Notes on the CPT theorem

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1. Introduction

In relativistic quantum field theory, the CPT theorem is a collection of results showing that the dynamics are invariant under a transformation interpreted as a combination of parity, time reversal, and the exchange of matter and antimatter. Invariance under CPT is connected to many other important results in the foundations of quantum field theory, such as the spin-statistics connection.[1] These notes thus review some of the history and philosophy of the CPT theorem.

2. Prehistory of the CPT theorem

The development of CPT really tracks the historical development of quantum field theory, going right back to the discovery of the Klein-Gordon and Dirac equations and their puzzling negative energy solutions, as noted by Swanson (2014, 2019). Quantum field theory arguably began with the observation that the Hamiltonian,

$$H = \sqrt{P^2 + m^2} \tag{1}$$

leads to a non-hyperbolic relativistic wave equation: wave packets spread faster than the speed of light! And although these are avoided in the Dirac equation (and the Klein-Gordon equation), both famously introduce states with unbounded negative energy.

Modern quantum field theory solves the problem by reinterpreting these curious 'negative energy matter' states, as *antimatter with positive energy.* But what assures that this is possible in general? It requires the dynamics to be invariant under an exchange of matter and antimatter. But there is a famous history of events that causes trouble for the strategy.

Parity violation. In the early theory of strong interactions, a particle state with a given mass and lifetime was observed to decay into two different outgoing states: sometimes two pions (positive and neutral), and sometimes three pions (two positive and one negative). One of these decay products was invariant under the parity

[1]Schwinger (1951) appears to have assumed CPT-invariance in his argument for the spin-statistics connection, which motivated the first attempted proof by Lüders (1954) (see Lüders and Zumino; 1957, fn.8). A classic modern treatment is Streater and Wightman (1964); see Greaves (2008), Bain (2016), and Swanson (2018, 2019) for recent philosophical appraisals.

transformation and the other was not:

$$P\pi^+\pi^0 = \pi^+\pi^0$$
$$P\pi^+\pi^+\pi^- = -\pi^+\pi^+\pi^-.$$

So, it was assumed by parity invariance that they must have originated from different in-going states, which were referred to as θ and τ.

Of course, we didn't really know the dynamics of these interactions yet: the standard model was still decades from being invented. So, what does parity invariance mean in this context? The ideas was to draw on the following.[2]

Fact 1. *Let S be a scattering matrix, and $R : \mathcal{H} \to \mathcal{H}$ be a unitary bijection. If a decay $\psi^{in} \to \psi^{out}$ with non-zero amplitude $\langle \psi_{out}, S\psi_{in} \rangle \neq 0$ satisfies either,*

(1) *(in but not out) $R\psi^{in} = \psi^{in}$ but $R\psi^{out} = -\psi^{out}$, or*
(2) *(out but not in) $R\psi^{out} = \psi^{out}$ but $R\psi^{in} = -\psi^{in}$,*

then,

(3) *(R-violation) $RS \neq SR$.*

Thus, even without knowing the dynamics, we can identify symmetry violation under R = parity with the presence of in-states and out-states that transform in different ways. So, to preserve parity symmetry, decay products that transformed differently under parity would have to come from different in-going states, referred to as θ and τ. A famous puzzle then arose (the '$\theta - \tau$ puzzle') as to why these states had the same mass and lifetime.

Lee and Yang (1956, p.254) controversially suggested that the problem could be solved by accepting parity violation. This proposal was dramatically confirmed when Chien-Shiung Wu et al. (1957) proved that the decay of the Cobalt-60 atom violates parity. This solved the θ-τ puzzle: the two particles were in fact one and the same particle, now known as a K meson. But it also introduced a further puzzle: this same interaction failed to be invariant under the exchange of matter and antimatter, ruining the interpretation of negative energy states proposed above!

The response of most physicists was to postulate that the combined transformation by parity and charge conjugation (interpreted as matter-antimatter exchange), or CP, must remain a symmetry of the fundamental laws. A number of elegant CP invariant models were developed, including by Weinberg (1958). James Cronin later described the situation:

> "It just seemed evident that CP symmetry should hold. People are very thick-skulled. We all are. Even though parity had been overthrown a few years before, one was quite confident about CP symmetry." (Cronin and Greenwood; 1982, p.41)

[2]Exercise: Prove this; or see Roberts (2015).

This of course led to the following dramatic turn of events.

CP violation. Neutral kaons are characterised by their decay into three pions, one neutral and two of opposite charges. (The neutral pion would not ionise in a spark chamber and so was invisible, but its trajectory could be calculated from the trajectories of the other two by conservation of momentum.) This decay is compatible with CP-invariance: the neutral kaon K_L and the three-pion state both change sign under CP. In contrast, a two-pion state is left unchanged by CP, and so a decay into just two pions would imply CP-violation. So, Cronin and Fitch set out to check whether they could show that, to a high degree of accuracy, no CP-violating two-pion decay events could be found.

After a long analysis of all the photographs, they found that to the contrary a small but unmistakable number of long-lived neutral kaons decayed into two pions, violating CP and time reversal symmetry. The event only occured in about one out of every 500 decays, but was still a clear signature. They immediately began checking their result, and discussing them with colleagues at Brookhaven. After explaining it to their colleague Abraham Pais over coffee, Pais reported that, "[a]fter they left I had another coffee. I was shaken by the news" (Pais; 1990).

Thus, we again lost the argument of a correspondence between matter and antimatter! Happily, it is restored by the presence of CPT symmetry. One might be concerned here about our success rate: we were wrong about parity, and then about CP. Can we really trust CPT symmetry? This is not clear. However, unlike the earlier arguments, there are arguments for CPT symmetry that draw only on very general assumptions about the structure of relativistic quantum field theory. So, perhaps it stands a better chance.

3. Interlude: The representation view

What does it mean to be a time translation or spatial translation in quantum theory? A Hilbert space theory by itself is just an abstract states space: so, we need a way to 'tie it down' to spacetime to make sense of these concepts. One way to do this is through a representation of a spacetime symmetry group. For local physics, that's given by the Poincaré group, the group of isometries of Minkowski spacetime. will use the following nomenclature:

- *Lorentz group* $L_+^\uparrow$: Lie group of boosts and rigid rotations about a point in $\mathbb{R}^4$. Isomorphic to $SO(1,3)$.
- *Poincaré group* $\mathcal{P}_+^\uparrow$: Lorentz group together with rigid spacetime translations (it is sometimes called the *inhomogeneous* Lorentz group). Defined by $\mathcal{P}_+^\uparrow := \mathbb{R}^4 \rtimes SO(1,3)$ with $(a_1,\Lambda_1)(a_2,\Lambda_2) := (a_1+\Lambda(a_2),\Lambda_1\Lambda_2)$.

- *Complete Poincaré group* $\mathcal{P}$: Also includes the group of discrete transformations $D = \{I, \tau, p, p\tau\}$ as a subgroup, which is isomorphic to the Klein four-group. Defined by $\mathcal{P} := \mathcal{P}_+^\uparrow \rtimes D$ with $(P_1, d_1)(P_2, d_2) := (P_1 \cdot d_1(P_2), d_1 d_2)$.

Note that it is implicit in the use of a semidirect product for these definitions is that the Lorentz group acts as *automorphisms* on $\mathbb{R}^4$, and that the discrete transformations D act as automorphisms on the Poincaré group.[3]

Now, a philosophical question: what makes a curve $s \mapsto U_s\psi$ through Hilbert space a spatial translation? What distinguishes it from a curve $t \mapsto U_t\psi$ that we might wish to call a time translation? It is hard to say at the level of an abstract state space alone. But, following Wigner (1939), we can tie these curves down to spacetime through a representation of a group of space and time translations, viewed as a 'homomorphic copy' of these transformations inside the Hilbert space.

More generally, viewing the Poincaré group as capturing the essential local structure of spacetime, we say can say: what it means to be a dynamical theory is that we have a representation of time translations; and, what it means to be a theory with spatial translations is that we have a representation of spatial translations. In other words, to have quantum theory 'on spacetime' is to have a representation, which is to say a strongly continuous homomorphism,

$$\varphi : g \mapsto U_g \tag{2}$$

from the Poincaré group $\mathcal{P}$ to the unitaries and antiunitaries on a Hilbert space. The fact that it is a continuous homomorphism means that we have captured all the 'essential structure' of spacetime. We can therefore understand the correspondence as identifying which Hilbert space transformations correspond to the various spacetime transformations.

Existence and non-existence of discrete symmetries. The discrete transformations of the Poincaré group are by definition automorphisms of $\mathcal{P}_+^\uparrow$. That is, at the level of the Poincaré group, transformations like time reversal τ, parity p and parity-time reversal $p\tau$ are all symmetries. In particular, in an appropriate reference frame, they transform each time translation $t \in \mathcal{P}$ as $\tau t \tau^{-1} = (p\tau)t(p\tau)^{-1} = -t$ and $ptp^{-1} = t$. As a result, a unitary-antiunitary representation of these transformations (being a homomorphism) will always satisfy,

$$P\mathcal{U}_t P^{-1} = \mathcal{U}_t, \qquad (PT)\mathcal{U}_t(PT)^{-1} = T\mathcal{U}_t T^{-1} = U_{-t}, \tag{3}$$

where $P := U_p$, $PT := U_{p\tau}$, and $T := U_\tau$. Writing $\mathcal{U}_t = e^{-itH}$, one can check that these properties are equivalent to preserving the Hamiltonian H, which is to say they

[3] For an introduction to semidirect products, see Robinson (1996, §1.5); for a related group theoretic analysis of the Poincaré group see Varadarajan (2007, §IX.2), and for applications to quantization theory see Landsman (1998, esp. §2.2 and Part IV).

are symmetries of the dynamics. So, the very existence of a representation of these group elements guarantees that they are symmetries. How then could it be that an element of the Poincaré group like parity or time reversal can fail to be a symmetry?

The answer is: *a representation of these transformations may not exist.* Given a quantum theory 'tied down' to spacetime by a representation of $\mathcal{P}_+^\uparrow$, it remains a question whether it is possible to extend that representation to all of $\mathcal{P}$. What the symmetry violating experiments of the mid-twentieth century show is that for some physical interactions, such an extension does not exist. This is a subtle but interesting conceptual point: strictly speaking, parity violation is not the failure of invariance under the parity operator P, and time reversal is not the failure of invariance under the time reversal operator — rather, symmetry violation occurs if and only if a representation of these operators fails to exist at all. This has interesting philosophical consequences: viewing a representation as what characterises the meaning of a symmetry transformation on Hilbert space, it follows that in symmetry-violating physical systems, the corresponding transformation not only fails to be a symmetry: it fails to have any meaning on Hilbert space at all.

Understanding this subtlety will help us to see a curiosity in early proofs of the CPT theorem, which seem to have assumed the separate existence of transformations interpreted as C, P and T.

3.1. **Charge conjugation and covering groups.** If we view the meaning of a transformation on Hilbert space as given by a representation, where does 'charge conjugation', or the exchange of matter and antimatter, get its meaning? There is a similar story here, which makes use of the universal covering group of the Lorentz group $L_+^\uparrow$, which is $\mathrm{SL}(2,\mathbb{C})$, and of the Poincaré group $\mathcal{P}_+^\uparrow$, which I will denote $\mathrm{ISL}(2,\mathbb{C})$ (with 'I' for 'Inhomogeneous').

A *covering group* G for a Lie group H is just a one that can be mapped onto H by a continuous homomorphism. So, if we are interested in the representations of a group H, then we can always represent the same Lie group structure using a covering group, although it may have some degeneracy: multiple covering group elements in G may get mapped to the same group element in H. As a result, on the representation view, a covering group for the Poincaré group is at least as suitable.

Given this, we can replace the Poincaré group with its universal covering group: for a Lie group, this is the unique simply connected group with the same Lie algebra, and subsumes all other the other covering groups. In the case of $\mathcal{P}_+^\uparrow$, this can be shown to be isomorphic to $\mathrm{ISL}(2,\mathbb{C})$, the group of linear transformations of $\mathbb{C}^2$ with unit determinant. This covering group is doubly degenerate, in that a continuous homomorphism $\phi : \mathrm{ISL}(2,\mathbb{C}) \to \mathcal{P}_+^\uparrow$ is a maps a pair of elements $P, -P$ to the same element of the Poincaré group, and is thus sometimes called the 'double covering group' of $\mathcal{P}_+^\uparrow$.

The vector space $V = \mathbb{C}^2$ on which $\mathrm{ISL}(2,\mathbb{C})$ acts in this context is called a (2-component) *spinor space*, and an element of it is called a (2-component) *spinor* or *spinorial vector*. (NB: the familiar 4-component spinors associated with the Dirac γ matrices can be constructed as pairs of 2-component spinors; see Wald (1984, Chapter 13).) It's worth a very brief digression on this, to introduce the notion of charge conjugation, although I will omit many interesting details in the theory of spinors: see Wald (1984) for a detailed treatment.

Spinors can be written in Penrose's abstract index notation as ξ^A, using a capital letter to distinguish it from real vectors. The dual covector is written $\xi_A \in V^{dual}$. Spinor space admits an operation of *complex conjugation*, which can be viewed as a map from $c : V \to V^{conj}$ the (isomorphic) complex conjugate vector space. We use a 'dotted' index A' to represent the complex conjugate index, in that,

$$\overline{\xi^A} = \bar{\xi}^{A'} \tag{4}$$

The dual dotted covectors are similarly written $\xi_{A'}$. We can proceed in the usual way to construct arbitrary 'spinorial tensors' of arbitrary undotted and dotted indices. And, conjugation on general spinorial tensors is then defined in a similar way.

The new operation of complex conjugation provides an abstract, group-theoretic way to describe the operation of *charge conjugation*, as an involution that is an automorphism of $ISL(2,\mathbb{C})$, but which leaves the Poincaré group itself fixed. In sketch, this is available because the elements of $\mathrm{ISL}(2,\mathbb{C})$ can be written in the form L^A_B. A representation of the Poincaré group turns out to be given by writing these elements in the form $L^A_B \bar{L}^{A'}_{B'}$. Seeing this requires a detailed argument that I will omit; see Wald (1984) which is of course preserved by complex conjugation. Since these elements are obviously invariant under complex conjugation, this operation provides a natural way to understand the meaning of charge conjugation at the group theoretic level.

Thus: to understand charge conjugation, we move from a representation of the Poincaré group $\mathcal{P}$ to a representation of its universal covering group $ISL(2,\mathbb{C})$. A representation of *charge conjugation* is then a representation of the charge conjugation automorphism on $ISL(2,\mathbb{C})$, similar to the way parity and time reversal are representations of automorphisms on $\mathcal{P}$.

4. Early arguments for CPT symmetry

The first attempted proof of a CPT theorem was given by Lüders, who referred to the transformation as "time reversal of the second kind". Sharpened arguments were soon given by Pauli (1955) and Lüders (1957).

This approach to quantum field theory, common in many textbooks, deserves a brief interlude. It begins with a classical Lagrangian field theory on Minkowski spacetime, defined using some sequence of classical fields $\phi^1_a, \phi^2_b, \phi^3_c, \ldots$, where each lower index denotes a set of tensor or spinor indices. We define arbitrary spinorial tensor in terms of some number m of undotted indices and n of dotted indices, and view 'charge conjuation' as the application of the conjugation map on those fields.

Philosophers have sometimes puzzled about how it is that charge conjugation in the CPT theorem could be connected to Lorentz invariance.[4] The connection between spinors and the Lorentz group helps to make the connection a little more plausible.

To begin, with a classical field theory in hand, the hope is generally that for each spacetime point x, one can define a collection of (typically unbounded) quantum field operators on a Hilbert space associated with the classical fields:

$$\Phi^1_a(x), \Phi^2_b(x), \Phi^3_c(x), \ldots. \tag{5}$$

This project is again facilitated by the fact that the symmetry group of spinor space is $ISL(2, \mathbb{C})$, the doubly-degenerate covering group of the 'restricted' Poincaré group of boosts, rotations and spacetime translations, whose unitary Hilbert space representations have been classified following the work of Wigner (1939). If all goes well, then the dynamics is given by the classical Lagrangian can be associated with such a unitary representation, although this is a big 'if', corresponding to one of the great mathematical challenges of Lagrangian quantum field theory. But 'if' all goes well, then a representation of the subgroup of spacetime translations $(\mathbb{R}^4, +)$ will be well-defined, and the subgroup of translations along timelike or null curves will determine the unitary dynamics.

The Lüders-Pauli approach (and many texbooks) describe the CPT theorem in this context by defining three separate operators: a unitary conjugation operator C, a unitary parity operator P, and an antiunitary time reversal operator T. On the representation view, these all get their meaning from their role in a unitary-antiunitary representation by the corresponding transformations on $ISL(2, \mathbb{C})$. The particular form of each operator on a Hilbert space thus depends on the representation that one chooses.[5] However, the composition Θ of all three, an antiunitary operator called the 'CPT operator', turns out to take a particularly simple form:

$$\Theta\Phi(x)\Theta^{-1} = (-1)^m(-i)^{(n+m) \bmod 2}\Phi(-x)^*, \tag{6}$$

where m and n represent the number of dotted and undotted spinor indices, respectively. The proof of the CPT Theorem then establishes that, given the way these fields appear in the unitary dynamics, that,

$$\Theta\mathcal{U}_t\Theta^{-1} = \mathcal{U}_{-t}, \tag{7}$$

[4]Cf. Greaves (2010), Arntzenius (2011) and Bain (2016, Chapter 3).

[5]Streater and Wightman (1964, §1-3) give examples in a number of different representations of $ISL(2, \mathbb{C})$. Since $ISL(2, \mathbb{C})$ is the covering group of the Poincaré group, its irreducible representations are the same as those of the Poincaré group (which shares the same Lie algebra), labelled by the Casimir operators m^2 for 4-momentum and S^2 for squared angular momentum (Wigner; 1939).

which is to say that the CPT operator Θ is a dynamical symmetry, and in particular a time reversing one.

Unfortunately, the conceptual clarity of these early approaches to the CPT Theorem, like the old formalism for quantum field theory, stumbled. In the first place, as is now well-known, the field operators $\Phi(x)$ at a spacetime point were not well-defined: it does not correspond to any well-defined Hilbert space operator, sometimes glossed by the concern that its measurement would require infinite energy. It must rather be 'smeared', in a sense I will describe shortly. In the second place, the approach gives no reason to think that a representation of the group elements corresponding to C, P and T exist. When such a representation does exist, the properties of $ISL(2, \mathbb{C})$ require it to be the case that P is a dynamical symmetry because parity commutes with time translations. The same goes for conjugation C and time reversal T. So, by assuming that a representation of these operators exist, the Lüders-Pauli approach implicitly assumes that all three are individually dynamical symmetries! This assumption trivialises the fact that their composition is a symmetry.

Worse: the assumption was dramatically proven to be false by the discovery of C-violation by Wu et al. (1957). As a result, *no representation of parity or matter-antimatter conjugation exists*. And of course the situation became worse with the discovery of CP violation as well. Some authors have noted this problem, such as Sachs (1987, §11.2):

> Before the discovery of parity violation in 1956, the definitions of the transformations P, T, and C appeared much more straightforward because the invariance of all interactions was taken as a basic assumption. P violation by the weak interactions showed that this assumptino is untenable.... In such a situation there is a question whether it is meaningful to assert the *existence* of a *kinematic* (i.e., independent of the interaction) transformation associated with the violated symmetry. (Sachs; 1987, p.267)

Although Sachs suggests a strategy for getting around this problem, by 'carrying over' the representation of each of these operators from a context in which it is a symmetry to one in which it is not, a conceptually clear application of this strategy is not forthcoming. His ultimate conclusion is that problems with the Lüders-Pauli approach to the CPT theorem remain. Happily, there is an easier way, which we will discuss next.

5. Jost's proof and The Great Misnomer

A first solution to the problems with the Lüders-Pauli approach to the CPT theorem — both the problematic field operators defined at a point and the existence problem for C, P and T — was floating around Princeton in these same years, in a widely-circulated manuscript by Arthur Wightman written[6] in 1951. Streater of his road to finding this paper:

> "I had asked [Abdus] Salam 'what *is* a quantised field', and received the answer 'Good; I was afraid you would ask me something I did not know. A quantised field, $\Phi(x)$ at the point x of space-time, is that operator assigned by the physicist using the correspondence principle, to the classical field ϕ at the point x'. I went away thinking about this; then I realised that what I needed was a statement of *which* operator is assigned by the physicist. I complained to P. K. Roy, who said that I should read Wightman's paper." (Streater; 2000)

There is room for debate as to whether Salam or Roy gave Streater better advice as to what a quantum field 'really is'.[7] However, Wightman's paper, published much later as a collaboration between Wightman and Gårding (1965), resolved the problem of how to define quantum field by associating each with a smooth Schwartz (1950, 1951) function f of on spacetime that falls of quickly at infinity, instead of with spacetime points directly. We can thus safely take $\Phi(x)$ to be a shorthand for this operator-valued distribution. Wightman's paper also proposed an axiomatic basis for quantum field theory now called the *Wightman axioms*, which led to the first rigorous approach to the CPT Theorem due to Jost (1957, 1965), who was at Princeton at the time.

By adopting the Wightman axioms, Jost's proof does make a concession, though different from that of Lüders and Pauli. There are no realistic models of the Wightman axioms that include interactions, and, Wightman fields that are invariant under local gauge symmetries is impossible in many senses.[8] However, although its scope is still narrow, it does offer a remarkable advantage over the approaches above, in deriving the CPT operator for quantum theories satisfying the Wightman axioms directly without needing to first pass through the individual operators. Jost's approach was generalised and adopted in most classic presentations (Streater and Wightman; 1964; Bogolubov et al.; 1990; Haag; 1996). He referred to his operator as 'Θ', only describing it as the composition of the other three operators only in theories for which they are a symmetry, which is the only scenario in which they exist.[9] This highlights what is really the Great Misnomer of CPT theorem discussions: the common but confusing

[6]As reported by Wightman (1996, p.174). See Rédei (2014, 2020) for a philosophical appraisal.

[7]Cf. the debate between Wallace (2006, 2011) and Fraser (2009, 2011); for conciliatory replies see Swanson (2017) and Rédei (2020).

[8]See Strocchi (2013, §7.3) for a summary, and Swanson (2017) for a philosophical appraisal of the give-and-take on various approaches to QFT.

[9]For example, he writes: "This involution is in theories which are invariant under time inversion T, space reflection P and matter-antimatter conjugation C represented by the product TCP" (Jost; 1965, p.100).

practice of referring to the transformation as 'the CPT operator.' In fact, it is no such thing: the operator Θ is independently constructed, and is only the composition of three separate operators in representations in which C, P and T each individually exist as symmetries. Since the language is so pervasive I will sometimes continue to refer to Θ as CPT. But the reader should be wary that it is a convenient misnomer.

Jost's proof of the CPT theorem culminates with the statement that for local quantum field theories associated with representations of $ISL(2,\mathbb{C})$ among the Wightman fields $\Phi(x)$, the covering group of the restricted Lorentz group associated with spinors, an operator Θ exists with the properties associated with CPT (Equation (6)). The fact that Θ is a dynamical symmetry immediately follows: the Wightman field representation of $ISL(2,\mathbb{C})$ of course includes a subgroup of spacetime translations $\mathcal{U}_a$ such that $\mathcal{U}_a\Phi(x)\mathcal{U}_a^* = \Phi(x+a)$ for each spacetime translation a. The Θ transformation properties for fields therefore imply that $\Theta U_a\Phi(x) = U_{-a}\Theta\Phi(x)$, from which it follows that $\Theta U_a\Theta^{-1} = U_{-a}$. This relation holds in particular for each one-parameter subgroup $\mathcal{U}_t$ of translations that are timelike, as well as for the subgroup of all timelike translations. Therefore, Θ is a time reversing dynamical symmetry.

It is worth mentioning that there are other rigorous approaches to the CPT theorem using different techniques. The field operators $\Phi(x)$ are not themselves observables, but rather the generators of an algebra of observables in each bounded spacetime region. So, an alternative 'algebraic' approach to quantum field theory due to Haag and Kastler (1964) skips directly to formulating observables in bounded spacetime regions on an axiomatic basis. A proof of the CPT theorem in this formalism was first given by Guido and Longo (1995). This approach has the particular advantage of allowing a definition of charge conjugation in a way that is more closely connected to gauge group structure, by drawing on the profound 'DHR/BF' superselection theory developed by Doplicher et al. (1971, 1974) and by Buchholz and Fredenhagen (1982).

6. Time reversing symmetries and time's arrow

If one is looking for a general symmetry that 'turns around time', then this is in general always possible in a quantum system with an ordinary unitary dynamics. This can be seen in the following.

Fact 2. *Every representation $t \mapsto \mathcal{U}_t$ of time translations $(\mathbb{R},+)$ amongst the automorphisms of a Hilbert space $\mathcal{H}$ (a strongly continuous unitary representation) admits a representation of time reversal $t \mapsto -t$ as a dynamical symmetry, given by an antiunitary operator T satisfying $T\mathcal{U}_tT^{-1} = \mathcal{U}_{-t}$.*

Proof. Let H be the self-adjoint operator such that $\mathcal{U}_t = e^{-itH}$, assured to exist by Stone's theorem, with spectrum $\Lambda \subseteq \mathbb{R}$. Let H_s be its spectral representation on $L^2(\Lambda)$, in that $H_s\psi(x) = x$ for all ψ in its domain, and $VHV^{-1} = H_s$ for some unitary $V : \mathcal{H} \to L^2(\Lambda)$ (cf. Blank et al.; 2008, §5.8). If K is the conjugation operator on $L^2(\Lambda)$, in that $K\psi = \psi^*$ for all $\psi \in L^2(\Lambda)$, then $[K, H_s] = 0$, since for all ψ in the domain of H_s we have $KH_sK^{-1}\psi(x) = x\psi(x) = H_s\psi(x)$. Thus, $T := V^{-1}KV$ is the desired antiunitary operator, since our definitions imply that $[T, H] = 0$, and hence $T\mathcal{U}_tT^{-1} = e^{T(-itH)T^{-1}} = e^{itTHT^{-1}} = e^{itH} = \mathcal{U}_{-t}$. □

What are the implications of this fact, and of the CPT theorem, for the arrow of time? In one sense, if we just focus on the time translations $\mathcal{U}_t$, it means that a symmetry of the time translations $\mathcal{U}_t \mapsto \mathcal{U}_{-t}$ can always be found. In this sense, time in quantum theory is quite generally symmetric: there is no strong arrow of time here.

On the other hand, one might also wish to view time symmetry as associated with a particular group element, such as the time reversal group element $\tau \in \mathcal{P}$, or the corresponding transformation in $\mathrm{ISL}(2, \mathbb{C})$. As the history we saw at the outset shows, we have no guarantee of that.

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