23. To summarize: I have tried to show that the popular mathematical refutation of Zeno's paradoxes will not do, because it simply assumes that Achilles can perform an infinite series of acts. By using the illustration of what would be involved in counting an infinite number of marbles, I have tried to show that the notion of an infinite series of acts is self-contradictory. For any material thing, whether machine or person, that set out to do an infinite number of acts would be committed to performing a motion that was discontinuous and therefore impossible. But Achilles is not called upon to do the logically impossible; the illusion that he must do so is created by our failure to hold separate the finite number of real things that the runner has to accomplish and the infinite series of numbers by which we describe what he actually does. We create the illusion of the infinite tasks by the kind of mathematics that we use to describe space, time, and motion.

Notes

1 *Grundgesetze der Arithmetik*, 2 (1903), §124. Or see my translation in *Translations from the Philosophical Writings of Gottlob Frege* (Oxford, 1952), p. 219.
2 Or class or set or aggregate, etc.
3 An alternative arrangement would be to have three similar machines constantly circulating three marbles.
4 Somebody might say that if the marble moved by Beta eventually shrank to nothing there would be no problem about its final location.
5 Cf. Peirce: "I do not think that if each pebble were broken into a million pieces the difficulty of getting over the road would necessarily have been increased; and I don't see why it should if one of these millions – or all of them – had been multiplied into an infinity" (*Collected Papers* [Cambridge, Mass., 1931], 6.182).

15 A Contemporary Look at Zeno's Paradoxes: An Excerpt from *Space, Time, and Motion**

Wesley C. Salmon

The Paradoxes of Motion

Our knowledge of the paradoxes of motion comes from Aristotle who, in the course of his discussions, offers a paraphrase of each. Zeno's original formulations have not survived.[1]

* From Wesley C. Salmon, *Space, Time, and Motion* (Minneapolis: University of Minnesota Press, 1980). Reprinted by permission of the author.

(1) *Achilles and the Tortoise.* Imagine that Achilles, the fleetest of Greek warriors, is to run a footrace against a tortoise. It is only fair to give the tortoise a head start. Under these circumstances, Zeno argues, Achilles can never catch up with the tortoise, no matter how fast he runs. In order to overtake the tortoise, Achilles must run from his starting point A to the tortoise's original starting point T_0 (see figure 3). While he is doing that, the tortoise will have moved ahead to T_1. Now Achilles must reach the point T_1. While Achilles is covering this new distance, the tortoise moves still farther to T_2.

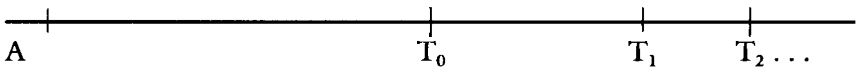

Figure 3

Again, Achilles must reach this new position of the tortoise. And so it continues; whenever Achilles arrives at a point where the tortoise *was*, the tortoise has already moved a bit ahead. Achilles can narrow the gap, but he can never actually catch up with him. This is the most famous of all of Zeno's paradoxes. It is sometimes known simply as "The Achilles."

(2) *The Dichotomy.* This paradox comes in two forms, progressive and regressive. According to the first, Achilles cannot get to the end of any racecourse, tortoise or no tortoise; indeed, he cannot even reach the original starting point T_0 of the tortoise in the previous paradox. Zeno argues as follows. Before the runner can cover the whole distance he must cover the first half of it (see figure 4).

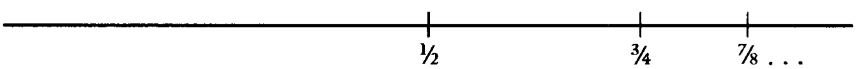

Figure 4

Then he must cover the first half of the remaining distance, and so on. In other words, he must first run one-half, then an additional one-fourth, then an additional one-eighth, etc., always remaining somewhere short of his goal. Hence, Zeno concludes, he can never reach it. This is the progressive form of the paradox, and it has very nearly the same force as Achilles and the Tortoise, the only difference being that in the Dichotomy the goal is stationary, while in Achilles and the Tortoise it moves, but at a speed much less than that of Achilles.

The regressive form of the Dichotomy attempts to show, worse yet, that the runner cannot even get started. Before he can complete the full

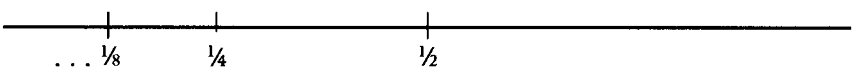

Figure 5

distance, he must run half of it (see figure 5). But before he can complete the first half, he must run half of that, namely, the first quarter. Before he can complete the first quarter, he must run the first eighth. And so on. In order to cover any distance no matter how short, Zeno concludes, the runner must already have completed an infinite number of runs. Since the sequence of runs he must already have completed has the form of a regression,

$$\ldots \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2},$$

it has no first member, and hence, the runner cannot even get started.

(3) *The Arrow.* In this paradox, Zeno argues that an arrow in flight is always at rest. At any given instant, he claims, the arrow is where it is, occupying a portion of space equal to itself. During the instant it cannot move, for that would require the instant to have parts, and an instant is by definition a minimal and indivisible-element of time. If the arrow did move during the instant it would have to be in one place at one part of the instant, and in a different place at another part of the instant. Moreover, for the arrow to move during the instant would require that during the instant it must occupy a space larger than itself, for otherwise it has no room to move. As Russell says, "It is never moving, but in some miraculous way the change of position has to occur ***between*** the instants, that is to say, not at any time whatever."[2] This paradox is more difficult to understand than Achilles and the Tortoise or either form of the Dichotomy, but another remark by Russell is apt: "The more the difficulty is meditated, the more real it becomes."

(4) *The Stadium.* Consider three rows of objects A, B, and C, arranged as in the first position of figure 6. Then, while row A remains at rest, imagine rows B and C moving in opposite directions until all three rows are lined up as shown in the second position. In the process, C_1 passes twice as many B's as A's; it lines up with the first A to its left, but with the second B to its left. According to Aristotle, Zeno concluded that "double the time is equal to half."

First Position

	A_1	A_2	A_3	
B_1	B_2	B_3		
		C_1	C_2	C_3

Second Position

A_1	A_2	A_3
B_1	B_2	B_3
C_1	C_2	C_3

Figure 6

Some such conclusion would be warranted if we assume that the time it takes for a C to pass to the next B is the same as the time it takes to pass to the next A, but this assumption seems patently false. It appears that Zeno had no appreciation of relative speed, assuming that the speed of C relative to B is the same as the speed of C relative to A. If that were the only foundation for the paradox we would have no reason to be interested in it, except perhaps as a historical curiosity. It turns out, however, that there is an interpretation of this paradox which gives it serious import.

Suppose, as people occasionally do, that space and time are atomistic in character, being composed of space-atoms and time-atoms of non-zero size, rather than being composed of points and instants whose size is zero.[3] Under these circumstances, motion would consist in taking up different discrete locations at different discrete instants. Now, if we suppose that the As are not moving, but the Bs move to the right at the rate of one place per instant while the Cs move to the left at the same speed, some of the Cs get past some of the Bs without ever passing them. C_1 begins at the right of B_2 and it ends up at the left of B_2, but there is no instance at which it lines up with B_2; consequently, there is no time at which they pass each other – it never happens.

It has been suggested that Zeno's arguments fit into an overall pattern.[4] Achilles and the Tortoise and the Dichotomy are designed to refute the doctrine that space and time are continuous, while the Arrow and the Stadium are intended to refute the view that space and time have an atomic structure. The paradox of plurality [not discussed here], also fits into the total schema. Thus, it has been argued, Zeno tries to cut off all possible avenues to escape from the conclusion that space, time, and motion are not real but illusory.

It is extremely tempting to suppose, at first glance, that the first three of these paradoxes at least arise from understandable confusions on Zeno's part about concepts of the infinitesimal calculus. It was in this spirit that the American philosopher C. S. Peirce, writing early in the twentieth century, said of Achilles that "this ridiculous little catch presents no difficulty at all to a mind adequately trained in mathematics and logic."[5] There is no reason to think he regarded any of Zeno's other paradoxes more highly.

We should begin by noting that, although the calculus was developed in the seventeenth century, its foundations were beset with very serious logical difficulties until the nineteenth century – when Cauchy clarified such fundamental concepts as functions, limits, convergence of sequences and series, the derivative, and the integral; and when his successors Dedekind, Weierstrass, et al., provided a satisfactory analysis of the real number system and its connections with the calculus. I am firmly convinced that Zeno's various paradoxes constituted insuperable difficulties for the calculus in its pre-nineteenth-century form, but that the nineteenth-century achievements regarding the foundations of the calculus provide means which go far toward the resolution of Zeno's paradoxes. Let us see what light these purified concepts can throw on the paradoxes of motion.[6]

The Sum of an Infinite Series

It is hard to guess how deep or subtle Zeno's actual reasoning was; experts differ on the point.[7] It may have been that Zeno's original version of Achilles and the Tortoise involved the following sort of argument: since Achilles must traverse an infinite number of distances, each greater than zero, in order to catch up with the tortoise, he can never do so, for such a process would take an infinite amount of time. Against this form of the argument Aristotle quite appropriately pointed out that the time span during which Achilles chases after the tortoise can likewise be subdivided into infinitely many non-zero intervals, so Achilles has infinitely many non-zero time intervals in which to traverse the infinitely many non-zero space intervals. But this response can hardly be adequate, for the question still remains: how can infinitely many positive intervals of time OR space add up to anything less than infinity? The answer to this question was not provided until Cauchy offered a satisfactory treatment of convergent series in the first half of the nineteenth century.

The first concept we need is the *limit* of an infinite sequence. An infinite sequence is simply an ordered set of terms $\{S_n\}$ which correspond in a one-to-one fashion with the positive integers – each term of the sequence being coordinated by the subscript n to a positive integer. The sequence is said to be *convergent* if it has a limit. To say that such a sequence has a limit means that there is some number L (the limit) such that the terms of the sequence become and remain arbitrarily close to that value as we run through the successive terms. More precisely, for any number ϵ greater than 0, there is some positive integer N such that for every term S_n with $n > N$, the difference between S_n and L is less than ϵ. In the sequence

$$1/2, 1/4, 1/8, \ldots, 1/2n, \ldots$$

the limit is 0, since the difference between the terms of the sequence and 0 is arbitrarily small for sufficiently large values of n. If, for example, we choose $\epsilon = 1/10$, by the time we reach the fourth term $S_4 = 1/16$, the difference between that term and L (= 0) is less than 1/10, and the difference remains less than 1/10 for every subsequent member of the sequence. For $\epsilon = 1/100$, $|S_n - 0|$ is less than ϵ for $n = 7$, and the difference remains less than 1/100 for every subsequent term. Similarly, ϵ may be chosen as small as we like, say 1/1,000,000 or 1/1,000,000,000, provided it is greater than zero, and there is some point in this sequence beyond which all remaining terms differ from L by less than ϵ. It is easy to show, by completely parallel reasoning, that the sequence

$$1/2, 3/4, 7/8, \ldots, 1 - 1/2n, \ldots$$

converges to the limiting value of 1.

After the concept of the limit of a sequence has been defined, it can be used to define the sum of an infinite *series*. An infinite series is simply an infinite sequence of terms which are related to one another by addition; for example,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2n}, + \ldots$$

Such a sum is not defined in elementary arithmetic, for ordinary addition is restricted to sums of finite numbers of terms, but this operation can be extended very naturally to an infinite series. In order to define the sum of an infinite *series*

$$s_1 + s_2 + s_3 + \ldots$$

we form the *sequence* of partial sums,

$$S_1 = s_1$$
$$S_2 = s_1 + s_2$$
$$S_3 = s_1 + s_2 + s_3$$
etc.

Each of these partial sums is a sum with a finite number of terms, and it involves only the familiar operation of addition from elementary arithmetic. We have already defined the limit of an infinite sequence. If the *sequence* of partial sums,

$$S_1, S_2, S_3, \ldots$$

has a limit, we say that the infinite *series*

$$s_1 + s_2 + s_3 + \ldots$$

is convergent, and we define its sum as the limit of the sequence of partial sums. This amounts to saying, intuitively, that the sum of a convergent infinite series is a number that can be approximated arbitrarily closely by adding up a sufficient (finite!) number of terms. Given this definition of the sum of an infinite series, it becomes perfectly meaningful to say that the infinitely many terms of a convergent series have a finite sum.

Both the first form of the Dichotomy and the Achilles paradoxes present us with infinite series to be summed. In the Dichotomy, for instance, it is shown that the runner, to cover a racecourse that is one mile in length, must cover the following series of non-overlapping distances:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

Each term of this series is greater than zero. We form the sequence of partial sums

$$S_1 = \frac{1}{2}$$
$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$
etc.

As we noted above, this sequence converges to the limit 1; that is the sum of this convergent infinite series. Achilles and the Tortoise is quite analogous. If Achilles can run twice as fast as the tortoise, and the tortoise has a head start of one-half of the course, the infinite series generated by Achilles running to each subsequent starting point of the tortoise is precisely the one we have just summed. To whatever extent these paradoxes raised problems about the intelligibility of adding up infinitely many positive terms, the nineteenth-century theory of convergent sequences and series resolved the problem.

Instantaneous Velocity

An initial reaction to the paradox of the Arrow might be the suspicion that it hinges on a confusion between the concepts of instantaneous motion and instantaneous rest. Perhaps Zeno did feel that the only way for an arrow to be at a particular place was to be at rest – that the notion of instantaneous non-zero velocity was illegitimate. If Zeno argued – we have no way of knowing whether he did or not – that at every moment of its flight the arrow is at some place in its trajectory, and hence at every moment of its flight it has velocity zero, then he would have been correct in concluding that its velocity during the whole course of its flight would be zero, rendering the arrow motionless. Nineteenth-century mathematics showed, however, that one of these assumptions is incorrect. It is entirely intelligible to attribute non-zero instantaneous velocities to moving objects when an instantaneous velocity is understood as a derivative – namely, the rate of change of position with respect to time. This derivative is defined as the limit of the average velocity during decreasing non-zero intervals of time. Suppose, for example, that the arrow flies at a uniform speed. We find that in one second it covers ten feet, in one-tenth of a second it covers one foot, in one-hundredth of a second it covers one-tenth of a foot, and so on. As we take these *average* velocities over decreasing finite time intervals which converge to an instant t_1, the average velocities approach a limit of ten feet per second, and this is, by definition, the instantaneous velocity of the arrow at t_1. The same can be said for every moment during its flight; it travels its whole course at ten feet per second, and its velocity at each moment is ten feet per second. If Zeno felt that the only intelligible instantaneous velocity is zero, nineteenth-century mathematics proved him wrong.

The infinitesimal calculus was, of course, developed in the seventeenth century, and it made use of instantaneous velocities. These were, unfortunately, considered to be infinitesimal distances covered in infinitesimal times. It was against such notions that Berkeley leveled his broadside in *The Analyst*,[8] characterizing infinitesimals as "ghosts of recently departed quantities." It is possible that Zeno's Arrow paradox was also directed against just such a conception. If we try to conceive of finite motion over a finite distance during a finite time as being composed of a large number of motions over infinitesimal distances during infinitesimal times, enormous confusion is likely to ensue. How much space does an arrow occupy during an infinitesimal time? Is it just as large as the arrow, or is it a wee bit larger? If it is larger, then how does the arrow get from one part of that space to another? And if not, then how can the arrow be moving at all? And how long is an infinitesimal time span? Does it have parts or not? If so, how can we characterize motion during its parts? If not, how can motion occur during this infinitesimal time? These are questions that Zeno and his fellow Greeks could not answer, and to which modern calculus prior to Cauchy had no satisfactory answer either. This is why I remarked earlier that nineteenth-century – not seventeenth-century – mathematics held an important key, in the concept of the derivative, to the resolution of Zeno's Arrow paradox.

Mathematical Functions

There is, however, still an underlying problem about instantaneous velocity. We have seen how such a concept can be defined intelligibly, but this definition makes essential reference to what is happening at neighboring instants. Instantaneous velocity is defined as a limit of a sequence of average velocities over finite time intervals; without some information about what happens in these intervals we can say nothing about the instantaneous velocity. If we know simply that the center of the arrow was at the point s_1 at time t_1 we can draw no conclusion whatever about its velocity at that instant. Unless we know what the arrow was doing at other times close to t_1 we *cannot* distinguish instantaneous motion from instantaneous rest. It was just this consideration, I believe, which led the philosopher Henri Bergson to say that Zeno's Arrow paradox calls attention to the absurd proposition "... that movement is made of immobilities."[9] Bergson concluded that the Arrow paradox proves that the standard mathematical characterization of motion must be wrong. We must look at this argument a little more closely.

In modern physics, motion is treated as a functional relationship between points of space and instants of time. The formula for the motion of a freely falling body, for example, is

$$x = f(t) = \tfrac{1}{2}gt^2.$$

Such formulas make it possible, by employing the function f, to compute the position x given a value of time t. But to understand this treatment of motion fully, it is necessary to have a clear conception of mathematical functions. Before the nineteenth century there was no satisfactory treatment of functions; functions were widely regarded as things which moved or flowed. Such a conception is of no help in attempting to resolve Zeno's paradoxes; on the contrary, Zeno's paradoxes of motion constitute severe difficulties for any such notion of mathematical functions. The situation was dramatically improved when Cauchy defined a function as simply a pairing of numbers from one set with numbers from another set. The numbers of the first set are the ***values of the argument***, sometimes called the ***independent variable***; the numbers of the second set (which need not be a different set) are the ***values of the function***, sometimes called the ***dependent variable***. For example, the function $F(x) = y = x^2$ pairs real numbers with non-negative real numbers. With the number 2 it associates the number 4, with the number −1 it associates the number 1, with the number ½ it associates the number ¼, and so forth. Now according to Cauchy, the mathematical function F simply *is* the set of all such pairs of numbers [namely, that shown in Table 1].

Table 1

x	$F(x) = x^2$
1	1
2	4
3	9
½	¼
⅓	1/9
−2	4
−1	1
etc.	etc.

Similarly, the function f used to describe the motion of a falling body is nothing more or less than a pairing of the values of the position variable x with values of the time variable t. At $t = 0$, $x = 0$; at $t = 1$, $x = 16$; at $t = 2$, $x = 64$. This is how we say, in mathematical language, that a body starting from rest in the vicinity of the surface of the earth and falling freely travels 16 feet in the first second, 48 feet in the next second, and so on.

Let us now apply this conception of a mathematical function to the motion of an arrow; to keep the arithmetic simple, let it travel at the uniform speed of ten feet per second in a straight line, starting from $x = 0$ at $t = 0$. At any subsequent time t, its position $x = 10t$. Accordingly, part of what we mean by saying that the arrow moved from point A ($x = 10$) to point B ($x = 30$) is simply that it was ***at*** A when $t = 1$, and it was ***at*** B when $t = 3$. When we ask how it got from A to B, the answer is that it occupied each of the intervening points x ($10 < x < 30$) at suitable times t ($1 < t < 3$) – that is, satisfying the equation $x = 10t$. For example, when $t = 2$, the arrow was at the point C ($x = 20$). When we ask how it got from A to C, the answer is again: by occupying the intervening positions at suitable times. Notice that this answer is ***not***: by zipping through the intervening points at ten feet per second. The requirement is that the arrow be ***at*** the appropriate point ***at*** the appropriate time – nothing is said about the instantaneous velocity of the arrow as it occupies each of these points. This approach has been appropriately dubbed "the at-at theory of motion." Once the motion has been described by a mathematical function that associates positions with times, it is then possible to differentiate the function and find its derivative, which in turn provides the instantaneous velocities for each moment of travel. But the motion itself is described by the pairing of positions with times alone. Thus, Russell was led to remark, "Weierstrass, by strictly banishing all infinitesimals, has at last shown that we live in an unchanging world, and that the arrow, at every moment of its flight, is truly at rest. The only point where

Zeno probably erred was in inferring (if he did infer) that, because there is no change, therefore the world must be in the same state at one time as at another. This consequence by no means follows. . . ."[10]

What Russell is saying is basically sound, although he does perhaps phrase it overdramatically. It is not that the arrow is "truly at rest" during its flight; rather, the motion consists in being *at* a particular point *at* a particular time, and regarding each individual position at each particular moment, there is no distinction between being at rest at the point and being in motion at the point. The distinction between rest and motion arises only when we consider the positions of the body at a number of different moments. This means that, aside from *being at* the appropriate places at the appropriate times, there is no *additional* process of *moving* from one to another. In this sense, there is no absurdity at all in supposing motion to be composed of immobilities.[11]

Although this way of viewing motion is, I believe, logically impeccable, it may be psychologically difficult to accept. Perhaps the problem can best be seen in connection with the regressive form of the Dichotomy paradox. Here we have Achilles at the starting point at the very moment at which the race begins. What, we ask, must he do first? Well, someone might say, first he has to run to the starting point of the tortoise. But that answer cannot be correct, for before he can do that, he must run to a point halfway between his and the tortoise's respective starting points. Before he can do that, however, he must get to a point halfway to the halfway point. And so on. We are off on the infinite regress. It seems that there is no first thing for him to do; whatever we suppose his first task to be, there is another that must be completed before he can finish it. There is, in other words, no first interval for him to cross. This conclusion is true. But it does *not* follow that Achilles cannot get started.

Consider the arrow once more. Suppose it is at point *C* midway in its flight path. When we ask how it gets from *C* to *B* we may be wondering, consciously or unconsciously, where it goes next – how it gets to the next point. But this question is surely illegitimate, for we are thinking of the arrow's path as a continuous one. Since the points in a continuum are densely ordered, there is no next point. Between any two distinct points there is another (and, hence, infinitely many). The question about Achilles, which we just considered in connection with the regressive Dichotomy, may arise from the same psychological source. We may feel that his first act must be to get to the point next to his starting point, but no such point exists. According to the at-at theory of motion, this fact is no obstacle to motion. Both space and time are regarded as continuous, and hence, densely ordered. True, there is no next point of space for Achilles to occupy, but also there is no next moment of time in which he must do so. For each moment of time there is a corresponding point, and for each spatial point there is a corresponding moment; nothing more is required.

The psychological compulsion to demand a next point or a next moment may arise from the fact that we do not experience time as a continuum of instants without duration, but rather, as a discrete series of specious presents, each of which lasts perhaps a few milliseconds. Aside from anthropomorphism, however, there is no reason to try to impose the discrete structure of psychological time upon the mathematical notion of time as a continuum, since the continuous conception has proved itself such an extremely fruitful tool for the description of physical motion.[12]

Limits of Functions

There is one final issue, arising out of the paradoxes of motion, that was significantly clarified by nineteenth-century foundations of mathematics. During the preceding two centuries, while the calculus floated on vague spatial and temporal intuitions, there was considerable controversy about the ability of a function to reach its limit. Some functions seemed to do so; others did not. It was all quite baffling. This puzzle relates directly to Zeno's paradoxes of Achilles and the Tortoise and the progressive form of the Dichotomy. Achilles seems capable of chasing the tortoise right up to the point of overtaking him, but can he reach that limiting point? Likewise, on the track by himself, Achilles seems capable of traversing the various fractional parts of the course right up to the finish line, but can he achieve that limit? Again, the definitions of functions and limits provided in the nineteenth century come nicely to the rescue. A limit is simply a number. A function is simply a pairing of two sets of numbers. If the limit happens to be one of the numbers in the set of values of the function, then the function does assume the limiting value for some value of its argument variable. If not, then the function never assumes the limiting value. No further question about the ability of a function to "reach" its limit can properly arise.

There can be no serious doubt that the aforementioned nineteenth-century mathematical developments went a long way in resolving the problems Zeno raised about space, time, and motion. The only question is whether there are any remaining problems associated with the paradoxes of motion. Beginning about 1950, a number of mathematically sophisticated writers, who were fully aware of the foregoing considerations, felt that an important problem still remained. One of the most articulate was Max Black, who argued that the analysis of Achilles' attempt to catch the tortoise into an infinite sequence of distinct runs introduces a severe logical difficulty.[13] The problem, specifically, is whether it even makes sense to suppose that anyone has completed an infinite sequence of runs. Black puts the matter forcefully and succinctly when he says that the mathematical operation of summing an infinite series will tell us where and when Achilles will catch the tortoise if he can catch the tortoise at all, but that is a big "if." There is, Black argues, a fundamental difficulty in supposing that he can catch the tortoise, for, he maintains, "the expression, 'infinite series of acts,' is self-contradictory."[14]

Black's argument is based upon consideration of a number of imaginary machines that transfer balls from one tray to another.[15] Suppose, for instance, that there are two machines, Hal and Pal, each equipped with a tray in front. When a ball is placed in Hal's tray, he moves it to Pal's tray; when a ball is placed in Pal's tray, he moves it to Hal's tray. They have a sort of friendly rivalry about getting rid of the balls. Suppose, further, that they are programmed in such a way that each successive transfer of the ball takes a shorter time; in particular, when the ball is first put into either tray, the machine takes ½ minute to move it to the other tray, next time it takes ¼ minute, next time ⅛ minute, and so forth. (Actually, it is more like a frantic compulsion to get rid of the ball; they carry the maxim "It is more blessed to give than to receive" to a ridiculous extreme.) We begin by putting a ball in Hal's tray, and he takes ½ minute to move it to Pal's tray. Pal then takes ½ minute to put it back in Hal's tray, during which time Hal is resting. Then Hal takes ¼ minute to transfer it to Pal's tray, while Pal is resting; in the next ¼ minute Pal returns it to Hal's tray while Hal rests. As the process goes on, the pace increases until we see just a blur, but at the end of two minutes it is over, and both machines come to rest. The ball has been transfered infinitely many times; in fact, each machine has made infinitely many transfers (and enjoyed infinitely many rest periods) during the two minutes.

Now, we must ask, where is the ball? Is it in Hal's tray? No, it cannot be in Hal's tray, because every time it was put in, Hal removed it. Is it in Pal's tray? No, because every time it was put there Pal removed it. Black concludes that the supposition that this infinite sequence of tasks has been completed leads to an absurdity.

Another hypothetical infinity machine – perhaps the simplest – is the Thomson lamp.[16] This lamp is of a common variety; it has a single push-button switch on its base. If the lamp is off and you push the switch, the lamp turns on; if the lamp is on and you push the switch, the lamp turns off. Now suppose that someone pushes the switch an infinite number of times; he accomplishes this by completing the first thrust in ½ minute, the second in ¼ minute, the third in ⅛ minute, much as the runner in the Dichotomy is supposed to cover the infinite sequence of distances in decreasing times. Consider the final state of the lamp after the infinite sequence of switchings. Is the lamp on or off? It cannot be on, for each time it was on it was switched off. It cannot be off, for each time it was off it was switched on.

The speed of switching demanded is, of course, beyond human capability, but we are concerned with logical possibilities, not "medical" limitations. Moreover, there are mechanical difficulties inherent in the speed required of Hal and Pal as well as Thomson's lamp, but we are not concerned with problems of engineering. Further, there is no use trying to evade the question by saying that the bulb would burn out or the switch would wear out. Even if we could cover such eventualities by technological advances, there remains a logical problem in supposing that an infinite sequence of switching (or ball transfers) has been achieved. The lamp must be both on and off, and also, neither on nor off. This is a thoroughly unsatisfactory state of affairs.

Black and Thomson are *not* maintaining that Achilles cannot overtake the tortoise and finish the race. We all know that he can, and to argue otherwise would be silly. Black is arguing that it is incorrect to *describe* either feat as "completing an infinite sequence of tasks," and Thomson draws a similar moral. They are suggesting that the paradoxes arise because of a misdescription of the situation.

These authors have focused upon a fundamental point. We must begin by realizing that no definition, by itself, can provide the answer to a *physical* problem. Take the simplest possible case, the familiar definition of arithmetical addition of two terms. We find, *by experience*, that it applies in some situations and not in others. If we have m apples in one basket and n oranges in another, then we will have $m + n$ pieces of fruit if we put them together in the same container. (Popular folklore notwithstanding, we obviously can "add" apples and oranges.) However, as is well known, if we have m quarts of alcohol in one bucket, and n quarts of water in another, we will not have $m + n$ quarts of solution if we put them together in the same container. The situation is simply another instance of the relation between pure and applied mathematics discussed in the preceding chapter [not included here]. We can define various mathematical operations within pure mathematics, but that is no guarantee of their applicability to the physical world. If such operations are to be applied in the description of physical facts we must determine empirically whether a given physical operation is an admissible interpretation of a given mathematical operation. We have just seen that the combining of apples and oranges in fruit baskets is a suitable counterpart of arithmetical addition, while the mixing of alcohol and water is not. A more significant example occurs in Einstein's special theory of relativity, where composition of velocities is seen not to be a physical counterpart of standard vector addition.

The same sort of question arises when we consider applying the (now standard) definition of the sum of an infinite series. Does a given physical situation correspond to a particular mathematical operation, in this case, the operation of summing an infinite series? Black concludes that the running of a race does not correspond to the summing of an infinite series, for the completion of an infinite sequence of tasks is a logical impossibility. Thus, the running of a race cannot correctly be described as completing an infinite sequence of tasks. This conclusion has far-reaching implications for modern science. If it is right, the usual scientific description of the racecourse as an infinitely divisible mathematical continuum is fundamentally incorrect. It may be a useful idealization for some purposes, but Zeno's paradoxes show that the description cannot be literally correct. The inescapable consequence of this view would seem to be that mathematical physics needs a radically different mathematical foundation if it is to deal adequately with physical reality.

Before accepting any such result, we must examine the infinity machines more closely. They do involve difficulties, but Black and Thomson have not identified them accurately. Consider Thomson's lamp. (The same considerations will apply to Black's infinity machines or any of the others.) Thomson has described a physical switching process that occupies one minute. Given that we begin at t_0 with the lamp off, and given that a switching occurs at $t_1 = \frac{1}{2}$, $t_2 = \frac{3}{4}$, and so on, we have a description that tells, for any moment *prior to* the time $T = 1$ (that is, one minute after t_0), whether the lamp is on or off. For $T = 1$, and subsequent times, it tells us nothing. For any time *prior to* T that the lamp is on, there is a subsequent time *prior to* T that the lamp is off, and conversely. But this does not imply that the lamp is both on and off at T; we can make any supposition we like without logical conflict. We have, in effect, a function defined over a half-open interval $0 \leq t < 1$, and we are asked to infer its value at $t = 1$. Obviously, there is no definite answer to such a question. If the function approached a limit at $t = 1$, it would be natural to extend the definition of the function by making that limit the value of the function at the end point. But the "switching function" describing Thomson's lamp has no such limit, so any extension we might choose would seem arbitrary.[17] The same goes for the position of the ball Hal and Pal pass back and forth. In the Dichotomy and the Achilles paradoxes, by contrast, the "motion function" of the runner does approach a limit, and this limit provides a suitably appealing answer to the question about the location of the runner at the conclusion of his sequence of runs.[18]

One cannot escape the feeling, however, that there are significant and as yet unmentioned differences between the infinite sequence of runs Achilles must make to catch the tortoise and the infinite sequence of ball transfers executed by Black's machines (or the infinite sequence of switch pushes required by the Thomson lamp). And there is at least one absolutely crucial difference. Consider the motion of the ball as it is passed back and forth between Hal and Pal. Say that the trays are three inches apart. Then the ball is made to traverse this *fixed* positive distance infinitely many times. In order to do so, it must travel an *infinite* distance in a finite length of time. Now, no one is interested in showing that Achilles can run an infinite distance in a finite amount of time – he is fast, but not that fast. The problem is to show how he can run a *finite* distance that can be subdivided into an infinite number of subintervals.

Achilles can make his run if he can achieve a fixed positive velocity; the ball which travels back and forth over the fixed distance between Hal and Pal must achieve velocities that increase without any bound. This difficulty could, of course, be repaired. Suppose we stipulate that the distances covered by the ball, like the distances Achilles must cover, decrease as the time available for each transit decreases. This can be done by making the trays of Hal and Pal move closer and closer during the two-minute interval, so that they coincide in the middle at the end of the infinite sequence of transfers. But now there is no problem at all about the position of the ball at the end – it is right in the middle in both trays! Similar considerations apply to the Thomson lamp. In order to accomplish a switching, the button must be moved a certain finite distance, say ⅛ inch. If this is done infinitely many times, the finger which pushes the button and the button itself must traverse an infinite total distance. A necessary, though not sufficient, condition for the convergence of an infinite series is that the terms converge to zero. In order to overcome this difficulty, the switch would have to be modified in some suitable way, in which case an answer can be given to the question regarding the final on–off state of the lamp.[19]

In the literature on Zeno's paradoxes of motion, especially that concerned with the infinity machines, a good deal of emphasis has been placed on the question of whether Achilles can be said to perform an infinite series of *distinct* tasks. When we divide up the racecourse into an infinite series of positive subintervals, it is often claimed, we are artificially breaking up what is properly considered one motion into infinitely many parts which – so the allegation goes – cannot be considered as individual tasks. In order to clarify this question, Adolf Grünbaum has given Achilles a fictitious twin – a doppelgänger – who runs a parallel racecourse, starting and finishing at the same time as the original Achilles.[20]

The new Achilles is a jerky runner. He starts out and runs the first half of the course twice as fast as his counterpart, and then stops and waits for him. When the slower one reaches the midpoint, the interloper runs twice as fast to the three-quarter mark, and again waits for the slower to catch up. He repeats the same performance for each of the remaining infinite series of subintervals. Grünbaum calls the original Achilles, who runs smoothly from start to finish, the *legato runner*; his new twin, who starts and stops, is called the *staccato runner*. The important facts about the staccato runner are: (1) He reaches the end of the course at the same time as the legato runner; if the original Achilles can run the course, so can the staccato runner. (2) The staccato runner takes a rest of finite (non-zero) duration between each of his infinite succession of runs; hence, there can be no question that he performs an infinite sequence of *distinct* runs. (3) The staccato runner (while he is running) runs at a fixed velocity which is simply twice that of his legato mate, so he is not involved in the kinds of ever-increasing velocities that were required in the unmodified Black and Thomson devices.

There is just one final feature of the staccato Achilles which might be a source of worry. Although he is not required to achieve indefinitely increasing velocities, he is required to do a lot of sudden stopping and starting, shifting instantaneously from velocity zero to velocity $2v$ (where v is the legato runner's velocity) and back again. This clearly involves infinite accelerations – and infinitely many of them. One could reasonably doubt the possibility of this degree of jerkiness. It turns out, however, that even the discontinuity in velocity is not a necessary feature of the staccato runner. The physicist Richard Friedberg has shown, by means of a complicated mathematical function, how to describe the motion of a more sophisticated (and less jerky!) staccato runner who covers *each* of the infinite sequence of subintervals by starting from rest, accelerating continuously to a maximum finite velocity, decelerating smoothly to rest, and remaining at rest for the required interval between runs. This staccato runner executes a motion conforming to a continuous function; his velocity (first derivative) and acceleration (second derivative) are continuous, as are all of the higher time-derivatives as well. Moreover, the peak velocities that occur in the successively shorter runs also decrease, converging to zero as the length of the run also converges to zero.[21] It is hard to see what kind of logical (or conceptual) objection can be raised against this kind of motion. But if the sophisticated staccato runner's series of tasks is feasible, so would be the motions of any of the appropriately modified infinity machines. The motion of the ball passed between Hal and Pal, for example, could be described by a combination of two such functions – the first would describe a sequence of motions from left to right with interspersed periods of rest; the second would consist of a similar sequence, but with the motions from right to left. The second set of motions would be executed during the periods of rest granted by the first function, and the first set of motions would occur during the rest periods granted by the second function.

It therefore appears that a suitably designed Hal–Pal pair of infinity machines are logically possible if the *legato* Achilles – the one we all granted from the beginning – can complete his ordinary garden-variety run.

The Discrete vs the Continuous

The infinitesimal calculus has long been – and still is – the basic mathematical tool in the description of physical reality. It employs variables that range over continuous sets of values, and the functions it deals with are continuous. Although the calculus has been completely "arithmetized," so that its *formal* development does not demand any geometrical concepts, it is still applied to phenomena that occur in physical space. Its applicability to spatial occurrences is achieved through analytic geometry, which begins with a one-to-one correspondence between the points on a line and the set of real numbers. The set of real numbers constitutes a continuum in the strict mathematical sense; consequently, the order-preserving one-to-one correspondence between the real numbers and the points of the geometrical line renders the line a continuum as well. If, moreover, the geometrical line is a correct representation of lines in physical space, then physical space is likewise continuous. Motion is treated, moreover, as a function of a continuous time variable, and the function itself is continuous. The continuity of the motion function is essential, for velocity is regarded as the first derivative of such a function, and acceleration as the second derivative. Functions which are not continuous are not differentiable, and hence they do not even have derivatives. Continuity is buried deep in standard mathematical physics. It is for this reason that we have concerned ourselves at length with the problems continuity gives rise to.[22]

A serious objection might be raised, however, to the view that the mathematical continuum provides a precise and literal representation of physical reality. Since physics customarily uses such idealizations as frictionless planes, point-masses, and ideal gases, the argument could go, it might be reasonable to suppose that the mathematical continuum is another idealization that is convenient for some purposes, but does not provide a *completely* accurate description of space, time, and motion. There is, in addition, ample precedent for treating magnitudes that are known to be discrete as if they were continuous. The law of radioactive decay, for example, employs a continuous exponential function even though it is universally acknowledged that the phenomenon it describes involves discrete disintegrations of individual atoms. Where very large finite numbers of entities are involved, the fiction of an infinite collection is often a convenient one which yields good approximations to what actually happens. In electromagnetic theory, for another example, the infinitesimal calculus is used extensively in dealing with charges, even though all the evidence points to the quantization of charges. It has sometimes been suggested that these considerations hold the solution to Zeno's paradoxes. For instance, the physicist P. W. Bridgman has said, "With regard to the paradoxes of Zeno . . . if I literally thought of a line as consisting of an assemblage of points of zero length and of an interval of time as the sum of moments without duration, paradox would then present itself."[23]

Although I am in complete agreement with the claim that physics uses idealizations to excellent advantage, it does not seem to me that this provides any basis for an answer to Zeno's paradoxes of plurality or motion. The first three paradoxes of motion purport to show a priori that motion, if it occurs, must be discontinuous. Indeed, Zeno's intention, as far as we can tell, seems to have been to prove a priori that motion cannot occur. With the exception of a very few metaphysicians of the stripe of F. H. Bradley, most philosophers would admit that the question of whether anything moves must be answered on the basis of empirical evidence, and that the available evidence seems overwhelmingly to support the affirmative answer. Given that motion is a fact of the physical world, it seems to me a further empirical question whether it is continuous or not. It may be a very difficult and highly theoretical question, but I do not think it can be answered a priori. Other philosophers have disagreed. Alfred North Whitehead believed that Zeno's paradoxes support the view that motion is atomistic in character, while Henri Bergson seemed to hold an a priori commitment to the continuity of motion.[24] It seems to me that considerable importance attaches to the analysis of Zeno's paradoxes for just this reason. Space and time may, as some physicists have suggested, be quantized, just as some other parameters, such as charge, are taken to be.[25] If this is so, it must be a conclusion of sophisticated physical investigation of the spatio-temporal structure of the atomic and subatomic domains. A priori arguments, such as Zeno's paradoxes, cannot sustain any such conclusion. The fine structure of space–time is a matter for theoretical physics, not for a priori metaphysics, physicists and philosophers alike notwithstanding. The result of our attempts to resolve Zeno's paradoxes of motion is not a proof that space, time, and motion are continuous; the conclusion is rather that for all we can tell a priori it is an open question whether they are continuous or not.

Before we finally leave Zeno's paradoxes, something should be said about the view of space, time, and motion as discrete quantities. The historical evidence suggests that some of Zeno's arguments were directed against this alternative; that is a plausible interpretation of the Stadium paradox at any rate. Zeno seems to have realized that, if space and time both have discrete structure, there is a standard type of motion that must always occur at a fixed velocity. If, for instance, an arrow is to fly from position *A* to position *B* in as nearly continuous a fashion as is possible in discrete space and time, then it must occupy adjacent space atoms at adjoining atoms of time. In other words, the standard velocity would be one atom of space per atom of time. To travel at a lesser speed, the arrow would have to occupy at least some of the space atoms for more than one time atom; to travel at a greater speed, the arrow would have to skip some of the intervening space atoms entirely, never occupying them in the course of the trip. All of this sounds a bit strange, perhaps, but surely not logically contradictory; this is the way the world might be. Moreover, it is possible, as Zeno's original Stadium paradox shows, for two arrows to pass one another traveling in opposite directions without ever being located next to one another. Imagine two paths, located as close together as possible in our discrete space, between *A* and *B*. Let one arrow travel one of these paths from *A* to *B*, while the other travels the other path from *B* to *A* (see figure 7). Suppose that the arrow traveling the upper track leaves *A* and occupies the first square on the left, while the arrow traveling the lower track leaves *B* at the same (atomic) moment of time, occupying the first square on the right end of his path. Let each arrow move along its track at the rate of one square for each atom of time. At the fourth moment, the upper arrow is just to the left of the lower arrow; at the next moment, the upper arrow is just to the right of the lower arrow. At no moment are they side-by-side – they get past one another, but there is no event which qualifies as the passing (if we mean being located side-by-side traveling in the opposite directions). This is strange perhaps, but again, it is hardly logically impossible.

A	1	2	3	4	5	6	7	8	B
	8	7	6	5	4	3	2	1	

Figure 7

The mathematician Hermann Weyl has, however, posed a basic difficulty for those who would like to quantize space.[26] If we think of a two-dimensional space as being made up of a large number of tiles (something like figure 7), we get into immediate trouble over certain geometrical relations. Suppose for example, that we have a right triangle *ABC* in such a space (see figure 8). Consider, first, the tiles drawn with solid lines. If the positions *A*, *B*, and *C* represent the respective corner tiles, then we see that the side *AB* is four units long, the side *AC* is four units long, and the hypotenuse *BC* is also four units long. The

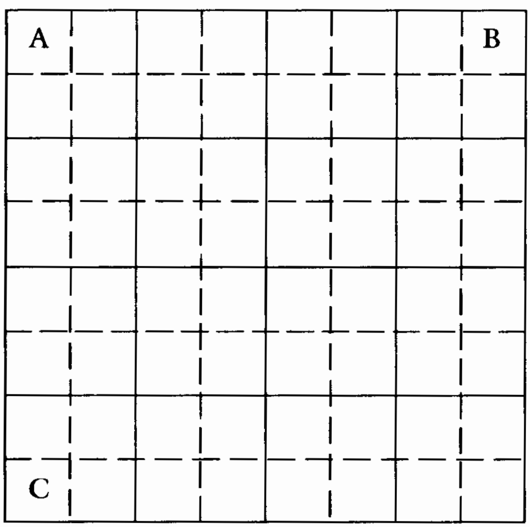

Figure 8

Pythagorean theorem says, however, that the square of the hypotenuse equals the sum of the squares of the other two sides. This means that a right triangle with two legs of four units each should have a hypotenuse about 5⅔ units long. The Pythagorean theorem is at least approximately true in physical space, as we have found by much experience. The result based upon tile-counting does not begin to approximate the correct result.

This example shows something important about approximations. It is easy to see that discontinuous motion in discrete space and time would be difficult to distinguish from continuous motion if our space and time atoms were small enough. It might be tempting to suppose that our geometrical relations would approach the accustomed ones if we make our tiles small enough. This, unfortunately, is not the case, as you can see by taking the finer grid in Figure 8 given by the broken and solid lines together. Instead of 16 tiles, we now have 64 tiles covering the same region of space. But looking at our triangle *ABC* once more, we see that all three sides are now 8 units long. No matter how small we make the squares, the hypotenuse remains equal in length to the other two sides. No wonder this is sometimes called the "Weyl tile" argument![27] This is one case in which transition to very small atoms does not help at all to produce the needed approximation to the obvious features of macroscopic space. It shows the danger of assuming that such approximation will automatically occur as we make the divisions smaller and smaller.

It is important to resist any temptation to account for the difficulty by saying that the diagonal distance across a tile is longer than the breadth or height of a tile, and that we must take that difference into account in ascertaining the length of the hypotenuse of the triangle. Such considerations are certainly appropriate if we are thinking of the tiles as subdivisions of a continuous background space possessing the familiar Euclidean characteristics. But the basic idea behind the tiles in the first place was to do away with continuous space and replace it by discrete space. In discrete space, a space atom constitutes one unit, and that is all there is to it. It cannot be regarded as properly having a shape, for we cannot ascribe sizes to parts of it – it has no parts.

Now, I do not mean to argue that there is no consistent way of describing an atomic space or time. It would be as illegitimate to try to prove the continuity of space and time a priori as it would be to try to prove their discreteness a priori. But, in order to make good on the claim that space and time are genuinely quantized, it would be necessary to provide an adequate geometry based on these concepts. I am not suggesting that this is impossible, but it is no routine mathematical exercise, and I do not know that it has actually been done.[28]

Notes

1 These formulations are taken from Wesley C. Salmon, *Zeno's Paradoxes* (Indianapolis: Bobbs-Merrill 1970), pp. 8–12.

2 Bertrand Russell, *Our Knowledge of the External World* (New York: W. W. Norton, 1929), p. 189.

3 See J. O. Wisdom, "Achilles on a Physical Racecourse," reprinted in Salmon, *Zeno's Paradoxes.*

4 See G. E. L. Owen, "Zeno and the Mathematicians," reprinted in Salmon, *Zeno's Paradoxes.*
5 Charles Hartshorne and Paul Weiss, eds, *The Collected Papers of Charles Sanders Peirce* (Cambridge, Mass.: Harvard University Press, 1935), § 6. 177–184.
6 For an excellent discussion of these developments see Carl B. Boyer, *The History of the Calculus and its Conceptual Development* (New York: Dover Publications, 1959).
7 Zeno's paradoxes pose enormous problems in historical scholarship; for some of the details see Gregory Vlastos, "Zeno of Elea," in the *Encyclopedia of Philosophy*, ed. Paul Edwards (New York: Macmillan and Free Press, 1967).
8 Reprinted in James R. Newman, ed., *The World of Mathematics* (New York: Simon and Schuster, 1956).
9 Henri Bergson, *Creative Evolution*, trans. Arthur Mitchell (New York: Holt, Rinehart and Winston, 1911), relevant passages reprinted in Salmon, *Zeno's Paradoxes*, quotation from p. 63.
10 Bertrand Russell, *The Principles of Mathematics*, 2nd edn (New York: W. W. Norton, 1943), p. 347.
11 The contrary view, that this is indeed an absurdity, is based upon the elementary fallacy of composition. This is the only non-trivial, non-artifical instance of this fallacy I have ever encountered.
12 For detailed and enlightening discussions of the relations between "physical time" and "psychological time," see Adolf Grünbaum, "Relativity and the Atomicity of Becoming," *Review of Metaphysics*, IV (1950–1), pp. 143–86.
13 Max Black, "Achilles and the Tortoise," *Analysis*, XI (1950–1), pp. 91–101; reprinted in Salmon, *Zeno's Paradoxes.*
14 Ibid., p. 72 in Salmon.
15 The idea of an infinity machine was first suggested by Hermann Weyl, *Philosophy of Mathematics and Natural Science* (Princeton, N.J.: Princeton University Press, 1949). See Salmon, *Zeno's Paradoxes*, p. 201, for relevant quotation.
16 James Thomson, "Tasks and Super-Tasks," *Analysis*, XV (1954–5), pp. 1–13; reprinted in Salmon, *Zeno's Paradoxes.*
17 The "switching function" may be defined as follows: let 1 represent the "on-state" of the lamp, and let 0 represent the "off-state." This function has a determinate value for each value of $t < 1$, but it fluctuates infinitely often between 0 and 1 in any neighborhood of $t = 1$; hence, it has no limit at $t = 1$.
18 The arguments of this paragraph were given by Paul Benacerraf, "Tasks, Super-Tasks, and the Modern Eleatics," *Journal of Philosophy*, LIX (1962), pp. 765–84; reprinted in Salmon, *Zeno's Paradoxes.*
19 This analysis of infinity machines and their modifications is due to Adolf Grünbaum, "Modern Science and Zeno's Paradoxes of Motion," Part II, in Salmon, *Zeno's Paradoxes*, pp. 218–44.
20 Ibid., Part I, "The Zenonian Runners," pp. 204–18.
21 See Salmon, *Zeno's Paradoxes*, pp. 215–16, for the details.
22 A continuous function is, intuitively, one that can be plotted by means of a line that has no gaps in it – one that can be drawn without lifting the pencil from the paper on which the function is being plotted. For a respectable mathematical treatment of the concept of continuity, in terms requiring no previous mathematical training beyond high school, see Richard Courant and Herbert Robbins, *What is Mathematics?* (New York: Oxford University Press, 1941), Ch. VI.
23 P. W. Bridgman, "Some Implications of Recent Points of View in Physics," *Revue Internationale de Philosophie*, III (1949), p. 490; quoted by Grünbaum, see Salmon,

Zeno's Paradoxes, p. 177.

24 See Salmon, *Zeno's Paradoxes*, pp. 16–20, for discussion of metaphysical interpretations of these paradoxes; see pp. 59–66 for a famous passage from Bergson.

25 See Grünbaum, "Modern Science and Zeno's Paradoxes of Motion," Part III, in Salmon, *Zeno's Paradoxes*, pp. 244–60, for an assessment of the extent to which the quantization of space and time has been accomplished.

26 Weyl, *op. cit.*, p. 43. See Salmon, *Zeno's Paradoxes*, p. 175, for the relevant quotation.

27 "Weyl" is pronounced like "vile."

28 See Peter D. Asquith, *Alternative Mathematics and Their Status*, Ph.D. dissertation, Indiana University, 1970.

16 Grasping the Infinite*

José A. Benardete

Once upon a time, long ago, a great controversy broke out among the Gumquats, plunging that ancient, benighted people into a state of confusion perilously close to civil war. A young hero had arisen to challenge some of the most deeply cherished beliefs of the tribe. With all the truculence of youth and ambition, he insisted that, contrary to received opinion, it must be admitted that there are a definite number of leaves in the jungle, a definite number of fish in the ocean, a definite number of stones in the valley. It was a profound mistake, he argued, to suppose that the stones in the valley were really innumerable, uncountable, numberless, indeed so plentiful as to be quite *without number*. The Gumquats at this time were fortunate enough to possess a decimal system of counting, but they rarely had any occasion to count beyond 100. Ancient records were on hand to prove that the highest that anyone had ever counted, in all the recorded history of the tribe, was to the number 488. This number was popularly regarded with almost sacred awe, it was chanted during the holy festivals, and it was held highly unlikely that anyone would ever count beyond it. It seemed to represent the very limit of human achievement.

To the horror of the old, to the delight of the young, our hero gathered the whole tribe together and undertook to break the spell of superstition under which they languished. In full view of all, he proceeded to count up to 200, then on to 300, 400, and as he moved on to 486, 487, 488! a great hush fell upon the tribe, 489! – the young burst forth with cheers, the old clapped their hands upon their ears, refusing to listen to this transgression. Our hero was unable to reach 500. Spears and rocks were being hurled in all directions. It was

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