Advanced Topics in Mathematics

T.Srisuma

October 2007

Content and Plan of the Lectures

1. Introduction, why we bother about abstract mathematics

2. Metric Spaces and Continuity *(<2 hours)*

3. Probability Theory *(<1.5 hours)*

4. Correspondences and Fixed Point Theorems *(<1.5 hours)*

Information Sheet

These lectures are designed to:

- introduce some basic concepts of abstract mathematics
- prepare students for a few advanced materials in the Advanced courses
- provide a collection of useful results, which may serve as a useful reference in the future

From these lectures, you are (generally) not expected to:

- be able to understand everythings right away if it is the first time you see such material
- be able to learn a great deal of abstract mathematics by simply attending the lectures

When attending these lectures keep in mind that:

- these sort of things are not necessarily indispensible for graduate studies, but they can be very useful in studying and researching in economics

1 Metric Spaces

A metric space is a pair (X, d), X is a set (whose elements we call points) together with a metric d, (X, d). The metric $d : X \times X \to \mathbb{R}^+$ satisfies

1. $0 \leq d(x, y) < \infty$, and $d(x, y) = 0 \Leftrightarrow x = y$ (finite and positive definiteness).
2. $d(x, y) = d(y, x)$ (symmetry).
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The metric has only a few of the most fundamental properties of a distance function for any pair of points in an abstract set (*without mentioning any algebraic aspects*). The metric d measures the distance between points in M. A notion of a metric space is therefore very general, they only are a collection of elements where the distance between elements are well defined.

Here are some metric spaces,

Example 1 $X = \mathbb{R}^n$, *and* $d(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$

Perhaps less familiar metric spaces are

Example 2 *(Discrete metric space) For some set M with the metric d defined by*

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

Example 3 *Let X be a set of all real sequences, with the metric d defined by*

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

Example 4 $X = C[a, b]$ *is the set of continuous functions defined on* $[a, b]$, *and* $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$

Example 5 *$X = C[a,b]$, and $d(f,g) = \int_a^b |f(x) - g(x)|\, dx$*

Some practice questions,

Exercise 6 *Verify the property of metric spaces of the examples above*

Exercise 7 *Consider a metric space with $X = \mathbb{R}^2$ and $d(x,y) = \sum_{i=1}^{2} |x_i - y_i|$, first show that it is a metric space then sketch $B_1(0)$,ie open ball centred around 0 with radius 1*

Exercise 8 *Show whether $d(x,y) = (x-y)^2$ define a metric on the set $\mathbb{R}$*

Exercise 9 *Let l^∞ denote the set of all bounded real sequences and $d(x,y) = \sup_{i\in\mathbb{N}} |x_i - y_i|$,show (l^∞, d) is a metric space. A subspace of a metric space $\left(Y, \widetilde{d}\right)$ of (X,d) is obtained if we take $Y \subset X$ and define $\widetilde{d} = d|_{Y\times Y}$. $\widetilde{d}$ is called the metric **induced** on Y by d. If A is the set of all sequences with 0's or 1's, what is the induced metric on A*

When there is no confusion with the choice of a metric, we will refer a metric space (X,d) simply as X.

In this abstract space, the notion of limits and continuity can be discussed, we now recall these concepts from the September Course. For the remaining of this section we assume to be working on a metric space (X,d) unless explicitly stated otherwise.

Definition 10 *The sequence (x_n) in X converges to a limit $l \in X$ if for each $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for each $n > N$, $d(x_n, l) < \varepsilon$.*

Example 11 *Show that a limit of a convergent sequence is unique.*

Definition 12 *(y_n) is a subsequence of a sequence (x_n) if for all n, $y_n = x_{k_n}$ for some $k_n \in \mathbb{N}$ such that $1 \leq k_1 \leq k_2 \leq \ldots$*

Example 13 *If (x_n) is a convergent sequence in X to a limit l, every subsequence converges to the same limit, l.*

Next we introduce some notions from (general) Topology.

Definition 14 *The set $S \subset X$ is open if for each $x \in S$, there exists some $\varepsilon > 0$ for which $B_\varepsilon(x) \subset S$.*

Definition 15 *The set $S \subset X$ is closed if its complement $S^c = X \backslash S$ is open.*

Using DeMorgan's law, we can show that any union of open sets remain open and finite intersections of closed sets remain close. Here we give an equivalent alternative definition of closedness defined in terms of sequences.

Definition 16 *The set $S \subset X$ is closed if every sequence (x_n) in S that converges in X, the limit is in S.*

Definition 17 *The set $S \subset X$ is* compact *if every sequence in S has a convergent subsequence in S.*

Example 18 *Is $\mathbb{Q}$ open, closed and/or compact? What about $\mathbb{R}$?*

Theorem 19 (Bolzano Weierstrass) *$S \subset \mathbb{R}^n$ is compact iff it is closed and bounded.*

The "only if" direction ($\Longrightarrow$) is intuitively clear; the "if" direction ($\Longleftarrow$) is more interesting as closed and bounded subsets of more general metric spaces are not necessarily compact. For example, consider the metric space with rational numbers, $\mathbb{Q}$. Is $[3,4] \cap \mathbb{Q}$ closed in $\mathbb{Q}$? It is since its complement is open in $\mathbb{Q}$. But clearly $[3,4] \cap \mathbb{Q}$ is not compact, as the sequence $3, 3.1, 3.14, 3.145, \ldots$ has no convergent subsequence in $[3,4] \cap \mathbb{Q}$.

Here we give 2 equivalent definitions of continuity in metric spaces.

Definition 20 *Let X_1 and X_2 be metric spaces. The function $f : X_1 \to X_2$ is* continuous *if for each $\varepsilon > 0$ and $x \in X_1$ there exists $\delta > 0$ and any $y \in X_1$ such that $d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.*

Definition 21 *Let X_1 and X_2 be metric spaces. The function $f : X_1 \to X_2$ is* continuous *if any open set $S \subset X_2$, $f^{-1}(S) \subset X_1$ is open.*

Where $f^{-1}(\cdot)$ denotes the pre-image of a set, in the example above $f^{-1}(S) = \{x \in X_1 \mid f(x) \in S\}$.

Example 22 *Show that the composition of continuous functions is continuous, i.e. if M, N, and P are metric spaces and $f : M \to N$ and $g : N \to P$ are both continuous, then $g \circ f$ is also continuous.*

Definition 23 *Let X_1 and X_2 be metric spaces. The function $f : X_1 \to X_2$ is* uniformly continuous *if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in X_1$, $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.*

Clearly, uniform continuity implies continuity.

Example 24 *Show x^2 on $\mathbb{R}$ is continuous but not uniformly continuous. What about on $[0,1]$?*

Continuity crops up everywhere in economic theory. One place is in fixed-point theorems that are useful for showing the existence of equilibria. (Intuitively, equilibrium is a point from which no one wants to deviate: people's best response to that point is that point.)

Definition 25 *Given a metric space X and map $f : X \to X$, x is said to be a fixed point of f if $f(x) = x$*

Example 26 *Let $f(x) = \sqrt{x}$ on $\mathbb{R}$. $\{0, 1\}$ are fixed points of f.*

Proposition 27 (Brouwer's Fixed-Point in $\mathbb{R}$) *If $f : [0,1] \to [0,1]$ is continuous, then it has a fixed point, i.e. there exists $p \in [0,1]$ for which $f(p) = p$.*

Proof of Brouwer in $\mathbb{R}$. Define $g(x) = x - f(x)$, and note $g(0) \leq 0 \leq g(1)$. We claim that for some $p \in [0,1]$, $g(p) = 0$. If not, then $[0,1] = \{p \in [0,1] : g(p) > 0\} \cup \{p \in [0,1] : g(p) < 0\}$; since g is continuous, each of these sets is open, and therefore $[0,1]$ is too, which we know to be false. Hence, for some $p \in [0,1]$, $f(p) = p$. ■

Brouwer's Theorem extends the result to the unit ball in R^n, but proving it is much harder.

Proposition 28 *If $f : X \to \mathbb{R}$ is continuous, and $S \subset X$ is compact, then f attains a minimum and a maximum on S.*

Proof. It can be shown that a continuous map of a compact set is also compact, therefore we know $f(A) \subset \mathbb{R}$ is compact. By Heine-Borel, $f(A)$ is closed and bounded and so contains its minimum and maximum. ■

We next briefly mention normed and inner product spaces. Very quick recollection of the concepts of *vector spaces* and associated concepts in linear algebra: Vector space (over a field, say, $\mathbb{R}$) is a non-empty set X closed under two algebraic operations, *vector addition* and *scalar multiplication*. (There should be no confusion so we will denote both the null vector and the zero number by 0). Metric spaces are very general, often we deal with spaces where we would like there to include natural geometric aspects of Euclidean space of length and/or the concept of orthogonality.

A normed space is a pair $(X, \|\cdot\|)$, where X is a vector space and a map $\|\cdot\| : X \longrightarrow \mathbb{R}$, called norm, such that for all $x, y \in X$

1. $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$ (positive definiteness).
2. $\|\alpha x\| = |\alpha| \|x\|$ (absolute homogeneity).
3. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Notice how we can trivially generate a metric on X by defining $d(x, y) = \|x - y\|$, (X, d) is a metric space induced by the normed space $(X, \|\cdot\|)$. Here are some example of normed spaces,

Example 29 $X = \mathbb{R}^n$, *and* $\|x\| = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2}$

Example 30 $X = l^p$ *for some* $p \in \mathbb{N}$ *(a set of infinite real sequences such that* $\left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} < \infty$*) and* $\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$

Example 31 $X = l^\infty$, *and* $\|x\| = \sup_{i \in \mathbb{N}} |x_i|$

Example 32 $X = C[a, b]$, *and* $\|f\| = \sup_{x \in [a,b]} |f(x)|$

Example 33 *$X = C[a,b]$, and $\|f\| = \int_a^b |f(x)|\,dx$*

The notion of the norm generalises the elementary concept of the length of a vector to a more general setting. The purpose to an innder product is to generalises the geometric aspect to more abstract spaces so we can talk about orthogonality and projection.

An inner product space is a pair $(X, \langle\cdot,\cdot\rangle)$, where X is a vector space and a map $\langle\cdot,\cdot\rangle : X \times X \longrightarrow \mathbb{K}$, such that for all $x, w, y \in X$

1. $\langle x, x\rangle \in \mathbb{R}$ and $\langle x, x\rangle \geq 0$
2. $\langle x, x\rangle = 0 \Leftrightarrow x = 0$ (positive definiteness).
3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (conjugate symmetry).
4. $\langle \alpha x + \beta w, y\rangle = \alpha\langle x, y\rangle + \beta\langle w, y\rangle$ (sesquilinearity, linear in first argument and conjugate-linear in the second)

Again, notice how we can trivially generate a norm on X by defining $\|x\| = \sqrt{\langle x, x\rangle}$, $(X, \|\cdot\|)$ is a normed space induced by the inner product space $(X, \langle\cdot,\cdot\rangle)$.

Example 34 *$X = \mathbb{C}^n$, and $\langle x, y\rangle = \displaystyle\sum_{k=1}^{n} x_k \overline{y_k}$*

Example 35 *$X = l^2_{\mathbb{C}}$ (a set of infinite complex sequence such that $\left(\displaystyle\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2} < \infty$) and $\langle x, y\rangle = \displaystyle\sum_{k=1}^{\infty} x_k \overline{y_k}$*

Example 36 *$X = C[a,b]$, and $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$*

2 Probability Theory

Consider a non-empty set Ω.

Definition 37 *A family $\mathcal{F}$ of subsets of Ω is said to be a σ-field if:*

1. $\emptyset \in \mathcal{F}$
2. if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
3. any countable sequence $(A_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{F}$, $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$

Exercise 38 *For a family $\mathcal{F}$ of subsets of Ω, show that if $A, B \in \mathcal{F}$ then*

1. $A \cap B \in \mathcal{F}$
2. $A \backslash B \in \mathcal{F}$

We call such pair $(\Omega, \mathcal{F})$ measurable space. In particular, we are interested in the probability space, defined by the triple $(\Omega, \mathcal{F}, \mathbb{P})$, so need to introduce $\mathbb{P}$.

Definition 39 *A function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure on $(\Omega, \mathcal{F})$ if:*

1. $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$
2. for any sequence of (pairwise) disjoint sets $(A_n)_{n\in\mathbb{N}} \in \mathcal{F}$, $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$

Naturally, we intepret $\mathcal{F}$ as set of events and the probability measure assigns a probability to the event.

Exercise 40 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A, A_i, B, B_i \in \mathcal{F}$ for all i. Show that following hold.*

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
2. for $A \subseteq B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$
3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
4. $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mathbb{P}(\cup_{i=1}^{n} A)$
5. if $A_1 \subseteq A_2 \subseteq ...$ is an increasing nested sequence of sets in $\mathcal{F}$, then $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mathbb{P}(A_i)$

6. define $B_i = A_i^c$ as in 5. and establish an analogous result for an decreasing nested sequence of sets in $\mathcal{F}$

Definition 41 *Borel field on $\mathbb{R}$, denoted by $\mathcal{B}$,is the smallest σ-field containing all half open intervals in $\mathbb{R}$. And Borel sets are elements of $\mathcal{B}$.*

Definition 42 *A function $X : \Omega \to \mathbb{R}^*$ is $\mathcal{F}$-measurable if for any $B \in \mathcal{B}$, $X^{-1}(B) \in \mathcal{F}$*

Exercise 43 *Show that X is $\mathcal{F}$-measurable if and only if $\{\omega \in \Omega | X(\omega) \leq a\} = \{X \leq a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$*

Definition 44 *A random variable is a function $X : \Omega \to \mathbb{R}^*$, which is $\mathcal{F}$-measurable.*

Definition 45 *An event $\{X \in B\}$, for some random variable X and a Borel set B, occurs almost surely (a.s.) if $\mathbb{P}(B) = 1$.*

Therefore, statements in probability theory with respect to a particular probability measure can only be made upto a.s. In a more general measure space, the almost sure notion is replaced by almost everywhere (a.e.).

Example 46 *Consider a probability space $([0, 1), \mathcal{B}, \mathbb{P}_L)$ and a random variable $Y = \mathbf{1}_{\mathbb{Q}}$. Then $Y = 15Y^2$ a.s.*

Definition 47 *The distribution function of a random variable X is defined by*

$$F_X(t) = \mathbb{P}(\{X \leq t\}) \text{ for all } t \in \mathbb{R}$$

Exercise 48 *Consider modelling tossing a fair coin twice. Let's have $\Omega = \{HH, HT, TH, TT\}$ and $\mathcal{F}$ be the set of all subsets of Ω, how many elements does $\mathcal{F}$ contain? Let X be the number of heads. Show that X is $\mathcal{F}$-measurable and use it to find its distribution function.*

Definition 49 *A simple random variable is a function $X : \Omega \to \mathbb{R}$ such that $X = \sum_{i=1}^{m} b_i \mathbf{1}_{B_i}$ for some constants $b_1, ..., b_m$ and sets $B_1, ..., B_m$ in $\mathcal{F}$.*

Definition 50 $\mathbb{E}_0(X)$, *the expectation of a simple non-negative random variable* $X = \sum_{i=1}^{m} b_i \mathbf{1}_{B_i}$ *is*

$$\mathbb{E}_0(X) = \sum_{i=1}^{m} b_i \mathbb{P}(B_i)$$

Definition 51 $\mathbb{E}(X)$ *the expectation of a non-negative random variable* X *is*

$$\mathbb{E}(X) = \sup\{\mathbb{E}_0(Y) \mid Y \leq X,\ Y \text{ simple}\}$$

Note, it can be proven that for any non-negative random variable X, there exists a sequence (X_n) of non-negative simple functions increasing pointwise to X. By monotone convergence theorem, we can now define expectation of any random variable as a Lebesgue integral.

Definition 52 $\mathbb{E}(X)$ *the expectation of a general random variable* X *is*

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}\left(X^+\right) - \mathbb{E}\left(X^-\right) \\ &= \int_\Omega X d\mathbb{P} \end{aligned}$$

where we defined $X^+ = \max\{X, 0\}$ *and* $X^- = \max\{-X, 0\}$

Compare this to the Riemann integrals.

3 Correspondences

Let X and Y be metric spaces and $\mathcal{P}(Y)$ be the set of subsets of Y. A correspondence $f : X \to \mathcal{P}(Y)$; that is, f maps points in X to subsets of Y. For illustrational purposes, we will focus for the case when $X, Y \subset \mathbb{R}$.

Definition 53 *The correspondence $f : X \to \mathcal{P}(Y)$ is* closed valued *if for each $x \in X$, $f(x)$ is closed in Y.*

Definition 54 *The correspondence $f : X \to \mathcal{P}(Y)$ is* compact valued *if for each $x \in X$, $f(x)$ is compact in Y.*

Definition 55 *The correspondence $f : X \to \mathcal{P}(Y)$ is* convex valued *if for each $x \in X$, $f(x)$ is a convex set.*

Definition 56 *The* graph *of the correspondence $f : X \to \mathcal{P}(Y)$ is the set $G = \{(x, y) \in X \times Y : y \in f(x)\}$. f has* closed graph *if G is closed in $X \times Y$.*

The graph of a correspondence is exactly what you would expect it to be.

Now, let's generalise the notion of continuity to correspondences.

Definition 57 *Given $X \subset \mathbb{R}^m$ and a closed set $Y \subset \mathbb{R}^k$, the correspondence $f : X \to \mathcal{P}(Y)$ is* upper hemicontinuous *(uhc) if it has a closed graph and the images of compact sets are bounded.*

In many applications, the range space Y is compact then uhc properties reduce to having a closed graph.

Definition 58 *Given $X \subset \mathbb{R}^m$ and a compact set $Y \subset \mathbb{R}^k$, the correspondence $f : X \to \mathcal{P}(Y)$ is* lower hemicontinuous *(lhc) if for every sequence $x_n \to x \in X$ with $x_n \in X$ for all n, and every $y \in f(x)$, we can find a sequence $y_n \to y$ and an interger N such that $y_n \in f(x_n)$. for $n > N$.*

A correspondence is continuous if it is both uhc and lhc.

Exercise 59 *Analyse the following correspondences*

1. $X = [0, 2]$ and

$$f(x) = \begin{cases} \{1\} & x \in [0,1) \\ [0,2] & x \in [1,2] \end{cases}$$

2. $X = [0, 2]$ and

$$f(x) = \begin{cases} \{1\} & x \in [0,1] \\ [0,2] & x \in (1,2] \end{cases}$$

3. $X = [1, 3]$ and $f(x) = (x - 1, x + 1)$.

Exercise 60 *Show that if a correspondence has closed graph, then it is closed valued. Prove or disprove the converse.*

Theorem 61 (Kakutani's Fixed Point Theorem) *Let $X \subset R^n$ be compact and convex. If $f : X \to P(X)$ is a uhc correspondence that has nonempty, convex, and compact valued, then f has a fixed point (i.e. for some x, $x \in f(x)$).*

Example 62 $X = [0, 2]$ *and*

$$f(x) = \begin{cases} \{1\} & x \in [0,1) \\ \{0\} & x \in [1,2] \end{cases}$$

Example 63 $X = [0, 2]$ *and*

$$f(x) = \begin{cases} \{2\} & x \in [0,1) \\ \{0,2\} & x = 1 \\ \{0\} & x \in (1,2] \end{cases}$$

Example 64 $X = [0, 2]$ *and*

$$f(x) = \begin{cases} \{2\} & x \in [0,1) \\ [0,2] & x = 1 \\ \{0\} & x \in (1,2] \end{cases}$$

More generally we can define upper and lower hemicontinuity with abstract metric spaces X and Y. Recall that for a function $f : X \to Y$, f is continuous on X if for any $x \in X$ and for each open $V \subset Y$ for which $f(x) \in V$, there exists an open $U \ni x$ such that $f(U) \subset V$.

Definition 65 *The correspondence* $f : X \to \mathcal{P}(Y)$ *is* upper hemicontinuous *(uhc) at* x *if for each open* $V \subset Y$ *for which* $f(x) \subset V$*, there exists an open* $U \ni x$ *such that* $f(U) \subset V$*.*

The correspondence is uhc if it is uhc at all $x \in X$.

Definition 66 *The correspondence* $f : X \to \mathcal{P}(Y)$ *is* lower hemicontinuous *(lhc) at* x *if for each open* $V \subset Y$ *for which* $f(x) \cap V \neq \varnothing$*, there exists an open* $U \ni x$ *such that for each* $x' \in U$*,* $f(x') \cap V \neq \varnothing$*.*

The correspondence is lhc if it is lhc at all $x \in X$.

It is easy to see from the definition above that if our correspondence is just a function $f : X \to Y$, both the upper hemicontinuous and lower hemicontinuous definitions above are equivalent to the regular continuous definition for a function.

Exercise 67 *Show that a continuous function is continuous when viewed as a correspondence (that is, it is both lhc and uhc).*