GNS and all that: a rough guide to algebras and states

Butterfield, Caulton and Roberts

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This document begins with general motivations for the algebraic approach to quantum theory. Then it expounds the basics of this approach's treatment of: algebras, representations and states. This will set us up for later discussion of various topics including: (i) spontaneous symmetry breaking and classical observables (which are well illustrated by the infinite spin-chain); (ii) the Unruh effect.

Contents

1 Three reasons for algebraic quantum theory

There are three broad reasons to "go beyond" the familiar Hilbert space quantum theory, which we will call: *generalization, superselection*, and *infinity* (i.e. infinite systems). Of course these reasons seem to prompt different sorts of "going beyond". But in fact algebraic quantum theory fulfils all three motivations with a single framework. (These reasons are independent of the issues facing how to formulate quantum field theory on curved spacetimes. For how those issues prompt the algebraic approach, cf. e.g. Wald (1994, Section 4.2 p. 57f. and Section 4.5, p. 73f. Here, we just note that these issues relate to all three of these reasons.)

These reasons are:

(i) *Generalization:* in the Heisenberg picture, observables are centre-stage, with states used only to fix expectation values. So maybe we can generalize quantum theory by postulating an algebra of observables with states as abstract linear expectation functionals?

(ii) *Superselection* in Hilbert space quantum theory can involve algebras smaller than $\mathcal{B}(\mathcal{H})$ with a centre (classical observables: with superselection sectors as their eigenspaces); and correspondingly, states that are represented mathematically by vectors can give the same statistics as a mixture. That is: if A with spectral projectors A_m commutes with C with spectral projectors C_n, $[A_m, C_n] = 0$, then: (i) the state represented by ρ, where $\rho = |\psi\rangle\langle\psi|$ and so ρ is a vector state; and (ii) the state represented by $\sum_n C_n \rho C_n$ both give the same statistics for A. That is: $\mathrm{tr}(\rho A_m) = \mathrm{tr}((\sum_n C_n \rho C_n) A_m)$. Cf. the quantum no-signalling theorem.

(iii) *Infinite systems:* The Stone-von Neumann theorem says (roughly speaking) that any irreducible representation of the CCRs for finitely many degrees of freedom is unitarily equivalent to the Schrödinger representation. Similarly, for the Jordan-Wigner theorem about the CARs and unitary equivalence to the Pauli representation. So their collective spirit is: "for finite canonical quantum systems, Hilbert space is enough". These theorems fail (i.e. fail to apply) for infinite systems. And there are good reasons why (some of) the resulting (continuum-many!) inequivalent representations of these infinite systems are physically significant.

We already saw an example in Section 4 of *The Quantization of Linear Dynamical Systems II: Infinite Systems*: the infinite spin-chain with its unitary inequivalence of the representations built from different vacua. At the end of Section 3 below (about operator topologies), we will return to this—and especially to our closing comment (vi) in that Section 4. We will see that the unitary inequivalence is really a matter of how when we represent a C^*-algebra in a Hilbert space, we can define a weak topology on the set of the algebra's representatives, and then close the set in this weak topology; and the new operators so generated (which don't live in the C^* algebra) can have different spectra in different representations—so that the representations *cannot* be unitarily equivalent.

Another important example of the physical significance of different representations is that in quantum statistical mechanics, we need states from inequivalent representations if we are to represent phase transitions. For in order to represent the coexistence of more than one phase of a system at the same temperature, we need more than one appropriate equilibrium states. And for a finite system, there is in any one representation at a given temperature, only one equilibrium state. Namely, the Gibbs equilibrium state, given by $\rho = e^{-\beta H}/\mathrm{Tr}(e^{-\beta H})$. For discussions that makes precise these remarks, including the constraints on the notion of equilibrium state (viz. as a KMS state), see e.g. Emch (1972, p. 206: remarks on Theorem II.2.12; Sewell 1986, Section 3.1, p. 50f; Sewell 2002, Section 5.3, p. 113f.).

Reason (i) suggests an abstract postulational approach. Reason (ii) suggests studying the Hilbert space representations of sub-algebras of $\mathcal{B}(\mathcal{H})$. Reason (iii) suggests defining infinite tensor products of (a) Hilbert spaces, and (b) operators on them.

We will adopt the first, postulational, approach; though linking in due course to the other two approaches. This approach soon yields a theorem, the GNS theorem, that secures Hilbert space representations for our abstract conception of states and algebras, generalized from the Heisenberg picture. *Prima facie*, this theorem suggests that there is after all "no news" to be had by the generalization. But in fact not so: the theorem sheds light on, even "controls", the "sea" of representations of various algebras envisaged by reason (ii), and the "pond" of inequivalent representations of a single (infinite-system) algebra envisaged by reason (iii). So indeed, a theorem worth putting centre-stage ...

So in successive subsections, we will:

(a) define a variety of algebras (Section 2), and associated notions such as operator topologies (Section 3); this exposition will enlarge on some items in our *Hilbert space review*, especially its Sections 4 and 5;

(b) outline the idea of a representation of an algebra in a Hilbert space (Section 4);

(c) outline the algebraic notion of a state; this will include (i) presenting the GNS theorem in an elementary, purely algebraic, form; and (ii) commenting on how this does not "collapse" algebraic QT back to Hilbert space QT; (Section 5).

2 Algebras, $*$-algebras, C^* algebras, von Neumann algebras

2.1 Preview of this Section

This Section augments our *Hilbert space Review*, especially its Section 4, on operator algebras. This Subsection gives a preview by briefly glossing five main ideas, labelled here as (1) to (5). Later Subsections and Sections will give more details. (For example, Section 5 will augment our *Hilbert space Review*'s Section 5, on states.)

(1): A *topological vector space*: more especially, a normed vector space V; the metric (and so topology) induced by a norm; a Cauchy sequence in a normed vector space; the completeness of such a space (and so called a *Banach space*).

Any metric space X can be isometrically embedded in a complete metric space $\overline{X}$, whose points are equivalence classes of Cauchy sequences in X.

Similarly, the ideas of inner product spaces, and their completeness, and so Hilbert space.

(2): *Linear functionals* on a normed vector space; continuity of such a functional; boundedness of such a functional, and so the norm

$$||f|| = \sup\{\ ||f(v)||\ \mid\ v \in V, ||v|| = 1\} \tag{1}$$

of a bounded functional. Then: a functional f is bounded iff it is continuous.

The continuous/bounded linear functionals on a normed vector space V, equipped with the norm (1), form a Banach space, V^*, the *dual* of V.

(We will discuss the definitions of various topologies on these spaces in Section 2.7 (around eq. 10) and Section 3.)

(3): *Algebras*:– The above notions carry over to algebras, which we will introduce formally in Section 2.2 et seq.. (We will in general assume the algebra has a 1). Thus for a *topological* *algebra*, we require that the vector space addition and the ring multiplication, ie the maps $(A, B) \mapsto A + B$ and $(A, B) \mapsto AB$, are continuous.

It is then natural to define a *Banach algebra* as a topological algebra which (as a normed vector space) is a Banach space. But in fact we will also require that the norm obeys, or is topologically equivalent to one that obeys:

(i) $||1|| = 1$ and

(ii) $||AB|| \leq ||A||\, ||B||$; i.e. the norm is *sub-multiplicative.*

By the way: The ideas mentioned so far imply a representation theorem (Kadison and Ringrose Prop. 3.1.2, p. 173-174):—

For every Banach algebra $\mathcal{A}$ there exists a Banach space V such that $\mathcal{A}$ is isomorphic and isometric to a closed sub-algebra of the algebra $\mathcal{B}(V)$ of all bounded operators $V \to V$ with the operator norm

$$||T|| \;:=\; \sup\{\, ||Tv|| \;\mid\; v \in V, ||v|| = 1\}.$$

(4): *Involution.* This is a map $^* : \mathcal{A} \to \mathcal{A}$ subject to conditions that abstract from the adjoint in Hilbert space. Then we have the ideas of: a *-algebra $\mathcal{A}$; normal and self-adjoint elements of $\mathcal{A}$; projections. And so: a Banach *-algebra, i.e. a normed *-algebra that is complete in norm. Thus finally, we get the central notion of a ...

(5) C^**-algebras.* Thus: A C^**-algebra* is a Banach *-algebra with a sub-multiplicative norm, that also obeys the C^* conditions

$$||A^*|| = ||A|| \;\text{ and }\; ||A^*A|| = ||A||^2 \,. \tag{2}$$

But C^**-algebras* are not the only kind of algebra we need. And not just because of *unbounded* quantities like position and momentum. For even if in physics, we are willing to handle unbounded quantities in terms of their spectral projections, an algebraic approach needs algebras that are rich in projections. But C^*-algebras do not in general contain non-trivial projections: not even the spectral projections of their self-adjoint elements. This will be "remedied" in Section 3.7, by considering *von Neumann algebras*: a kind of C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space.

2.2 Algebras

First, recall that an *algebra* $\mathcal{A}$ over the field $\mathbb{C}$ is a set of elements such that:

1. *Addition.* $\forall A, B \in \mathcal{A}$, $\exists (A + B) \in \mathcal{A}$, where ($\forall C \in \mathcal{A}$):
   - (a) $A + B = B + A$ (commutativity);
   - (b) $A + (B + C) = (A + B) + C$ (associativity);
   - (c) $\exists \mathbf{0} \in \mathcal{A}$ such that $\forall A \in \mathcal{A}, A + \mathbf{0} = A$ (additive identity);
   - (d) $\forall A \in \mathcal{A}, \exists (-A) \in \mathcal{A}$ such that $A + (-A) = \mathbf{0}$ (additive inverses.).
2. *Scalar multiplication.* $\forall A \in \mathcal{A}$ and $\forall \alpha \in \mathbb{C}$, $\exists (\alpha A) \in \mathcal{A}$, where ($\forall \beta \in \mathbb{C}, \forall B \in \mathcal{A}$):
   - (a) $\beta(\alpha A) = (\beta\alpha)A$ (compatibility of scalar and field multiplication);
   - (b) $1A = A$ (scalar multiplication by identity);
   - (c) $\alpha(A + B) = \alpha A + \alpha B$ (distributivity 1);
   - (d) $(\alpha + \beta)A = \alpha A + \beta A$ (distributivity 2).

1 and 2 alone give $\mathcal{A}$ the structure of a *vector space*; by adding a *norm* and completing $\mathcal{A}$ in the norm, we obtain a *Banach space*: cf. the discussion of norms in Sections 2.4 and 3.

3. *Multiplication.* $\forall A, B \in \mathcal{A}$, $\exists(AB) := A \cdot B \in \mathcal{A}$, where $(\forall C \in \mathcal{A})$:
   (a) $A(BC) = (AB)C$ (associativity);
   (b) $A(B + C) = AB + AC$ (distributivity).
   (c) You *may* also have $AB = BA$ (commutativity); this makes the algebra *Abelian.*
   (d) There *may* also $\exists \mathbf{1} \in \mathcal{A}$ such that $A\mathbf{1} = \mathbf{1}A = A$ (multiplicative identity; this makes the algebra *unital*).

1 and 3 alone give $\mathcal{A}$ the structure of a *ring.*

If your set of physical quantities is an algebra, then arbitrary polynomials formed from quantities always yields quantities. In quantum theory, algebras are always unital and are typically non-Abelian. For an algebra without a unit there is a standard way of adjoining a unit, called *unitization*; cf. e.g. De Faria and de Melo 2010, Appendix B.1 p. 259, 287.

2.3 $*$-Algebras

A $*$-*algebra* $\mathcal{A}$ is an algebra also equipped with an *involution operator* $* : \mathcal{A} \to \mathcal{A}$, i.e. an operator such that:

4. *Involution.* $\forall A, B \in \mathcal{A}$, $\forall \alpha \in \mathbb{C}$:
   (a) $(A^*)^* = A$ (self-inverse);
   (b) $(A + B)^* = A^* + B^*$ (distributivity);
   (c) $(\alpha A)^* = \overline{\alpha} A^*$ (sesquilinear);
   (d) $(AB)^* = B^* A^*$.

It is orthodox to say that we need $*$-algebras do as to secure that (at least some) physical quantities have real-valued eigenvalues. Thus we say that an element $A \in \mathcal{A}$ is *self-adjoint* iff $A^* = A$. For any element $A \in \mathcal{A}$, we may identify its self-adjoint component as $X := \frac{1}{2}(A + A^*)$ and its anti-self-adjoint component as $Y := \frac{1}{2i}(A - A^*)$, so that $A \equiv X + iY$, where X and Y are both self-adjoint. We also say that a $*$-*algebra* $\mathcal{A}$ is self-adjoint iff $\mathcal{A}$ is closed under $*$.

We also define, for any $A \in \mathcal{A}$:

- A is *normal* iff $AA^* = A^*A$;
- A is an *isometry* iff $A^*A = \mathbf{1}$;
- A is *unitary* iff $AA^* = A^*A = \mathbf{1}$.

All this is of course just an abstract ("free of Hilbert space") version of what we said in Sections 2 and 3 of our *Hilbert space review.* But we recall that in Remark A of Paragraph 4 of Section 3 of that *Hilbert space review*, we reported Roberts' analysis of "why quantities are required to be represented by self-adjoint operators" ... so note that one could pursue that question also in the present abstract context ...

2.4 Norms, metrics, uniform convergence, the uniform operator topology and C^* algebras

The main notion to be defined will be a C^* algebra $\mathcal{A}$, which is a certain kind of $*$-algebra equipped with a *norm* $\|\cdot\| : \mathcal{A} \to \mathbb{R}_+$.

Any norm on any vector space is required to satisfy:

5. *Norm.* $\forall A, B \in \mathcal{A}, \forall \alpha \in \mathbb{C}$:

   (a) $\|A\| = 0$ if and only if $A = \mathbf{0}$ (positive-definiteness);

   (b) $\|\alpha A\| = |\alpha| \|A\|$;

   (c) $\|A + B\| \leqslant \|A\| + \|B\|$: the triangle inequality.

For a norm on an algebra, we also require that it is *sub-multiplicative*, i.e. $||AB|| \leq ||A|| \, ||B||$. This is also called the product inequality.

A norm defines a *metric* in the usual sense of a *metric space*, which allows you to define *convergence* for a sequence of elements. This allows us to give the vector space that has the norm, a *topology.* Recall from the preview, i.e. Section 2.1, the idea of a *Banach space*: a normed vector space in which all Cauchy sequences converge.

In particular, for an algebra or $*$-algebra $\mathcal{A}$, we get a topology:

- Given some sequence $\{A_n\}$ of elements and another element A, all in $\mathcal{A}$, $\{A_n\}$ *converges in the norm* $\|\cdot\|$ to A iff $\|A_n - A\| \to 0$ as $n \to \infty$.

$\{A_n\}$ is a *norm-wise Cauchy sequence* iff $\forall \epsilon > 0, \exists N_\epsilon$ such that, for all $j, k > N_\epsilon$, $\|A_j - A_k\| < \epsilon$.

You may then say that $\mathcal{A}$ is *complete in the norm* iff the limit of every norm-wise Cauchy sequence is also in $\mathcal{A}$. Equivalently, you may say that $\mathcal{A}$ is *closed in the uniform operator topology.* $\mathcal{A}$ is then a *Banach algebra or $*$-algebra.*

So, finally: a C^* *algebra* $\mathcal{A}$ is a $*$-algebra also equipped with a norm $\|\cdot\| : \mathcal{A} \to \mathbb{R}_+$ such that:

6. C^* *algebra conditions.* $\forall A, B \in \mathcal{A}$:

   (a) $\mathcal{A}$ is complete in the norm $\|\cdot\|$ (Banach condition 1);

   (b) $\|AB\| \leqslant \|A\|\|B\|$ (Banach condition 2: "sub-multiplicative");

   (c) $\|A^*A\| = \|A\|^2$ (C^* identity).

   Conditions 6(a) and 6(b) (with the norm conditions under 5 above) are necessary and sufficient to make $\mathcal{A}$ a *Banach $*$-algebra.* (They are also necessary and sufficient to make an algebra into a Banach algebra; and 6(a) is the condition on a vector space for it to be a Banach space.)

The C^* identity, combined with the norm being sub-multiplicative, i.e. $||AB|| \leq ||A|| \, ||B||$, readily implies that $||A^*|| = ||A||$. For some history of this definition and of alternative definitions, cf. Bratteli & Robinson, volume 1, p. 152. And cf. Haag (1992, Scction III.2.2, p. 119-121) for discussion of:

(i) how the C^* identity is motivated by the spectral radius formula for a Banach algebra with unit (cf. Section 2.5, and

(ii) how this plays a role in the Gelfand-Naimark representation theorem for abelian C^* algebras, i.e. in Theorem 2.2, below.

If your set of physical quantities form a C^* algebra, then you can not only define arbitrary polynomials and self-adjoint elements; you can also form *limits* of polynomials. For example, let A be any element of a unital C^* algebra $\mathcal{A}$ such that $\|A\| < 1$. Now consider the sequence $\{B_n \mid n \in \mathbb{N}\}$ where

$$B_n := \mathbf{1} + A + A^2 + \ldots + A^n. \tag{3}$$

Clearly, each $B_n \in \mathcal{A}$. Now, assuming without loss of generality that $j < k$,

$$\begin{aligned} \|B_k - B_j\| &= \|A^{j+1} + \ldots + A^k\| \\ &\leqslant \|A^{j+1}\| + \ldots + \|A^k\| \ , \quad \text{given condition 5(c) on norms generally;} \\ &\leqslant \|A\|^{j+1} + \ldots + \|A\|^k \ , \quad \text{given condition 6(b) on the } C^* \text{ algebra norm.} \end{aligned} \tag{4}$$

This last expression is a geometrical series. Using the general rule that $1+x^2+\ldots+x^n = \frac{1-x^{n+1}}{1-x}$, we have that

$$\|B_k - B_j\| \leqslant \frac{\|A\|^{j+1} - \|A\|^{k+1}}{1 - \|A\|}. \tag{5}$$

Since $\|A\| < 1$, we can clearly choose, for any given $\epsilon > 0$, an N_ϵ such that $\forall j, k > N_\epsilon$, $\|B_k - B_j\| < \epsilon$. That is: $\{B_n\}$ is a norm-wise Cauchy sequence. So the limit $B := \mathbf{1}+A+A^2+\ldots$ is also in $\mathcal{A}$. In fact, it can be proved that $B \equiv (\mathbf{1} - A)^{-1}$.

2.5 Spectra

Let $\mathcal{A}$ be a unital C^* algebra (in fact it need only be a unital Banach algebra). Then any $A \in \mathcal{A}$ for which there is no $B \in \mathcal{A}$ such that $AB = BA = \mathbf{1}$—i.e., for which there is no inverse—is called *singular*. This notion allows us to define the *spectrum* $\sigma(A)$ of any element A:

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid \lambda\mathbb{1} - A \text{ is singular}\}. \tag{6}$$

We also define A's *spectral radius* as $\nu(A) := \sup_{\lambda\in\sigma(A)} |\lambda|$. The *spectral radius formula* for a Banach algebra with unit states that $\nu(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$; (Gel'fand 1941; Haag 1992, Thm. 2.2.3, p. 120; de Faria & de Melo, Lemma B.16, p. 264). Using this formula, we may prove that if A is normal, then $\|A\| = \nu(A)$.

If we now use the C^* identity (condition 6(c), above), then for any $A \in \mathcal{A}$,

$$\|A\|^2 = \|A^*A\| = \nu(A^*A). \tag{7}$$

But norms are always positive, so $\|A\| = \sqrt{\nu(A^*A)}$. The upshot is that, in any C^* algebra $\mathcal{A}$, the elements' norms are *determined* by the algebraic structure of $\mathcal{A}$.

2.6 Concrete C^* algebras

An absolutely central example of a C^* algebra is the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on some Hilbert space $\mathcal{H}$, in which

$$\|A\| := \sup_{\langle\psi|\psi\rangle=1} \|A|\psi\rangle\| \tag{8}$$

The norm defined by the RHS defines a topology on $\mathcal{B}(\mathcal{H})$, which is called $\mathcal{H}$'s *uniform topology*: cf. Section 3 for details.

So a *concrete C^* algebra* is a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, for some $\mathcal{H}$, that is closed in $\mathcal{H}$'s uniform operator topology. In fact, these concrete C^* algebras are in a sense "universal" examples; cf. Theorem 2.1 below.

Note that, in the case $\mathcal{H} = L^2(\mathbb{R})$, position Q and momentum P do not belong to $\mathcal{B}(L^2(\mathbb{R}))$, since they are unbounded, i.e. "have infinite norm". The same goes for field operators $\Phi(x)$ and number operators $N(f)$ in QFT.

An important Theorem allows us to give a concrete characterisation of a C^* algebra; (here "concrete" means: in terms of the Hilbert space). To state it, we define some jargon.

We say that, a $*$-*morphism* is a map $\pi : \mathcal{A} \to \mathcal{B}$ between C^* algebras such that: (i) $\pi(\alpha_1 A_1 + \alpha_2 A_2) = \alpha_1 \pi(A_1) + \alpha_2 \pi(A_2)$; (ii) $\pi(A_1 A_2) = \pi(A_1)\pi(A_2)$; (iii) $\pi(A^*) = \pi(A)^*$. A $*$-*isomorphism* is a $*$-morphism that is one-to-one and onto. If the co-domain of π, $\mathcal{B}$, is a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then we of course use the jargon of representations: we say that π is a *representation* of $\mathcal{A}$ on $\mathcal{H}$. A representation is automatically continuous, and contracting: $\|\pi(A)\| \leq \|A\|$. If a representation π is one-one(i.e. injective, i.e. $\pi(A) = 0$ only if $A = 0$), then we say π is *faithful.* A faithful representation preserves the norm: $\|\pi(A)\| = \|A\|$. (We will review more jargon of representation theory in Section 4.) Then our theorem is:

Theorem 2.1. (Kadison & Ringrose 1997, thm. 4.5.6: p. 281; Bratteli & Robinson, vol. 1, thm. 2.1.10, p. 24, pp. 54-60; Araki 1999, Thm B.2, p. 209)
Any C^* algebra $\mathcal{A}$ is $*$-isomorphic to a norm-closed, self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, for some $\mathcal{H}$. In other words: Any C^* algebra $\mathcal{A}$ has a faithful representation on some Hilbert space $\mathcal{H}$.

As we have just stated it, Theorem 2.1 suggests that there is "no escape" from Hilbert space. That is, it prompts the question whether the algebraic approach will really generalize Hilbert space quantum theory. But this theorem is in fact part of the important *GNS theorem/construction/representation* (named after Gelfand, Naimark and Segal, in view of their papers Gelfand and Naimark (1943) and Segal (1947). Details are given in Section 5.2.

For now, the point is just that this construction builds a Hilbert space, in which $\mathcal{A}$ gets a representation, by using the action on $\mathcal{A}$ of a state (a linear expectation functional) ω. Thus there is a GNS representation, i.e. a Hilbert space, carrying a representation of $\mathcal{A}$, for each state ω. As a result, the Hilbert space $\mathcal{H}$ mentioned in Theorem 2.1 is *"huge"*. For it is the direct sum of all the carrier spaces for the GNS representations for each pure state. It is called the *universal representation* (Araki 1999, p. 210). (It is provably faithful, and so there is a *-isomorphism.) Thus as we said near the end of Section 1: one main benefit of the algebraic approach will be to "control the vast sea" of representations.

Another central example, this time of an *Abelian C^** algebra, is the algebra $C_0(X)$ of those continuous complex-valued functions (under pointwise addition and multiplication) on some locally compact space X that (i) *vanish at infinity* in the sense that for each $f \in C_0(X)$ and any $\varepsilon > 0$, there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for all x in the set-complement of K, i.e. $X - K$; and that (ii) are equipped with the supremum norm, i.e.

$$\|f\| := \sup_{x \in X} |f(x)|, \tag{9}$$

Again, these concrete abelian C^* algebras are in a sense "universal" examples; cf. Theorem 2.2.

Theorem 2.2. (Kadison & Ringrose 1997, thm. 4.4.3: p. 270; Bratteli & Robinson, vol. 1, thm. 2.1.11A, 2.1.11B, p. 24, 62-63; Haag 1992, thm. 2.2.4, p. 121)
Any Abelian C^* $\mathcal{A}$ algebra is $*$-isomorphic to the algebra $C_0(X)$ of continuous complex functions on X that vanish at infinity, for some locally compact Hausdorff space X. And X is compact iff $\mathcal{A}$ contains the identity.

The proof of this theorem, originally by Gelfand and Naimark, involves some deep ideas: for example that the set X can be taken as comprising multiplicative linear functionals on the algebra $\mathcal{A}$, and so X can be "built out" of $\mathcal{A}$ ... These ideas deserve a Section of their own ...

2.7 The Gelfand-Naimark representation theorem for abelian C^* algebras

We will expound some details of Theorem 2.2, emphasising the ideas:

(i) (from physics) that points in a phase space X represent pure states of the system, i.e. assignments to each quantity (each function on phase space) of a value in its spectrum, the assignments of course preserving addition and multiplication; and :

(ii) (from mathematics) that there can be a duality ("mutual informativeness") between algebra and geometry (called 'Gelfand duality').

The rough idea:— Recall that in the theory of algebras (rings), ideals play a role similar to normal subgroups in group theory: namely in making the quotient structure well-defined. Then notice that in an algebra of complex functions on some set X (under pointwise addition and multiplication), $f : X \to \mathbb{C}$: for any $x \in X$, the set of functions that vanish at x, i.e. $\{f : f(x) = 0\}$ is an ideal. That is: it is a linear subspace that is closed under multiplication by an arbitrary function. (In fact it is a maximal ideal, i.e. not contained in any other proper ideal. In our abelian setting, all ideals will be two-sided, i.e. closed under both left-multiplication and right-multiplication, by an arbitrary element.) The idea thus arises that maybe the *abstract* algebraic structure of this algebra of functions, in particular the structure of its ideals, or maybe just its maximal ideals, encodes information about the set X. Maybe we could even recover from the algebraic structure the full realization of it as a set of functions on X. Indeed so! In particular, there will be a bijection between:

(a) points of X, which will act as pure states on the algebra of quantities (cf. (i) above); here we will see the neat idea that we think of $f(x)$—not, as usual, as f acting on a given x—but instead as x acting on f: we think of x as the *evaluation map* on the set of functions: $f \mapsto f(x) \in \mathbb{C}$ (called 'Gelfand transform'); and :

(b) maximal ideals of the algebra (even the abstract algebra!) of functions $f : X \to \mathbb{C}$.

To develop this, we will mostly follow de Faria and De Melo 2010, Appendix B.1 to B.3, p. 260-268; with some additions from Bratteli and Robinson, and from Haag. The topological aspects will be treated in more detail in Section 3, especially its Part (A)(3).

We begin with comments about Banach algebras (labelled (A) to (C)), and only later add the assumption of a C^* algebra.

(A): First, we note that for a Banach algebra our remark above, about ideals making quotient structures well defined, is made precise as:

Theorem:— If $\mathcal{A}$ is a Banach algebra and $I \subset \mathcal{A}$ is a closed proper 2-sided ideal, then $\mathcal{A}/I$ is a Banach algebra. (ibid. Proposition B.6).

(B): Now we introduce the all-important idea of a *character* on a Banach algebra—which for an abelian C^*-algebra will turn out to be nothing other than a pure state. For a Banach algebra $\mathcal{A}$, a character is a continuous multiplicative linear functional: i.e. a continuous algebra homomorphism $\phi : \mathcal{A} \to \mathbb{C}$ such that for all $A, B \in \mathcal{A}, \phi(AB) = \phi(A)\phi(B)$. (We will be concerned with abelian algebras, so that $AB = BA$; and by the way, for C^*-algebras, continuity is automatic.) The set of all non-zero characters of a Banach algebra $\mathcal{A}$ is called the *Gelfand spectrum*, $\sigma(\mathcal{A})$ of $\mathcal{A}$. It is a subset of $\mathcal{A}^*$, the dual of $\mathcal{A}$.

Here we recall (or note! cf. Haag 1992, p. 124-125) that the dual V^* of any complex Banach space V is the set of all bounded $\mathbb{C}$-valued linear functionals on V. V^* is a Banach space with respect to the supremum norm, i.e. $||\phi|| := \sup_{v \in V} |\phi(v)| \, ||v||^{-1}$. In particular, $\mathcal{A}^*$ is a Banach space with respect to the supremum norm.

(C): In addition to the norm topology, V^* has another important topology, the *weak* topology.* This is important both mathematically (in this Section) and physically (in Section 5.5) in connection with the suggestion that physical equivalence must allow for experimental errors and so must be less demanding than unitary equivalence—a suggestion made precise by *Fell's theorem* (1960).

Again, there will be more details about operator topologies in Section 3. (See especially Theorem 3.1 and its discussion, in Section 3.2, about semi-norms as a source of topologies.) But in short:

For any finite set $v_1, \dots v_n \in V$ and any $\varepsilon > 0$, one defines a neighbourhood of the origin in V^* by:

$$\mathcal{N}(\varepsilon; v_1, \dots v_n) = \{\phi \in V^* \, : \, |\phi(v_k)| < \varepsilon \;\text{ for } k = 1, \dots, n\,\}. \tag{10}$$

Then the linear structure on V allows one to translate these neighbourhoods of the origin to define neighbourhoods of arbitrary points.

Thus these neighbourhoods, for arbitrary finite n and arbitrary elements $v_1, \dots v_n \in V$, define the *weak* topology* on V^* induced by V. Then we have:

Theorem (Banach-Alaoglu):— The closed unit ball in V^*, i.e. $\{\phi \in V^* : ||\phi|| \leq 1\,\}$ is compact in the weak* topology induced by V.

Returning to our main narrative:— We equip $\mathcal{A}^*$ with the weak* topology, and thereby $\sigma(\mathcal{A})$ with the induced topology; so characters get given the supremum norm $||\phi||$ as above. Then it is straightforward to show that for a complex Banach algebra $\mathcal{A}$ with unit, any non-zero character has norm 1, so that in fact $\sigma(\mathcal{A})$ is a subset of the unit sphere of $\mathcal{A}^*$. In terms of states as linear expectation functionals that are normalized i.e. that map the unit to 1 (for this algebraic definition of state, recall Section 5 of our Hilbert space review, or cf. Section 5.1 below): any character is a state—in this algebraic sense.

Besides, a sharper statement will be an easy corollary of Theorem 5.2 below in Section 5.3. This theorem will say that the GNS representation of a C^*-algebra that is defined by a state is irreducible (so that by Schur's lemma: the operators in the representation have trivial commutant: cf. Section 4 of our Hilbert space review) iff the state is pure. Namely, the corollary will be that the characters are precisely the pure states. (The condition of multiplicativity of values, $\phi(AB) = \phi(A)\phi(B)$, corresponds to the GNS representation being irreducible; cf. e.g. Bratteli and Robinson Corollary 2.3.21, p. 58.)

It is also straightforward, using the Banach-Alaoglu theorem, to show that $\sigma(\mathcal{A})$ is a compact Hausdorff space in the weak* topology. (ibid, B.9).

Now let $\mathcal{M}$ be the set of maximal ideals of a complex abelian Banach algebra with unit, $\mathcal{A}$. Then it is straightforward to show:—

Theorem:— The map: $\Phi : \sigma(\mathcal{A}) \ni \phi \mapsto \ker(\phi) \in \mathcal{M}$ is a bijection from the Gelfand spectrum $\sigma(\mathcal{A})$ to the set $\mathcal{M}$ of maximal ideals. (ibid, B.10).

This is already reminiscent of the statement, in the "rough idea" paragraph above, that there is a bijection between points (of the "phase space" X) and maximal ideals of an associated algebra ...

To go further, we now add the assumption that our algebra is a C^* algebra. This amounts to assuming that the norm on the algebra obeys the C^* identity. (Cf. item 6 in Section 2.4, and the discussion following it that refers to Haag (1992, Section III.2.2, p. 119-122).)

We then have the result, which justifies our using the word 'spectrum' in the name 'Gelfand spectrum':—

Theorem:— Let $\mathcal{A}$ be an abelian C^* algebra with unit, and $A \in \mathcal{A}$. Then a complex number $\lambda \in \sigma(A)$ iff there is a character $\phi \in \sigma(\mathcal{A})$ such that $\phi(A) = \lambda$. (ibid. B.21).

(Proof sketch: If $\lambda \in \sigma(A)$, then $A - \lambda\mathbb{I}$ is not invertible. So the ideal $I := (A - \lambda\mathbb{I})\mathcal{A}$ is proper. Let $J \supset I$ be a maximal ideal. Then by the Theorem just above (B.10), there is a character $\phi \in \sigma(\mathcal{A})$ with $\ker(\phi)= J$, so that $\phi(A - \lambda\mathbb{I}) = 0$, i.e. $\phi(A) = \lambda$. The converse is equally short!)

We can now give the final statement of Theorem 2.2. It will make precise the idea of the bijection (indeed isometry) between the C^*-algebra and the continuous complex-valued functions on the Gelfand spectrum of the algebra—so that this spectrum is the "natural phase space" encoded in the structure of the algebra.

We simply associate with each $A \in \mathcal{A}$ the functional $\hat{A} \in C(\sigma(\mathcal{A}))$ defined by:

$$\hat{A}(\phi) := \phi(A), \text{ for all } \phi \in \sigma(\mathcal{A}). \tag{11}$$

The map $A \mapsto \hat{A}$ is called 'Gelfand transform'; (as we mentioned in the "rough idea" paragraph, above).

Thus we have, as our final statement of Theorem 2.2:—

Theorem 2.3. de Faria & de Melo, Theorem B.22, p. 266
If $\mathcal{A}$ is an abelian C^*-algebra with unit, the Gelfand transform $\mathcal{A} \ni A \mapsto \hat{A} \in C(\sigma(\mathcal{A}))$ is a *-isomorphism, and an isometry, of $\mathcal{A}$ onto the compact Hausdorff space of continuous complex functions on the Gelfand spectrum $\sigma(\mathcal{A})$.

The proof is straightforward. The main idea, additional to what we have stated so far, is that to prove that the map is onto, one needs the *Stone-Weierstrass theorem*. We recall that Weietrstrass proved (in 1885) that any function in the space $C[a, b]$ of continuous functions on a real closed interval $[a, b] \subset \mathbb{R}$ can be uniformly approximated arbitrarily well by a polynomial. That is: the set of polynomials is dense in $C[a, b]$, equipped with the uniform topology (cf. Section 3). This was generalized by Stone, replacing:

(i) the interval $[a, b]$ by: any compact Hausdorff space X; and

(ii) the polynomials by: the algebra generated by any subset S of the continuous complex-valued functions $C(X, \mathbb{C})$ that *separates* X in the sense that for distinct points $x_1 \neq x_2$ in X there is a function $f \in S$ such that $f(x_1) \neq f(x_2)$.

One main application of Theorems 2.2 and 2.3 is to give (yet another!) proof of the spectral theorem for bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. The idea of course is that if we are given an abelian concrete C^*-algebra with unit, $\mathcal{A}$, then we can think of $\mathcal{A}$ as the functions of—the algebra generated by—a self-adjoint bounded operator. This amounts, roughly speaking, to there being a finite measure space (X, μ) such that there is a unitary isometry U between the space of square-integrable functions $L^2(X, \mu)$ and $\mathcal{H}$ that represents $\mathcal{A}$ as multiplication: i.e. $U : L^2(X, \mu) \to \mathcal{H}$ is such that for every element T of $\mathcal{A}$, $U^*TU : L^2(X, \mu) \to L^2(X, \mu)$ is a multiplication operator. For details, see de Faria and de Melo Appendix B.3, p. 268-271.

3 Operator topologies

The discussion so far—especially Section 2.7—prompts a brief outline of rival operator topologies. Cf. also the closing paragraph 3 of Section 4 of our *Hilbert space Review*: this Section expands on that paragraph. There will be three main stages. (A): We begin with orienting remarks about topologies on vector spaces, including normed spaces, in general; this is independent of operator algebras and Hilbert space (Sections 3.1 to 3.3). Then (B): we settle on Hilbert space and define four standard topologies (Sections 3.4 to 3.5). This puts us in a good position to (C): give some examples of convergence in one topology but not another (Section 3.6), and finally to discuss von Neumann algebras (Section 3.7).

3.1 Some notions and jargon

This Section gives remarks about: (1) the (sometimes confusing) jargon of "strong", "weak", "fine", "coarse", and "neighbourhoods"; (2) the need for nets rather than sequences.

(1): One leading idea here will be that a topology being *strong* or *fine* means that convergence according to it is a *logically strong* condition. This corresponds to there being *more* open sets; and so—on the usual understanding of a topology as specified by a set of open sets—a stronger topology is *bigger*, as a set of open sets. For having more open sets makes convergence—i.e. "for any open set, you eventually enter it and remain forever"—logically stronger. Convergence being logically stronger means that there are fewer limit points; and this means that requiring closure in the topology—requiring a set to be closed—is a relatively *weak* condition. (Again, this corresponds to there being more open sets. For closed sets are just the complements of open sets, and taking complements is a bijection on the set of all subsets. So there being more open sets is equivalent to there being more closed sets.)

Similarly for the comparative adjectives, of course. A topology τ_1 being *stronger* or *finer* than another τ_2 means that, with topologies defined as usual as a set of open sets: $\tau_2 \subset \tau_1$. We also say that τ_2 is *weaker* or *coarser* than τ_1. (As Kelley (1955, p. 38) remarks: unfortunately, the opposite usage also occurs!)

Here is an example of this leading idea which will be important to us. In Section 3.7 (and elsewhere), we will emphasise, in connection with von Neumann algebras, the idea of an algebra being closed in *weak* topologies, in particular in the weak topology just discussed in Section 2.7. Thus von Neumann algebras are *defined* as being closed in the weak topology. So *mutatis mutandis*! ... since convergence in a weak topology (i.e. with fewer open sets) is a *logically weak* condition, there are more limit points, and so an algebra being closed in a weak topology (i.e. the algebra containing the limit points of any convergent sequence) is a *logically strong* condition. So von Neumann algebras are a special subset of C^* algebras.

Another heuristic remark: for any $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ is a C^* algebra, but many other C^* algebras can be proper subalgebras of $\mathcal{B}(\mathcal{H})$. You can think of this as being due to the fact that the uniform operator topology is particularly *strong* or *fine*; i.e. convergence in the uniform operator topology is a logically strong condition. So again: there are fewer limit points, and accordingly there are more closed sets, and requiring closure is a relatively weak condition.

Finally, a remark about *neighbourhoods*, compared with open sets. We have so far discussed topologies in terms of open sets. We recall that one can instead use neighbourhoods. For we shall shortly present topologies in terms of neighbourhoods. That is: a *neighbourhood* of a point x in a topological space X is a set $U \subset X$ which contains an open set containing x. It follows that: (i) a set is open iff it contains a neighbourhood of each of its points; (ii) the set of neighbourhoods of a point is closed under taking supersets, and under finite intersection (Kelley ibid.). Besides, one can present a topology as a system of neighbourhoods, instead of as a system of open sets. That is: one says that a family of subsets subject to appropriate conditions, including for example being closed under taking supersets, and under finite intersection—but otherwise arbitrary—is a system of neighbourhoods. Given such a family, the open sets are then recovered using (i) above. That is, one defines a set to be open if it is a neighbourhood of each of its elements. If the system of neighbourhoods is in fact the neighbourhoods of a given topology originally presented in terms of open sets, then the defined notion of openness coincides with the original one; (for details, cf. Kelley 1955, B., p. 56).

(2): Usually convergence is discussed in terms of sequences; and we will do this also. But we should sketch how *in general* topology needs a notion of convergence in terms of *nets*, not *sequences*, of elements of the space. In short: the usual use of sequences depends on the assumption that the topological space in question is *first countable*, i.e. every element has a countable basis of neighbourhoods. The details are as follows.

We note that a *directed set* D is a partially ordered set (a poset) in which for any two elements x, y there is a third z that is greater than or equal to both, i.e. $x \leq z$ and $y \leq z$. Then a net $\{x_\alpha : \alpha \in D\}$ on any set S is a map $x : D \to S$ where D is a directed set. Of course, if D is the natural numbers, we get the usual notion of a sequence of points in S. A net $\{x_\alpha\}$ on a topological space X is said to *converge* to $x \in X$ iff for every neighbourhood U of x there is an $\alpha_U \in D$ such that $x_\alpha \in U$ for every $\alpha \geq \alpha_U$. Then we have the general result (Kelley 1955, Theorem 2.2, p. 66):

In any topological space X, a set $S \subset X$ is closed iff for every net on S (so with any directed set D as its domain of definition) that converges to $x \in X$, we have that to $x \in S$.

If (but only if!) X is first countable, i.e. every element $x \in X$ has a countable basis of neighbourhoods, then this is equivalent to convergence by sequences. Every metric space is first countable; (think of the rational-radius balls around x). But we will need to deal with topological spaces that are not metric spaces (and not metrizable, i.e. their topology is not equivalent to one given by any metric).

3.2 Semi-norms as a source of topologies

We define the notion of a *semi-norm* on a vector space, and discuss how an appropriate family of semi-norms induces a *locally convex* topology on a topological vector space.

The notion of a *semi-norm* simply relaxes the "only if" in condition (a) in the definition of norm that we gave in item 5. in Section 2.4. There, we stated the definition of norm for algebras; but of course it applies to any vector space.

That is: a semi-norm on a vector space V over $\mathbb{R}$ or $\mathbb{C}$ is a real-valued function on V, $v \mapsto ||v||$ such that:

(a): $||v|| = 0$ if $v = 0$;
(b): for any scalar α: $||\alpha v|| = |\alpha| \, ||v||$
(c): the triangle inequality: $||v + w|| \leq ||v|| + ||w||$.

It follows that a semi-norm is positive-valued, i.e. $||v|| \geq 0$; and that a norm is a semi-norm such that $||v|| > 0$ if $v \neq 0$.

To avoid confusion with the two familiar norms (i.e. as defined on vectors in a Hilbert space, and on operators on a Hilbert space, respectively), we shall write semi-norms—which for us will always be defined on operators in an algebra—with a p. This notation is usual.

Semi-norms provide an important general way to define topologies, through a system of neighbourhoods, not just on algebras but on any topological vector space. So in this Section and Section 3.3, we will present details; (even though, once we specialise to Hilbert space in Section 3.4 onwards, only the second and third of our four topologies are presented using semi-norms). We follow the *maestri*, Kadison and Ringrose (1997; especially pp. 13-17, 28-31, 35, 43-47,113-115).

Recall that a *topological vector space* is simply a vector space endowed with some topology for which the addition and scalar multiplication are continuous. (This topology is sometimes called the *initial topology*, to distinguish it from other topologies it is useful to define.) Since the continuous function $v \mapsto v + v_0 : V \to V$ has a continuous inverse function $v \mapsto v - v_0$, a set of vectors $U \subset V$ is a neighbourhood of $0 \in V$ iff $v_0 + U$ is a neighbourhood of v_0. So the topology on V can be defined by specifying a base of neighbourhoods of 0. We shall sometimes do this. For example, recall the definition in (2) of Section 3.1 of a net $\{x_\alpha\}$ converging to a point $x \in X$. In a topological vector space V, we can say that a net $\{v_\alpha\}$ converges to $v \in V$ iff for any neighbourhood U of 0, there is an index α_0 such that $v_\alpha \in v + U$ whenever $\alpha \geq \alpha_0$.

A *locally convex space* is a topological vector space in which the topology has a basis consisting of convex sets. Similarly, a *locally convex topology* on a vector space V is a topology with which V becomes a locally convex space.

Semi-norms are a "universal" way to define locally convex topologies. For we have:

Theorem 3.1. Kadison and Ringrose, Theorem 1.2.6, p. 17
Let V be a (real or complex) vector space, and Γ a family of semi-norms that separates the points of V in the sense that if $v \in V$ with $v \neq 0$, there is a $p \in \Gamma$ with $p(v) \neq 0$. Then there is a locally convex topology on V in which for each $v_0 \in V$: the family of all sets

$$V(v_0 : p_1, \dots, p_m; \varepsilon) := \{v \in V \,:\, p_j(v - v_0) < \varepsilon \ (j = 1, \dots, m)\} \tag{12}$$

(where $\varepsilon > 0$ and $p_1, \dots, p_m \in \Gamma$) is a base of neighbourhoods of v_0. With this topology, each of the semi-norms in Γ is continuous. Moreover, each locally convex topology on V arises in this way from a suitable family of semi-norms.

An important "source" of semi-norms is *linear functionals*. Let V be a real or complex vector space and $\mathcal{F}$ a family of linear functionals on V (i.e. linear maps from V to the scalar field $\mathbb{R}$ or $\mathbb{C}$). Assume that $\mathcal{F}$ separates the points of V in the sense of Theorem 3.1: i.e. for any non-zero vector $v \in V$, there is a $\rho \in \mathcal{F}$ with $\rho(v) \neq 0$. Then any $\rho \in \mathcal{F}$ defines a semi-norm p_ρ by $p_\rho(v) := |\rho(v)|$. Applying Theorem 3.1 to the separating family of semi-norms $\{p_\rho : \rho \in \mathcal{F}\}$, the locally convex topology thus defined on V is called the *weak topology induced on* V by $\mathcal{F}$. It is denoted $\sigma(V, \mathcal{F})$. Thus in this topology, each vector $v_0 \in V$ has a base of neighbourhoods of the form

$$V(v_0 : p_{\rho_1}, \dots, p_{\rho_m}; \varepsilon) := \{v \in V \,:\, |\rho_j(v) - \rho_j(v_0)| < \varepsilon \ \ (j = 1, \dots, m)\} \tag{13}$$

(where $\varepsilon > 0$ and $\rho_1, \dots, \rho_m \in \mathcal{F}$)

Returning to Remark (1) of Section 3.1, about the jargon of 'strong', 'weak' etc.: we can now justify calling $\sigma(V, \mathcal{F})$ 'weak', as follows. Since $|\rho(v) - \rho(v_0)| < \varepsilon$ whenever $v \in V(v_0 : p_\rho; \varepsilon)$, each of the linear functionals $\rho \in \mathcal{F}$ is continuous relative to the topology $\sigma(V, \mathcal{F})$. But now consider any topology τ on V such that each linear functional in $\mathcal{F}$ is τ-continuous. Then all the sets $V(v_0 : \rho_1, \dots, \rho_m; \varepsilon)$ are τ-open. So $\sigma(V, \mathcal{F})$ is a coarser, i.e. weaker, i.e. smaller topology than τ. That is: $\sigma(V, \mathcal{F})$ is the coarsest, i.e. weakest, i.e. smallest topology on V relative to which each element of $\mathcal{F}$ is continuous.

The most important case of the above paragraph is when V is a locally convex space. This implies that the set of continuous linear functionals on V (continuous with respect to the initial topology), which is called the *continuous dual space* of V and is denoted $V^\sharp$, separates the points of V (Kadison and Ringrose 1997: Corollary 1.2.11, p. 21).

Accordingly, the topology $\sigma(V, V^\sharp)$ is called *the weak topology* on V. It is the coarsest topology on V that makes each element of $V^\sharp$ continuous.

We now consider the "dual of the dual" (as so often!). For any vector space V, and linear functionals ρ on it, the equation (reminiscent of the Gelfand transform in equation 11 in Section 2.7)

$$\hat{v}(\rho) := \rho(v) \tag{14}$$

defines a linear functional on the algebraic dual space V'. If V is a locally convex space, equation 14 defines a linear functional on the continuous dual space $V^\sharp$. Then the set

$$\hat{V} := \{\hat{v} \,:\, v \in V\} \tag{15}$$

is a subspace of the algebraic dual space of $V^\sharp$. And $\hat{V}$ separates the points of $V^\sharp$. Then $\sigma(V^\sharp, \hat{V})$ is, using the above terminology, the weak topology on $V^\sharp$ (induced by $\hat{V}$). (Often the $\hat{}$ is omitted and one writes $\sigma(V^\sharp, V)$.)

With the usual use of * to indicate taking a dual, $\sigma(V^\sharp, \hat{V}) = \sigma(V^\sharp, V)$ is called *the weak* topology* on $V^\sharp$. Thus each ρ_0 in $V^\sharp$ has a base of neighbourhoods consisting of sets of the form

$$\{\rho \in V^\sharp \ : \ |\rho(v_j) – \rho_0(v_j)| < \varepsilon \ \ (j = 1, \ldots, m)\} \tag{16}$$

where $\varepsilon > 0$ and $v_1, \ldots, v_m \in V$. It can be shown that the weak* continuous linear functionals on $V^\sharp$ are exactly the elements of $\hat{V}$; (Kadison and Ringrose 1997: Prop 1.3.5, p. 31).

3.3 Normed spaces

Now we specialise to the case of *normed spaces*. So now the family Γ of semi-norms consists of a single norm: which we as usual write as $|| \cdot ||$. As we have often noted: in any normed vector space $(V, || \cdot ||)$, the norm induces a metric by $d(v, w) := ||v – w||$; and this metric defines a topology as usual, called the *norm topology*. But now, with Theorem 3.1 in mind, we note that this familiar topology can be derived as in that theorem, namely with the family Γ consisting just of the single norm $|| \cdot ||$. In this topology, each vector $v_0 \in V$ has a base of neighbourhoods of the "open-ball" form

$$V(v_0 : || \, ||; \varepsilon) := \{v \in V \ : \ ||v - v_0|| < \varepsilon\} = \{v \in V \ : \ d(v, v_0) < \varepsilon\} . \tag{17}$$

One readily proves that for two normed spaces V, W a linear map $T : V \to W$ is continuous iff it is bounded, i.e. there is a non-negative real number C such that $||Tv|| \leq C||v||$ for all $v \in V$. And this is equivalent to each of:

(a): $\sup \{||Tv||/||v|| \ : \ v \in V, v \neq 0\} < \infty$; and
(b): $\sup \{||Tv|| \ : \ v \in V, ||v|| = 1\} < \infty$;

and when these conditions are satisfied the suprema in (a) and (b) are equal to the least real number C in the above condition. (We stated this, more briefly, for Hilbert spaces in Paragraph 2, Section 2 of our *Hilbert space review*; this more general formulation is in Kadison and Ringrose 1997: Theorem 1.5.5., p. 40.)

We denote by $||T||$ the (equal and possibly infinite) suprema in this result, and call it *the (operator) bound* of T.

One readily shows that the set $\mathcal{B}(V, W)$ of all bounded linear operators from a normed space V to another W (on the same scalar field!) is a vector space, with the operator bound as a norm. Besides, it is readily shown (ibid., Theorem 1.5.6, p. 41) that if W is a Banach space (i.e. is complete in its norm), then $\mathcal{B}(V, W)$ is a a Banach space, i.e. is complete in the operator norm.

Of course, in the case where $W = V$, we write $\mathcal{B}(V)$ for $\mathcal{B}(V, V)$. One readily checks that $\mathcal{B}(V)$ is an associative algebra with a unit (viz. the identity map 1), and with the operator bound as a norm, obeying $||1|| = 1$ and $||ST|| \leq ||S|| ||T||$. We recall that these are exactly the conditions for being a *Banach algebra*, laid out in items 5. and 6. of Section 2.4.

So now we turn to linear functionals on normed spaces. First of all, we infer, by taking the space W two paragraphs above to be the ground field $\mathbb{R}$ or $\mathbb{C}$, that a linear functional ρ on a normed space V is continuous iff it is bounded; with the bound being the familiar supremum bound:

$$||\rho|| := \sup\{|\rho(v)|/||v|| \ : \ v \in V, v \neq 0\} = \sup\{|\rho(v)| \ : \ v \in V, ||v|| = 1\} . \tag{18}$$

So the continuous dual space $V^\sharp$, defined (as above) as the vector space of all continuous linear functionals on V, coincides—according as the ground field is $\mathbb{R}$ or is $\mathbb{C}$—with $\mathcal{B}(V, \mathbb{R})$ or $\mathcal{B}(V, \mathbb{C})$. It is a Banach space, with the norm given be equation 18. We call it the *Banach dual space* of V.

Now we again consider the "dual of the dual". We again use the idea that an argument of a function defines the evaluation functional on a space of functions, that we saw in equation 14 above, and in Section 2.7's definition of the Gelfand transform (equation 11). But now in the context of normed spaces, we can deduce more about the map $v \mapsto \hat{v}$. We have

Theorem 3.2. Kadison and Ringrose, Theorem 1.6.4, p. 45
If V is a normed space, the $v \mapsto \hat{v}$ defined by the equation

$$\hat{v}(\rho) := \rho(v)\,,\ \rho \in V^\sharp \tag{19}$$

is an isometric isomorphism from V onto the subspace $\hat{V} := \{\hat{v} : v \in V\}$ of the second dual space $V^{\sharp\sharp}$.

Accordingly, $\hat{V}$ is called the *natural image* of V in $V^{\sharp\sharp}$. We can now readily prove (Kadison and Ringrose, Theorem 1.6.5(i), p. 45) the Banach-Alaoglu Theorem cited in Section 2.7: which, we recall, says (in our present notation):
The closed unit ball in $V^\sharp$, i.e. $\{\rho \in V^\sharp : ||\rho|| \leq 1\}$ is compact in the weak* topology on $V^\sharp$ induced by V (which we write as $\sigma(V^\sharp, \hat{V})$ or as $\sigma(V^\sharp, V)$.

In general, $\hat{V} \neq V^{\sharp\sharp}$. (Examples are the l_1 and l_∞ Banach spaces; cf. Kadison and Ringrose, Ex. 1.9.24, p. 70. Of course for V finite-dimensional there is equality; recall the canonical, i.e. basis-independent isomorphism of a vector space and its second dual.) But if in fact, $\hat{V} = V^{\sharp\sharp}$, we say that V is *reflexive*. So a reflexive normed space V is automatically a Banach space since it is isometrically isomorphic to the Banach dual space $V^{\sharp\sharp}$. In fact, the necessary and sufficient condition for a normed space V to be reflexive is that *its* closed unit ball, i.e. $\{v \in V : ||v|| \leq 1\}$, is compact in the weak topology on V: (ibid., Theorem 1.6.7, p. 47).

So let us sum up this discussion of linear functionals on normed spaces: more specifically, of the continuous dual space $V^\sharp$, and to the properties of the weak topology $\sigma(V, V^\sharp)$ on V, and of the weak* topology $\sigma(V^\sharp, V) \equiv \sigma(V^\sharp, \hat{V})$ on $V^\sharp$. The main points have been that:

(a): in a natural way, $V^\sharp$ is a *Banach space*;

(b): V is isometrically isomorphic to a subspace of the second dual space $V^{\sharp\sharp} := (V^\sharp)^\sharp$;

(c): V is the whole of $V^{\sharp\sharp}$ iff its closed unit ball, $\{v \in V : ||v|| \leq 1\}$, is compact in the weak topology on V.

3.4 The case of Hilbert space: the strong topology

We now apply Theorem 3.1 to our familiar setting of a Hilbert space $\mathcal{H}$, and its set of bounded operators $\mathcal{B}(\mathcal{H})$. We first note that any $\psi \in \mathcal{H}$ defines a semi-norm on $\mathcal{B}(\mathcal{H})$ by:

$$p_{|\psi\rangle}(A) = ||A(|\psi\rangle)||\ , \quad \text{for all } A \in \mathcal{B}(\mathcal{H}); \tag{20}$$

and this family of semi-norms separates the points of $\mathcal{B}(\mathcal{H})$, in the sense of Theorem 3.1. So this family defines a locally convex topology on $\mathcal{B}(\mathcal{H})$, which is called the *strong operator topology*. Each $A_0 \in \mathcal{B}(\mathcal{H})$ has a base of neighbourhoods consisting of sets

$$V(A_0 : |\psi_1\rangle, \ldots, |\psi_m\rangle; \varepsilon) := \{\, A \in \mathcal{B}(\mathcal{H}) \,:\, ||(A - A_0)|\psi_j\rangle|| < \varepsilon \ (j = 1, \ldots, m)\,\} \tag{21}$$

(where $\varepsilon > 0$ and $|\psi_1\rangle, \ldots, |\psi_m\rangle \in \mathcal{H}$).

We can describe this topology in terms of convergence of a net as discussed in (2) of Section 3.1, as follows. A net $\{A_\alpha\}$ (α ranging over some directed set D) converges to A_0 in $\mathcal{H}$'s strong operator topology iff: for all $|\psi\rangle \in \mathcal{H}$ and any positive ε, there is an index α_0 such that for all $\alpha \geq \alpha_0$:

$$p_{|\psi\rangle}(A_\alpha - A_0) = ||(A_\alpha - A_0)|\psi\rangle|| < \varepsilon \ . \tag{22}$$

In other words: $\{A_\alpha\}$ is strong operator convergent to A_0 iff for any $|\psi\rangle \in \mathcal{H}$, $\{A_\alpha|\psi\rangle\}$ converges to $\{A_0|\psi\rangle\}$ in the norm topology on $\mathcal{H}$.

3.5 Summary of four topologies on $\mathcal{B}(\mathcal{H})$

We summarise the preceding Subsections by presenting four standard topologies on the set $\mathcal{B}(\mathcal{H})$, i.e. the bounded linear operators on a Hilbert space $\mathcal{H}$: in terms of their definitions of convergence.

Some orienting remarks:— For the first (which is strongest in the sense discussed in (1) of Section 3.1), convergence can be defined in terms of sequences. For the other three, one needs nets. For the second and third one uses a family of semi-norms. The fourth topology is called the *ultraweak* topology: we have not yet discussed this topology, since it invokes the notion of a *state* (especially density matrices), which we will take up in Section 5. We will discuss the ultraweak topology in Section 5.4. Here, we will only define it. As we will see, the definition uses the trace: which is of course linear, rather than obeying (b) and (c) in the definition of semi-norm in Section 3.2.

We also remark that *for the most part*, the topologies earlier in this list of four are stronger/finer (larger, as a set of open sets) than topologies later in the list. So as discussed in (1) of Section 3.1: convergence in a topology earlier in the list implies convergence in a topology later in the list, but not vice versa. The next Subsection will give examples. We note that all four topologies coincide iff $\mathcal{H}$ is finite-dimensional.

The exception to this "for the most part" statement is that the ultraweak topology is stronger/finer (larger, as a set of open sets) than the weak topology. (Mnemonic: think of "ultra" as "beyond" or "bigger", so larger as a set of open sets; rather than "more" as in "more weak".) Thus we have as statements of subsethood for sets of open sets:

(i): weak $\subset$ strong $\subset$ uniform;

(ii): weak $\subset$ ultraweak $\subset$ uniform;

(iii): ultraweak and strong are incomparable: neither is a subset of the other.

To put (i) and (ii) in terms of implication relations about convergence:

(i'): uniform convergence implies strong convergence: which implies weak convergence; (while if $\mathcal{H}$ is infinite-dimensional, the converse implications fail);

(ii'): uniform convergence implies ultraweak convergence: which implies weak convergence; (while if $\mathcal{H}$ is infinite-dimensional, the converse implications fail).

A final orienting remark concerns the general contrast between *uniform convergence* ("the same ε does for all points") and *pointwise convergence* ("different εs at different points"). It will be clear that in the uniform topology, convergence is indeed a uniform notion; while strong, weak and ultraweak convergence are all forms of pointwise convergence.

- *Uniform.* (Also known as *norm topology.*) $\{A_n\}$ converges to A in $\mathcal{H}$'s uniform topology iff $\sup_{\langle\psi|\psi\rangle=1} ||(A_n - A)|\psi\rangle|| \to 0$ as $n \to \infty$.

- *Strong.* The strong topology is defined in terms of the family of semi-norms $\{p_{|\psi\rangle} : |\psi\rangle \in \mathcal{H}\}$, with $p_{|\psi\rangle}(A) := ||A|\psi\rangle||$. We say that a net $\{A_\alpha\}$ (α ranging over some directed set

D) converges to A in $\mathcal{H}$'s strong topology iff, for all $|\psi\rangle \in \mathcal{H}$: the net $p_{|\psi\rangle}(A_\alpha)$ converges to $p_{|\psi\rangle}(A)$. That is: iff $p_{|\psi\rangle}(A_\alpha - A) := ||(A_\alpha - A)|\psi\rangle||$ converges to 0.

- *Weak.* The weak topology is defined in terms of the family of semi-norms $\{p_{|\psi\rangle,|\phi\rangle} : |\psi\rangle, |\phi\rangle \in \mathcal{H}\}$, with $p_{|\psi\rangle,|\phi\rangle}(A) := \langle\psi|A|\phi\rangle$. We say that a net $\{A_\alpha\}$ (α ranging over some directed set D) converges to A in $\mathcal{H}$'s weak topology iff, for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$: the net $p_{|\psi\rangle,|\phi\rangle}(A_\alpha)$ converges to $p_{|\psi\rangle,|\phi\rangle}(A)$. That is: iff for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, $|\langle\psi|(A_\alpha - A)|\phi\rangle| \to 0$.
- *Ultraweak.* The ultraweak topology is defined in terms of the family of (*not semi-norms!*) $\{p_\rho : \rho \in \mathcal{T}(\mathcal{H})\}$, where $\mathcal{T}(\mathcal{H})$ is the set of positive trace 1 operators ("density operators") on $\mathcal{H}$, with p_ρ defined by $p_\rho(A) := \mathrm{Tr}(\rho A)$. Then we say that a net $\{A_\alpha\}$ (α ranging over some directed set D) converges to A in the ultraweak topology iff, for all $\rho \in \mathcal{T}(\mathcal{H})$: the net $\mathrm{Tr}(\rho A_\alpha)$ converges to $\mathrm{Tr}(\rho A)$.

3.6 Examples illustrating inequivalent topologies

Recall the summary of the relations between Section 3.5's four topologies, in its statements (i) to (iii) and (i') to (ii'). To illustrate, here are three examples showing the failure of the converse implications. Happily, we can exhibit such examples with sequences—so we do not have to think about nets.

Example of strong, but not uniform, convergence:
Let $\mathcal{H} = l^2(\mathbb{N})$ and let $\{P_i\}$ be a complete orthogonal set of projections. Then define

$$S_n := \sum_{i=1}^{n} P_i. \tag{23}$$

The sequence $\{S_n\}$ converges in the strong, but not the uniform, operator topology; in fact it converges to $\mathbf{1}$. That is: $\{S_n\}$ converges in the strong operator topology to $\mathbf{1}$, since for any $|\psi\rangle \in \mathcal{H}$,

$$||(S_n - \mathbf{1})|\psi\rangle|| = \left|\left|\sum_{j=n+1}^{\infty} P_j|\psi\rangle\right|\right| \to 0 \quad \text{as} \quad n \to \infty. \tag{24}$$

(For $|\psi\rangle$ must be square-summable, i.e. $\langle\psi|\psi\rangle = \sum_{j=0}^{\infty} ||P_j|\psi\rangle||^2 < \infty$, which requires that $\sum_{j=n+1}^{\infty} ||P_j|\psi\rangle||^2 \to 0$ as $n \to \infty$.)

But $\{S_n\}$ does not converge in the uniform operator topology, since

$$\|(S_n - S_{n-1})\| = \|P_n\| = 1. \tag{25}$$

Another example of strong, but not uniform, convergence:
Consider the Hilbert space $L^2(X)$ of square-(Lebesgue-)integrable complex functions on some compact Hausdorff space X. The abelian C^* algebra $C(X)$ (quantum-heuristically: the algebra of continuous functions of the operator representing "position" $x \in X$) acts multiplicatively on $L^2(X)$. That is: for any $f \in C(X)$ and any $\psi \in L^2(X)$, we define: $(f\psi)(x) := f(x)\psi(x)$. (Recall Section 2.7, especially its last paragraph.)

Now, for any interval $\Delta \subset X$, define the *characteristic function* χ_Δ:

$$\chi_\Delta(x) = \begin{cases} 1 & \text{for } x \in \Delta; \\ 0 & \text{for } x \notin \Delta. \end{cases} \tag{26}$$

Clearly, no χ_Δ is a continuous function; i.e. $\chi_\Delta \notin C(X)$. But we can still think of the χ_Δs as operators acting multiplicatively. On this understanding, the χ_Δs are projection operators: i.e., $\chi_\Delta^2 = \chi_\Delta = \chi_\Delta^*$. Again, think of the spectral family, and the spectral projectors, of the position operator Q in our previous discussions.

We can also think of each χ_Δ as being ever more closely approximated by a sequence of continuous functions $\{\chi_\Delta^n \mid n \in \mathbb{N}\}$—e.g., by trapezoid functions whose "slopes", just to the left and to the right of Δ, become increasingly vertical. So consider how these continuous functions act multiplicatively on $L^2(X)$. For each $\psi \in L^2(X)$, we have $||(\chi_\Delta^n - \chi_\Delta)\psi|| \to 0$ as $n \to \infty$. So the $\{\chi_\Delta^n\}$ converge to χ_Δ in the strong operator topology.

But $\{\chi_\Delta^n\}$ is *not* a Cauchy sequence in the uniform operator topology. For each χ_Δ^n, we can find some normalised state ϕ_n whose support lies in χ_Δ^n's support but is disjoint from Δ.

Example of weak, but not uniform, convergence:
Consider again the infinite spin-chain. So we here return to Section 4 of *The Quantization of Linear Dynamical Systems II: Infinite Systems.* Recall its closing argument for the unitary *inequivalence* of the representations built from excitations of different "vacua". The argument turned on the different values, in the infinite limit, of the expectation values of polarization. So now we spell out how this is a matter of weak, but not uniform, convergence.

Consider the representation $\mathcal{H}_{(0,0)}$, in which the "vacuum" has all spin-sites spin-up. Define

$$m_{z,N}^{(0,0)} := \frac{1}{2N+1} \sum_{k=-N}^{N} \sigma_k^{(0,0)}(z) \tag{27}$$

The sequence $\{m_{z,N}^{(0,0)}\}$ converges in the weak, but not the uniform, operator topology. Indeed $m_{z,N}^{(0,0)} \to \mathbf{1}$ as $N \to \infty$ in the weak operator topology. But for each N, we can find a state $|\{s_k\}_{k=-N}^N\rangle$ such that

$$\left|\left|\frac{1}{2N+1} \sum_{k=-N}^{N} \sigma_k^{(0,0)}(z) |\{s_k\}_{k=-N}^N\rangle\right|\right| \leqslant \frac{1}{N}, \tag{28}$$

so that $\left|\left|(m_{z,N}^{(0,0)} - \mathbf{1})|\{s_k\}_{k=-N}^N\rangle\right|\right| \geqslant 1 - \frac{1}{N} \not\to 0$ as $N \to \infty$.

So the weak closure of the quantities governed by the CARs for the infinite spin-chain captures $m_{z,\infty}^{(0,0)}$. But the uniform closure does not. It follows that the C^* algebra generated by the quantities governed by the CARs for the infinite spin-chain does not contain $m_{z,\infty}^{(0,0)}$.

This example motivates the next Section on *von Neumann algebras.* As we said in item (vi) of our closing comment in Section 4 of *The Quantization of Linear Dynamical Systems II: Infinite Systems*: we will see this argument in the context of the facts that:

(i) the *representations* of a C^*-algebra are given within a Hilbert space, which allows us to define a weak topology;

(ii) we can close the set of the C^*-algebra's representatives in this weak topology, defining von Neumann algebras; and

(iii) the new operators so generated (which don't live in the C^*-algebra) have different spectra in different representations; so

(iv) they cannot be unitarily equivalent.

3.7 von Neumann algebras and the double commutant theorem

Unlike other algebras, von Neumann algebras are characterised essentially in a concrete way. A *von Neumann, or W^* algebra* is a $*$-algebra that is a subalgebra of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$, *which is* *closed in the weak operator topology.*

Since closure in the weak operator topology entails closure in the uniform operator topology, any von Neumann algebra is a C^* algebra. (Note that closure in the weak operator topology entails closure in the uniform operator topology' just because: if one topology τ_1 is weaker than another τ_2, i.e. $\tau_1 \subset \tau_2$ as sets of open sets, then also being closed in τ_1 implies being closed in τ_2.) But there are C^* algebras that are not von Neumann algebras: for example, the C^* algebra generated by the quantities governed by the CARs for the infinite spin-chain, which does not contain $m_{z,\infty}^{(0,0)}$. The von Neumann algebra generated by the representation of these CARs built from the vacuum state $\Omega_{(0,0)}$ *does* contain that quantity.

More generally, we are willing to handle unbounded observables in terms of their spectral projections; so an algebraic approach needs algebras rich in projections. But (as we saw above) C^* algebras do not in general contain non-trivial projections: not even the spectral projections of their self-adjoint elements. This is "remedied" by considering von Neumann algebras.

So we now expand on the discussion in Section 4 of *Hilbert space review.*

For any self-adjoint algebra $\mathcal{A} \leqslant \mathcal{B}(\mathcal{H})$, the *commutant* $\mathcal{A}'$ is defined by

$$\mathcal{A}' := \{B \in \mathcal{B}(\mathcal{H}) \mid [A, B] = 0, \forall A \in \mathcal{A}\}. \tag{29}$$

Then we have:

Theorem 3.3. *[Double Commutant Theorem]* (von Neumann; Kadison and Ringrose 1997 Thm. 5.3.1, p. 326)
The strong and weak closures of a self-adjoint algebra $\mathcal{A}$ coincide, and they coincide with $\mathcal{A}''$.

It follows that any von Neumann algebra $\mathcal{R}$ may also be characterised by $\mathcal{R} = \mathcal{R}''$.
Examples:

- $\mathcal{B}(\mathcal{H})$. Note that $\mathcal{B}(\mathcal{H})' = \{\alpha \mathbf{1} \mid \alpha \in \mathbb{C}\}$.
- Any maximal abelian subalgebra $\mathcal{A} < \mathcal{B}(\mathcal{H})$. In this case, $\mathcal{A}' = \mathcal{A}$, so *a fortiori* $\mathcal{A}'' = \mathcal{A}$. (*More specific example:* the Abelian algebra $C(X)$ of continuous complex-valued functions on some compact Hausdorff space X.)

Another perspective: (This was the perspective we adopted in item 3 of Section 4 of *Hilbert space review*: so this paragraph summarizes that item.)
Ask yourself: which bounded operators on $\mathcal{H}$ should be considered to be functions of a given set $\mathcal{A}$ of (in general, non-commuting) self-adjoint operators? For the case where $\mathcal{A}$ is Abelian, we have: $A \in \mathcal{B}(\mathcal{H})$ is a function of the elements of $\mathcal{A}$ iff $A \in \mathcal{A}''$. (This is a theorem by P. Jordan; we cited it in ibid. as Theorem 20.1 from T. Jordan (1969, p.70).) So the double commutant theorem is the non-commutative generalization of this result.

The *centre* $\mathfrak{Z}_{\mathcal{A}}$ of an algebra $\mathcal{A}$ is defined by

$$\mathfrak{Z}_{\mathcal{A}} := \{B \in \mathcal{A} \mid [A, B] = 0, \forall A \in \mathcal{A}\}. \tag{30}$$

Note that $\mathfrak{Z}_{\mathcal{A}} \subseteq \mathcal{A}'$, since $\mathcal{A}'$ may contain elements not in $\mathcal{A}$.

A von Neumann algebra is a *factor* iff its centre is trivial. The zoology of von Neumann factors was begun by Murray and von Neumann (1936), and is still an area of contemporary research.

4 Representations of algebras

A *representation* of an abstract C^* algebra $\mathcal{A}$ is a $*$-morphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and may be specified by the ordered pair $(\pi, \mathcal{H})$. Since $*$-morphisms preserve norm structure, $\pi[\mathcal{A}]$ is *also* a (concrete) C^* algebra.

$\mathcal{A}$, being an abstract C^* algebra, is closed in the uniform operator topology. But $\pi[\mathcal{A}]$, "living in" $\mathcal{H}$, may be closed in weaker topologies. In particular, we may construct the von Neumann algebra $\pi[\mathcal{A}]''$.

- A representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ is *faithful* iff $\pi(A) = 0$ implies $A = \mathbf{0}$, for all $A \in \mathcal{A}$.

- A representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ is *irreducible* iff the only closed subspaces of $\mathcal{H}$ invariant under $\pi[\mathcal{A}]$ are $\{\mathbf{0}\}$ and $\mathcal{H}$ itself.

  If $(\pi, \mathcal{H})$ is reducible, then *it is not always true that* π may be decomposed into orthogonal components: $\pi = \oplus_i \pi_i$, $\mathcal{H} = \oplus_i \mathcal{H}_i$, where each $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$. Such strange representations arise in AQFT, but there is a simple example: consider the algebra of matrices
  $$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \tag{31}$$
  on $\mathbb{C}^2$, where $\alpha \in \mathbb{C}$. The subspace of vectors of the form $\begin{pmatrix} \lambda \\ 0 \end{pmatrix}$ is invariant under the algebra, and is clearly a proper subspace of $\mathbb{C}^2$, but there is no other non-trivial invariant subspace. It is true, however, that any unitary representation of a group is *completely reducible.*

- Two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{K})$ are *unitarily equivalent* iff there exists a one-to-one, invertible, norm-preserving linear map $U : \mathcal{H} \to \mathcal{K}$ such that
  $$U\pi(A)U^{-1} = \pi'(A), \quad \text{for all } A \in \mathcal{A}. \tag{32}$$

- Two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{K})$ of $\mathcal{A}$ are *disjoint* iff every subrepresentation ρ of π is unitarily inequivalent to every subrepresentation ρ' of π'. We may write $\pi \perp \pi'$.

- Two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{K})$ of $\mathcal{A}$ are *quasi-equivalent* iff: for every subrepresentation ρ of π, there is some subrepresentation ρ' of π' such that ρ is unitarily equivalent to ρ'; and *vice versa.* We may write $\pi \sim \pi'$. Quasi-equivalence is a weakening of unitary equivalence to allow for multiplicity. For irreducible representations, quasi-equivalence collapses back into unitary equivalence. Disjointness and quasi-equivalence are mutually exclusive, but they are not jointly exhaustive; i.e., two representations may be *neither* disjoint *nor* quasi-equivalent—unless, of course, the representations are both irreducible.

- Take two representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{K})$ of $\mathcal{A}$. π and π' are quasi-equivalent iff there exists a $*$-isomorphism $\alpha : \pi[\mathcal{A}] \to \pi'[\mathcal{A}]$ such that
  $$\alpha(\pi(A)) = \pi'(A), \quad \text{for all } A \in \mathcal{A}. \tag{33}$$
  (In fact, α uniquely extends to a $*$-isomorphism between the von Neumann algebras $\pi[\mathcal{A}]''$ and $(\pi'[\mathcal{A}])''$.).

5 Algebraic states and the GNS Theorem

5.1 Algebraic states

A *state* ω on a $*$-algebra $\mathcal{A}$ is a normalized positive linear functional on $\mathcal{A}$. That is: $\omega(\mathbf{1}) = 1$, and $\omega(A^*A)$ is real and non-negative for all $A \in \mathcal{A}$. We denote the set of states on $\mathcal{A}$ by $\mathcal{S}(\mathcal{A})$.[1]

This entails that $\omega(A^*) = \overline{\omega(A)}$, and the Cauchy-Schwarz inequality:

$$\omega(A^*A)\omega(B^*B) \geqslant |\omega(B^*A)|^2. \tag{34}$$

(See Kadison & Ringrose, Prop. 2.1.1, pp. 76-7.) Some more important comments about states:

- Any state ω on a C^* algebra $\mathcal{A}$ is *(norm-)continuous*. That is: if $A_n \to A$ in the C^* norm, then $\omega(A_n) \to \omega(A)$.
- $\mathcal{S}(\mathcal{A})$ is convex. That is: if ω_1 and ω_2 are states, so is $\omega := \lambda\omega_1 + (1-\lambda)\omega_2$, for any $\lambda \in [0, 1]$. The extremal elements of $\mathcal{S}(\mathcal{A})$ are called *pure states*; all other states (i.e. states ω for which $\omega = \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$ for some $\omega_1, \omega_2 \in \mathcal{S}(\mathcal{A})$ such that $\omega_1 \neq \omega_2$) are called *mixed states*.

Any density operator on some Hilbert space $\mathcal{H}$, which carries a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, defines a unique algebraic state ω by the condition

$$\omega(A) = \mathrm{Tr}(\rho\pi(A)), \quad \text{for all } A \in \mathcal{A}. \tag{35}$$

But we can also go in the other direction: i.e. given an algebraic state on $\mathcal{A}$, we may realise it as a state in a concrete Hilbert space; this is the topic of the GNS Theorem.

Here we should of course recall two "predecessor theorems", that also start with the idea of assuming a state ω is a linear expectation functional, and then proving that it is given by a density operator, in the manner of eq. 35. Namely: von Neumann's "no hidden variables" theorem (1932), and Gleason's theorem (1957), which we discussed in *Hilbert space Review*: in Paragraph 9 of Section 3, and Paragraph 5 of Section 5, respectively. But both these "predecessor theorems" assume that the quantities, that are the arguments of the expectation functional, are given to us as operators on a Hilbert space—whereas the GNS Theorem will construct the Hilbert space from the algebra $\mathcal{A}$, and the algebraic state (linear expectation functional) ω.

5.2 The GNS Theorem

First, we say that $|\xi\rangle \in \mathcal{H}$ is *cyclic* w.r.t. the representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ iff $\{\pi(A)|\xi\rangle \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$. Then

Theorem 5.1 (The Gel'fand-Naimark-Segal (GNS) Theorem)**.** *Araki 1999, Section 2.3, pp. 33-42; Bratteli & Robinson, vol. 1, thm. 2.3.16, p. 56; Kadison and Ringrose 1997, Thms. 4.5.2, 4.5.3, pp. 278-280; Emch 2000, thm. I.14, p. 73-75, 80-81 ; Haag 1992, thm. 2.2.10, p. 122-124; de Faria & de Melo, Theorem B.42, p. 274)*

[1] Many presentations of the GNS theorem define positivity of ω by using the spectral theory of C^*-algebras to: (i) say that an element A of a C^*-algebra $\mathcal{A}$ is positive iff A is self-adjoint and $\mathrm{sp}(A) \subset \mathbb{R}^+$; and then (ii) define ω as positive if $\omega(A) \geqslant 0$ for any positive A.

Our definition not only avoids spectral theory, but also simplifies the GNS theorem in that, to define the representation space, we do not need to quotient by the ideal consisting of all $A \in \mathcal{A}$ such that $\omega(A^*A) = 0$. Thus we here follow Kadison and Ringrose 1997, Thms. 4.5.1-4.5.2, pp. 275-279. Araki (1999, Section 2.3, pp. 33-42) is a very clear exposition of the approach that quotients by the ideal (cf. his eq. 2(2.38) and (2.44) on pp. 35-36).

Compare also the exposition and illuminating example by Bryan Roberts *The GNS theorem for Pauli operators*, which also performs the quotient, and so is closer to these other presentations.

Let ω be a state on a $*$-algebra $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{H}_\omega$, a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ and a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ which is cyclic w.r.t. π such that, for all $A \in \mathcal{A}$,

$$\omega(A) = \langle \Omega_\omega | \pi(A) | \Omega_\omega \rangle. \tag{36}$$

Besides, this construction is unique up to unitary equivalence in the obvious sense. Namely: any other cyclic representation π' of $\mathcal{A}$ on a Hilbert space $\mathcal{H}'$, with a unit vector $\Omega' \in \mathcal{H}'$ such that the state ω is equal to the "Born rule expression in $\mathcal{H}'$", i.e. equal to $\langle \Omega' | \pi'(A) | \Omega' \rangle$, is unitarily equivalent to the representation π, with cyclic vector Ω_ω. That is: there is a isomorphism U from $\mathcal{H}_\omega$ to $\mathcal{H}'$ such that:

$$|\Omega'\rangle = U|\Omega_\omega\rangle \; ; \; \pi'(A) = U\pi(A)U^* \; . \tag{37}$$

Proof sketch: Define $\mathcal{H}_\omega$ to be a subspace of the dual space $\mathcal{L}(\mathcal{A})$ of linear functionals on $\mathcal{A}$. First, one checks that the map $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{L}(\mathcal{A}))$ defined by

$$\text{For all } f \in \mathcal{L}(\mathcal{A}) \text{ and } A, B \in \mathcal{A} : \; (\pi(A)f)(B) := f(BA) \tag{38}$$

is linear and a ring homomorphism, $\pi(AB) = \pi(A)\pi(B)$.

To define the inner product, we restrict attention to the subspace of $\mathcal{A}'$

$$\mathcal{H}_\omega := \{\pi(A)\omega \mid A \in \mathcal{A}\} \; . \tag{39}$$

(Since $\mathcal{H}_\omega$ is clearly invariant under any $\pi(B)$, some authors define π to have $\mathcal{H}_\omega$ as a codomain, and signal this by writing π_ω.)

The inner product is defined by

$$\langle \pi(A)\omega \mid \pi(B)\omega \rangle := \omega(A^*B) \; . \tag{40}$$

(40) is well-defined since

$$\omega(A^*B) = \overline{\omega(B^*A)} = (\pi(B)\omega)(A^*) \tag{41}$$

shows (i) that it depends on B only through $\pi(B)\omega$, and on A only through $\pi(A)\omega$ and (ii) conjugate linearity. (40) inherits linearity from ω. And the Cauchy-Schwarz inequality (34) implies that the inner product is strictly positive.

Since $\omega \in \mathcal{H}_\omega$, and $\langle \omega \mid \pi(A)\omega \rangle = \omega(1^*A) = \omega(A)$, we see that ω will do for Ω_ω. $\square$

Furthermore, the GNS representation associated with ω is unique up to unitary equivalence.

5.3 Hilbert space after all? No!

Reason 1: reducible representations:
(1): $\mathcal{H}_\omega$ is often "bigger" than the physics encoded in $\mathcal{A}$ and ω needs. In particular, not every linear transformation on $\mathcal{H}_\omega$ is of the form $\pi(A) \equiv \pi_\omega(A)$. Indeed, ω being a mixed state is *equivalent* to $\pi_\omega(\mathcal{A})$ being a *reducible* representation. For we have:

Theorem 5.2. (Araki 1999, Theorem 2.20, p. 39; Bratelli & Robinson 1987, Thm. 2.3.19 (p. 57))
The GNS representation $\pi_\omega(\mathcal{A})$ induced by the state ω is an irreducible representation of $\mathcal{A}$ iff ω is pure.

So being a vector state is not sufficient for being a pure state (unless the representation is irreducible). This is familiar from ordinary QM in the phenomenon of *superselection*.

Schur's Lemma entails that a representation's being irreducible, i.e. having no invariant subspaces, is equivalent to the commutant $\pi(\mathcal{A})'$ being trivial, i.e. comprising just multiples of the identity: $\pi(\mathcal{A})' = \mathbb{C}\mathbf{1}$. And so it is equivalent to the condition $\pi(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$ (see also Sakai 1971, thm. 1.12.9).

To make the same point more concretely, i.e. within the familiar framework: Let the $*$-algebra $\mathcal{A}$ be a set of operators on a Hilbert space $\mathcal{H}$ for the system, and let ω be given by a density operator ρ_ω on $\mathcal{H}$, i.e. $\omega(A) = \text{tr}(\rho_\omega A)$. Now let ρ_ω be of dimension greater than 1; it follows that it is a mixed state in the sense above.) Then by Theorem 5.2, the GNS space $\mathcal{H}_\omega$ is a reducible representation of $\mathcal{A}$, even if $\mathcal{A}$ acts irreducibly on our given $\mathcal{H}$. In physical terms, this is not so surprising: think of ρ_ω as an "improper mixture" obtained by tracing out the environment from a pure (vector) state on the composite of system plus environment.

Reason 2: inequivalent representations:
In many important cases, different states ω induce *unitarily inequivalent* representation spaces $\mathcal{H}_\omega$. *Example:* for an infinite spin-chain, which could function as a model of a ferromagnet: we have seen that different directions of global magnetization (spin-density) involve unitarily inequivalent Hilbert spaces.

Evidently, reason 2 bears on motivations (i) and (iii) (*Generalization* and *Infinity*) in Section 1: the key idea is that the system is characterized primarily by its algebra of observables. But we shall see that both reason 2 and reason 1 also bear on motivation (ii) (*Superselection*). There are of course many inter-linked issues here, but Theorem 5.2 is one important aspect.

5.4 Normal states, representations and folia

Let ω be a state on $\mathcal{A}$: $\omega \in \mathcal{S}(\mathcal{A})$. Which other states on $\mathcal{A}$ can be represented in ω's GNS representation? More specifically: which states on $\mathcal{A}$ have "well-behaved" representations w.r.t. $\pi_\omega(\mathcal{A})''$?

One precise notion of "well-behaved" is provided by *countable additivity*, which need not hold for states (unlike in ordinary QM). But we may define a state $\omega \in \mathcal{S}(\mathcal{R})$, on some von Neumann algebra $\mathcal{R}$, to be *normal* iff, for any countable set of orthogonal projectors $\{E_i\}$, we have

$$\omega\left(\sum_i E_i\right) = \sum_i \omega(E_i); \tag{42}$$

i.e., ω is normal iff it is countably additive. Normal states are the topic of Gleason's Theorem. (Recall (iii) in Paragraph 9 of Section 3 of the *Hilbert space Review.*) Here we report a generalization of this theorem:

Theorem 5.3. (Bratelli & Robinson 1987, thm. 2.4.21 (pp. 76-77)):
Any state ω on a von Neumann algebra $\mathcal{R}$ acting on the Hilbert space $\mathcal{H}$ is normal iff there is a density operator $\rho \in \mathcal{B}(\mathcal{H})$ (not necessarily in $\mathcal{R}$) such that $\omega(A) = Tr(\rho A)$ for all $A \in \mathcal{R}$.

Another notion of "well-behaved" comes from *ultraweak continuity.* A state ω on a von Neumann algebra $\mathcal{R}$ acting on some Hilbert space $\mathcal{H}$ is *ultraweakly continuous* iff, for any sequence $\{A_n\}$ where $A_n \in \mathcal{R}$ such that $A_n \to A$ in $\mathcal{H}$'s ultraweak operator topology, $\omega(A_n) \to \omega(A)$. (Note that this is a stronger condition that norm-continuity—which any state satisfies—since the ultraweak topology is coarser than the norm topology.) But, in fact, this notion of "well-bahaved" coincides with the first:

Theorem 5.4. (Bratelli & Robinson 1987, def. 4.13 Σ2.4.3)):
Any state ω on a von Neumann algebra $\mathcal{R}$ is normal iff ω is ultraweakly continuous.

We now link the discussions of abstract states on C^* algebras, and of normal states on von Neumann algebras, by considering representations: i.e. for any representation π, we may consider the concrete C^* algebra $\pi(\mathcal{A})$ (representing $\mathcal{A}$), and the von Neumann algebra $\pi(\mathcal{A})''$ it generates. So fix some representation $(\mathcal{H}, \pi)$ of a C^* algebra $\mathcal{A}$. Now any state $\omega \in \mathcal{S}(\mathcal{A})$ is called π-*normal* iff it satisfies $\omega(A) = \mathrm{Tr}(\rho\pi(A))$ for some density matrix $\rho \in \mathcal{B}(\mathcal{H})$. So the π-normal states on the C^* algebra $\mathcal{A}$ are the normal states on the von Neumann algebra $\pi(\mathcal{A})''$ (and *vice versa*).

We are interested in whether a state can be represented by a density matrix in another state's GNS representation. So we apply the above to GNS representations. Let $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ be ω's GNS representation, where $\omega \in \mathcal{S}(\mathcal{A})$. Then the π_ω-normal states in $\mathcal{S}(\mathcal{A})$ are all and only the states representable as density operators on $\mathcal{H}_\omega$.

This prompts the definition of a *folium*:[2]

Folia. A folium is a subset $\mathcal{F} \subseteq \mathcal{S}(\mathcal{A})$ such that:

(a) $\mathcal{F}$ is closed under convex combinations;

(b) $\mathcal{F}$ is complete in $\mathcal{S}(\mathcal{A})$'s uniform operator topology, defined $\forall \omega \in \mathcal{S}(\mathcal{A})$ by

$$\|\omega\| := \sup_{\|A\|=1} \omega(A); \tag{43}$$

(c) for all $\omega \in \mathcal{F}$ and all $A \in \mathcal{A}$, there is a state $\omega_A \in \mathcal{F}$, defined by

$$\omega_A(B) := \omega(A^*BA), \qquad \forall B \in \mathcal{A}. \tag{44}$$

Note that this condition is essential to have $\mathcal{F}$ closed under Lüders' rule conditionalisation. (Recall (i) and (ii) in Paragraph 9 of Section 3 of the *Hilbert space Review.*)

Now let ω be a state on the C^* algebra $\mathcal{A}$. We will say that the *folium associated with* ω is the set $\mathcal{F}_\omega \subseteq \mathcal{S}(\mathcal{A})$ of states expressible as density operators on ω's GNS representation π_ω; i.e. the set of π_ω-normal states. It may be checked that $\mathcal{F}_\omega$ is a folium in the sense above. Conversely, the GNS-construction implies that any folium $\mathcal{F} \subseteq \mathcal{S}(\mathcal{A})$ is of the form $\mathcal{F} = \mathcal{F}_\omega$ for some ω.

Call any two states quasi-equivalent/disjoint iff their GNS representations are. Then:

Theorem 5.5.
For any two states $\omega, \rho \in \mathcal{S}(\mathcal{A})$: (i) ω and ρ are quasi-equivalent iff $\mathcal{F}_\omega = \mathcal{F}_\rho$; (ii) ω and ρ are disjoint iff $\mathcal{F}_\omega \cap \mathcal{F}_\rho = \varnothing$.

Since any folium is a folium associated with a state, and vice versa, this theorem entails a bijective correspondence between quasi-equivalence classes of representations and folia in $\mathcal{S}(\mathcal{A})$.

5.5 Fell's Theorem

A state ω on $\mathcal{A}$ is *weak* approximated* by states in a set $S \subseteq \mathcal{S}(\mathcal{A})$ iff, for all $\epsilon > 0$, all $n \in \mathbb{Z}$ and all $A_1, ..., A_n \in \mathcal{A}$, there is state $\rho \in S$ such that

$$|\omega(A_i) - \rho(A_i)| \ < \ \epsilon, \quad \text{for} \ \ i = 1, \ldots, n. \tag{45}$$

Theorem 5.6 (Fell's Theorem)**.** *(Haag 1992, thm. 2.2.13, p. 125)*
Let π be a faithful representation of a C^*-algebra $\mathcal{A}$. Then every state in $\mathcal{S}(\mathcal{A})$ can be weak approximated by states in $\mathcal{F}_\pi$. In other words: $\mathcal{F}_\pi$ is weak*-dense in $\mathcal{S}(\mathcal{A})$.*

[2]Beware of the name 'folium': $\mathcal{S}(\mathcal{A})$ is *not* foliated by its folia, since a folium may contain subfolia, and a folium can be dense in the state space—see Fell's Theorem!

This can be glossed as: "Any finitely accurate experiments on the values of (a varying choice of finitely many) elements of any C^* algebra $\mathcal{A}$ can be modelled by some density operator that is associated with any faithful representation of $\mathcal{A}$."

For some discussion of this in the context of quantum field theory on curved spacetime, cf. Wald (1994: pp. 81-83, end of his Section 4.5).

6 References

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