# Time reversal and Boson-Fermion Superselection

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Earman (2008) classified the definitions of superselection rules. Here we will consider just his third and fourth definitions, assuming the presence of a Hilbert space $\mathcal{H}$ and a von Neumann algebra $\mathcal{M}$ of bounded operators on $\mathcal{H}$. Earman takes a superselection rule of types III and IV to be defined by the following properties, respectively.

> **SSR-III.** Two subspaces $\mathcal{H}^+$ and $\mathcal{H}^-$ of $\mathcal{H}$ satisfy the property that for all $\phi^+ \in \mathcal{H}^+$, $\phi^- \in \mathcal{H}^-$, and all $A \in \mathcal{M}$,
>
> $$\langle \phi^+, A\phi^- \rangle = \langle \phi^-, A\phi^+ \rangle = 0.$$
>
> **SSR-IV.** $\psi = \alpha\phi^+ + \beta\phi^-$ is a mixed state for all non-zero $\phi^+ \in \mathcal{H}^+$, $\phi^- \in \mathcal{H}^-$, and all $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$.

These properties are known to hold whenever $\mathcal{H}^-$ consists of fermion states and $\mathcal{H}^+$ of boson states. The following argument for this is an explication of the one given by Wick et al. (1952, p.103). It makes use of the fact that for fermionic systems, $T^2 = -1$, where $T$ is the time reversal operator; this condition in turn follows from the assumption that the direction of time is isotropic and therefore commutes with rotations (Roberts; 2012).

**Theorem.** *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra on the Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, and let $T^2$ be a unitary transformation. If:*

(1) *(boson space) $T^2\phi^+ = \phi^+$ for all $\phi^+ \in \mathcal{H}^+$;*
(2) *(fermion space) $T^2\phi^- = -\phi^-$ for all $\phi^- \in \mathcal{H}^-$;*
(3) *(reversal property) $[T^2, A] = 0$ for all $A \in \mathcal{M}$;*

*then $\langle \phi^+, A\phi^- \rangle = \langle \phi^-, A\phi^+ \rangle = 0$ (SSRIV), and $\psi = \alpha\phi^+ + \beta\phi^-$ is a mixed state for all non-zero $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$ (SSRIII).*

*Proof.* The first part is established by the observation that,

$$\begin{aligned}
\langle \phi^+, A\phi^- \rangle &= \langle T^2\phi^+, T^2A\phi^- \rangle && (T^2 \text{ is unitary}) \\
&= \langle T^2\phi^+, AT^2\phi^- \rangle && (3) \\
&= \langle \phi^+, A(-\phi^-) \rangle && (1, 2) \\
&= -\langle \phi^+, A\phi^- \rangle.
\end{aligned}$$

Thus $\langle\phi^+, A\phi^-\rangle = \langle\phi^-, A\phi^+\rangle = 0$. Now apply this to get,

$$\begin{aligned}\langle\psi, A\psi\rangle &= \langle\alpha\phi^+ + \beta\phi^-, A(\alpha\phi^+ + \beta\phi^-)\rangle \\ &= |\alpha|^2\langle\phi^+, A\phi^+\rangle + \alpha\beta^*\langle\phi^-, A\phi^+\rangle + \beta\alpha^*\langle\phi^+, A\phi^-\rangle + |\beta|^2\langle\phi^-, A\phi^-\rangle \\ &= |\alpha|^2\langle\phi^+, A\phi^+\rangle + |\beta|^2\langle\phi^-, A\phi^-\rangle.\end{aligned}$$

Adopt the notation, $\omega_\phi(A) := \langle\phi, A\phi\rangle$ for $\phi \in \mathcal{H}$. Then the above equation says that $\psi = \alpha\phi^+ + \beta\phi^-$ satisfies $\omega_\psi = \lambda\omega_{\phi^+} + (1-\lambda)\omega_{\phi^-}$ with $\lambda = |\alpha|^2$, where $\lambda$ is neither 0 nor 1. Since $\omega_{\phi^+} \neq \omega_{\phi^-} \neq \omega_\psi$, this means by definition that $\psi$ is a mixed state. □

Wick, Wightman and Wigner (1952) emphasise that this argument depends on time reversal invariance in some sense. Indeed, after Cronin and Fitch discovered time reversal symmetry violation in the weak sector in 1964, a replacement argument was published by Hegerfeldt, Kraus and Wigner (1968), which instead draws on the fact that the rotation operator $R_\pi$ through $\pi$ also satisfies $R_\pi^2 = -I$ for fermionic systems. Wigner, who introduced both time reversal and superselection into modern physics, was the common author in both articles. And yet, there is a sense in which this is puzzling: the presence of superselection sectors is kinematic, not dynamical. It should have nothing to do with the question of whether time reversal is a dynamical symmetry in commuting with the Hamiltonian. So, why was Wigner concerned enough to author a second article on the matter?

The answer, it seems, is that Wigner adopted a representation theory perspective on symmetry in quantum theory. Viewing time reversal not just as a formal Hilbert space operator, but as in the image of a representation of the Poincaré group, then it is indeed true that time reversal symmetry violation implies that there is no formal representation of the time reversal element of the full Poincaré group.

This is simply because the definition of the full Poincaré group $\mathfrak{P}$ includes the requirement that if $\tau$ is the group element corresponding to time reversal, then for each element $s_t$ corresponding to translation by $t$ along a timelike line, $\tau s(t)\tau^{-1} = s(-t)$. But if $\phi : \mathfrak{P} \to B(\mathcal{H})$ is a representation of the inhomogeneous Poincaré group amongst the bounded operators on a Hilbert space, then since the representation is a homomorphism,

$$\phi_\tau \phi_{s(t)} \phi_{\tau^{-1}} = \phi_{s(-t)},$$

or in more typical notation, $TU_tT^{-1} = U_{-t}$. Since $T$ is antiunitary, this holds if and only if the Hamiltonian generator $H$ commutes with $T$, which is the definition of time reversal invariance. Thus: a representation of the full Poincaré group requires time reversal invariance.

Wick et al. (1952) and Hegerfeldt et al. (1968) are concerned with the equivalent contrapositive: if time reversal invariance fails, then there is no Hilbert space representation in which a time reversal operator exists.

That interpretation is indeed what is suggested in the remarks of the authors. Wick et al. (1952) write that the proof of the above superselection properties holds “as long as a time inversion operator... exists” (p.103). And, Hegerfeldt et al. (1968) write that “[a]nother proof of the fermion superselection rule without the assumption of $T$ invariance is thus desirable” (p.2029).

## References

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