2

A POTPOURRI OF QUESTIONS AND ATTEMPTED ANSWERS

THE purpose of this chapter is to sketch the major problems and some of the major positions in the interpretive enterprise of the philosophy of mathematics. What questions must a philosophy of mathematics answer in order to illuminate the place of mathematics in the overall intellectual enterprise—in the ship of Neurath? What sorts of answers have been proposed?

1. Necessity and A Priori Knowledge

A casual survey of the sciences shows that mathematics is involved in many of our best efforts to gain knowledge. Thus, the philosophy of mathematics is, in large part, a branch of epistemology—that part of philosophy that deals with cognition and knowledge. However, mathematics at least appears to be different from other epistemic endeavours and, in particular, from other aspects of the pursuit of science. Basic mathematical propositions do not seem to have the contingency of scientific propositions. Intuitively, there do not have to be nine planets of the sun. There could have been seven, or none. Gravity does not have to obey an inverse—square law, even approximately. In contrast, mathematical propositions, like $7 + 5 = 12$ are sometimes held up as paradigms of *necessary truths*. Things just cannot be otherwise.

The scientist readily admits that her more fundamental theses might be false. This modesty is supported by a history of scientific revolutions, in which long-standing, deeply held beliefs were rejected. Can one seriously maintain the same modesty for mathematics? Can one doubt that the induction principle holds for the natural numbers? Can one doubt that $7 + 5 = 12$? Have there been *mathematical* revolutions that resulted in the rejection of central long-standing mathematical beliefs? On the contrary, mathematical methodology does not seem to be probabilistic in the way that science is. Is there even a coherent notion of the probability of a mathematical statement? At least prima facie, the epistemic basis of the induction principle, or '$7 + 5 = 12$', or the infinity of the prime numbers, is firmer, and different in kind, than that of the principle of gravitation. Unlike science, mathematics proceeds via *proof*. A successful, correct proof eliminates all rational doubt, not just all reasonable doubt. A mathematical demonstration should show that its premisses logically entail its conclusion. It is not possible for the premisses to be true and the conclusion false.

In any case, most thinkers agree that basic mathematical propositions enjoy a high degree of certainty. How can they be false? How can they be doubted by any rational being—short of a general sceptic who holds that everything should be doubted? Mathematics seems essential to any sort of reasoning at all. If, as part of a philosophical thought experiment, we entertain doubts about basic mathematics, is it clear that we can go on to think at all?

The phrase 'a priori' means something like 'prior to experience' or 'independent of experience'. It is an epistemic notion. Define a proposition to be *known a priori* if the knowledge is not based on any 'experience of the specific course of events of the actual world' (Blackburn 1994: 21). One may need experience in order to grasp the concepts involved in the proposition, but no other specific experience with the world. A proposition is known *a posteriori* or *empirically* if it is not known a priori. That is, a proposition is known a posteriori if the knowledge is based on experience of how the world unfolds. A true proposition is itself a priori if it can become known a priori, and a true proposition is a posteriori if it cannot—if experience with the world (beyond what is needed to grasp the concepts) is necessary in order to come to know the proposition.

Typical examples of a posteriori propositions are 'the cat is on the mat' and 'gravity approximately obeys an inverse-square law'. As we shall see (ch. 4, §3; ch. 8, §2), some philosophers hold that there is no a priori knowledge, but for the rest, typical a priori propositions include 'all red objects are coloured' and 'nothing is completely red all over and completely green all over at the same time'. Probably the most-cited examples are the propositions of logic and mathematics, our present focus. Mathematics does not seem to be based on observation in the way that science is. Again, mathematics is based on proof.

It is thus incumbent on any complete philosophy of mathematics to account for the at-least apparent necessity and a priority of mathematics. The straightforward option, perhaps, would be to articulate the notions of necessity and a priority, and then show how they apply to mathematics. Let us call this the 'traditional route'. It follows the maxim that things are as they seem to be. The burden on the traditional route is to show exactly what it is for something to be necessary and a priori knowable. In the present climate, no one can rightfully claim that these notions are sufficiently clear and distinct. If the philosopher is to invoke the twin notions of necessity and a priority, she must say what it is that is being invoked.

There is an important tension in the traditional picture. On that view, mathematics is necessary and knowable a priori, but mathematics has *something* to do with the physical world. As noted, mathematics is essential to the scientific approach to the world, and science is empirical if anything is—rationalism notwithstanding. So how does a priori knowledge of necessary truths figure in ordinary, empirical knowledge-gathering? Immanuel Kant's thesis that arithmetic and geometry are 'synthetic a priori' was a heroic attempt to reconcile these features of mathematics (see ch. 4, §2). According to Kant, mathematics relates to the forms of perception. It concerns the ways that we perceive the material world. Euclidean geometry concerns the forms of spatial intuition, and arithmetic concerns the forms of spatial and temporal intuition. Mathematics is thus necessary because we cannot structure the world in any other way. We *must* perceive the world through these forms of intuition. No other forms are available to us. Mathematical knowledge is a priori since we do not need any particular experience with the world in order to grasp the forms of perceptual intuition.

It is a gross understatement that Kant's views were, and remain, influential, but his views on mathematics were seen to be problematic, almost from the start. The Kantian may be guilty of trading some difficult problems and obscure notions like a priority and necessity for some even more difficult problems concerning intuition. Alberto Coffa (1991) points out that a major item on the agenda of western philosophy throughout the nineteenth century was to account for the (at least) apparent necessity and a priori nature of mathematics, and the applications of mathematics, without invoking Kantian intuition. This agenda item is alive today.

Another option is for the philosopher to argue that mathematical principles are not necessary or a priori knowable, perhaps because no propositions enjoy these honours. Some empiricists find this non-traditional option attractive, rejecting or severely limiting the a priori. Today this view is more popular than ever, mostly in North America under the influence of W. V. O. Quine's naturalism/empiricism (see ch. 1, §3 and ch. 8, §3). One burden on a philosopher who pursues this non-traditional option is to show why it *appears* that mathematics is necessary and a priori. One cannot simply ignore the long-standing belief concerning the special status of mathematics. That is, even if the traditional beliefs are mistaken, there must be something about mathematics that has led so many to believe that it is necessary and a priori knowable.

2. Global Matters: Objects and Objectivity

As noted in the previous chapter, the philosopher of mathematics immediately encounters sweeping issues. What, if anything, is mathematics about? How is mathematics pursued? How do we know mathematics? What is the methodology of mathematics, and to what extent is this methodology reliable? What do mathematical assertions mean? Do we have determinate and unambiguous conceptions of the basic mathematical concepts and ideas? Is mathematical truth bivalent, in the sense that every well-formed and unambiguous sentence is either determinately true or determinately false? What is the proper logic for mathematics? To what extent are the principles of mathematics objective and independent of the mind, language, and social structure of mathematicians? Is every mathematical truth knowable? What is the relation between mathematics and science that makes application possible?

Some of these questions, of course, are not limited to mathematics. From almost the beginning of recorded history, the basic metaphysical problem has been to determine what (if anything) ordinary language, or scientific language, is about, and philosophers have always wondered whether ordinary truth is independent of the human mind. Recently, the proper semantics and logic for ordinary discourse has become an important topic in philosophy, with philosophers venturing into linguistics. As noted in ch. 1, we must learn the lesson of rationalism and be careful when we extend conclusions concerning mathematics to the rest of language and the rest of the intellectual enterprise. And vice versa: we must be careful when extending conclusions about ordinary language and science to mathematics.

2.1. Object

One global issue concerns the subject-matter of mathematics. Mathematical discourse has the marks of reference to special kinds of objects, such as numbers, points, functions, and sets. Consider the ancient theorem that for every natural number n, there is a prime number $m > n$. It follows that there is no largest prime number, and so there are infinitely many primes. At least on the surface, this theorem seems to concern *numbers*. What are these things? Are we to take the language of mathematics at face value and conclude that numbers, points, functions, and sets exist? If they do exist, are they independent of the mathematician, her mind, language, and so on? Define *realism in ontology* to be the view that at least some mathematical objects exist objectively, independent of the mathematician.

Realism in ontology stands opposed to views like idealism and nominalism. The *idealist* agrees that mathematical objects exist, but holds that they depend on the (human) mind. He may propose that mathematical objects are constructs arising out of the mental activity of individual mathematicians. This would be a subjective idealism, analogous to a similar view about ordinary physical objects. Strictly speaking, from this perspective every mathematician has his or her own natural numbers, Euclidean plane, and so on. Other idealists take mathematical objects to be part of the mental fabric shared by all humans. Perhaps mathematics concerns the ever-present *possibility* of construction. This is an inter-subjective idealism, of sorts. All idealists agree on the counterfactual that if there were no minds, there would be no mathematical objects. Ontological realists deny the counterfactual, insisting that mathematical objects are independent of the mind.

Nominalism is a more radical denial of the objective existence of mathematical objects. One version holds that mathematical objects are mere linguistic constructions. In ordinary discourse we distinguish a given item, such as the author of this book, from a name of that item. Stewart Shapiro is not the same as 'Stewart Shapiro'. One is a person and the other a pair of words. Some nominalists deny this distinction concerning mathematical objects, suggesting that the number nine, for example, just is the corresponding numeral '9' (or 'nine', 'IX', etc.).[1] This is a variation of a more traditional nominalism concerning so called 'universals', like colours and shapes. That view, popular during the medieval period, has it that only names are universal. There is no more to an object being red than having the word 'red' correctly apply to (a name of) that object.

Today it is more common for a sceptic to deny the existence of mathematical objects than to construct them out of language. This mathematical nihilism is also called 'nominalism' (see ch. 9).

Some philosophers hold that numbers, points, functions, and sets are *properties* or *concepts*, distinguishing those from objects on some metaphysical or semantic grounds. I would classify these philosophers according to what they say about properties or concepts. For example, if such a philosopher holds that properties exist independent of language and the mind—a realism concerning properties—then I would classify her as a realist in ontology concerning mathematics, since she holds that mathematics has a distinctive subject-matter and this subject-matter is independent of the language and mind of the mathematician. Similarly, if a philosopher holds that numbers, say, are concepts and that concepts are

[1] There are ontological issues concerning such linguistic items as numerals. Some philosophers hold that they are abstract, eternal, acausal objects, much like what the ontological realist says about numbers. Numerals in this sense are called *types*. In contrast, numeral *tokens* are physical objects—hunks of ink, burnt toner, etc.—that exemplify the types. Unlike types, tokens are created and destroyed at will. For our nominalist to be an anti-realist in ontology concerning mathematics, she must deny the objective existence of types. This matter recurs several times below.

mental, then he is an idealist concerning mathematics, and if he is a traditional nominalist concerning properties or concepts, then he is a nominalist concerning mathematics.

Realism in ontology does not, by itself, have any ramifications concerning the nature of the postulated mathematical objects (or properties or concepts), beyond the bare thesis that they exist objectively. What are numbers like? How do they relate to more mundane objects like stones and people? Among ontological realists, the most common view is that mathematical objects are acausal, eternal, indestructible, and not part of space-time. After a fashion, mathematical and scientific practice support this, once the existence of mathematical objects is conceded. The scientific literature contains no reference to the location of numbers or to their causal efficacy in natural phenomenon or to how one could go about creating or destroying a number. There is no mention of experiments to detect the presence of numbers or determine their mathematical properties. Such talk would be patently absurd. Realism in ontology is sometimes called 'Platonism', because Plato's Forms are also acausal, eternal, indestructible, and not part of space-time (see ch. 3, §1).

The common versions of realism in ontology nicely account for the *necessity* of mathematics: if the subject-matter of mathematics is as these realists say it is, then the truths of mathematics are independent of anything contingent about the physical universe and anything contingent about the human mind, the community of mathematicians, and so on. So far, so good.

What of a priori knowledge? The connection with Plato might suggest the existence of a quasi-mystical connection between humans and the abstract and detached mathematical realm. This faculty, sometimes called 'mathematical intuition', supposedly leads to knowledge of basic mathematical propositions, such as the axioms of various theories. The analogy is with sense perception, which leads to knowledge of the external world. Kurt Gödel (1964) seems to have something like this in mind with his suggestion that some principles of set theory 'force themselves on us as true' (see ch. 8, §1). Since, presumably, the connection between the mind and the mathematical realm is independent of any sensory experience, the quasi-mystical manoeuvre would make mathematical knowledge a priori *par excellence*. Despite Gödel's authority, however, most contemporary philosophers reject this more or less direct mathematical intuition. The faculty is all but ruled out on the naturalist thesis of the human knower as a physical organism in the natural world (see ch. 1, §3). According to the naturalist, any epistemic faculty claimed by the philosopher must be subject to ordinary, scientific scrutiny. That is, a philosopher/scientist cannot invoke a direct connection between the mind and the mathematical universe until he has found a natural, scientific basis for it. Such a basis seems most unlikely if numbers, points, and so on are as eternal and acausal as the typical realist says they are. How does one go about establishing a link to such objects? So perhaps the Platonist has gone too far with this mind–mathematical connection via mathematical intuition. Sometimes, the 'platonism' of realism in ontology is written with a lower-case 'p', in order to temper the connection to Plato. The typical realist in ontology defends something like a Platonic ontology for mathematics, without a Platonic epistemology.

With the rejection of a quasi-mystical connection, however, the ontological realist is left with a deep epistemic mystery. If mathematical objects are part of a detached, eternal, acausal mathematical realm, how is it possible for humans to gain knowledge of them? It is close to a piece of incorrigible data that we do have at least some mathematical knowledge, whatever this knowledge comes to. If realism in ontology is correct, mathematical knowledge is knowledge of an abstract, acausal mathematical realm. How is this knowledge possible? How can we know anything about the supposedly detached mathematical universe? If our realist is also a naturalist, the challenge is to show how a physical being in a physical universe can come to know anything about abstract objects like numbers, points, and sets.

Let us turn to the anti-realisms. If numbers, for example, are creations of the human mind or are inherent in human thought, as idealists contend, then mathematical knowledge is, in some sense, knowledge of our own minds. Mathematics would be a priori to the extent that this self-knowledge is independent of sensory experience. Similarly, mathematical truths would be necessary to the extent that the structure of human thought is necessary. On views like this, the deeper problem is to square the postulated picture of mathematical objects and mathematical knowledge with the full realm of mathematics as practised. There are infinitely many natural numbers, and even more real numbers than natural numbers. The idealist must square our knowledge of natural and real numbers with the apparent finitude of the mind.

If mathematical objects are constructed out of linguistic items, then mathematical knowledge is knowledge of language. It is not clear what would become of the theses that mathematical truths are necessary and a priori knowable. That would depend on the nominalist's views on language. Mathematical knowledge would be a priori knowable to the extent that our knowledge of language is a priori. Here again, the main problem is one of reconciling the view with the full range of mathematics. Finally, if there are no mathematical objects, as some nominalists contend, then the philosopher must construe mathematical propositions as not involving reference to mathematical objects, or else the nominalist should hold that mathematical propositions are systematically false (and so not necessary) or vacuous. Similarly, our nominalist will have to construe mathematical knowledge in terms other than knowledge of mathematical objects, or else argue that there is no mathematical knowledge (and so no a priori mathematical knowledge) at all.

2.2. Truth

In light of the interpretative nature of philosophy of mathematics, and the trend of analytic philosophy generally, it is natural to turn our attention to the *language* of mathematics. What do mathematical assertions mean? What is their logical form? What is the best semantics for mathematical language? Georg Kreisel is often credited with shifting the focus from the existence of mathematical objects to the *objectivity* of mathematical discourse. Define *realism in truth-value* to be the view that mathematical statements have objective truth-values, independent of the minds, languages, conventions, and so on of mathematicians.

The opposition is *anti-realism in truth-value*, the thesis that if mathematical statements have truth-values at all, these truth-values are dependent on the mathematician. One version of truth-value anti-realism is that unambiguous mathematical statements get their truth-values in virtue of the human mind or in virtue of actual or possible human mental activity. On this view, we *make* some propositions true or false, in the sense that the structure of the human mind is somehow constitutive of mathematical truth. The view here is an idealism in truth-value, of sorts. It does not follow that we *decide* whether a given proposition is true or false, just as an idealist about physical objects holds that we do not decide what perceptions to have.

Part of what it is for mathematical statements to be objective is the possibility that the truth of some sentences is beyond the abilities of humans to know this truth. That is, the realist in truth-value countenances the possibility that there may be unknowable mathematical truths. According to that view, truth is one thing, knowability another. The truth-value anti-realist might take the opposite position, arguing that all mathematical truths are knowable. If, in some sense, mathematical statements get their truth-values in virtue of the mind, then it would be reasonable to contend that no mathematical truth lies beyond the human ability to know: for any mathematical proposition Φ, if Φ is true then at least in principle, Φ can become known.

There is a similar battle-line along the semantic front. The realist in truth-value presumably holds that mathematical language is *bivalent*, in the sense that each unambiguous sentence is either determinately true or determinately false. Bivalence seems to be part and parcel of objectivity (so long as vagueness or ambiguity is not part of the picture). Many anti-realists demur from bivalence, arguing that the mind and/or the world may not determine, of every unambiguous mathematical sentence, whether it is true or false. If, as suggested above, the anti-realist holds that all truths are knowable, then modesty would counsel against bivalence. It is arrogant to think that the human mind is capable of determining, of every unambiguous mathematical sentence, whether it is true or false. Some anti-realists take their view as entailing that classical logic must be replaced by intuitionistic logic, which amounts to a philosophically based demand for revisions in mathematics (see ch. 1, §2 and ch. 7).

A second, more radical version of anti-realism in truth-value is that mathematical assertions lack (non-trivial, non-vacuous) truth-values altogether. Strictly speaking, it would follow that there is no mathematical knowledge either, so long as we agree that 'Φ is known' entails 'Φ is true'. If this anti-realist does not wish to attribute massive error and confusion to the entire mathematical and scientific community, then she needs an account of what passes for mathematical knowledge. If mathematics is not a knowledge-gathering activity, then what is it? Presumably, this radical anti-realist in truth-value agrees that mathematics is a significant and vitally important part of the intellectual enterprise, and so she needs an account of this significance. If good mathematics is not true mathematics (since the sentences do not have non-trivial, non-vacuous truth-values), then what is good mathematics?

There is a prima facie alliance between realism in truth-value and realism in ontology. Realism in truth-value is an attempt to develop a view that mathematics deals with objective features of the world. The straightforward way to interpret the language of mathematics is to take it at face value, and not opt for a global reinterpretation of the discourse. Prima facie, numerals are singular terms, proper names. The linguistic function of singular terms is to denote objects. So, if the language is to be taken literally, then its singular terms denote something. Numerals denote numbers. If non-trivial sentences containing numerals are true, then numbers exist. The truth-value realist further contends that some of the sentences are objectively true—independent of the mathematician. The *ontological* thesis that numbers exist objectively may not directly follow from the semantic thesis of truth-value realism. There may be objective truths about mind-dependent entities. However, the objective existence of mathematical objects is at least suggested by the objective truth of mathematical assertions.

This perspective recapitulates half of a dilemma proposed in Paul Benacerraf's 'Mathmatical Truth' (1973), an article that continues to dominate contemporary discussion in the philosophy of mathematics. One strong desideratum is that mathematical statements should be understood in the same way as ordinary statements, or at least respectable scientific statements. That is, we should try for a uniform semantics that covers ordinary/scientific language as well as mathematical language. If we assume that some sort of realism in truth-value holds for the sciences, then we are led to realism in truth-value for mathematics, and an attempt to understand mathematical assertions at face value—the same way that ordinary scientific assertions are understood. Another motivation for the desideratum comes from the fact that scientific language is thoroughly intertwined with mathematical language. It would be awkward and counter-intuitive to provide separate semantic accounts for mathematical and scientific language, and yet another account of how the discourses interact.

This leads to our two realisms, in ontology and truth-value. According to the two views, mathematicians mean what they say and most of what they say is true. In recent literature on philosophy of mathematics, Gödel (1944, 1964), Penelope Maddy (1990), Michael Resnik (1997), and myself (Shapiro 1997) are thoroughgoing realists, holding both realism in ontology and realism in truth-value (see chs. 8 and 10).

We now approach the other horn of Benacerraf's dilemma. Our realisms come with seemingly intractable epistemological problems. From the realism in ontology, we have the objective existence of mathematical objects. Since mathematical objects seem to be abstract and outside the causal nexus, how can we know anything about them? How can we have any confidence in what the mathematicians say about mathematical objects? This is a prime motivation to seek an alternative to one or other of the realisms. Benacerraf argues that anti-realist philosophies of mathematics have a more tractable line on epistemology, but then the semantic desideratum is in danger. The dilemma, then, is this: the desired continuity between mathematical language and everyday and scientific language suggests the two realisms, but this leaves us with seemingly intractable epistemic problems. We must either solve the problems with realism, give up the continuity between mathematical and everyday discourse, or give up the prevailing semantical accounts of ordinary and scientific language.

There is another close alliance between what I call idealism in ontology and idealism in truth-value. The former contends that numbers, for example, are dependent on the human mind. This at least suggests that mathematical truth is also dependent on the mind. The same goes for the other sorts of anti-realisms. Whatever one says about numbers at least suggests something similar about mathematical truth. On the contemporary scene Hartry Field (1980), Michael Dummett (1973, 1977), and the traditional intuitionists L. E. J. Brouwer and Arend Heyting are thorough-going anti-realists, concerning both ontology and truth-value. Field holds that mathematical objects do not exist and that mathematical propositions have only vacuous truth-values (see ch. 9, §1). The traditional intuitionists are mathematical idealists (see ch. 7, §2).

Despite the natural alliances, a survey of the literature reveals no consensus on any logical connection between the two realist theses or their negations. Perhaps the Benacerraf dilemma leads some to different approaches. Each of the four possible positions is articulated and defended by established and influential philosophers of mathematics.

A relatively common programme today, pursued by Charles Chihara (1990) and Geoffrey Hellman (1989), is realism in truth-value combined with a thorough (nominalist) anti-realism in ontology (see ch. 9, §2, ch. 10, §3). The goal is to account for the objectivity of mathematical discourse without postulating a specifically mathematical ontology. Numbers do not exist (or may not exist), but some of the propositions of arithmetic are objectively true. Of course, these views demand that ordinary mathematical statements should not be understood literally, at face value. Advocates of this perspective suggest alternative interpretations of mathematical discourse, and then hold that, so interpreted, mathematical statements are objectively true or objectively false. I only know of one prominent example of a realist in ontology who is an anti-realist in truth-value, Neil Tennant (1987, 1997, 1997a). He holds, with Frege, that some mathematical objects exist objectively (as a matter of necessity), but he joins Dummett as a *global* truth-value anti-realist, holding that all truths, and not just all mathematical truths, are knowable.

Advocates of these 'mixed' views grasp the first horn of the Benacerraf dilemma, since they entail that mathematical discourse does not have the same semantics as ordinary and scientific discourse (assuming some sort of realism for the latter). Of course, there is no denying the extensive interconnections between the discourses. Hellman, for example, shows how mathematical discourse, properly reinterpreted, does fit in smoothly with scientific discourse, while Tennant (1997) argues that the discourses are complementary in important ways.

3. The Mathematical and the Physical

The interactions between mathematics and science are extensive, going well beyond those few branches sometimes called 'applied mathematics'. The rich and varied roads connecting mathematics and science run in both directions. As Nicolas Goodman (1979: 550)

THINKING ABOUT MATHEMATICS

The Philosophy of Mathematics

STEWART SHAPIRO

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Bangkok Buenos Aires Cape Town Chennai
Dar es Salaam Delhi Hong Kong Istanbul Karachi Kolkata
Kuala Lumpur Madrid Melbourne Mexico City Mumbai Nairobi
São Paulo Shanghai Taipei Tokyo Toronto

Published in the United States
by Oxford University Press Inc., New York

First published 2000

British Library Cataloguing in Publication Data
Data available

Library of Congress Cataloging in Publication Data
Data available

ISBN 0-19-289306-8

5 7 9 10 8 6

Typeset in Dante by
RefineCatch Limited, Bungay, Suffolk
Printed in Great Britain by
Biddles Ltd., King's Lynn, Norfolk