkable. Starting with only the pair $(\mathcal{A}, \varphi)$, we're going to apply an algorithm that uses these objects to build the following structures.

(1) A 2-dimensional vector space $\mathcal{H}$;
(2) An inner-product $\langle , \rangle$ that makes $\mathcal{H}$ a Hilbert space; and
(3) A set of bounded operators on $\mathcal{H}$ that is isomorphic to $\mathcal{A}$.

In other words, we're going to build a Hilbert space representation of $(\mathcal{A}, \varphi)$ algorithmically, using only the known facts about $(\mathcal{A}, \varphi)$. That is the GNS construction. In a sense, we'll show that a representation of $\mathcal{A}$ was "living" in the pair $(\mathcal{A}, \varphi)$ all along. The construction proceeds in three steps.

Step 1: A vector space. The first step is to observe that the algebra $\mathcal{A}$ is itself already a complex vector space under addition. This is part of the definition of an algebra: it is a vector space under its built-in operations of addition and scalar multiplication.

Step 2: A semi-inner product. We'll work out the structure of this vector space shortly. But first, observe that one can use the state φ we defined above to construct something that is *almost* our desired inner product $\langle , \rangle$. Let us make the definition,

$$\langle A, B \rangle := \varphi(A^* B).$$

This product is obviously antilinear in the first argument and linear in the second. Since $\varphi(A^*B)^* = \varphi(B^*A)$, this product also satisfies conjugate symmetry, $\langle A, B \rangle = \langle B, A \rangle^*$. We also know that $\langle A, A \rangle \geq 0$, since $\varphi(A^*A) \geq 0$ whenever φ is positive, and a state is by definition positive. A product with these properties is sometimes called a *semi-inner product.*

However, quantum mechanics requires an *inner product*, which means that $\langle , \rangle$ would have to satisfy the positive-definiteness condition, $\langle A, A \rangle = 0$ only if $A = 0$. That doesn't hold in our case. Just look at $A = \sigma_x + i\sigma_y$. It is certainly not zero. And yet it is an easy exercise[2] to check that, according to the commutation relations on the first page and our definition of φ, $\langle A, A \rangle = 0$. This operator A is non-zero, but still has zero "length" from the perspective of the inner product. This indicates a sense in which our vector space is not appropriate for our inner product.

Step 3: A better vector space with an inner product. In order to reduce our vector space to something more appropriate for our inner product, we must "quotient out" the redundancy in length above. Let us

[2]*Solution:* From the commutation relations we find that $A^*A = (\sigma_x - i\sigma_y)(\sigma_x + i\sigma_y) = \sigma_x^2 + i(\sigma_x\sigma_y - \sigma_y\sigma_x) + \sigma_x^2 = 1 + i(2i\sigma_z) + 1 = 2 - 2\sigma_z$. But φ is linear and defined to satisfy $\varphi(\sigma_z) = 1$. Thus we have $\langle A, A \rangle := \varphi(A^*A) = \varphi(2 - 2\sigma_z) = 2 - 2\varphi(\sigma_z) = 2 - 2 = 0$.

write ψ_A to denote an equivalence class of elements that includes $A \in \mathcal{A}$, where equivalence is defined by,

$$\psi_A = \psi_B \text{ iff } \varphi((A-B)^*(A-B)) = 0.$$

The vectors ψ_A together form a vector space. Why? The answer is a little bit subtle. The set $\ker\varphi := \{A \,|\, \varphi(A^*A) = 0\}$ is called the *kernal* of φ. One can show using a straightforward application of the Cauchy-Schwarz inequality that this set is in fact a left-ideal of A (see e.g. Araki 1999, p.36). This implies that the space of equivalence classes of the form $\psi_A = \{B \,|\, (A-B) \in \ker\varphi\}$ is a quotient space, which can be written $\mathcal{A}/\ker\varphi$. The vectors satisfying the equivalence relation above are the elements of that quotient (vector) space.

Let us call completion of this vector space $\mathcal{H}$, suggestively — we can now see that it is a Hilbert space. The problematic element $\sigma_x + i\sigma_y$ that gave rise to the "redundancy" above is now identical to the 0 vector in the new vector space $\mathcal{H}$. Indeed, this new vector space is constructed in such a way that the same product $\langle , \rangle$ defined on $\mathcal{H}$ automatically becomes positive definite, and is therefore an inner product. So, since $\mathcal{H}$ is defined to be complete, it follows that it is a Hilbert space.

Step 4: Dimension of the Hilbert space. Our Hilbert space $\mathcal{H}$ is two-dimensional, as one would expect given the single-fermion algebraic relations that define $\mathcal{A}$. To see this, we simply observe that (σ_y, σ_z) form an orthonormal basis for $\mathcal{H}$. They span the space because their algebraic combinations produce the generators $\sigma_x, \sigma_y, \sigma_z$ of the algebra $\mathcal{A}$. Neither ψ_{σ_y} nor ψ_{σ_z} are zero as elements of $\mathcal{H}$, and indeed $\langle \sigma_y, \sigma_y \rangle = \langle \sigma_z, \sigma_z \rangle = 1$. They are also orthogonal, since,

$$\langle \sigma_y, \sigma_z \rangle = \varphi(\sigma_y^* \sigma_z) = \varphi(\sigma_y \sigma_z) = \varphi(i\sigma_x) = i\varphi(\sigma_x) = 0,$$

since $\varphi(\sigma_x) = 0$ by our definition. Thus, the underlying vector space of $\mathcal{H}$ is actually just the complex vector space of two dimensions, i.e. it is $\mathbb{C}^2$.

Step 5: Operators on the Hilbert space. Now we just need to see that we can find bounded operators on this Hilbert space that are isomorphic to the Pauli spin algebra $\mathcal{A}$. But our construction provides a neat way to do this. For each vector $\psi_A \in \mathcal{H}$ and each operator $B \in \mathcal{A}$, let us define a mapping $\hat{B} : \mathcal{H} \to \mathcal{H}$ given by the definition,

$$\hat{B}\psi_A := \psi_{BA}.$$

The resulting algebra of operators on $\mathcal{H}$ are isomorphic to $\mathcal{A}$, since $\hat{B}\hat{C}\psi_A = \psi_{BC(A)} = (\widehat{BC})\psi_A$. In other words, we have constructed a representation of the algebra $\mathcal{A}$ on $\mathcal{H}$.

To see a little more concretely how this looks, we can think through the value of $\langle \psi_{\sigma_z}, \sigma_y \psi_{\sigma_z} \rangle$. By our definition we know that $\sigma_y \psi_{\sigma_z} = \psi_{\sigma_y \sigma_z} =$

$-\psi_{i\sigma_x} = i\psi_{\sigma_x}$. One can check using the same arguments as above that,

$$\langle \psi_{\sigma_z}, \sigma_y \psi_{\sigma_z} \rangle = i\langle \psi_{\sigma_z}, \psi_{\sigma_x} \rangle = 0.$$

This says that the expectation value of the σ_y observable for a system in the σ_y state is zero, as expected. It is also easy to verify a number of further (expected) facts: that ψ_{σ_z} is a z-eigenstate for σ_z with eigenvalue 1, etc.

That is the essence of the GNS representation theorem. We begin with only algebraic facts about the Pauli spin operators, and construct a Hilbert space representation on this basis alone. Of course, we already knew that we have a representation of the Pauli spin algebra in terms of the famous matrices $\sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $\sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Why do we need this new representation? The reason is that the GNS technique for constructing a representation for the Pauli relations is perfectly general. We could have begun with an *arbitrary* algebra $\mathcal{A}$, and we would still be able to find a representation.

There are further interesting facts that one can infer about the GNS representation, such as that it is irreducible whenever φ is pure, and that the vector ψ_I corresponding to the identity in $\mathcal{A}$ is cyclic in the algebra. I leave these consequences as exercises.

References

Araki, H. (1999). *Mathematical theory of quantum fields*, Oxford: Oxford University Press.