UNITARY INEQUIVALENCE

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Background Reading: Baker and Halvorson (2013),

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1. Introduction

Spontaneous symmetry breaking occurs when a ground state is not preserved by some symmetry of the system. For example, the so-called Mexican Hat Potential is a toy model with multiple ground states are related by a rotation (Figure 1). Or, in the Higgs mechanism, degenerate ground states in the electroweak interaction are related by an $SU(2) \times U(1)$ symmetry.

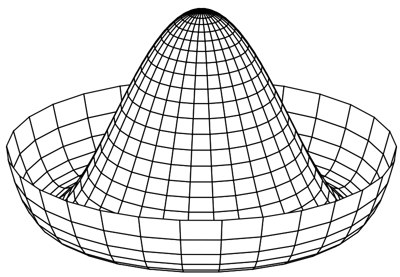

Figure 1. Degenerate ground states in a Mexican hat potential.

A precise description of spontaneous symmetry breaking requires unitarily inequivalent Hilbert space representations, because the vacuum state in a given irreducible representation is generally unique. But that seems to give rise to an apparent paradox: by Wigner's theorem, every symmetry can be implemented by a unitary operator. Shouldn't that all representations related by a symmetry be unitarily equivalent — and hence, shouldn't spontaneous symmetry breaking be impossible?

Baker and Halvorson propose to resolve this paradox by just thinking carefully about the definitions involved. Let's review them now.

2. Unitary equivalence

When do two mathematical descriptions refer to the same physical situation? It's not always easy to answer this. For example, arch-relationist Leibniz argued that two descriptions related by a spatial flip are physically equivalent; arch-substantivalists Newton and Clarke disagreed. Or: Einstein initially failed to understand that the apparent $r = R_S$ singularity at the black hole event horizon in Schwarzschild coordinates (t, r, θ, ϕ) can be removed by switching to Kruskal coordinates, while still representing the same physical spacetime.

Let's try to understand when two representations of a quantum system refer to the same statistical situation. Suppose we describe the kinematics of a quantum system with a preferred vacuum state using the triple $(\mathcal{H}, \mathcal{A}, \Omega)$, where $\mathcal{H}$ is a Hilbert space (like Fock space), $\mathcal{A}$ is an algebra of bounded operators on $\mathcal{H}$, and $\Omega \in \mathcal{H}$ is a preferred vector on $\mathcal{H}$, which we interpret as the vacuum. Usually, Ω is assumed to be such that the set of vectors $\{A\Omega \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$ (in the Hilbert space norm), to capture the fact that every state is accessible through some operation on the vacuum. A vector in $(\mathcal{H}, \mathcal{A}, \Omega)$ is called *cyclic*.

When do two such quantum systems describe the same statistical possibilities? The standard answer is: when they are related by a unitary[1] intertwiner or *unitarily equivalent*.

Definition 1 (Unitarily equivalence)**.** Two quantum systems $(\mathcal{H}, \mathcal{A})$ and $(\mathcal{H}', \mathcal{A}')$ are *unitarily equivalent* iff there exists a bijection $\alpha : \mathcal{A} \to \mathcal{A}'$ and a unitary or antiunitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that $\alpha(A) = UAU^*$ for all $A \in \mathcal{A}$.

Why is this the right standard of equivalence in quantum theory? It can be motivated in different ways, depending on how we construe the predictions of quantum theory. Here are two (for more see Aniello, 2018). First, we can use the notion of a *transition probability* $|\langle \phi, \psi \rangle|^2$. If we view the self-adjoint operators as observables, then the spectral theorem assigns each observable a set of basis vectors $\phi_1, \phi_2, \ldots,$,

[1]Recall that if $A : \mathcal{H} \to \mathcal{H}'$ is a linear operator, then for all $\psi' \in \mathcal{H}$ there exists a $\phi \in \mathcal{H}$ such that $\langle \psi', A\chi \rangle_{\mathcal{H}'} = \langle \phi, \chi \rangle_{\mathcal{H}}$. This ϕ is denoted $\phi = A^*\psi'$, which defines an operator $A^* : \mathcal{H}' \to \mathcal{H}$ called the *adjoint*. A *unitary operator* is a linear operator U such that $U^*U = I_{\mathcal{H}}$ and $UU^* = I_{\mathcal{H}'}$.

which each vector ϕ_i in the basis set interpreted as a possible outcome in an experiment. Given a system prepared in the state ψ, the transition probability $|\langle\phi_i, \psi\rangle|^2$ then gives the probability that the prepared state ψ will be measured in the state ϕ_i. Thinking of the predictions of quantum theory as transition probabilities of this kind, we might adopt the following as a reasonable standard of equivalence:

Definition 2 (equal transition probabilities)**.** Two quantum systems $(\mathcal{H}, \mathcal{A})$ and $(\mathcal{H}', \mathcal{A}')$ have *equal transition probabilities* iff there is a bijection $\alpha : P_1(\mathcal{H}) \to P_1'(\mathcal{H}')$ between their one-dimensional projections that preserves transition probabilities, in that if $\psi \in \mathcal{H}$ denotes a unit vector contained in the projection E_ψ, and $\psi^\alpha \in \mathcal{H}'$ denotes a unit vector contained in $\alpha(E_\psi)$, then,

$$|\langle\psi^\alpha, \phi^\alpha\rangle|^2 = |\langle\psi, \phi\rangle|^2 \text{ for all } E_\psi, E_\phi \in P_1(\mathcal{H}).$$

A version of Wigner's theorem says that this property holds if and only if unitary equivalence does (Aniello, 2018):

Proposition 1. *$(\mathcal{H}, \mathcal{A})$ and $(\mathcal{H}', \mathcal{A}')$ are unitarily equivalent if and only if they have equal transition probabilities.*

Proof. Assuming unitary equivalence, define $\alpha(\psi) = U\psi$ to immediately find that $|\langle\alpha(\psi), \alpha(\phi)\rangle|^2 = |\langle U\psi, U\phi\rangle|^2 = |\langle\psi, \phi\rangle|^2$. Conversely, assume equal transition probabilities. Then by Wigner's theorem there exists a unitary or antiunitary operator U such that $\alpha(\psi) = U$. Associating each ψ with a one-dimensional projection E in $\mathcal{A}$, we find that the map $E \mapsto UEU^*$ gives rise to bijection from the projections in $\mathcal{A}$ to those in $\mathcal{A}'$, which extends uniquely to a bijection $A \mapsto UAU^*$ from $\mathcal{A}$ to $\mathcal{A}'$. □

Another way to look at the predictions of quantum theory is in terms of the structure of the density matrices. Recall that a density operator ρ — a positive semidefinite operator of trace 1 in general, or a 'density matrix' in finite dimensions — is associated with a *mixed state* if and only if $\rho = \epsilon\rho_1 + (1-\epsilon)\rho_2$ for some $\epsilon \in (0, 1)$ and density matrices $\rho_1 \neq \rho_2$. Otherwise, it is called a *pure state.* A mapping that preserves the density operator structure is called a *density operator automorphism* or a

Kadison automorphism. This is another general sense in which we might say that two quantum systems can be equivalent: they can have the same density operator structure. We formulate this as follows:

Definition 3 (equivalent density operators)**.** Two quantum systems $(\mathcal{H}, \mathcal{A})$ and $(\mathcal{H}', \mathcal{A}')$ have *equivalent density operators* iff there is a bijection Φ from the density matrices of one to the density matrices of the other that preserves convex structure, $\Phi(\epsilon\rho_1 + (1-\epsilon)\rho_2) = \epsilon\Phi(\rho_1) + (1-\epsilon)\Phi(\rho_2)$ for all ρ_1, ρ_2 and for all $\epsilon \in [0, 1]$.

We then have the following result due to Kadison (1965).

Proposition 2. *Two quantum systems* $(\mathcal{H}, \mathcal{A})$ *and* $(\mathcal{H}', \mathcal{A}')$ *have equivalent density operators iff they are unitarily equivalent.*

An even more general way to look at a quantum system is as an abstract unital C^* algebra $\mathfrak{A}$, together with its set of 'states', where a *state* $\omega : \mathcal{A} \to \mathbb{C}$ is any linear function that is positive ($\rho(A^*A) \geq 0$ for all $A \in \mathfrak{A}$) and satisfies $\omega(I) = 1$. The self-adjoint elements of $\mathfrak{A}$ are again interpreted as observables, while the states $\omega \in S_{\mathcal{A}}$ have the general properties associated with the expectation value of an observable in ordinary quantum mechanics; thus, we refer to $\omega(A)$ as the 'expectation value' of A in the state ω.

Some have argued that this is the appropriate level of generality for describing quantum theory:

> "The proper sophistication, based on a mixture of operational and mathematical considerations, gives however a unique and transparent formulation within the framework of the phenomenology described; the canonical variables are fundamentally elementsin an abstract algebra of observables, and it is only relative to a particular state of this algebra that they become operators in Hilbert space." (Segal, 1959)

Moreover, this 'sophisticated' level of description comes with a natural notion of 'statistical equivalence', which is to say that their states make the same assignments to observables in $\mathfrak{A}$:

Definition 4 (equal abstract expectation values)**.** Two C^* algebras $\mathfrak{A}$ and $\mathfrak{A}'$ have *equal expectation values* iff there is a bijection between their states $\alpha : \omega \mapsto \omega'$ and between their elements $\beta : A \mapsto A'$ that preserves expectation values, $\omega'(A') = \omega(A)$ for all $A \in \mathfrak{A}$ and all states ω on $\mathfrak{A}$.

Remarkably, this abstract perspective on equivalence comes apart from the ones formulated above in terms of representations. To see this, we first note that two C^* algebras have equal expectation values iff they are related by a C^*-algebra isomorphism. This was proved by Roberts and Roepstorff (1969):

Proposition 3. *Two C^* algebras $\mathfrak{A}$ and $\mathfrak{A}'$ have equal expectation values iff they are related by a $*$-isomorphism.*

However, when $*$-isomorphic algebras (with equal abstract expectation values) are given concrete Hilbert space representations, it does *not* follow that the representations are unitarily equivalent. To see this, recall first:

Definition 5 (unitary implementability)**.** If $(\mathcal{H}, \mathcal{A})$ is a Hilbert space representation of $\mathfrak{A}$ defined by $\pi : \mathfrak{A} \to \mathcal{A}$, then a $*$-isomorphism $\alpha : \mathfrak{A} \to \mathfrak{A}$ is *unitarily implementable* iff there exists a Hilbert space $\mathcal{H}'$ and a unitary $U : \mathcal{H} \to \mathcal{H}'$ such that $\pi(\alpha(A))$ defines a representation of $\mathfrak{A}$, which is unitarily equivalent to the first: $\pi(\alpha(A)) = U\pi(A)U^*$ for all $A \in \mathcal{A}$.

As it turns out, a 'symmetry' in the sense of a $*$-isomorphism may not be unitarily implementable: this is indeed often taken to be the definition of spontaneous symmetry breaking (cf. Sewell, 2002; Earman, 2003). So, the failure of a $*$-isomorphism to be unitarily implementable implies that the representations are not unitarily equivalent. This is our first insight about spontaneous symmetry breaking: two abstract C^* algebra descriptions of a quantum systems may have equal abstract expectation values, and so be $*$-isomorphic — but when they are not unitarily implementable, then they their representations are not unitarily equivalent, and so have different density matrix structure, and different transition probabilities.

However, this isomorphism does still allow a certain expression of Wigner's theorem, and so a puzzle remains. This is the subject of the next section.

3. Baker and Halvorson's resolution

An apparent paradox threatens: spontaneous symmetry breaking is associated with two quantum systems $(\mathcal{H}, \mathcal{A}, \Omega)$ and $(\mathcal{H}', \mathcal{A}', \Omega)$ that are related by a symmetry, but still manage to be unitarily inequivalent, in order for there to be some sense in which two systems have an inequivalent vacuum. Baker and Halvorson make this precise in the form of the following fact.

Proposition 4 ('Representation Wigner Theorem')**.** *Let $(\mathfrak{A}, \omega)$ be a C^* algebra and state, with an automorphism $\alpha : \mathfrak{A} \to \mathfrak{A}$. Let $(\mathcal{H}, \pi, \Omega)$ be a GNS representation for ω, and let $(\mathcal{H}', \pi', \Omega')$ be a GNS representation for $\omega \circ \alpha^{-1}$. Then there is a unique unitary operator $W = W_{\pi,\pi'} : \mathcal{H} \to \mathcal{H}'$ such that $W\Omega = \Omega'$, and $W\pi(\alpha^{-1}(A)) = \pi'(A)W$ for all $A \in \mathfrak{A}$.*

The proof of this is an easy corollary of Wigner's theorem: found in Baker and Halvorson's appendix. Note the subtle difference between the conclusion of this theorem and unitary equivalence. Given two representations $(\mathcal{H}, \pi(\mathfrak{A}))$ and $(\mathcal{H}', \pi'(\mathfrak{A}))$, unitarily equivalence means that there exists a bijection $\beta : \pi(\mathfrak{A}) \to \pi'(\mathfrak{A})$ given by a unitary intertwiner $W : \mathcal{H} \to \mathcal{H}'$, i.e.

$$\beta(\pi(A)) = \pi'(A) = W\pi(A)W^* \tag{1}$$

for all $A \in \mathfrak{A}$. In contrast, this theorem only concludes that there is an intertwiner 'up to an automorphism' of the original algebra, in that,

$$\beta(\pi(A)) = \pi' \circ \alpha(A) = W\pi(\alpha(A))W^*. \tag{2}$$

So, although we can conclude from this that, since $\alpha(\mathfrak{A}) = \mathfrak{A}$, a $*$-automorphism allows one to construct a unitary W with the property that $\pi'(\mathfrak{A}) = W\pi(\mathfrak{A})W^*$, this does not entail that $\pi'(A) = W\pi(A)W^*$ for all A, as would be required by unitary equivalence.

This leads Baker and Halvorson to the following path forward: the automorphism $\alpha : \mathfrak{A} \to \mathfrak{A}$ on a C^*-algebra gives rise to a sister-automorphism $\alpha' : \omega \to \omega'$ that acts on states, defined by, $\alpha'(A) := \alpha(A)$. They then propose the following solution to the paradox:

> *It is possible for an automorphism $\alpha : \mathfrak{A} \to \mathfrak{A}$ to be be unitarily implemented as it acts on states, but not as it acts on operators.*

They propose that the former is the result of Wigner's theorem, while the latter occurs in the context of spontaneous symmetry breaking. So, given two quantum systems $(\mathcal{H}, \pi(\mathfrak{A}), \Omega)$ and $(\mathcal{H}', \pi'(\mathfrak{A}), \Omega')$ that are representations of the same C^*-algebra $\mathfrak{A}$, Baker and Halvorson point out that, for each symmetry α (a $*$-automorphism) of $\mathfrak{A}$, there is a unitary operator U such that α' acting on states is implemented by a unitary operator $W : \mathcal{H} \to \mathcal{H}$. This W may still have the effect of mapping the two representations to each other, in that for any fixed $\psi \in \mathcal{H}$, the following two sets of vectors in $\mathcal{H}$ are the same:

$$\{W^*\pi'(A)W\psi \mid A \in \mathfrak{A}\} = \{\pi(A)\psi \mid A \in \mathfrak{A}\}, \tag{3}$$

and W is unitary. However, W still may not relate the operators in the two algebras element-by-element, as we have seen above.

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